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A Horizontal Edges Bound for the Independence Number of a Graph

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A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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Master of Science

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Acknowledgment

I would like to thank my husband, Ben, for all his support. I would also like to thank my family for eveything they have done for me. In addition, I want to thank Dr. Larson for all the time he spent helping me with everything.

Contents

Abstract

A HORIZONTAL EDGES BOUND FOR THE INDEPENDENCE NUMBER OF A GRAPH

By Michelle Lynn Grigsby, Master of Science.

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

Virginia Commonwealth University, 2012.

Director: Craig E. Larson, Assistant Professor, Department of Mathematics and Applied Mathematics.

The independence number α of a graph is the size of a maximum independent set of vertices. An independent set is a set of vertices where every pair of vertices in non-adjacent. This number is known to be hard to compute. The bound we worked with is defined as $\varepsilon = max_{v \in V}\{e(v) - eh(v)\}$. $e(v)$ is the number of vertices at even distance from v. $eh(v)$ is the number of edges both of whose endpoints are at even distance from v. ε can be calculated in polynomial time. Siemion Fajtlowicz proved that $\alpha \geq \varepsilon$ for any graph. We worked to characterize graphs where $\alpha = \varepsilon$.

Preliminaries

This sections gives important definitions that will be used. We begin with an independent set and the independence number, α, since we worked with a lower bound on α.

DEFINITION 1.1. An *independent set*, I, is a set of vertices of a graph where any pair of vertices, $a, b \in I$, are not adjacent.

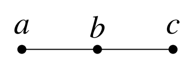


Figure 1.1: An example of an independent set for this graph is $I = \{a, c\}$.

DEFINITION 1.2. The *independence number* $\alpha = \alpha(G)$ of a graph G is the cardinality of a maximum independent set.

Figure 1.2: For this graph $\alpha = 2$.

DEFINITION 1.3. The *distance* between two vertices, a and b, $d(a,b)$ is the length of the shortest path from a to b. A *path* is a sequence of vertices $v_1, v_2, \ldots, v_n$ such that v_i is adjacent to v_{i+1} for $i = 1, \ldots, n-1$.

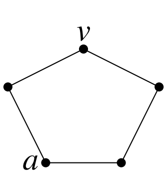


Figure 1.3: For this graph, $d(a, v) = 2$.

DEFINITION 1.4. The set $D_i(v)$ of a graph G is the set of vertices at distance i from a vertex v.

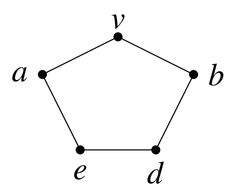


Figure 1.4: For this graph the set $D_0(v) = \{v\}$, $D_1(v) = \{a,b\}$, and $D_2(v) = \{d,e\}$.

DEFINITION 1.5. A *v-horizontal edge* is an edge ab such that $d(v,a) = d(v,b)$.

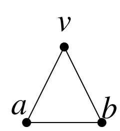


Figure 1.5: The edge ab is a v-horizontal edge since $d(v,a) = d(v,b) = 1$.

DEFINITION 1.6. An *even horizontal edge* is a v-horizontal edge, ab, such that $d(v,a)$ is even.

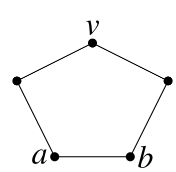


Figure 1.6: For this graph, $d(v,a) = d(v,b) = 2$. So the edge ab is an even-horizontal edge to v.

DEFINITION 1.7. An *odd-horizontal edge* is a v-horizontal edge, ab, such that $d(v,a)$ is odd.

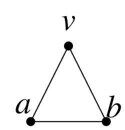


Figure 1.7: For this graph, $d(v,a) = d(v,b) = 1$. So the edge ab is an odd-horizontal edge to v.

DEFINITION 1.8. $E(v)$ is the set of vertices at even distance from v in a graph G. $e(v)$ is the cardinality of $E(v)$. Also, $E_G(v)$ is used when refering to a specific graph G.

