The Axiom of Determinacy

Samantha Stanton
Virginia Commonwealth University

Follow this and additional works at: https://scholarscompass.vcu.edu/etd

Part of the Physical Sciences and Mathematics Commons

© The Author

Downloaded from
https://scholarscompass.vcu.edu/etd/2189

This Thesis is brought to you for free and open access by the Graduate School at VCU Scholars Compass. It has been accepted for inclusion in Theses and Dissertations by an authorized administrator of VCU Scholars Compass. For more information, please contact libcompass@vcu.edu.
This is to certify that the thesis prepared by Samantha Stanton titled “The Axiom of Determinacy” has been approved by his or her committee as satisfactory completion of the thesis requirement for the degree of Master of Science.

__________________________________________
Dr. Andrew Lewis, College of Humanities and Sciences

__________________________________________
Dr. Lon Mitchell, College of Humanities and Sciences

__________________________________________
Dr. Robert Gowdy, College of Humanities and Sciences

__________________________________________
Dr. John Berglund, Graduate Chair, Mathematics and Applied Mathematics

__________________________________________
Dr. Robert Holsworth, Dean, College of Humanities and Sciences

__________________________________________
Dr. F. Douglas Boudinot, Graduate Dean

Date
The Axiom of Determinacy

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

Samantha Stanton
Master of Science

Director: Dr. Andrew Lewis, Associate Professor, Department Chair
Department of Mathematics and Applied Mathematics

Virginia Commonwealth University
Richmond, Virginia
May 2010
Acknowledgment

I am most appreciative of Dr. Andrew Lewis. I would like to thank him for his support, patience, and understanding through this entire process. I thank him for sharing his knowledge with me and for guiding me through the understanding, and writing and composing processes that are necessary for a successful thesis. I feel most fortunate to have him as my thesis advisor. I would like to thank Dr. Lon Mitchell for his help and support through the writing and composing of the thesis. I am happy he is one of my committee members and that I was able to learn from him and work with him in this process. I thank him for his help with LaTeX. I would like to thank Dr. Robert Gowdy for being one of my committee members. I appreciate his willingness to work with me not having known me beforehand. I appreciate everyone who has worked with me, giving up their time to help me have a pleasant experience.
## Contents

Abstract \hspace{2cm} iv

1 Introduction: Background Concepts and Definitions \hspace{2cm} 1

2 Baire Space \hspace{2cm} 17

3 Introduction to Games (AD) \hspace{2cm} 20

4 Uncountable Sets and the Continuum Hypothesis: AD and AC \hspace{2cm} 26

5 Property of Baire: AD and AC \hspace{2cm} 31

Bibliography \hspace{2cm} 45

Appendices \hspace{2cm} 46

A Code List 46

Vita 47
Abstract

THE AXIOM OF DETERMINACY

By Samantha Stanton, Master of Science.

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

Virginia Commonwealth University, 2010.

Director: Dr. Andrew Lewis, Associate Professor, Department Chair, Department of Mathematics and Applied Mathematics.

Working within the Zermelo-Frankel Axioms of set theory, we will introduce two important contradictory axioms: Axiom of Choice and Axiom of Determinacy. We will explore perfect polish spaces and games on these spaces to see that the Axiom of Determinacy is inconsistent with the Axiom of Choice. We will see some of the major consequences of accepting the Axiom of Determinacy and how some of these results change when accepting the Axiom of Choice. We will consider 2-player games of perfect information wherein we will see some powerful results having to do with properties of the real numbers. We will use a game to illustrate a weak proof of the continuum hypothesis.
In this section, we will review some topological concepts that will help to prepare us for theorems we will encounter in future sections. We will work under the Zermelo-Frankel Axioms listed below. We will introduce another Axiom in the chapter on Games. These first eight axioms were created in 1908 by Ernest Zermelo and were later refined by Adolf Fraenkel and Thoralf Skolem. The axioms are each strings of logical symbols and formulas.

**Definition 1.1.** An **axiom** is any mathematical statement that serves as a starting point from which other statements are logically derived. Logical Axioms are statements that are taken to be universally true.

**Definition 1.2.** A **set** is a collection of distinct objects called the members (or elements) of a set. A set is considered an object itself.

**Example 1.3.** \( A = \{0, 1, 2, 3\} \). We have \( 0 \in A \), \( 1 \in A \), \( 2 \in A \), and \( 3 \in A \).

The elements of \( A \) then are 0, 1, 2, and 3.

**Example 1.4.** \( B = \{\{0\}, \{1\}, \{2\}\} \). We have \( \{0\} \in B \), \( \{1\} \in B \), and \( \{2\} \in B \). The elements of \( B \) are sets themselves.

We can have sets where the members or elements are numbers, sets, or functions.

Zermelo-Frankel (ZF=0-8) Axioms:

Axiom 0: Set Existence: \( \exists x(x = x) \)
This axiom says that our universe is non-void. There is a set that exists.

**Axiom 1: Extensionality:** \( \forall x, \forall y (\forall z (z \in x \iff z \in y) \rightarrow x = y) \)

This axiom says that a set is determined by its members or elements. Given sets \( x \) and \( y \), \( x \) and \( y \) are the same set if and only if they have the same elements.

**Axiom 2: Foundation:** \( \forall x [\exists y (y \in x) \rightarrow \exists y (y \in x \land \exists z (z \in x \land z \in y))] \)

This axiom says that every nonempty set \( x \) has an element \( y \) such that \( x \) and \( y \) are disjoint sets. That is to say \( \exists z \) such that \( z \in x \) and \( z \notin y \) or \( z \in y \) and \( z \notin x \).

**Axiom 3: Comprehension Scheme:** For each formula \( \phi \) with free variables among \( x, y, w_1, \ldots, w_n \),
\[
\forall z, \forall w_1, \ldots, w_n \exists y \forall x (x \in y \iff x \in z \land \phi)
\]

This axiom says that if \( \phi \) is some formula then for any set \( z \) there exists a set \( y \) that contains all the \( x \in z \) that have the formula \( \phi \). If a collection is a subcollection of a given set, then the collection does exist. Notice also that this axiom expresses one idea but yields an infinite collection of axioms, one for each \( \phi \).

So far with these axioms we can create the set which contains no elements (empty-set) and see that there is no universal set being the "set of all sets". The next axioms 4-8 help to establish which collections do form sets.

**Axiom 4: Pairing:** \( \forall x \forall y \exists z (x \in z \land y \in z) \)

This axiom says that for a given set \( x \) and \( y \), there is a set \( z \) containing both. Using Axiom 3 \( \{x, y\} \) exists since \( \{x, y\} = \{w \in z : w = x \text{ or } w = y\} \).

**Axiom 5: Union:** \( \forall F \exists A \forall Y \forall X (x \in Y \land Y \in F \rightarrow x \in A) \)

This axiom says that given a family of sets, \( F \), there exists a set \( A \) such that for each \( Y \in A \), \( Y \subseteq A. \cup F = \{x \in A : x \in Y \text{ and } x \in A \text{ for some } A \in F\} \).

**Axiom 6: Replacement Scheme:** For each formula \( \phi \) with free variables among \( x, y, A, w_1, \ldots, w_n \),
\[
\forall A \forall w_1, \ldots, w_n [\forall x \in A \exists ! y \phi \rightarrow \exists Y \forall x \in A \exists y \in Y \phi]
\]

This axiom shows that new sets (not necessarily subsets of existing sets) can be defined using a relationship \( \phi \). For example, the replacement axiom is used twice when defining
the cartesian product: \( A \times B = \{(x, y) : x \in A \text{ and } y \in B \} \). Notice Axiom 6, like Axiom 3, expresses one idea but yields an infinite collection of axioms, one for each \( \phi \).

Axiom 7: Infinity: \( \exists x(0 \in x \land \forall y \in x(S(y) \in x)) \) where \( S(y) = y \cup \{y\} \)

This axiom says that there is an infinite set.

Axiom 8: Power Set: \( \forall x \exists y \forall z(z \subseteq x \rightarrow z \in y) \)

This axiom says that for any set \( x \), there exists a set \( y \) containing each subset of \( x \). So we can define the power set \( P(x) = \{z \in y : z \subseteq x\} \).

For this next axiom we need to define some types of orderings.

**Definition 1.5.** A **partial order** is a relation \( R \) over a set \( P \) which is reflexive, antisymmetric, and transitive:

\[
\forall a, b, c \in P \\
1.) aRa (reflexivity) \\
2.) If \( aRb \) and \( bRa \) then \( a = b \). (antisymmetry) \\
3.) If \( aRb \) and \( bRc \) then \( aRc \). (transitivity)
\]

**Example 1.6.** Let the partially ordered set be \( \mathbb{N} \) and the relation \( R \) be divisibility.

**Example 1.7.** Let the partially ordered set be the power set of the naturals and the relation \( R \) be subset inclusion.

**Definition 1.8.** A **total order** is a relation \( R \) on a set \( X \) which is antisymmetric, transitive, and total:

\[
\forall a, b, c \in X \\
1.) If \( aRb \) and \( bRa \) then \( a = b \). (antisymmetry) \\
2.) If \( aRb \) and \( bRc \) then \( aRc \). (transitivity) \\
3.) Either \( aRb \) or \( bRa \). (totality)
EXAMPLE 1.9. Notice in example 1.6, we do not have a total order since for example 2 and 3 cannot be compared since 2 does not divide 3 and 3 does not divide 2.

EXAMPLE 1.10. Notice in example 1.7, we do not have a total order since for example \{1, 2, 3\} and \{1, 2, 4\} cannot be compared since \{1, 2, 3\} \not\subseteq \{1, 2, 4\} and \{1, 2, 4\} \not\subseteq \{1, 2, 3\}.

EXAMPLE 1.11. Let the totally ordered set be \( \mathbb{R} \) and the relation \( R = \leq \).

DEFINITION 1.12. Given a partially ordered set \( B \), the **least element** of \( A \subseteq B \) is \( x \in A \) such that \( \forall y \in A \) with \( y \neq x \), \( x R y \).

DEFINITION 1.13. A **well-order** on a set \( S \) is a total order on \( S \) with the property that every nonempty subset of \( S \) has a least element.

Axiom 9: Choice: \( \forall A \exists R (R \text{ well-orders } A) \)

This axiom says that every set can be well-ordered.

Sometimes the Axiom of Choice is defined to give a choice function such that on an arbitrary number of nonempty sets, we can map a nonempty set to an element inside the set. It can be shown that the Axiom of Choice is equivalent to the Well-ordering Theorem. Here Kunen defines the Axiom of Choice as above. This way when we assume the Axiom of Choice, we will assume our set can be well-ordered. When Axiom 9 is not included, the system is called ZF. The Axiom of Choice is not provable in ZF, meaning assuming ZF and using legitimate rules of inference, we cannot derive the Axiom of Choice (Kunen). ZFC is the name of the system using axioms 0-9.

DEFINITION 1.14. A **metric** on a set \( X \), is a function \( d : X \times X \to \mathbb{R} \) with the following properties:

1.) \( d(x, y) \geq 0 \ \forall x, y \in X \) : equality holds iff \( x = y \)

2.) \( d(x, y) = d(y, x) \ \forall x, y \in X \)

3.) (triangle inequality) \( d(x, y) + d(y, z) \geq d(x, z) \ \forall x, y, z \in X \)
\((X, d)\) denotes a metric space, a set \(X\) with a metric \(d\) on \(X\).

**Example 1.15.** In \((\mathbb{R}^n, d)\), given \(x, y \in \mathbb{R}^n\) such that \(x = (x_1, x_2, \ldots, x_n)\) and \(y = (y_1, y_2, \ldots, y_n)\) let \(d(x, y) = |x \cdot y| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}\). Here \(d\) defines the **Usual Metric** or the **Euclidean Metric** on \(\mathbb{R}^n\).

**Example 1.16.** In \((\mathbb{R}^n, d)\), given \(x, y \in \mathbb{R}^n\) such that \(x = (x_1, x_2, \ldots, x_n)\) and \(y = (y_1, y_2, \ldots, y_n)\) let \(d(x, y) = \max\{|x_i - y_i| : 1 \leq i \leq n\}\). Here \(d\) defines a metric called the **Square Metric** on \(\mathbb{R}^n\).

**Example 1.17.** In \((\mathbb{R}^N, d)\), given \(x, y \in \mathbb{R}^N\) such that \(x = (x_1, x_2, \ldots, x_n)\) and \(y = (y_1, y_2, \ldots, y_n)\) let \(d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{|x_n - y_n|, 1\}\). Here \(d\) defines a metric on \(\mathbb{R}^N\).

**Example 1.18.** In \((X, d)\) where \(X\) is any set and \(x, y \in X\), let

\[
\begin{align*}
    d(x, y) &= \begin{cases} 
    0 & \text{if } x = y \\
    1 & \text{if } x \neq y
    \end{cases}
\end{align*}
\]

Here \(d\) defines a metric on \(X\) called the **Discrete Metric**.