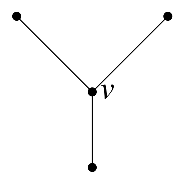


Figure 1.8: For this graph, $e(v) = 1$.

DEFINITION 1.9. $O(v)$ is the set of vertices at odd distance from v. $o(v)$ is the cardinality of $O(v)$. $O_G(v)$ is used when refering so a specific graph G.

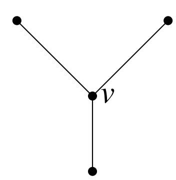


Figure 1.9: For this graph, $o(v) = 3$.

DEFINITION 1.10. $EH(v)$ is the set of v-even horizontal edges. $eh(v)$ is the cardinality of $EH(v)$.

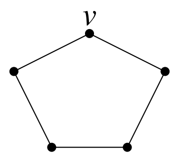


Figure 1.10: For this graph, $eh(v) = 1$.

DEFINITION 1.11. For any graph $G = (V, E)$, $\varepsilon = max_{v \in V}\{e(v) - eh(v)\}$.

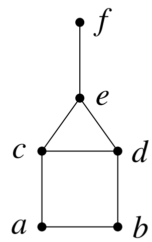


Figure 1.11: For this graph, $e(a) - eh(a) = 2$, $e(b) - eh(b) = 2$, $e(c) - eh(c) = 3$, $e(d) - eh(d) = 3$, $e(e) - eh(e) = 2$, $e(f) - eh(f) = 2$. Since $\varepsilon = max\{e(v) - eh(v)\}$ we get $\varepsilon = 3$.

DEFINITION 1.12. An *induced subgraph* of a graph G is a graph on a subset V_1 of $V(G)$ and the edges from G between the vertices of V_1. This graph is denoted $G[V_1]$.

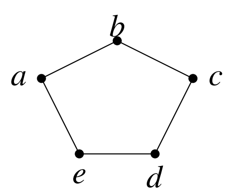


Figure 1.12: G

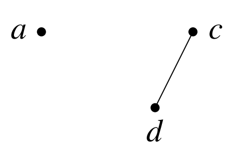


Figure 1.13: The above graph is the induced subgraph of 1 on the set $\{a, c, d\}$.

DEFINITION 1.13. A graph G is a *split graph* if the vertex set V can be partitioned into sets V_1, V_2 so that the graph induced on the vertices of V_1, $G[V_1]$ is an independent set and $G[V_2]$ is complete, where the graph induced on a set of vertices is the graph with the vertices in the set and the edges between only the vertices in the set.

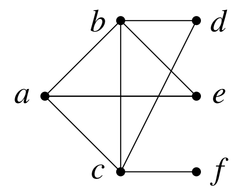


Figure 1.14: For this graph, $V_1 = \{a, b, c\}$ is the complete graph on three vertices, K_3 and $V_2 = \{d, e, f\}$ is an independent set.

DEFINITION 1.14. A *complete split graph* is a split graph such that all the vertices in V_1 are adjacent to every vertex in V_2.

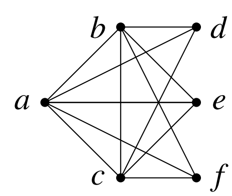


Figure 1.15: Again, $V_1 = \{d, e, f\}$ is an independent set and $V_2 = \{a, b, c\}$ is the complete graph on three vertices, K_3. α of a complete split graph equals $|V_1|$.

DEFINITION 1.15. A *complete graph*, K_n is the graph on n vertices with an edge between ever pair of vertices.

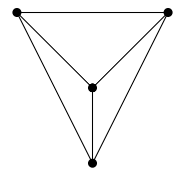
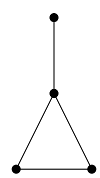


Figure 1.16: This graph is the complete graph K_4.

DEFINITION 1.16. A graph is *unicyclic* if it contains a single cycle. A *cycle* is a sequence of vertices $v_1, v_2, ..., v_n$ such that v_i is adjacent to v_{i+1} for $i = 1, ..., n-1$ and v_1 is adjacent to v_n .



DEFINITION 1.17. A *bipartite* graph is a graph G where the vertices can be partitioned into two sets V_1 and V_2 where each set is an independent set.

DEFINITION 1.18. A *complete bipartite graph*, $K_{m,n}$ is a bipartite graph with bipartition (A, B) where $|A| = m$ and $|B| = n$, with an edge between any vertex in one set of the bipartition to a vertex in the other set.

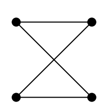


Figure 1.17: This is the complete bipartite graph, $K_{2,2}$.

DEFINITION 1.19. A *matching*, M, in a graph is a set of edges which are independent, that is, no pair of edges in the matching have a common endpoint. A *maximum matching* is a largest matching for a graph.

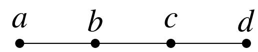


Figure 1.18: For this graph, $\{bc\}$ is a matching but it is not a maximum matching, whereas, $\{ab, cd\}$ is a maximum matching.

1.1 The Independence Number

In general the independence number α, of a graph is hard to compute. The independence number of a graph is the size of a maximum independent set of vertices. Since α is hard to compute the goal is to find bounds on this number to estimate α. Many people have tried to find bounds that approximate α well. The naive algorithm for computing the independence number looks at every subset of the vertices and there are 2^n of them, where n is the number of vertices of the graph. The algorithm with the best wosrt-case bound is due to Robson [9] and can be computed in order $2^{.276n}$ time which is still exponential.

There are known formulas for the independence number of the common graph classes. The following are examples of these common graph classes along with the formulas for computing α, where n is the number of vertices of the graph.

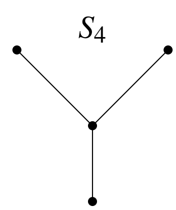


Figure 1.19: The stars, S_n. Here $\alpha(S_4) = 3$ and $\alpha(S_n) = n - 1$.

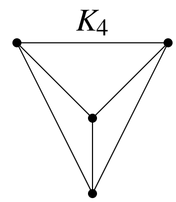


Figure 1.20: The complete graphs, K_n. Here $\alpha(K_4) = 1$ and $\alpha(K_n) = 1$.

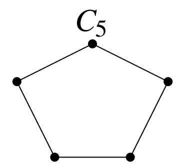


Figure 1.21: The cycles, C_n. Here $\alpha(C_5) = 2$ and $\alpha(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

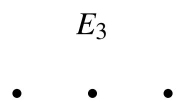


Figure 1.22: The empty graphs, E_n. Here $\alpha(E_3) = 3$ and $\alpha(E_n) = n$.

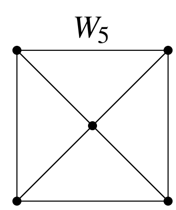


Figure 1.23: The wheels, W_n. Here $\alpha(W_4) = 1$ and $\alpha(W_n) = \left\lfloor \frac{n-1}{2} \right\rfloor$.

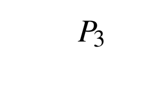


Figure 1.24: The path, P_n. Here $\alpha(P_3) = 2$ and $\alpha(P_n) = \left\lceil \frac{n}{2} \right\rceil$.

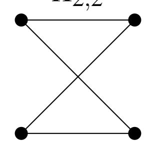


Figure 1.25: The complete bipartite graphs, $K_{m,n}$. Here $\alpha(K_{2,2}) = 2$ and $\alpha(K_{m,n}) = max\{m,n\}$.

The bound that we explore is an adaptation of a bound that was conjectured by a computer program called Graffiti. This program was created by Siemion Fajtlowicz to make conjectures about graph theory. These conjectures are discussed in the manuscript "Written on the Wall" [4]. The original conjectured bound was $\alpha \geq \max_{v\in V}\{|O(v)|\} - \min_{v\in V}\{|H(v)|\}$. The benefit of this bound is that it is fully distance related. Since distance is easy to compute, this made a bound on α easy to compute. A counterexample to the original bound was found by Peter Puget. Thus, this bound was modified by Siemion Fajtlowicz to $\alpha \geq \varepsilon = \max_{v\in V(G)}\{e(v) - eh(v)\}$ for any graph G.