**Example 1.19.** In \((X, d)\) where \(X = \prod_{n=0}^{\infty}X_n\) given \((X_0, d_0), \ (X_1, d_1)\)… metric spaces and \(x, y \in X\) such that \(x = (x_0, x_1, \ldots)\) and \(y = (y_0, y_1, \ldots)\), let

\[
    d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{d_n(x_n, y_n), 1\}\]

Here \(d\) defines a metric on \(X\) called the **Product Metric**.

Metric properties 1, 2, and 3 can be checked to see that \(d\) defines a metric in each of the above examples.

**Definition 1.20.** On a set \(X\), a **topology** is a collection \(\tau\) of subsets of \(X\) having the following properties:

1.) \(\emptyset\) and \(X\) are in \(\tau\).
2.) The union of any subcollection of \( \tau \) is in \( \tau \).
3.) The intersection of the elements of any finite subcollection of \( \tau \) is in \( \tau \).

If \( X \) is a set and \( \tau \) is a topology, \( (X, \tau) \) is a topological space.

**Definition 1.21.** Given \((X, d), x, y \in X\) and \( \varepsilon > 0 \), the **epsilon-ball** (\( \varepsilon \)-ball centered at \( x \)), denoted \( B_d(x, \varepsilon) \), is the set \( \{ y \mid d(x, y) < \varepsilon \} \).

**Definition 1.22.** Given \( \varepsilon > 0 \), the **epsilon-neighborhood** of \( A \subseteq X \) is the set \( U(A, \varepsilon) = \{ x \mid d(x, A) < \varepsilon \} \). This is the union of the open epsilon-balls \( B_d(a, \varepsilon) \) for \( a \in A \). In other words given \((X, d)\) with \( \varepsilon > 0 \) an \( \varepsilon \)-neighborhood of a point \( a, a \in A \) is the set of all points whose "distance" (given by the metric) from \( a \) is less than \( \varepsilon \).

This is denoted \( N(a, \varepsilon) \).

**Definition 1.23.** Given \((X, \tau), U \subseteq X\) is open if \( U \) belongs to the collection \( \tau \).

In terms of Neighborhoods, if given a metric that induces \( \tau \), \( U \subseteq X \) is open in \( X \) if for any point \( u \in U \), there is an \( \varepsilon \)-neighborhood \( N \) such that \( u \in N \) and \( N \subseteq U \).

**Definition 1.24.** A topological space whose topology is induced by a metric is called a metrizable space.

**Definition 1.25.** If \( X \) is a set, a **basis** for a topology on \( X \) is a collection \( B \) of subsets of \( X \) (called **basis elements**) such that:

1.) For each \( x \in X \) there is at least one basis element \( B \) such that \( B \in B \) with \( x \in B \).
2.) If \( x \in B_1 \cap B_2 \) where \( B_1, B_2 \) are basis elements, then there is a basis element \( B_3 \) such that \( x \in B_3 \) and \( B_3 \subseteq B_1 \cap B_2 \).

**Definition 1.26.** Given \((X, \tau)\) and \( A \subseteq X \), the **interior** of \( A \), denoted \( \text{int}(A) \), is the union of all open sets contained in \( A \).

**Definition 1.27.** Given \((X, \tau_1)\) and \((Y, \tau_2)\), the **product topology** \( \tau \) is the topology having as a basis the collection of sets of the form \( U \times V \) where \( U \subseteq X \), \( U \) open, and \( V \subseteq Y \), \( V \) open.
open. \((X \times Y, \tau)\) denotes the product space.

**Definition 1.28.** Let \(X\) be a set. The collection of all subsets of \(X\) is a topology on \(X\) called the **discrete topology**. In other words, every subset of \(X\) is open. And the collection of all singletons of \(X\) is a basis.

**Definition 1.29.** Let \(B\) be the collection of all open intervals in the real line, \((a, b) = \{x | a < x < b\}\). The topology generated by \(B\) is called the **usual topology** on \(\mathbb{R}\).

**Definition 1.30.** Let \(X\) be a space and \(u \in X\). The complement of \(u\), denoted \(u^c\) is \(X \setminus u = \{y | y \in X, y \neq u\}\). Similarly, if \(V \subseteq X\), \(V^c = X \setminus V = \{y | y \in X, y \notin V\}\).

**Definition 1.31.** Given \((X, d)\), if \(A \subseteq X\) and \(x \in X\), \(x\) is a **limit point** of \(A\) if \(\forall \varepsilon > 0 \ \exists y \in B_d(x, \varepsilon) \cap A\) where \(y \neq x\).

**Definition 1.32.** A set \(U\) is **closed** if \(U^c\) is open.

Equivalently, in \((X, d)\), a set \(U\) is closed if and only if it contains all of its limit points.

**Definition 1.33.** Given \((X, \tau)\) and \(A \subseteq X\), the **closure** of \(A\), denoted \(\text{cl}(A)\), is the intersection of all closed sets that contain \(A\).

**Definition 1.34.** Given \((X, \tau)\) and \(A \subseteq X\), the **boundary** of \(A\), denoted \(\text{bd}(A)\), is equal to \(\text{cl}(A) \cap \text{cl}(X \setminus A)\).

The following characterization of compactness is usually stated as a theorem after defining compactness using "open covers" but we state it here as a definition.

**Definition 1.35.** Given \((X, \tau)\) then \(X\) is **compact** if and only if for every collection \(C\) of closed sets in \(X\) having the finite intersection property, \(\bigcap_{C \in C} C \neq \emptyset\).

**Definition 1.36.** Given \((X, d)\), a point \(x\) of a set \(X\) is called an **isolated point** of \(X\) if there exists a neighborhood of \(x\) that does not contain any other points of \(X\).
Definition 1.37. Given a set $X$, we define an $\omega$-tuple of elements of $X$ to be a function $f : \omega \to X$. This function is also called a sequence of elements of $X$.

Definition 1.38. The value of $f$ at $i$, denoted $f_i$ or $f(i)$ is called the $i$th coordinate of $f$.

Definition 1.39. Given $(X, d)$, a sequence $(x_n)$ of points in $X$ is a Cauchy sequence in $(X, d)$ if given $\varepsilon > 0 \exists N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $n, m \geq N$.

Definition 1.40. Given $(X, \tau)$ and $(x_n)$ s.t $x_1, x_2, ... \in X$ $(x_n)$ converges to $x_0 \in X$ if $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that $\forall n \geq N, x_n \in N(x, \varepsilon)$. The sequence $(x_n)$ is said to be a convergent sequence.

Definition 1.41. Given $(X, d)$, a sequence $(x_n)$ in $X$ and $x \in X$, if $(x_n)$ converges to $x$, denoted $x_n \to x$ or $\lim_{n \to \infty} (x_n) = x$ if $d(x_n, x) \to 0$ as $n \to \infty$, then $x$ is called the limit of $(x_n)$.

Definition 1.42. The metric space $(X, d)$ is complete if every Cauchy sequence in $X$ converges in $X$.

Example 1.43. $\mathbb{R}^n$ with the usual metric and the square metric is complete since every Cauchy sequence converges in $\mathbb{R}^n$ (Munkres).

Example 1.44. $\mathbb{Q}$ with the usual metric is not complete since there is for example a sequence of rational numbers that converges to $\pi$ which is not a rational number.

Example 1.45. The set $(0,1)$ is not complete with the usual metric since the sequence $\left\{ \frac{1}{n+1} \right\}_{n=1}^{\infty}$ converges to $0$ which is not in $(0,1)$.

Example 1.46. Any closed interval of $\mathbb{R}$ is complete with the usual metric.

Example 1.47. $\mathbb{C}$ with the usual metric is complete.

Definition 1.48. A topological space $(X, \tau)$ is completely metrizable if $\tau$ is induced by a complete metric.
DEFINITION 1.49. A set $A$ is **finite** if there is a bijection (1-1 and onto) $f : A \rightarrow \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.

We say $A$ has cardinality $n$, denoted $|A| = n$.

EXAMPLE 1.50. If $A = \emptyset$ then $|A| = 0$.

DEFINITION 1.51. A set $A$ is **infinite** if it is not finite.

DEFINITION 1.52. A set $A$ is **countably infinite** if there is a bijection $f : A \rightarrow \mathbb{N}$.

DEFINITION 1.53. A set $A$ is **countable** if it is finite or countably infinite. A set $A$ is countable if and only if there is an injective mapping $f : A \rightarrow \mathbb{N}$.

EXAMPLE 1.54. The set of integers is countable.

EXAMPLE 1.55. The set of rationals is countable.

EXAMPLE 1.56. The cartesian product of a finite number of countable sets is countable.

EXAMPLE 1.57. The prime numbers are countably infinite so countable.

EXAMPLE 1.58. Any subset of a countable set is countable.

DEFINITION 1.59. If a set $A$ is not countable then $A$ is **uncountable**. An uncountable set is an infinite set that contains "too many" elements to be countable. A set $A$ is uncountable if there is no injective mapping $f : A \rightarrow \mathbb{N}$.

EXAMPLE 1.60. The irrationals are uncountable.

EXAMPLE 1.61. The reals are uncountable.

EXAMPLE 1.62. Any interval in the reals is uncountable.

EXAMPLE 1.63. Let’s consider the Cantor set. This is how the Cantor set is constructed.

Begin with $[0, 1] \subseteq \mathbb{R}$.

Step 1: Delete the middle third open interval $(\frac{1}{3}, \frac{2}{3})$ such that $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ remain.
Step 2: Delete the middle third open interval from each of the remaining closed intervals in step 1 so that the deleted open intervals are \((\frac{1}{3}, \frac{2}{3})\) and \((\frac{7}{9}, \frac{8}{9})\). This way \([0, \frac{1}{3}], [\frac{2}{3}, \frac{4}{3}], [\frac{2}{3}, \frac{7}{9}],\) and \([\frac{8}{9}, 1]\) remain.

Step n: After the \((n-1)\)st step there will be \(2^{n-1}\) closed intervals remaining. Delete the middle third open interval from each of these \(2^{n-1}\) closed intervals.

Continue in this manner to create a subset of \([0, 1]\). The numbers that remain after these steps are carried out indefinitely are those in the Cantor set. This cantor set contains no open interval and is a closed set since it is the intersection of closed sets. Here we note that the Cantor set is uncountable.

**Definition 1.64.** A \(A \subseteq X\) is **dense** in \(X\) if \(\text{cl}(A) = X\).

**Example 1.65.** \(Q\) is dense in \(R\) since \(\text{cl}(Q) = R\).

**Example 1.66.** \(Q^n\) is dense in \(R^n\).

**Example 1.67.** \(R - Q\) is dense in \(R\) since \(\text{cl}(R - Q) = R\).

**Definition 1.68.** A space \(X\) is **separable** if it has a countable dense subset.

**Example 1.69.** \(R\) is separable since \(Q\) is a countable, dense subset of \(R\).

**Definition 1.70.** A set \(B\) is **nowhere dense** if \(\text{int}(\text{cl}(B)) = \emptyset\).

**Example 1.71.** The natural numbers form a nowhere dense set in \(R\) with the usual metric.

**Example 1.72.** Any finite subset of \(R\) is nowhere dense (McDonald, Weiss).

**Example 1.73.** The Cantor Set is an example of an uncountable set which is nowhere dense (McDonald, Weiss).

**Example 1.74.** \(Q\) is not nowhere dense in \(R\) since \(\text{cl}(Q) = R\) which has nonempty interior.

**Example 1.75.** For any closed set \(F\), \(F \setminus \text{int}(F)\) is nowhere dense.
DEFINITION 1.76. If \( f : A \to B \) is a bijective (injective and surjective) function, then \( f^{-1} : B \to A \) is the inverse function of \( f \). The function \( f^{-1} \) is defined by letting \( f^{-1}(b) = a \) for the unique \( a \in A \) such that \( f(a) = b \).

DEFINITION 1.77. Given topological spaces \( X \) and \( Y \), \( f : X \to Y \) and \( x \in X \), \( f \) is continuous at \( x \) if \( \forall \) open \( V \supseteq f(x), \exists \) open \( U \supseteq x \) such that \( f(U) \subseteq V \).

DEFINITION 1.78. Given topological spaces \( X \) and \( Y \), \( f : X \to Y \) is called continuous if it is continuous at every \( x \in X \).

Equivalently, \( f : X \to Y \) is continuous if and only if \( f^{-1}(V) \) is open (closed) in \( X \) for every open (closed) set \( V \in Y \).

DEFINITION 1.79. Given \( f : X \to Y \), \( f^{-1} \) is continuous if \( \forall U \subseteq X \), \( U \) open, the inverse image of \( U \) under \( f^{-1} : Y \to X \) is open in \( Y \).