1.2 The Even Horizontal Bound

A very important fact about ε is that it can be computed efficiently. In this section we will describe one algorithm for computing ε.

For any graph G, the distance between any pair of vertices, v and w can be computed efficiently. One algorithm for computing this distance runs in $O(n^3)$ time, where n is the number of vertices in G. This algorithm is known as Dijkstra's Shortest Path algorithm [12]. Using this algorithm we can find the distance matrix, D, where $D_{ij} = d(v_i, v_j)$. Since there are n^2 entries in D, this matrix can be computed in$O(n^5)$ time.

Now, for any vertex $v_i \in V(G)$, $e(v_i)$ is defined as the number of vertices at even distance from v_i. This can now be found by checking which entries in the i^{th} row of D are even. The following is an example.

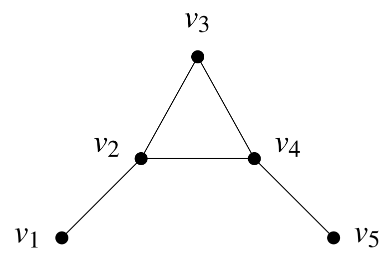


Figure 1.26: G

The distance matrix, D, for this graph is

$$\begin{bmatrix} 0 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 2 & 2 & 1 & 0 \end{bmatrix}$$

For this graph, $E(v_1) = \{v_1, v_3, v_4\}$.

Also, the adjacency matrix, A, for this graph, G, is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Where $A_{ij} = 0$ if v_i is not adjacent to v_j and $A_{ij} = 1$ if v_i is adjacent to v_j.

Now, the even horizontal edges with respect to a vertex v can be found simply by considering the graph induced on $E(v)$.

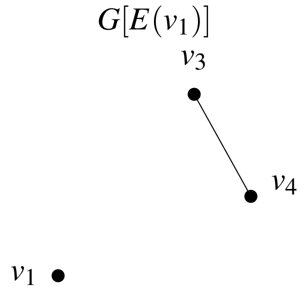


Figure 1.27: $EH(v_1) = \{v_3v_4\}$ and $eh(v_1) = |EH(V_1)| = 1$.

Thus, $e(v_1) - eh(v_1) = 3 - 1 = 2$. This quantity can now be calculated in the same way for the remaining vertices. After doing this, we find that $e(v_3) - eh(v_3) = 3$ and $\varepsilon(G) = max\{e(v) - eh(v)\} = 3$.

Here we found formulas for ε for some of the common graph classes. For all the following graph classes the formulas for α and ε were equal.

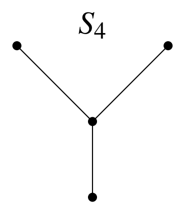


Figure 1.28: The stars, S_n. Here $\varepsilon(S_4) = 3$ and $\varepsilon(S_n) = n - 1$.

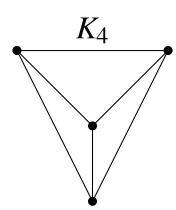


Figure 1.29: The complete graphs, K_n. Here $\varepsilon(K_4) = 1$ and $\varepsilon(K_n) = 1$.

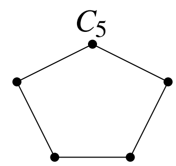


Figure 1.30: The cycles, C_n. Here $\varepsilon(C_5) = 2$ and $\varepsilon(C_n) = \lfloor \frac{n}{2} \rfloor$.

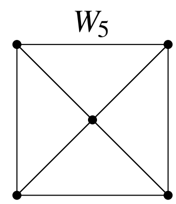


Figure 1.31: The wheels, W_n. Here $\varepsilon(W_5) = 2$ and $\varepsilon(W_n) = 2$ for all $n > 4$.

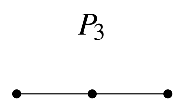


Figure 1.32: $\varepsilon(P_3) = 2$ and $\varepsilon(P_n) = \lceil \frac{n}{2} \rceil$.

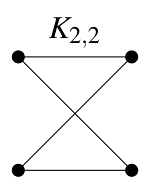


Figure 1.33: The complete bipartite graphs, $K_{m,n}$. Here $\varepsilon(K_{2,2}) = 2$ and $\varepsilon(K_{m,n}) = max\{m,n\}$.

1.2.1 The Difference Between α and ε can be arbitrarily large

Is there an example of a family of graphs where ε is a bad bound for α?

A family of graphs which demonstrates that the difference between α and ε can be arbitrarily large.

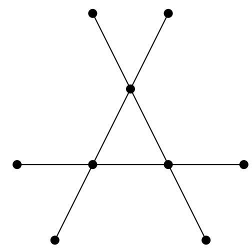


Figure 1.34: This is the graph of G_2.

Let G_r be a triangle with r pendants adjacent to each vertex. The pendents of G_r form a maximum independent set. So $\alpha = 3r$. There are two types of vertices in G_r, pendents and triangle vertices. For a pendent vertex v, $|E(v)| - |EH(v)| = (r+2) - 1 = r+1$. For each triangle vertex w, $|E(w)| - |EH(w)| = (2r+1) - 0 = 2r+1$. So $\alpha - \varepsilon = 3r - (2r+1) = r - l$, where $r-1$ will go to ∞.

1.2.2 Basic Facts About ε

The main goal we focused on was to characterize graphs where $\alpha = \varepsilon$. To begin doing this we worked to learn about the properties of the ε bound and how we could work with the sets $E(v)$ and $EH(v)$.

LEMMA 1.20. For any graph G, $\alpha(G) \geq n(G) - e(G)$ where $n(g)$ is the number of vertices of G and $e(G)$ is the number of edges in G.

Proof. Let G_i be a component of G. We know $\alpha(G_i) \geq 1$. Then $e(G_i)$ is at least the number of edges of any spanning tree of G_i. So $e(G_i) \geq n(G_i) - 1$. Hence, $n(G_i) - e(G_i) \leq 1$. We also know, $\alpha(G) \geq \sum \alpha(G_i) \geq$ the number of components of G $\geq \sum 1 \geq \sum[n(G_i) - e(G_i)] = \sum n(G_i) - \sum e(G_i) = n(G) - e(G)$. □

So, $\alpha(G[D_i(v)]) \geq n(G[D_i(v)]) - e(G[D_i(v)]) = n(G[D_i(v)]) - |EH(v) \cap E(D_i(v))|$ since $E(D_i(v))$ is contained in $EH(v)$.

THEOREM 1.21. For any fixed vertex v of a graph G, $\alpha \geq e(v) - eh(v)$. Similarly, $\alpha \geq |O(v)| - |OH(v)|$.

Proof. Let $D_i(v)$ be the set of vertices at distance i from v. Then, $E(v) = D_0(v) \cup D_2(v) \cup \ldots$. So $\alpha(G) \geq \sum \alpha(G[D_{2i}(v)])$ since $\sum \alpha(G[D_{2i}(v)])$ is the size of an independent set because each $D_{2i}(v)$ is disjoint. Then $\alpha(G) \geq \sum \alpha(G[D_{2i}(v)]) \geq \sum[n(G[D_{2i}(v)]) - e(G[D_{2i}(v)])] = \sum(n(G[D_{2i}(v)])) - \sum(e(G[D_{2i}(v)])) = |E(v)| - |EH(v)|$. The second inequality comes from Lemma 1.20. □

THEOREM 1.22. (Fajtlowicz [4]) For any graph G, $\alpha \geq \varepsilon$.

This is a direct consequence of Theorem 1.21.