DEFINITION 1.80. Given \( (X, \tau_1) \) and \( (Y, \tau_2) \), a function \( f : X \to Y \) is a homeomorphism if it has the following properties:

1.) \( f \) is a bijection (1-1 and onto).
2.) \( f \) is continuous.
3.) \( f^{-1} \) is continuous.

EXAMPLE 1.81. The function \( f(x) = 2x \) is a homeomorphism of the interval \((0, 1)\) onto the interval \((0, 2)\) with the usual metric in \( \mathbb{R} \).

EXAMPLE 1.82. The function \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = 3x + 1 \) is a homeomorphism.

EXAMPLE 1.83. It can be shown that for \((a, b), (c, d) \subset \mathbb{R} \) there exists a homeomorphism \( f : (a, b) \to (c, d) \) (McDonald, Weiss).

EXAMPLE 1.84. The composition of any two homeomorphisms, is a homeomorphism (Srivastava).
**Definition 1.85.** Given two sets $X$ and $Y$, $f : X \to Y$ is a **homomorphism** if $f$ preserves the algebraic operations on the set $X$.

**Example 1.86.** Consider $(\mathbb{N}, +)$. The map $f : \mathbb{N} \to \mathbb{N}$ by $f(x) = 3x$ is a homomorphism since $f(a + b) = 3(a + b) = 3(a) + 3(b) = f(a) + f(b)$.

**Example 1.87.** Consider $(\mathbb{R}, +)$ and $(\mathbb{R}, \cdot)$. The map $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = e^x$ is a homomorphism since $f(a + b) = e^{a+b} = e^a \cdot e^b = f(a) \cdot f(b)$.

**Definition 1.88.** A function $f : X \to Y$ is an **isomorphism** if it is a bijective map such that $f$ and $f^{-1}$ are homomorphisms.

**Example 1.89.** Consider $(\mathbb{R}, +, \cdot)$. The map $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x$ is an isomorphism since $f$ is 1-1 and onto and $f = f^{-1}$ with $f(a + b) = a + b = f(a) + f(b)$ and $f(ab) = ab = f(a) \cdot f(b)$.

**Definition 1.90.** Natural Number representation: We can think of the natural numbers as a set whose elements are also sets. $\mathbb{N} = (0, 1, 2, ...).$ Another way of writing out the Naturals is below:

- $0 = \emptyset$
- $1 = \{0\} = \{\emptyset\}$
- $2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$
- $3 = \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
- $4 = \{0, 1, 2, 3\}$

and inductively,

- $n + 1 = \{0, 1, 2, 3, ..., n\}$

To understand the construction of the natural numbers we will define an ordinal and see the relationship each natural number has with its specific ordinal.

**Definition 1.91.** A set $X$ is **transitive** if and only if every element of $X$ is a subset of $X$. 
In other words, $X$ will be transitive if whenever $x \in X$ and $y \in x$ then $y \in X$.

**Example 1.92.** $X_0 = 0 = \emptyset$, $X_1 = \{0\}$, $X_2 = \{0, \{0\}\}$, $X_3 = \{\{\{0\}\}, \{0\}, 0\}$ are transitive sets.

Consider $X_3$. Notice that $\{0\} \in X_3$ and $0 \in \{0\}$ and $0 \in X_3$. Also, $\{\{0\}\} \in X_3$ and $\{0\} \in \{\{0\}\}$ and $0 \in X_3$.

**Example 1.93.** $X = \{\{0\}\}$ is not transitive since $\{0\} \in X$ and $0 \in \{0\}$ but $0 \notin X$.

**Definition 1.94.** A set $x$ is an **ordinal** if and only if $x$ is transitive and well-ordered by $\in$ where $\in$ means $\alpha < \beta$ if and only if $\alpha \in \beta$.

**Example 1.95.** The set $2 = \{0, 1\} = \emptyset, \emptyset\}$ is an ordinal since 2 is transitive and well-ordered by $\in$. We can see that $1 \in 2$ and $0 \in 1$ and $0 \in 2$ giving transitivity for 2. We can see that the ordinal $2 > 1$ since $1 \in 2$.

**Example 1.96.** $A = \{1, 2, 5\}$ is not an ordinal since $5 \in A$ and $3 \in 5$ but $3 \notin A$. Here $A$ fails to be transitive.

**Definition 1.97.** If $\alpha$ is an ordinal, then $S(\alpha) = \alpha \cup \{\alpha\}$.

**Example 1.98.** If $\alpha = 3$ then $S(3) = 3 \cup \{3\} = \{0, 1, 2, 3\} = 4$.

**Definition 1.99.** The ordinal $\alpha$ is a **successor ordinal** if and only if $\exists \beta$, $\beta$ an ordinal, such that $\alpha = S(\beta) = \beta + 1$.

**Definition 1.100.** Each $n \in \mathbb{N}$ is an **ordinal**. If $\alpha$ is one of these natural numbers, then $\alpha$’s successor is also an ordinal.

**Definition 1.101.** An ordinal $\alpha$ is a **limit ordinal** if and only if $\alpha \neq 0$ and $\alpha$ is not a successor ordinal. A limit ordinal $\alpha$ is the supremum of all the previous ordinals.

**Definition 1.102.** Define a **natural number** (an ordinal $\alpha$) by induction: $0$ is a natural number.
If $n$ is a natural number then $S(n)$ is a natural number.

The natural numbers form an "initial segment" of the ordinals and are obtained by applying $S$ to $\emptyset$ a finite amount of times.

**Definition 1.103.** The ordinal $\omega$ is the set of all natural numbers. It is the least limit ordinal since it is not a successor ordinal.

We can think of $\omega$ as the "size" of the Naturals and as the "size" of any set of numbers which is countably infinite (has an injective mapping into the naturals).

In set theory the Natural numbers are constructed as sets such that each natural number is the set of all smaller or preceding natural numbers. Since it is true that every natural number is an ordinal, then by definition, every natural number is a well-ordered set. It can be shown that every well-ordered set with a finite number of elements is order isomorphic to a unique natural number.

**Definition 1.104.** $\text{ON}$ is the collection of all ordinals.

In general we have,

- $0 = \{ \alpha : \alpha \text{ is an ordinal and } \alpha < 0 \} = \emptyset$
- $1 = \{0\}$
- $2 = \{0, 1\}$
- ...
- ...
- $k + 1 = \{0, 1, \ldots, k\} \ldots$
- ...
- $\omega = \{0, 1, \ldots, k, k + 1, \ldots\} = \aleph_0$ which is a limit ordinal.
- $\omega + 1 = \{0, 1, \ldots, k, k + 1, \ldots, \omega\}$
- ...
- $\omega + k + 1 = \{0, 1, \ldots, \omega, \ldots, \omega + k\}$ which is a successor ordinal.
\[ \omega + \omega = \omega \times 2 = \{0, 1, \ldots, \omega, \ldots, \omega + k, \ldots\} \]

\[ \omega \times \omega = \{0, 1, \ldots, \omega, \ldots, \omega \times 2, \ldots, \omega \times k, \ldots\} \] which is a limit of limits.

\[ \omega \times \omega + 1 \]

\[ \omega_1 = \{\alpha : \alpha \text{ is a countable ordinal }\} = \mathbb{R}_1 \]

\[ \omega_2 \]

\[ \omega_\omega \]

\[ \omega_{\omega_\omega} \]

**Definition 1.105.** A **cardinal** is a special kind of ordinal. These are ordinals that do not have an injective mapping from themselves to an ordinal number that precedes them.

**Example 1.106.** The ordinal \( \omega \) is a cardinal since it cannot be mapped injectively to an ordinal that precedes it.

**Example 1.107.** The ordinal \( \omega + 1 \) is not a cardinal since it can be mapped injectively to \( \omega \) which is an ordinal that precedes it. To see this, define \( f : \omega + 1 \to \omega \) by recalling that \( \omega + 1 = \omega \cup \{\omega\} \) such that \( \omega \in \omega + 1 \).

\[
f(\alpha) = \begin{cases} 
\alpha + 1 & : \alpha < \omega \\
0 & : \alpha = \omega 
\end{cases}
\]

It is interesting to see that the collection ON is not a set. Kunen proves that there does not exist a set \( z \) such that \( \forall x, \text{ if } x \text{ is an ordinal then } x \in z \). If there were such a set \( z \) containing all of the ordinals then since every ordinal contains only other ordinals, we would have that
every element of the collection of ordinals is also a subset (transitivity). If ON were a set, it would have to be an ordinal itself because it fits the definition of an ordinal. Then we would have to have ON as an element of itself since ON contains all ordinals, but there is no such ordinal that is an element of itself. This is the Burali-Forti Paradox.
Baire Space

**Definition 2.1.** Given $X$ such that $X = \prod_{i \in I} X_i$, (Cartesian product of the topological spaces $X_i$), the **infinite product topology** on $X$ is the topology with the open sets that are unions of sets of the form $\prod U_i$, with $U_i$ open in $X_i$ and $U_i \neq X_i$ only finitely often.

**Definition 2.2.** The **Baire space** is $\omega^\omega = \omega \times \omega \times \omega \ldots = \{ f \mid f : \omega \to \omega \}$. This space is the set of functions from $\omega$ to $\omega$ or the set of all tuples of $\omega$-length whose entries are elements of $\omega$. We use the discrete topology on $\omega$ and the product topology on the Baire Space.

**Definition 2.3.** If $A \subseteq \mathbb{R}$ and $A \neq \emptyset$ then $A$ is said to be **perfect** if it is closed and has no isolated points.

**Definition 2.4.** A **Polish space** is a separable, completely metrizable topological space. A Polish space is one that is homeomorphic to a complete metric space that has a countable dense subset. Polish spaces such as the Cantor set and $\mathbb{R}$ and the Baire Space are separable, complete, and perfect.

**Example 2.5.** Any countable discrete space with the discrete topology is a Polish Space.

**Example 2.6.** The real line $\mathbb{R}$, $\mathbb{R}^n$, and $[0, 1] \subset \mathbb{R}$ with the usual topologies are Polish Spaces (Srivastava).

The Baire Space is a perfect polish space. Bogachev shows that the Baire space $\omega^\omega$ with the product topology is homeomorphic to the irrational numbers in $(0, 1)$ with the usual
topology. Since the Baire Space is homeomorphic to the irrationals we could consider an \( \omega \)-tuple to be a representation of a real number. Bunch shows that there is a continuous onto function \( f : \mathbb{R} \rightarrow X \) where \( X \) is any Polish space. This way there is a replica of any Polish space \( X \) in the Baire Space.

**Definition 2.7.** Given \( \alpha \in \omega^\omega \), the **restriction** on \( \alpha \)'s domain, denoted \( \alpha|_n \) is the \( n \)-tuple \( \beta \) such that \( \beta = \alpha|_n \).

**Definition 2.8.** A tuple \( \beta \) is an **initial segment** of a tuple \( \alpha \in \omega^\omega \) if \( \beta = \alpha|_n \) for some \( n \in \mathbb{N} \).

**Definition 2.9.** \( \omega^{< \omega} \) is the set \( \{ f \mid f : n \rightarrow \omega \text{ for some } n \in \omega \} \).

**Definition 2.10.** Consider \( \alpha = \langle a_0, a_1, \ldots \rangle \) and \( \beta = \langle b_0, b_1, \ldots \rangle \) where \( \alpha, \beta \in \omega^\omega \). The metric for the Baire space \( (\omega^\omega, d) \) is \( d(\alpha, \beta) = \frac{1}{n+1} \) where \( n \) represents the left most coordinate of \( \alpha \) and \( \beta \) where the values of \( \alpha \) and \( \beta \) no longer agree. This would mean \( a_0 = b_0, a_1 = b_1, \ldots a_{n-1} = b_{n-1} \) but that \( a_n \neq b_n \).

**Example 2.11.** Let \( \alpha = (0, 1, 2, 3, 3 \ldots) \) and \( \beta = (0, 1, 2, 4, \ldots) \). Then we have \( d(\alpha, \beta) = \frac{1}{3+1} = \frac{1}{4} \).

The Baire space is often represented as the infinite paths through the tree of finite sequences of natural numbers.

**Definition 2.12.** If \( u \in \omega^{< \omega} \) then **length**(\( u \)) is the number of coordinates of \( u \).

**Example 2.13.** Let \( u = (1, 2, 5, 9, 12, 14) \). We have that length(\( u \)) is 6.

**Definition 2.14.** \( T \) is a **tree** on \( \omega \) if \( T \subseteq \omega^{< \omega} \) such that if \( u \in \omega^{< \omega}, u \in T \) and if \( v \in \omega^{< \omega}, v \subseteq u \) then \( v \in T \). \( v \subseteq u \) if there is an \( n \leq \) length(\( u \)) such that \( u|_n = v \). In other words \( T \) consists of finite length strings of Natural numbers and \( T \) is closed downward.

**Definition 2.15.** An element \( f \) in the Baire space is an **infinite branch** if \( \forall n \in \omega, f|_n \in T \).
**Definition 2.16.** The **body of a tree** $T$ is a collection of infinite branches of the tree denoted $[T]$, which is a subset of the Baire space.

**Theorem 2.17.** In the Baire space a set is closed if and only if it is the body of a tree.

**Proof.** ($\Leftarrow$) Let $T$ be a tree. Show $[T]$ is closed by showing that its complement is open.

Assume $f \notin [T]$. We want an open set $U$ such that $f \in U$ and $U \cap [T] = \emptyset$. Since $f \notin [T]$ there exists $n$ such that $f|_n \notin T$. So there is no $g \in [T]$ such that $g|_n = f|_n$. Consider $N(f|_n)$ where $N(f|_n) = \{f \in \omega^\omega : f|_n = f\}$. Let $U = N(f|_n)$. It witnesses that $f \notin [T]$. So $[T]$ is closed.

($\rightarrow$) Assume $C$, a subset of the Baire space is closed. Define $T = \{f|_n : f \in C \land n \in \omega\}$. This is a tree and $C \subseteq [T]$. We must show that $[T] \subseteq C$ so that $C = [T]$. Assume $f \in [T]$.

Show $f \in C$. We need a sequence $\langle f_n \rangle_{n \in \omega}$ such that $f_n \in [T]$ and $f_n \to f$. This way since $C$ is closed, it will contain its limit point $f$ so that $f \in C$. For each $n$, since $f|_n \in T$, there is a $g_n \in C$ s.t $g_n|_n = f|_n$. Consider $\langle g_n \rangle$. We claim $g_n \to f$ putting $f \in C$. For $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\epsilon > \frac{1}{N}$ and $\forall n \geq N, d(g_n, f) \leq \frac{1}{n+1} \leq \frac{1}{N+1} < \frac{1}{N} < \epsilon$. \hfill $\square$
Chess, checkers, and tic-tac-toe are a few simple examples of games we are probably familiar with in which 2 players are actively involved and have perfect information. By perfect information we mean that player I and player II are able to view every move made by each player and no game-play information is hidden or kept secret. Similarly, with the games used to present set-theoretic results, our games will be 2-player games of perfect information. Certainly you may have experienced playing a game with another person where the game resulted in a tie and there was not a single winner. You might wonder what situations and what types of 2-player games warrant ties, i.e have no single winner. The Axiom of Determinacy (AD) was introduced by Jan Mycielski and Hugo Steinhaus.

Axiom of Determinacy (AD): It refers to certain two-player games of length $\omega$ with perfect information. Every such game in which both players choose natural numbers is determined; Either player I or player II has a winning strategy.

We will show that the Axiom of Determinacy implies that all subsets of the real numbers have the property of Baire and the perfect set property which we will define later. It can also be shown that with the Axiom of Determinacy, all subsets of the real numbers are Lebesgue measurable. When assuming the Axiom of Determinacy we will work in ZF with AD. Just as the Axiom of Choice cannot be derived from assuming ZF, the Axiom of Determinacy cannot be derived from ZF. Since the Axiom of Determinacy (AD) is inconsistent with the Axiom of Choice (AC), these axioms are said to be contradictory. We will see later that when the Axiom of Determinacy shows all subsets of the real numbers have the property of
Baire, the Axiom of Choice shows there is a subset of the real numbers which doesn’t have the property of Baire. We will see other examples of these two contradictory axioms later.

In preparation for showing what the Axiom of Determinacy can prove, we will try to understand what the Axiom of Determinacy states by first defining how a game is constructed, what the objectives of the game are for each player, and what strategies each player employs in order to achieve their objectives for the game.

**Definition 3.1.** The set $X^\omega = \{ f | f : \omega \to X \}$.

**Definition 3.2.** Let $X$ be a non-empty set. Given $A \subseteq X^\omega$ (of infinite sequences from $X$), we have a 2 person game $G = G_X(A)$ where $A$ is the "set of reward" known as the **payoff** for $G = G_X(A)$.

The game $G = G_X(A)$ identifies the space $X$ and the set of interest $A$.

**Definition 3.3.** A game $G_X(A)$ is said to be **determined** if there is a guaranteed winner. In games of perfect information with 2 players, the Axiom of Determinacy says either player I or player II will have a winning strategy in $G_X(A)$ for every $A$ when the players are playing natural numbers.

A typical structure for a game is to begin with a set $\omega$. Let $A \subseteq \omega^\omega$ so that if $\alpha \in A$, then $\alpha(n) \in \omega$ for every $n$. Player I’s moves are recorded on the coordinates with even subscripts and Player II’s moves are recorded on the coordinates with odd subscripts. Player I will choose an $a_0 \in \omega$ and Player II will respond to $a_0$ with $a_1 \in \omega$. The players will continue to choose elements from $\omega$. A subgame can be viewed at any time that will display the previous moves. In this case a subgame will be $\alpha |_n$ for some $n$ where $\alpha$ is a play of the game. If the subgame is of even length, it is then player I’s move next. If the subgame is of odd length, it is then player II’s move next. This is a process that continues theoretically...
forever and can be recorded in an infinite string $\alpha$. Each player uses a strategy mentioned below in hopes of winning the game on $A$. In the end either $\alpha \in A$ or $\alpha \notin A$. If $\alpha \in A$ then player I wins. If $\alpha \notin A$ then player II wins.

**Definition 3.4.** A **strategy** for a player in $G_X(A)$ will be a function that will take as an input the play of the game thus far, and as an output will give the player whose turn it is next, an appropriate play. The domain for the function is all the possible sequences of finite length with elements from $X$ of appropriate even or odd length (even for player I’s strategy function $\sigma$ and odd for player II’s strategy function $\tau$).

**Definition 3.5.** **Player I’s strategy** is denoted $\sigma$ which is a function that takes an even-length input and gives out the next play for player I. Player I wins $G_X(A)$ if the outcome of the game $\alpha \in A$. Player I follows/plays $\sigma$ in the game $G$ if the resulting play $\alpha = (a_0, a_1, a_2, a_3, \ldots)$ satisfies

- $a_0 = \sigma(\emptyset)$
- $a_2 = \sigma(a_0, a_1)$
- \[ \vdots \]
- $a_n = \sigma(a_0, a_1, \ldots a_{n-1})$ for $n$ even.

**Definition 3.6.** **Player II’s strategy** is denoted $\tau$ which is a function that takes an odd-length input and gives out the next play for player II. Player II wins $G_X(A)$ if the outcome of the game $\alpha \notin A$. Player II follows/plays $\tau$ in the game $G$ if the resulting play $\alpha = (a_0, a_1, a_2, a_3, \ldots)$ satisfies

- $a_1 = \tau(a_0)$
- $a_3 = \tau(a_0, a_1, a_2)$
- \[ \vdots \]
- $a_n = \tau(a_0, a_1, \ldots a_{n-1})$ for $n$ odd.

When Player I plays $\sigma$ and Player II plays $\tau$, the resulting play $\alpha = (a_0, a_1, a_2, \ldots)$ =
\[ \sigma \ast \tau = (\sigma(\emptyset), \tau(a_0), \sigma(a_0, a_1), \tau(a_0, a_1, a_2), \ldots). \]

**Definition 3.7.** For Player I \( \sigma \) is a **winning strategy** if for every \( \tau \) (Player II’s strategy), \( \sigma \ast \tau = \alpha \in A \).

This means Player I wins playing \( \sigma \) regardless of what Player II plays.

**Definition 3.8.** For Player II \( \tau \) is a **winning strategy** if for every \( \sigma \) (Player I’s strategy), \( \sigma \ast \tau = \alpha \notin A \).

This means Player II wins when playing \( \tau \) regardless of what Player I plays.

**Definition 3.9.** If \( s = (a_0, a_1, \ldots a_{n-1}) \) and \( t = (b_0, b_1, \ldots b_{m-1}) \) then the **concatenation**, denoted \( s \hat{\ast} t \) is \( (a_0, a_1, \ldots a_{n-1}, b_0, b_1, \ldots b_{m-1}) \).

**Definition 3.10.** If \( A \subseteq \omega^\omega \) and \( u = (a_0, a_1, \ldots a_{n-1}) \) is a sequence of even length, the **subgame of** \( A \) at \( u \), denoted \( A(u) \) is \( A(u) = \{ f \in \omega^\omega : (a_0, a_1, \ldots a_{n-1}, f(0), f(1), \ldots) \in A \} = \{ f \in \omega^\omega : u \hat{f} \in A \} \).

Let’s consider some simple examples of games on \( \omega^\omega \).

**Example 3.11.** Let \( A = \{ \alpha | \forall n \exists m : n, m \in \omega \text{ and } \alpha(2m) < \alpha(2n + 1) \} \) where \( A \subseteq \omega^\omega \). Player II can easily win this game by ensuring that the end result of the plays \( \alpha \notin A \). Regardless of what move Player I makes first \( (\alpha(0) = a_0 = x \text{ for some } x \in \omega) \), Player II can play the next move \( \alpha(1) = a_1 = 0 \) and continue on everyone of his moves to play 0 for that coordinate of \( \alpha \) so that for \( n = 0 \), there does not exist \( m \in \omega \) such that \( \alpha(2m) < \alpha(1) = 0 \) since for any \( m, 2m \) will be even and \( \alpha(2m) \) will be a play Player I made which could only be 0 or a number larger than 0.

**Example 3.12.** Let \( A = \{ \alpha = (a_0, a_1, a_2, \ldots) | \{a_n\}_n \text{ is not a monotone increasing sequence} \} \) where \( A \subseteq \omega^\omega \). Player I can easily win this game by ensuring that the end result of the plays \( \alpha \in A \). After Player I plays some \( a_0 \) and Player II plays some \( a_1 \), Player I simply plays some
$a_2 \in \omega$ such that $a_2 < a_1$. This way $\alpha|_2$ has coordinates that are not monotone increasing so $\alpha$ cannot have coordinates that are monotone increasing.

**Theorem 3.13.** Let $A \subseteq X^\omega$ and suppose $u = (a_0, a_1, a_2, \ldots, a_{n-1})$ is a finite sequence from $X$ of even length. If Player II does not win the game $A(u)$, then there is some $a$ such that $\forall b$, Player II does not win $A(u^\sim(a, b))$.

**Proof.** Assume for contradiction that player II does not win $A(u)$ but that for each $a$, $\exists b^a$, $\tau^a$ (winning strategy) for player II in $A(u^\sim(a, b))$. Using the axiom of choice, let $f : a \rightarrow (b^a, \tau^a)$ be a function which assigns to each $a$, such a $b^a$ and a $\tau^a$. Now player II can win $A(u)$ by responding to player I’s first move $a_0$ by $b^{a_0}$ and then following $\tau^{a_0}$ as if he were playing in $A(u^\sim(a_0, b^{a_0}))$. But we assumed player II does not win $A(u)$ so it must be that player II does not win $A(u^\sim(a, b))$. \hfill \square

We will discuss two particular games $G_X^*(A)$ and $G_X^{**}(A)$ to see the structure of each game and how player I and player II move appropriately within the rules of the game. We will use $G_X^*(A)$ for the Perfect Subset Game Theorem and $G_X^{**}(A)$ to show that every subset of the reals has the Property of Baire.

The Game $G_X^*(A)$:
Let $X$ be a set. Let $A \subseteq X^\omega$. Player I is able to play a string of elements from $X$ of any length. This creates a string of finite length whose elements along each coordinate come from $X$. Player II is able to play only one element from $X$ which will be recorded following player I’s finite amount of elements of $X$. Player I and player II will continue in this manner to create $\alpha$ the result of the game. For example if player I began with $a_0, a_1 \ldots a_n$ where each $a_i, 0 \leq i \leq n \in X$ then player II could only respond with some $a_{n+1}$ where $a_{n+1} \in X$. Player I could then continue with $a_{n+2}, \ldots, a_{n+m}$ for $m < \omega$. Player II could only respond with some $a_{m+1}$ where $a_{m+1} \in X$. 
This next game created around 1928 by S. Mazur is called the **Banach Mazur Game**. It will be used later to discuss a property of the real numbers. 

The Game $G^*_X(A)$:

Let $X$ be a Polish space with a standard countable enumeration of a neighborhood basis for each of its elements. Each neighborhood will be open and have non-empty interior. Fix $A \subseteq X$. Player I and Player II alternate turns. Player I begins choosing some natural number $a_0$ corresponding to a neighborhood and then player II responds by choosing some natural number $a_1$ corresponding to a neighborhood such that $\text{cl}(N(a_0)) \supseteq \text{cl}(N(a_1))$ and $\text{rad}(N(a_1)) \leq (1/2)\text{rad}(N(a_0))$. Player I then chooses some natural number $a_2$ for a neighborhood such that $\text{cl}(N(a_1)) \supseteq \text{cl}(N(a_2))$ and $\text{rad}(N(a_2)) \leq (1/2)\text{rad}(N(a_1))$. The players continue in this manner and in doing so define a unique point $x \in \cap \text{cl}(N(a_i))$. This will be the case since,

1. we have created a nested, descending chain of closed sets: $\text{cl}(N(a_0)) \supseteq \text{cl}(N(a_1)) \supseteq \text{cl}(N(a_2)) \ldots$ and
2. $\text{cl}(N(a_0))$ is compact.