PROPOSITION 1.23. For a bipartite graph $\max_v\{e(v) - eh(v)\} = \max_v\{e(v)\}$.

Proof. Since there are no odd cycles in bipartite graphs and even horizontal edges only occur when an odd cycle is present. So $|EH(v)| = 0$. □

PROPOSITION 1.24. If G is a connected bipartite graph, then the vertices at even and odd distance from a vertex v in $V(G)$ form a partition of the vertices into two independent sets.

Proof. Let $v \in V(G)$ then the sets $E(v)$ and $O(v)$ form a partition because there can not be a vertex at both even and odd distance from v. Since G is bipartite, there are no odd cycles and no even horizontal edges. Then there are no edges between vertices in $E(v)$ since there are no horizontal edges. Also, there are no edges between vertices in $O(v)$ since there are no odd cycles. Hence both sets $E(v)$ and $O(v)$ are independent sets. □

LEMMA 1.25. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ and, for all i, $x_i \geq y_i$ and $x_i \geq 1$
then, for all i, $x_i = y_i$.

Proof. Suppose there exists a j such that $x_j > y_j$ then $\sum_{i=1}^{n} x_i = \sum_{i=1}^{i=j-1} x_i + x_j + \sum i = j+1^n x_i$ $> \sum_{i=1}^{i=j-1} y_i + y_j + \sum_{i=j+1}^{n} y_i$, which is a contradiction. Hence no such j exists. $\square$

LEMMA 1.26. If G is connected, then $\alpha(G) = n(G) - e(G)$ if and only if G is a K_1 or a K_2.

Proof. For a connected graph, $e \geq n-1$. So, $n - e \geq n - (n-1) = 1 \leq \alpha$. So, $\alpha = n - e$ implies $\alpha = 1$. Hence, G is a K_1 or K_2. $\square$

LEMMA 1.27. For any graph G, $\alpha(G) = n(G) - e(G)$ if, and only if, every component of G is a K_1 or K_2.

Proof. Let $G_1, ..., G_k$ be the components of G. Then $\alpha(G) = \sum_{i=1}^{k} \alpha(G_i)$. Suppose $\alpha(G) = n(G) - e(G)$. Then $\alpha = \sum_{i=1}^{k}[n(G_i) - e(G_i)]$. Since $\alpha(G_i) \geq n(G_i) - e(G_i)$ for all i, it follows from Lemma 1.25 that $\alpha(G_i) = n(G_i) - e(G_i)$. Then by the Lemma 1.26 each component, G_i, is a K_1 or a K_2.

Conversely, assume every component of G is a K_1 or a K_2. Let A be the set of K_1s and B be the set of K_2s. Then $\alpha(G) = \sum \alpha(G_i)$
$= \sum_A \alpha(G_i) + \sum_B \alpha(G_i) = \sum_A [n(G_i) - e(G_i)] + \sum_B [n(G_i) - e(G_i)]$
$= \sum n(G_i) - \sum e(G_i) = n(G) - e(G)$. $\square$

The Main Conjecture

While trying to characterize graphs where $\alpha = \varepsilon$ we came up with a conjecture in which it would be easy to determine whether or not a graph is of this form. The conjecture says that α will equal ε if an only if for an ε-maximizing vertex v, the graph induced on the vertices at even distance from v consists of isolated vertices or paths of length one and there exists a matching M that saturates the vertices at odd distance from v. This conjecture is still open as it has not been proved and no counterexample has been found.

CONJECTURE 2.1. For any graph G, $\alpha = \varepsilon$ if and only if, for every ε-maximizing vertex v, the components of the graph induced on $E(v)$ are K_1s and K_2s and there is a matching that saturates $O(v)$.

For any graph G and any set $X \subseteq V(G)$, it can be checked in polynomial-time whether there is a matching that saturates X. This can be computed using Edmonds' algorithm for finding a maximum matching of a graph [8]. Let $X \subseteq V(G)$. We need only consider the graph induced on $X \cup N(X)$, since any edge which saturated a vertex of X will be included in the induced edges. Let $G' = G[X \cup N(X)]$. Find a maximum matching M for G'. If M saturates X in G' then M saturates X in G. If M does not saturates X in G' then no matching saturates X in G.