These closed neighborhoods have the finite intersection property since the finite intersection will equal the smallest $\text{cl}(N(a_i))$ and $\text{cl}(N(a_i)) \neq \emptyset$. The infinite intersection is then nonempty since $\text{cl}(N(a_0))$ is compact and the collection of all of the closed neighborhoods has the finite intersection property. So for this $\alpha = x \in \cap \text{cl}(N(a_i))$, if $x \in A$ then Player I wins. If $x \notin A$ then Player II wins. Also, a player loses if the rules of the game are violated. If a player chose a point that did not adhere to the specific size of the neighborhood around that point in reference to the previous player’s neighborhood and point, then that would be considered as a violation.
Uncountable Sets and the Continuum Hypothesis: AD and AC

Continuum Hypothesis (CH): $2^\omega = \omega_1$.

Generalized Continuum Hypothesis (GCH): $\forall \alpha (2^{\omega_\alpha} = \omega_{\alpha+1})$.

The continuum hypothesis remains today an intriguing hypothesis. In 1877 the mathematician Georg Cantor began discussing the "size" of infinite sets. When set theory first began, the central questions of set theory were that of the continuum hypothesis and the generalized continuum hypothesis. The name stems from the continuum for the real numbers. And the idea is that there is no cardinality inbetween that of the Natural numbers $\omega$ and the Real numbers $2^\omega$, so that $\omega_1 = 2^\omega = |\mathbb{R}|$. The generalized continuum hypothesis is a generalization of the continuum hypothesis that says for an infinite cardinal, there is no set that has a cardinality inbetween that infinite cardinal’s cardinality and the cardinality of the power set of that cardinal. In other words, if any $\lambda$ is an infinite cardinal, there does not exist a cardinal $\kappa$ with $\lambda < \kappa < 2^\lambda$. In 1938 Kurt Godel produced a "model" of ZFC which satisfied the generalized continuum hypothesis. His model doesn’t show though, assuming ZFC, CH and GCH can be derived. Later in 1963 Paul Cohen was able to create a method of forming "models of set theory" and "models" of ZFC (called forcing). The method of forming models does not satisfy CH. As a consequence CH and GCH can neither be proved nor disproved using ZFC.
**Theorem 4.1.** Perfect Subset Game

AD implies that if $A \subseteq 2^\omega$ and $|A| > \omega$, then $|A| = 2^\omega$ (Continuum Hypothesis).

**Proof.** We will use the game $G_X^\omega(A)$ and assuming the Axiom of Determinacy we will show that

1.) Player I wins if and only if $A$ has a perfect subset and

2.) Player II wins if and only if $A$ is countable.

1.)

$(\rightarrow)$

Assume Player I wins. Show $\forall P \subseteq A$, $P$ is perfect. We will show if Player I uses $\sigma$ to win then the collection of plays against $\sigma$ is perfect. Let $P = \{ \alpha : \alpha$ is a play in a game against $\sigma \}$. These are the plays Player II makes. To show $P$ is perfect, we need to show $P$ is closed and has no isolated points. If a sequence of plays $\alpha_i \rightarrow \alpha$, we want to show $\alpha \in P$ so that $P$ contains all of its limit points and is then closed. We show that Player I and Player II can "cooperate" to play $\alpha$. Since $\sigma$ is a function Player I has a fixed first move, so $\sigma(\emptyset) = (s_0, \ldots, s_{m_0-1}) = \alpha|_{m_0}$ for each $i$ since each $\alpha_i$ is a play of the game. By convergence, for each $i$, $\alpha_i|_{m_0} = \alpha|_{m_0}$ so the play of the game so far is $\alpha|_{m_0}$. Player II plays the next coordinate of $\alpha : \alpha(m_0)$ so the play of the game so far is $\alpha|_{m_0+1} = \alpha_j|_{m_0+1}$ for infinitely many $j$ by convergence, say $j > N$ for some large enough $N$. This means $\alpha|_{m_0+1} = \alpha_j|_{m_0+1}$ for $j \geq N$. Player I plays using $\sigma : \sigma(\alpha|_{m_0+1}) = (s_{m_0+1}, \ldots, s_{m_1-1}) = (\alpha(m_0+1), \ldots, \alpha(m_1-1)) = \alpha|_{m_1}$. So $\sigma(\alpha|_{m_0+1}) = \sigma(\alpha_j|_{m_0+1}) = (\alpha_j(m_0+1), \ldots, \alpha(m_1-1))$ for each $j \geq N$. So the play of the game so far is $\alpha|_{m_1}$. Player II plays the next coordinate of $\alpha : \alpha(m_1)$. And we continue in this manner. So for every $n$, the play of the game is $\alpha|_n$ making the final play of the game $\alpha$. So $\alpha \in P$ which is thus closed.

To see $P$ has no isolated points, suppose $\alpha \in P$. Now assume for contradiction that $\alpha$ is isolated. Then $\exists N$ s.t $\forall n \geq N \not\exists \beta \in P$ such that $\beta|_n = \alpha|_n$. So in the game in which $\alpha$ is
played, consider when the play of the game so far is of length $n$ for some $n > N$ and it is Player I’s turn to play. It must be that $\alpha|_n \sim \sigma(\alpha|_n) = \alpha|_m$ for some $m > n > N$. Now let Player II play $1 - \alpha(m)$ resulting in a play of the game $\beta \in P$ since $\sigma$ wins for Player I and every play by Player I must be to a splitting node for $\alpha$ otherwise Player II would win because $\alpha \notin A$. Here $\beta|_m = \alpha_m$ which is a contradiction. Since $P$ is closed with no isolated points, $P$ is perfect and from Corollary 2.A.2 (Moschovakis) $|P| = 2^{\aleph_0}$.

(←)

Assume $C \subseteq A$, $C$ perfect. Show Player I wins. So $C$ is closed with no isolated points. Let Player I pick some $\alpha \in C$ and play to the "splitting nodes" of $\alpha$ since $\alpha$ is not isolated. This means $\exists \beta \in C$ such that $\beta \neq \alpha$ but $\beta|_n = \alpha|_n$ for some $n > 0$. Let Player I now play $\alpha|_n$ as the first play. Player II will then make the next move. Regardless of what Player II plays, Player I can again play to another splitting node of either $\alpha$ or $\beta$ since neither is isolated. The game continues in this manner. Player I wins since $\alpha \in C \subseteq A$.

2.)

(←)

Assume $A$ is countable. Show Player II wins. Let $A = \{\alpha_n : n \in \omega\}$. Player II will diagonalize out of $A$ so that the end result of the game $\alpha$ will not be in $A$. On each move Player II makes, Player II will ensure that on the $n$th turn, the play is different from $\alpha_n$. For example if Player II’s $n$th play is $\alpha(m)$ then Player II plays $\alpha(m) = 1 - \alpha_n(m)$. This way $\alpha|_m \neq \alpha_n|_m$ and so $\alpha \neq \alpha_n$ for any $\alpha_n \in A$. So Player II wins.

(→)

Assume Player II wins using $\tau$. Show $A$ is countable. If $\alpha \in A$, then let $G$ be the set of initial segments of $\alpha$ that appear in a game against $\tau$ when it is Player I’s turn to play. Since $\tau$ is a function, if $\alpha|_n \in G$ then $\tau(\alpha|_{n-1}) = \alpha(n - 1)$ making the play of the game at that point $\alpha|_n$ otherwise $\alpha|_n \notin G$ (i.e $\alpha|_n = \alpha|_{n-1} \sim \tau(\alpha|_{n-1})$ so $\alpha(n - 1) = \tau(\alpha|_{n-1}))$. Now $\exists N$ such that $\alpha|_N = (a_0, ... ,a_{n-1})$ is the longest such initial segment in $G$. If there is no
such maximal good play then at each play Player I can play "one short" of a longer good sequence \( \alpha|_m = (a_0, ... a_{m-1}) \). This would mean Player I would play so that the play of the game after his move is \( \alpha|_{m-1} = (a_0, ... a_{m-2}) \). It must be that \( \tau((a_0, ... a_{m-2})) = a_{m-1} \) since \( \tau \) is a function and \( \alpha|_m \) appears in a play of the game. So \( \alpha \) is a play of the game but this is a contradiction to \( \tau \) winning for Player II. So \( \alpha|_{N+1} \) determines \( \alpha \) for \( m > N \), \( \alpha(m) \) is what \( \tau \) would not play in response to \( \alpha|_m \). This means \( \alpha(N+1) = 1 - \tau(\alpha|_N \upharpoonright \alpha(N)) \) and in general \( \alpha(N+1) = 1 - \tau(\alpha|_N \upharpoonright (\alpha(N), ... \alpha(N+i-1))) \). So if \( \alpha \in A \) there is an \( s_\alpha \in G \) such that \( s_\alpha = \alpha|_{N+1} \) and if \( \alpha \neq \beta \) then \( s_\beta \neq s_\alpha \). So \( |A| \leq |\omega^\omega| = \omega \). So we have that \( A \) is countable.

If we assume \( |A| > \omega \) then \( A \) is uncountable and Player II wins the game on \( A \) since we assumed the Axiom of Determinacy. This means \( A \) has a perfect subset \( B \) with \( |B| = 2^\omega \) since a perfect set has cardinality equal to \( 2^\omega \). So \( |A| \geq 2^\omega \). Since \( A \subseteq 2^\omega \), we have that \( |A| = 2^\omega \).

**Theorem 4.2.** AC implies that there is an uncountable set \( A \subseteq \mathbb{R} \) with no perfect subset.

**Proof.** We want to show that there is an uncountable subset of the reals that contains no subset that is closed with no isolated points. We will construct sets \( A, B \subseteq \mathbb{R} \), \( A \cap B = \emptyset \), \( A \cup B = \mathbb{R} \), neither \( A \) nor \( B \) contains a perfect subset, and at least one is uncountable. Using the axiom of choice, let \( P = \{P_\xi : \xi < 2^\omega \} \) list the perfect sets since by the Axiom of Choice we can well-order the set \( P \). Let \( \mathbb{R} = \{x_\xi : \xi < 2^\omega \} \) list the real numbers since again using the Axiom of Choice we can well-order \( \mathbb{R} \). Now using transfinite recursion, define \( A_\xi, B_\xi \subseteq \mathbb{R} \) such that at each step \( \xi \), the following holds:

1.) \( \forall \alpha < \xi, A_\alpha \cap P_\alpha \neq \emptyset \) and \( B_\alpha \cap P_\alpha \neq \emptyset \)

2.) \( \forall \alpha < \xi, A_\alpha \cap B_\alpha = \emptyset \)

3.) \( \forall \alpha < \beta < \xi, A_\alpha \subseteq A_\beta \) and \( B_\alpha \subseteq B_\beta \)

4.) \( \forall \alpha < \xi, |A_\alpha| < 2^\omega \) and \( |B_\alpha| < 2^\omega \)
Given the construction up to $\xi$:

If $P_\xi \cap (\bigcup_{\alpha<\xi} A_\alpha) \neq \emptyset$ and $P_\xi \cap (\bigcup_{\alpha<\xi} B_\alpha) \neq \emptyset$ then make $A_\xi = \bigcup_{\alpha<\xi} A_\alpha$ and $B_\xi = \bigcup_{\alpha<\xi} B_\alpha$.

We can check to see that 1-4 still hold. Now without loss of generality, if $P_\xi \cap (\bigcup_{\alpha<\xi} A_\alpha) = \emptyset$ and $P_\xi \cap (\bigcup_{\alpha<\xi} B_\alpha) = \emptyset$ then since $|P_\xi| = 2^{\omega}$ (*) and $|\bigcup_{\alpha<\xi} (A_\alpha \cup B_\alpha)| < 2^{\omega}$, we can pick points in $\mathbb{R}$ say $x_n, x_v$ of minimal index such that $x_n, x_v \in P_\xi$. Let $A_\xi = (\bigcup_{\alpha<\xi} A_\alpha) \cup \{x_n\}$ and $B_\xi = (\bigcup_{\alpha<\xi} B_\alpha) \cup \{x_v\}$. We can check to see that 1-4 still hold. Now let $A = \bigcup_{\alpha<2^\omega} A_\alpha$ and $B = \mathbb{R} \setminus A \supseteq \bigcup_{\alpha<2^\omega} B_\alpha$. This way $A \cap B = \emptyset$ and $A \cup B = \mathbb{R}$. At least one of either $A$ or $B$ is uncountable since $\mathbb{R}$ is uncountable. Neither $A$ nor $B$ has a perfect subset since if $D$ is a perfect set, then it would have been ordered among $P$ and $D = P_\xi$ for some $\xi$ and $A \cap P_\xi \supseteq A_\xi \cap P_\xi \neq \emptyset$ and $B \cap P_\xi \supseteq B_\xi \cap P_\xi \neq \emptyset$. So $P_\xi \not\subseteq A$ and $P_\xi \not\subseteq B$.

(*)$|P_\xi| = 2^{\omega}$. If we let $C = \{[i, i+1] : i \in \mathbb{Z}\}$ then each $I_i \in C$ is perfect since it is closed and has no isolated points. There are $2^{\omega}$ countable unions of elements from $C$. To see this use $\alpha \in 2^{\omega}$ to represent a countable union of elements from $C$. Let $f$ map an $I_i \to i$ and let $g$ map $i \to 2i$ if $i > 0$ otherwise map $i \to -2i + 1$. $A 1$ for $\alpha(j)$ represents the interval mapped to $j \in \mathbb{N}$ being included in the union. So there are at least $2^{\omega}$ perfect sets. And there are $2^{\omega}$ open sets, since each can be expressed as a countable union of the countable basis of open intervals with rational endpoints.

$\square$
The following definitions are attributed to R. Baire who formulated them in 1899.

**Definition 5.1.** A set $B$ is a **meager** set if it can be written as the countable union of nowhere dense sets. If $B$ is meager then it is of first category.

**Example 5.2.** The set $\mathbb{Q}$ with the usual topology is meager in $\mathbb{Q}$ with the usual topology since $\mathbb{Q}$ can be written as the countable union of rational numbers, each being a nowhere dense set since each rational number is a singleton and is closed.

**Definition 5.3.** A set $B$ with $B \subseteq \mathbb{R}$ is of **second category** if it is not of first category.

**Definition 5.4.** Let $A$ and $B$ be sets. The **symmetric difference** of $A$ and $B$ is denoted $A \Delta B$. $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

**Definition 5.5.** A set $P$ has the **property of Baire** if $\exists P^*, P^*$ open, such that $P \Delta P^*$ is meager.

If a set $P$ has the property of Baire, then we can think of it as being "almost" open since its symmetric difference with an open set $P^*$ contains a countable number of nowhere dense sets. These nowhere dense sets do not contain "many" elements and the countable union of these "small" sets is "small".

**Example 5.6.** Given $(X, \tau)$, if $A \in \tau$ then $A$ has the property of Baire.

To see this we need an open $B$ such that $A \Delta B$ is meager. Take $B = A$ so that $A \Delta B = (A \setminus A) \cup (A \setminus A) = \emptyset$. We know the $\emptyset$ is meager since it can be written as itself, which is a countable nowhere dense set since $\text{int}(\text{cl}(\emptyset)) = \emptyset$. 

Property of Baire: AD and AC
EXAMPLE 5.7. Any set $A$ which is meager has the property of Baire.

To see this we need an open $B$ such that $A \Delta B$ is meager. Take $B = \emptyset$ so that

$$A \Delta B = (A \setminus \emptyset) \cup (\emptyset \setminus A) = A$$

and $A$ is meager.

This next theorem (The Baire Category Theorem) is also attributed to R. Baire and can be phrased in many ways. Since the reals make up a complete metric space, the Baire Category Theorem shows that the complement of any set of the first category in the reals is dense, no interval in $\mathbb{R}$ is of first category, and the intersection of any sequence of dense open sets is also dense.

THEOREM 5.8. Baire Category Theorem: In a complete metric space, no open ball is meager.

Proof. Let $X$ be a complete metric space. Let $B, A_n \subseteq X$. Assume for contradiction that we have an open ball which is meager. Let $B = \bigcup_n A_n$ with $B$ an open ball and each $A_n$ nowhere dense. The $\text{cl}(A_0)$ contains no open ball since it is nowhere dense. Therefore $B \setminus \text{cl}(A_0)$ is nonempty and open since $B \setminus \text{cl}(A_0) = B \cap (\text{cl}(A_0))^\complement$ and the intersection of two open sets is open. So $\exists x_0$ so that we can choose an open ball $B(x_0, \varepsilon_0) = B_0$ with $\text{cl}(B_0) \subseteq B \setminus \text{cl}(A_0)$ with radius $\varepsilon_0 < 1$. In general we can choose $B_{n+1} = B(x_{n+1}, \varepsilon_{n+1})$ such that $\text{cl}(B_{n+1}) \subseteq B_n \setminus \text{cl}(A_{n+1})$ with $\varepsilon_{n+1} < (1/2^k)\varepsilon_n$. Is there an $x$ such that $x \in \bigcap_n B_n$? If $x \in \bigcap_n B_n$ then $x \notin \text{cl}(A_n) \forall n$ since $\forall n, B_n \cap A_n = \emptyset$. So $x \notin \bigcap_n A_n$ and $x \notin \bigcap_n \text{cl}(A_n)$. But $B_0 \subseteq B = \bigcup_n A_n \rightarrow x \in \bigcap_n B_n, x \notin B_0$ so $x \in \bigcap_n A_n$. We find if we assume $x \in \bigcap_n B_n, x \notin \bigcap_n B_n$. Now assume $x \notin \bigcap_n B_n$ i.e $\bigcap_n B_n = \emptyset$. The sequence $\langle x_n \rangle$ is Cauchy since $x_k, x_{k+1} \in B_k \rightarrow d(x_k, x_{k+1}) < (1/2^k)$ and by construction the radius of $B_{k+1}$ is less than the radius of $B_k$.

Since $X$ is complete, $\langle x_n \rangle \rightarrow x$ for some $x \in B$. Now $\forall n, x \in \text{cl}(B_n)$ because the tail of the sequence $T = \{x_k : k \geq n\} \subseteq B_n$ and if $U$ is open, $x \in U$, then $U \cap T = \emptyset$ since $\langle x_n \rangle \rightarrow x$. So $x \in \bigcap_n \text{cl}(B_n)$. But each $\text{cl}(B_n) \cap \text{cl}(A_n) = \emptyset$ so $x \notin \bigcap_n \text{cl}(A_n)$. Therefore $x \notin B$ since $B = \bigcup_n A_n$. But $x \in \bigcap_n \text{cl}(B_n) \subseteq B$. This gives a contradiction. \qed
A corollary to the Baire Category Theorem is below.

**Corollary 5.9.** Every completely metrizable space $X$ is of second category in $(X, \tau)$.

**Proof.** Let $X$ be a completely metrizable space. Suppose for contradiction that $X$ is of first category in itself. We can choose a sequence of closed and nowhere dense sets $F_n$ such that $X = \bigcup_n F_n$. Then the sets $U_n = X \setminus F_n$ are dense and open since each $F_n$ is closed implies that $F_n^c = X \setminus F_n$ is open. Now $\bigcap_n U_n = \emptyset$ which is not dense since $\text{cl}(\emptyset) = \emptyset \neq X$ since the emptyset is both open and closed. This gives us that the intersection of countably many dense open sets is not dense which contradicts the Baire Category Theorem. So $X$ is of second category in itself. \qed

**Example 5.10.** The irrationals are of second category.

To see this assume $\mathbb{R} \setminus \mathbb{Q}$ is of first category. Then we can write it as the countable union of sets which are nowhere dense: $\mathbb{R} \setminus \mathbb{Q} = \bigcup_n \{F_n\}$ where each $F_n$ is nowhere dense. Now we know $\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = \bigcup_n \{F_n\} \cup \{q_n\}$ for $q \in \mathbb{Q}$. This says $\mathbb{R}$ is of first category but this contradicts the Baire Category Theorem. So the irrationals are of second category with the usual metric.

It can be shown that any subset of a set of first category is of first category and the countable union of a family of first category sets is of first category (Oxtoby).

**Definition 5.11.** Let $X$ be a set. A nonempty collection $\mathcal{A}$ is a **$\sigma$-algebra** if it has the following properties:

1.) If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$.
2.) If $(A_n)_n \subseteq \mathcal{A}$ then $\bigcup_n A_n \in \mathcal{A}$.

Therefore a $\sigma$-algebra is closed under complementation, countable unions and countable intersections (using DeMorgan’s laws).
DEFINITION 5.12. Given $(X, \tau)$ and $B \subseteq X$, $B$ is a **Borel set** if it is contained in the smallest $\sigma$-algebra containing $\tau$.

DEFINITION 5.13. Given $(X, \tau)$, $X$ is called a **Baire Space** if no nonempty open subset of $X$ is of first category (in $X$ or in itself).

EXAMPLE 5.14. $\mathbb{R}$ is a Baire Space and so is of second category in itself.

EXAMPLE 5.15. The Cantor set is a Baire Space and so is of second category in itself.

The next theorem shows that every Borel subset of a metrizable space has the property of Baire meaning that Borel sets and Baire Spaces coincide when we have a metrizable space. This will give us that every subset of the reals has the property of Baire.

THEOREM 5.16. Every Borel set has the property of Baire.

*Proof.* Assume we have a Borel set $P$. Show $P$ has the property of Baire:

1.) $P$ is open. $P$ has the property of Baire. Just take $P^* = P$ s.t $(P \setminus P^*) \cup (P^* \setminus P) = \emptyset \cup \emptyset = \emptyset$ and $\emptyset$ is closed so $\text{cl}(\emptyset) = \emptyset$ with empty interior thus nowhere dense.

2.) $P$ is closed. Let $P^* = \text{int}(P) = \{ x \in P : \exists U \text{ open, } x \in U \subseteq P \}$. We claim $P \setminus P^*$ is nowhere dense. Since $P^* \subseteq P$, $P \Delta P^* = (P \setminus P^*) \cup (P^* \setminus P) = (P \setminus P^*) \cup \emptyset = P \setminus P^*$ so then $P \Delta P^*$ is nowhere dense so $P \Delta P^*$ is meager (once we prove our claim).

(proof of claim): Assume $\setminus P^*$ is not meager and $\exists$ open $U \subseteq \text{cl}(P \setminus P^*)$. Since $P \setminus P^* \subseteq P$ because $P^* \subseteq P$, we have $\text{cl}(P \setminus P^*) \subseteq \text{cl}(P) = P$ since $P$ is closed. Now $U \subseteq \text{cl}(P \setminus P^*) = P$ so $U \subseteq P$ but then since $U$ is open, $U \subseteq \text{int}(P) = P^*$ by definition. This contradicts that $U \subseteq \text{cl}(P \setminus P^*)$ since $U \cap (P \setminus P^*) = \emptyset$. If $x \in U$ we find that $x \notin \text{cl}(P \setminus P^*)$ so $x \notin U$. This proves $P \Delta P^*$ is meager.

3.) Show the property of Baire holds for complements. Let $x \in P^* \Delta Q$

$\leftrightarrow x \in (P^c \setminus Q) \cup (Q \setminus P^c)$

$\leftrightarrow x \in (P^c \cap Q^c) \cup (Q \cap P)$
\[ \leftrightarrow x \in (Q^c \cap P^c) \cup (P \cap Q) \]
\[ \leftrightarrow x \in (Q^c \setminus P) \cup (P \setminus Q^c) \]
\[ \leftrightarrow (P \setminus Q^c) \cup (Q^c \setminus P) \]
\[ \leftrightarrow x \in P \Delta Q^c \]

Assume P has the property of Baire and U is open and \( P \Delta U \) is meager. Show \( \exists \) an open set such that the symmetric difference of \( P^c \) and this open set is meager. Let \( Q = U^c \) where \( U^c \) is closed so that \( Q \) is closed. \( Q^c = U \). So if \( P \Delta U \) is meager then \( P \Delta Q^c \) is meager. From above \( P^c \Delta Q \) is meager and we have \( Q \) is closed. The \( \text{bd}(Q) = Q \setminus \text{int}(Q) \) is nowhere dense since \( \text{bd}(Q) \subseteq Q \) and by definition we cannot have an open \( V \subset \text{bd}(Q) = Q \setminus \{ U \text{ open} : U \subseteq Q \} \). Then \( P^c \Delta \text{int}(Q) = (P^c \setminus \text{int}(Q)) \cup (\text{int}(Q) \setminus P^c) \)
\[ \subseteq (P^c \setminus Q) \cup (Q \setminus P^c) \cup \text{bd}(Q) \]
\[ \subseteq P^c \Delta Q \cup \text{bd}(Q) \]. Since \( (P^c \setminus Q) \) takes out some of the \( \text{bd}(Q) \), \( \text{bd}(Q) \) is added back, and \( (\text{int}(Q) \setminus P^c) \subseteq (Q \setminus P^c) \)

\( P^c \Delta Q \) and \( \text{bd}(Q) \) are meager so \( P^c \Delta \text{int}(Q) \) can be written as the union of two meager sets and is thus meager. The \( \text{int}(Q) \) is open and is what we needed to show \( P^c \) has the property of Baire.

4.) Show the property of Baire holds for countable unions.

Assume \( \forall n < \omega, P_n \) has the property of Baire and \( \exists \) open sets \( U_n \) such that \( P_n \Delta U_n \) is meager. Let \( U = \bigcup_n U_n \) which is open. Let \( P = \bigcup_n P_n \). We want to find an open set \( (U) \) such that \( P \Delta U \) is meager. \( P \Delta U = \bigcup_n P_n \Delta \bigcup_n U_n = \bigcup_n (P_n \Delta U_n) \) which is the countable union of meager sets which is meager.

\((*)\)If \( x \in P \Delta U \) then \( x \in \bigcup_n (P_n \Delta U_n) \). If \( x \in P \Delta U \) then a.) \( x \in \bigcup_n P_n \setminus \bigcup_n U_n \) or b.) \( x \in \bigcup_n U_n \setminus \bigcup_n P_n \).

a.) \( x \in \bigcup_n P_n \) and \( x \notin \bigcup_n U_n \) so \( x \in P_{n_0} \setminus U_{n_0} \in P_{n_0} \Delta U_{n_0} \) for some \( n_0 \).

b.) \( x \in \bigcup_n U_n \) and \( x \notin \bigcup_n P_n \) so \( x \in U_{n_0} \setminus P_{n_0} \in P_{n_0} \Delta U_{n_0} \) for some \( n_0 \).

In either case a.) or b.) \( x \in \bigcup_n (P_n \Delta U_n) \). Equivalent to \((*)\) is if \( x \notin \bigcup_n (P_n \Delta U_n) \) then \( x \notin P \Delta U \).
So there does not exist \( x \) such that \( x \in P \Delta U \).

**Definition 5.17.** A \( G_\delta \) set is the intersection of a countable number of open sets.

**Definition 5.18.** An \( F_\sigma \) set is the union of a countable number of closed sets.

In this next theorem the Axiom of Choice is used. Without the use of the Axiom of Choice, it is not possible to show the existence of a subset of real numbers that doesn’t have the property of Baire (Srivastava).

**Theorem 5.19.** There are sets of real numbers which do not have the property of Baire (using AC).

**Proof.** We will construct a set \( A \subseteq \mathbb{R} \) such that there does not exist an open set \( G \) such that \( A \Delta G \) is meager. Let \( \mathcal{B} = \{ (G,F) | G \text{ is open and Borel and } F \text{ is a meager } F_\sigma \} \). Any open \( G \) is a countable union of basic open sets and there are \( 2^{\omega} \) open sets and so there are also \( 2^{\omega} \) closed sets since each closed set is the complement of an open set. So there are \( 2^{2^{\omega}} = 2^{\omega} F_\sigma \) sets that are meager. We have that \( |\mathcal{B}| = |G||F| = 2^{\omega} \times 2^{\omega} = 2^{\omega} \). Now use the Axiom of Choice to well-order the reals and the set \( \mathcal{B} \) such that \( \mathbb{R} = \{ x_\xi | \xi < 2^{\omega} \} \) and \( \mathcal{B} = \{ (G_\xi,F_\xi) | \xi < 2^{\omega} \} \).

We will construct \( A \) such that \( A \Delta G_\xi = (A \setminus G_\xi) \cup (G_\xi \setminus A) \not\subseteq F_\xi \) for each \( \xi \). Assuming the construction, there is no open set \( G \) such that \( A \Delta G \) is meager since if \( A \Delta G \) is meager then \( A \Delta G = \bigcup_n \{ A_n \} \subseteq \bigcup_n \text{cl}(A_n) \) where \( A_n \) is nowhere dense and thus \( \text{cl}(A_n) \) is nowhere dense. But \( (G, \bigcup_n \text{cl}(A_n)) = (G_\xi,F_\xi) \) for some \( \xi \) so \( (A \setminus G_\xi) \cup (G_\xi \setminus A) \subseteq F_\xi \) which contradicts our construction. We will be done when we construct such a set \( A \). Now define by recursion, sets \( A_\xi \) and \( B_\xi \) such that \( |A_\xi|, |B_\xi| < 2^{\omega} \), \( A \cap B = \emptyset \), and \( (A_\xi \setminus G_\xi) \cup (G_\xi \setminus A) \not\subseteq F_\xi \). Let \( A_0, B_0 = \emptyset \). At the \( \xi \)th step either:

1.) \( \mathbb{R} \setminus (G_\xi \cup F_\xi) \) is uncountable and borel of cardinality \( 2^{\omega} \), so we put into \( A_\xi \) some element of this set so that \( A_\xi \setminus G_\xi \not\subseteq F_\xi \).

or
2.) \( \mathbb{R} \setminus (G_\xi \cup F_\xi) \) is countable and thus meager since it can be written as the countable union of singletons which are nowhere dense, and an \( F_\sigma \) set since singletons are closed. Let \( F'_\xi = F_\xi \cup (\mathbb{R} \setminus (G_\xi \cup F_\xi)) \) so \( F'_\xi \) is meager since it is the union of two meager sets. Clearly \( F_\xi \subseteq F'_\xi \). Since \( F'_\xi = F_\xi \cup (\mathbb{R} \setminus (G_\xi \cup F_\xi)) \), \( F'_\xi = \mathbb{R} \setminus G_\xi \), so \( \mathbb{R} = G_\xi \cup F'_\xi \). Since \( F_\xi \) is countable and \( \mathbb{R} \) is uncountable, \( G_\xi \) is uncountable so \( G_\xi \setminus F'_\xi \) is uncountable and borel so of cardinality \( 2^\omega \) so we can put a point \( x_\mu \) of this set into \( B_\xi \). Now \( x_\mu \notin F'_\xi \) since \( x_\mu \in G_\xi \setminus F'_\xi \) so \( x_\mu \notin F_\xi \) since \( F_\xi \subseteq F'_\xi \). Since \( x_\mu \in G_\xi \cap B_\xi \), \( G_\xi \cap B_\xi \nsubseteq F'_\xi \) since \( x_\mu \in G_\xi \cap B_\xi \) but \( x_\mu \notin F'_\xi \). Let \( A = \bigcup A_\xi \) and \( B = \mathbb{R} \setminus A \) so that \( \bigcup B_\xi \subseteq B \). We can check to see that \( A \cap B = \emptyset \) and \( (A_\xi \setminus G_\xi) \cup (G_\xi \setminus B_\xi) \nsubseteq F_\xi \) for each \( \xi \). If \( A \) has the property of Baire then there must exist \( \xi \) such that \( A \Delta G_\xi \subseteq F_\xi \) but \( A \Delta G_\xi = (A \setminus G_\xi) \cup (G_\xi \setminus A) \nsubseteq (A \setminus G_\xi) \cup (G_\xi \setminus B_\xi) \nsubseteq F_\xi \) since \( (A_\xi \setminus G_\xi) \subseteq F_\xi \) and \( (G_\xi \setminus B_\xi) \nsubseteq F_\xi \) by our construction. So \( A \Delta G_\xi \) is not meager. So the set \( A \subseteq \mathbb{R} \) constructed doesn’t have the property of Baire. \( \square \)

**Theorem 5.20.** (Banach-Mazur Game) Every set has the property of Baire (using AD).

**Proof.** Let \( X \) be a set. Let \( A \subseteq X \). We will show \( A \) has the property of Baire using two facts:

1.) Player II wins the game on \( A \) if and only if \( A \) is meager.

2.) Player I wins the game on \( A \) if and only if \( \text{cl}(\mathcal{N}(s)) \setminus A \) is meager for some \( s \).

We will then conclude why these two facts give us that \( A \subseteq \mathbb{R} \) has the Property of Baire.

1.)

\( \leftarrow \)

Assume \( A \) is meager. Show Player II wins. So \( A \subseteq \bigcup_n F_n \) where each \( F_n \) is nowhere dense \( (\text{int} (\text{cl}(F_n)) = \emptyset) \). We can assume without loss of generality that each \( F_n \) is closed with empty interior. Player II can play to avoid \( F_i \) on his \( i \)th play. Since \( \mathcal{N}(s_0) \setminus F_0 \) is open and nonempty, Player II can choose a \( \mathcal{N}(s_1) \) such that \( \text{rad} (\mathcal{N}(s_1)) \leq \frac{1}{2} \text{rad}(\mathcal{N}(s_0)) \) such that \( \text{cl}(\mathcal{N}(s_1)) \cap F_0 = \emptyset \). So on Player II’s "0th" play \( s_1 \), Player II avoids \( F_0 \) and on Player
II’s "1st" play $s_3$, Player II avoids $F_1$ since $N(s_2) \setminus F_1$ is open and nonempty Player II can choose $N(s_3)$ such that $\text{rad}(N(s_3)) \leq \frac{1}{2} \text{rad}(N(s_2))$ and such that $\text{cl}(N(s_3)) \cap F_1 = \emptyset$. We have that $A \subseteq \bigcup_n F_n = \bigcup_n \text{cl}(F_n)$ because each $F_n$ is closed. Thus $\cap_i \text{cl}(N(s_i)) \cap \bigcup_n F_n = \emptyset$, so $\cap_i \text{cl}(N(s_i)) \cap A = \emptyset$. We know $x \in \cap_i \text{cl}(N(s_i))$ but $x \not\in A$ and so Player II wins.

→

Assume Player II wins. We will show $A$ is meager. Let $x \in A$. Call a sequence $s_0, s_1, s_2, \ldots s_n$ of even length "good" if:

1. It is the initial part of some play where the restrictions have been followed,
2. Player II plays according to winning strategy $\tau$, and
3. $x \in \text{cl}(N(s_n))$.

Claim: There is a maximal "good" sequence, say $s_0, s_1, s_2, \ldots s_n$. If every "good" sequence had indefinitely many extensions, then $x$ would be a play of the game which is a contradiction to the fact that $\tau$ was a winning strategy for Player II. Now if $s_0, s_1, \ldots s_n$ is any even sequence, define,

$$B(s_0, s_1, \ldots s_n) = \cap \{\text{cl}(N(s_n)) \setminus N(\tau(s_0, \ldots s_n, s) : (1), (2) are satisfied\},$$

where

1. $\text{cl}(N(s)) \subseteq \text{cl}(N(s_n))$ and (2) $\text{rad}(N(s)) \leq \frac{1}{2} \text{rad}(N(s_n))$.

So at each step we are intersecting a set which is $\text{cl}(N(s_n))$ minus a possible played neighborhood by Player II (in response to a play by I). So this ball consists of plays that will not occur according to $\tau$, the winning strategy for II (i.e. Player II will never play "into" the ball). Now since $x \in A$, $x$ is never a play of the game according to $\tau$ since we assumed Player II wins and therefore the final play $x \not\in A$. So we have that $x \in B(s_0, s_1, \ldots s_n)$. We can think about it another way. We know that there is an $(s_0, s_1, \ldots s_n)$ which is a maximal "good" sequence for $x$. This means there is not an extension of this good sequence. So $x \in \text{cl}(N(s_n))$ and $x \not\in N(\tau(s_0, s_1, \ldots s_n, s))$ for any $s$ and so $x \in B(s_0, s_1, \ldots s_n)$. Now we will show that $B(s_0, s_1, \ldots s_n)$ is nowhere dense for any $n$. If $B(s_0, s_1, \ldots s_n)$ is not nowhere dense and there exists an open $N(s^*) \subseteq B(s_0, s_1, \ldots s_n)$ which could be a possible play by Player I, and then
there exists some \( s \) such that \( N(s) \subseteq N(s') \) with \( \rad(N(s)) \leq \frac{1}{2}\rad(N(s')) \). So \( N(s) \) satisfies the rules of the game and is therefore a possible play by Player II according to the strategy \( \tau \). But the construction of the ball \( B(s_0, s_1, \ldots, s_n) \) deleted all possible plays by Player II that used the strategy \( \tau \) and so we have our contradiction. Thus \( B(s_0, s_1, \ldots, s_n) \) is nowhere dense for any \( n \). To show that \( A \) is meager we show that \( A \) is a subset of the union of countably many nowhere dense sets which is meager, and any subset of a meager set is meager. We now know that \( B(s_0, s_1, \ldots, s_n) \) is nowhere dense for any \( n \). So \( A \subseteq \bigcup_n B(s_0, s_1, \ldots, s_n) \) since we chose \( x \in A \) (arbitrary) and found that \( x \in B(s_0, s_1, \ldots, s_n) \) for some \( n \).

2.)

\[ \rightarrow \]

Assume Player I wins the game on \( A \) with winning strategy \( \sigma \). We will show that \( \cl(N(s)) \setminus A \) is meager for some \( s \). If \( \sigma(\emptyset) = s_0 \), let Player II play with \( \sigma \) to win the game on \( \cl(N(s_0)) \setminus A \) showing then that \( \cl(N(s_0)) \setminus A \) is meager (because if Player II wins the game on \( \cl(N(s_0)) \setminus A \) then \( \cl(N(s_0)) \setminus A \) is meager by 1.) proven above.

Case a.): If Player I plays \( s \) first and \( (s_0, s) \) is legal meaning it doesn’t violate restrictions for the Banach-Mazur game then let Player I play \( s \), Player II play \( \sigma(s_0, s) \), Player I play \( s_2 \), Player II play \( \sigma(s_0, s, \sigma(s_0, s), s_2) \) then Player I play \( s_4 \), etc. so that the final play \( \alpha \in A \) since \( \sigma \) is a winning strategy for \( A \) and Player I wins if \( \alpha \in A \). So \( \alpha \notin \cl(N(s_0)) \setminus A \) so Player II wins the game on \( \cl(N(s_0)) \setminus A \) so that \( \cl(N(s_0)) \setminus A \) is meager for \( s_0 \).

Case b.): If Player I plays \( s \) first and \( (s_0, s) \) is not legal meaning it violates the rules of the Banach-Mazur game, then let Player II find a \( s' \) such that \( N(s') \) is small enough so that both \( (s_0, s') \) and \( (s, s') \) are legal meaning \( \cl(N(s')) \subseteq \cl(N(s_0)) \) and \( \rad(N(s')) \leq \frac{1}{2}\rad(N(s_0)) \) and such that \( \cl(N(s')) \subseteq \cl(N(s)) \) and \( \rad(N(s')) \leq \frac{1}{2}\rad(N(s)) \). This way let Player I play \( s \), Player II play \( \sigma(s_0, s') \), Player I play \( s_2 \), Player II play \( \sigma(s_0, s', \sigma(s_0, s'), s_2) \), Player I play \( s_4 \), etc. Since \( \sigma \) is a winning strategy for Player I in the game on \( A \), \( \alpha \in A \). So \( \alpha \notin \cl(N(s_0)) \setminus A \) so Player II wins the game on \( \cl(N(s_0)) \setminus A \). So \( \cl(N(s_0)) \setminus A \) is meager by
1.) Player II wins the game on \( A \).

Assume \( \text{cl}(N(s)) \setminus A \) is meager. We will show that Player I wins. Since \( \text{cl}(N(s)) \setminus A \) is meager, Player II wins the game on \( \text{cl}(N(s)) \setminus A \) which means Player II has a winning strategy \( \tau \) such that \( \alpha \notin \text{cl}(N(s)) \setminus A \) and therefore \( \alpha \in A \). So Player I will win the game on \( A \). To achieve this, let Player I begin with this such \( s \). Player II will play \( \tau(s) \), Player I will play \( \tau(s, \tau(s)) \), etc.

Now that 1.) and 2.) have been proven, we will assume the Axiom of Determinacy and show for a specific \( X = R \) and \( A \subseteq R \), that \( A \) has the property of Baire. We need to show \( \exists A^* \) open such that \( \Delta A^* = A \setminus A^* \cup A^* \setminus A \) is meager. Let \( A^* = \bigcup \{ N(s) \mid \text{cl}(N(s)) \setminus A \) is meager \}. We see that \( A^* \) is open. Now we will show:

a.) \( A^* \setminus A \) is meager and b.) Show \( A \setminus A^* \) is meager. \( A^* \setminus A = (\bigcup \{ N(s) \mid \text{cl}(N(s)) \setminus A \) is meager \}) \setminus A \subseteq (\bigcup \{ \text{cl}(N(s)) \mid (N(s)) \setminus A \) is meager \}) \setminus A = \bigcup (\{ \text{cl}(N(s)) \setminus A : \text{cl}(N(s)) \setminus A \) is meager \}). \) So \( A^* \setminus A \) is meager since it is the countable union (each \( s \) is an integer) of meager sets, each of which is the countable union of nowhere dense sets. Now to show \( A \setminus A^* \) is meager we will assume for contradiction that \( A \setminus A^* \) is not meager. We know,

1.) Player II wins the game on \( A \) if and only if \( A \) is meager, and

2.) Player I wins the game on \( A \) if and only if \( \text{cl}(N(s)) \setminus A \) is meager for some \( s \).

Now consider the game on \( A \setminus A^* \). By AD, either Player I or Player II wins this game. Player II cannot win the game on \( A \setminus A^* \) since then \( A \setminus A^* \) would be meager by 1.). So Player I wins the game on \( A \setminus A^* \). This means from 2.) that \( \exists s \) some integer, such that \( \text{cl}(N(s)) \setminus (A \setminus A^*) \) is meager. This gives \( \text{cl}(N(s)) \setminus A \subseteq \text{cl}(N(s)) \setminus (A \setminus A^*) \). So \( \text{cl}(N(s)) \setminus A \) is meager since it is a subset of a meager set. Now \( N(s) \subseteq A \) since \( A^* = \bigcup \{ N(s) \mid \text{cl}(N(s)) \setminus A \) is meager \} \).

Claim: \( N(s) \subseteq \text{cl}(N(s)) \setminus (A \setminus A^*) \) Case 1: \( A \cap A^* = \emptyset \) then \( N(s) \cap A = \emptyset \), since \( N(s) \subseteq A^* \). Thus \( N(s) \subseteq \text{cl}(N(s)) \setminus A = \text{cl}(N(s)) \setminus (A \setminus A^*) \).

Case 2: \( A \cap A^* \neq \emptyset \). Then \( (A \setminus A^*) \) does not contain \( N(s) \), so \( N(s) \subseteq \text{cl}(N(s)) \setminus (A \setminus A^*) \).
Thus the claim is true and $N(s)$ is meager. But $N(s)$ is open and $\mathbb{R}$ is a complete metric space, so by the Baire Category Theorem this open neighborhood cannot be meager. So our assumption that $A \setminus A^*$ is not meager is false. So $A \setminus A^*$ is meager and therefore $A \Delta A^* = A \setminus A^* \cup A^* \setminus A$ is the union of 2 meager sets. So $A$ has the property of Baire. Since $A \subseteq \mathbb{R}$ was arbitrary, every subset of the Reals has the Property of Baire. □

Theorem 5.21. The Gale-Stewart Theorem: AC implies $\forall X \neq \emptyset$, every closed subset of $X^\omega$ is determined.

Proof. We use the product topology on $X^\omega$ with $X$ discrete. Let $A \subseteq X^\omega$. If Player II wins then we are done. So assume Player II does not have a winning strategy in $A$. Show Player I has a winning strategy in $A$. By theorem 3.5, $\exists a_0$ such that $\forall b$ Player II cannot win the subgame $A(a_0, b)$. Let Player I first play $a_0$ and let Player II respond by playing some $a_1$. Now Player II cannot win $A(a_0, a_1)$. Again by theorem 3.5, $\exists a_2$ such that $\forall b$ Player II cannot win the subgame $A(a_0, a_1, a_2, b)$. Let Player I play this such $a_2$ and continue in this manner to play $\alpha$. At the end of the game, $\alpha$ is the result of the plays such that $\alpha = (a_0, a_1, a_2, \ldots)$ and we have that Player II cannot win $A(a_0, a_1, \ldots a_{n-1})$ for every even $n$. So $\exists \alpha_i \in X^\omega$ such that $\alpha_i(0) = a_0, \alpha_i(1) = a_1, \ldots \alpha_i(n - 1) = a_{n-1}$ with $a_i \in A$. Otherwise Player II could win $A(a_0, a_1, \ldots a_{n-1})$ by playing any such $a_n$ because then $\alpha \notin A$ and Player II would win. Since $A$ is closed by assumption, $A$ contains all of its limit points and $\alpha_i \to \alpha$ where $\alpha \in A$ so Player I wins. And so the game is determined. □

Moschovakis shows that the Axiom of Choice implies that for $X \neq \emptyset$, every $\Sigma^0_2$ subset of $X^\omega$ is determined where a $\Sigma^0_2$ set is one that contains all the open sets, closed sets, and sets that are the countable union of closed sets.

It is interesting to see what can be proven with certain assumptions. We end in seeing that the Axiom of Choice says that there is a game where neither Player I nor Player II have a winning strategy. On one hand it is not surprising that the Axiom of Choice would be able
to show this since so far we have seen the Axiom of Choice and Axiom of Determinacy contradicting one another. It is surprising though that the Axiom of Choice has so much "power" to contradict the statement for the Axiom of Determinacy. It would be interesting to assume the Axiom of Determinacy and construct a set that cannot be well-ordered. And this must be the case since we cannot assume both the Axiom of Choice and the Axiom of Determinacy.

**Theorem 5.22.** AC implies there is an undetermined game.

**Proof.** Assume AC. We can then well-order sets. We want to show there exists a set \( A \subseteq 2^\omega \) where the game on \( A \) is not determined (Player I nor Player II has a winning strategy). We will construct a game on \( A \). Let \( 2^\omega = \{ f \mid f : \omega \to 2 \} \) where \( 2 = \{ 0, 1 \} \) (the ordinal 2). So \( 2^\omega \) is the collection of all infinite strings where the coordinates either have a 0 or a 1. Since there are \( 2^\omega \) binary strings, we can well-order \( A \) which is a set of binary strings such that \( A = \{ \alpha_\xi \mid \xi < 2^\omega \} \). Since a strategy \( \sigma \) for Player is a function and has to respond to an input that is a binary string of any length \( n \leq \omega, n \) even, and there are \( 2^\omega \) binary strings, \( \sigma \) has \( 2^\omega \) possibilities. The same is true for strategy \( \tau \) for Player II (with \( n \leq \omega, n \) odd). So we can well-order all the \( \sigma \) possible strategies and well-order all the \( \tau \) possible strategies such that we have \( O = \{ \sigma_\xi \mid \xi < 2^\omega \} \) and \( T = \{ \tau_\xi \mid \xi < 2^\omega \} \). Now use induction to construct sets \( A_\xi \) and \( B_\xi \). Let \( A_0, B_0 = \emptyset \). And \( \forall \xi < 2^\omega \), at each stage \( \xi \), make sure the following are true:

1.) \( A_\xi \cap B_\xi = \emptyset, |A_\xi| < 2^\omega, |B_\xi| < 2^\omega \).

2.) \( \exists \tau \) (winning strategy for Player II) such that \( \sigma_\xi * \tau \in B_\xi \).

3.) \( \exists \sigma \) (winning strategy for Player I) such that \( \sigma * \tau_\xi \in A_\xi \).

Now using transfinite induction, assume 1.), 2.), and 3.) are true \( \forall \alpha < \xi \). Show for \( \xi \) 1.), 2.), and 3.) are true. Let \( C = \{ \sigma_\xi * \tau \mid \tau \in T \} \). Since \( |A_\xi| < 2^\omega \) and \( |C| = 2^\omega \) since \( |T| = 2^\omega \) and \( \sigma_\xi \) is fixed, \( \exists \tau_0 \) such that \( \sigma_\xi * \tau_0 \in C \) and \( \sigma_\xi * \tau_0 \notin A_\xi \). Take this \( \tau_0 \) and put it in \( B_\xi \). Let \( D = \{ \sigma * \tau_\xi \mid \sigma \in O \} \). Similarly, \( \exists \sigma_0 \) such that \( \sigma_0 * \tau_\xi \in D \) and \( \sigma_0 * \tau_\xi \notin B_\xi \) since \( |B_\xi| < 2^\omega \)
and $|D| = 2^{\omega}$. Take this $\sigma_0$ and put it in $A_\xi$. We can check to see 1.), 2.), and 3.) are true at this stage $\xi$. Let $A = \bigcup_\xi A_{\xi}$ and let $B = R \setminus A$. This way $\bigcup_\xi B_{\xi} \subseteq B$. Now we will show $A$ is not determined. Assume for a contradiction that $A$ is determined and assume Player I wins with some strategy $\sigma$. This means for some $\xi$, $\sigma_{\xi} = \sigma$. So $\sigma_{\xi}$ is the winning strategy for Player I. So $\forall \tau$ (Player II uses) $\sigma_{\xi} * \tau \in A$. But we know $\sigma_{\xi} * \tau \in B_{\xi}$ for some $\tau$ by our construction of $B_{\xi}$. Since $B_{\xi} \subseteq B$ and $B = R \setminus A$, $\sigma_{\xi} * \tau \notin A_{\xi} \forall \xi$ so $\sigma_{\xi} * \tau \notin A$. Since $\sigma_{\xi} * \tau \notin A$, and the game is determined by assumption Player II must win. But we assumed Player I wins so this is a contradiction. If we assume Player I wins we find out neither Player I nor Player II wins. The result is similar if we assume Player II wins from the beginning. So $A$ is not determined. \[\square\]
Bibliography
Bibliography


Appendix A

Code List
Vita

Samantha Renee Stanton was born on March 31, 1984 in Richmond, Virginia. She attended Chesterfield County Schools and graduated from Midlothian High School in 2002. In 2007 she graduated from Virginia Commonwealth University with a B.S in Math and a M.T in Secondary Math. After graduation Samantha taught middle school math for a year in Chesterfield County. She decided to further her education in Math as she returned to Virginia Commonwealth University for a M.S in Math in 2008. After graduation, her studies will continue.