For this conjecture, it is important to note that the conditions that need to be checked can all be found efficiently.

THEOREM 2.2. For any graph G if $\alpha = \varepsilon$ and v is an ε-maximizing vertex then the graph $G[E(v)]$ consists of components which are K_1s and K_2s.

Proof. Let v be a vertex which maximizes $e(v) - eh(v)$. Then $e(v) - eh(v) = |E(v)| - |EH(v)| = \sum |D_{2i}(v)| - \sum |EH_{2i}(v)|$
$= \sum(|D_{2i}(v)| - |EH_{2i}(v)|) = \sum[n(G[D_{2i}(v)]) - e(G[D_{2i}(v)])]$.
Assume $\alpha = e(v) - eh(v)$. By Lemma 1.20, $\alpha(G[D_{2i}(v)]) \geq n(G[D_{2i}(v)]) - e(G[D_{2i}(v)])$.
So, $\sum[n(G[D_{2i}(v)]) - e(G[D_{2i}(v)])] = e(v) - eh(v) = \alpha(G) \geq \sum \alpha(G[D_{2i}(v)])$
$\geq \sum n(G[D_{2i}(v)]) - e(G[D_{2i}(v)])$.
So, they all must be equal. Now, for all i, $\alpha(G[D_{2i}(v)]) = n(G[D_{2i}(v)]) - e(G[D_{2i}(v)])$ by Lemma 1.25. Hence, by Lemma 1.27, each $G[D_{2i}(v)]$ has components which are K_1s or K_2s. So $G[E(v)]$ has components which are K_1s or K_2s. □

CONJECTURE 2.3. For any graph G, $\alpha = \varepsilon$ if, and only if, for every ε-maximizing vertex v, the components of $G[E(v)]$ are K_1s and K_2s.

A counterexample to this conjecture is the Benzenoid in Figure 2.1. For this graph $G[E(v)]$ consists of only K_1s and $\alpha \geq 22$ but $\varepsilon = 21$. So $\alpha \neq \varepsilon$.

CONJECTURE 2.4. If G is a graph with ε-maximizing vertex v and the components of $G[E(v)]$ are K_1's or K_2's, then $\alpha(G) = e(v) - eh(v)$.

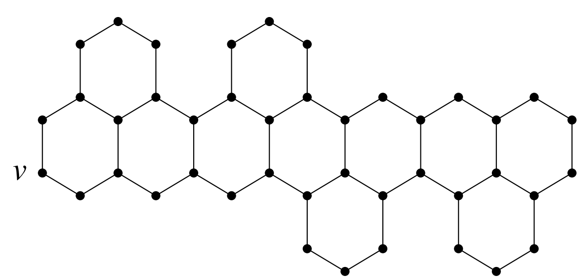


Figure 2.1: This graph is a benzenoid and is bipartite so for all v, $eh(v) = 0$. Here $\alpha \geq 22$ but $\varepsilon = 21$. So $\alpha \neq \varepsilon$. This is a counterexample to Conjecture 2.4. [8]

THEOREM 2.5. If $\alpha = \varepsilon$ then, for any ε-maximizing vertex v, there is a matching which saturates $O(v)$.

Proof. Assume $\alpha = \varepsilon$. Let v be an ε-maximizing vertex. By Lemma 1.27, we know $G[E(v)]$ consists of components which are K_1s and K_2s. So if we take all the vertices from the K_1s and one vertex from each of the K_2s in $G[E(v)]$ we will get an independent set with ε vertices. So, we get a maximum independent set, I, that is contained in $E(v)$. Now, by Theorem 2.9 we know that any maximum matching of G saturates V/I. Since I is contained in $E(v)$, then $O(v)$ is contained in V/I. Hence any maximum matching must saturate $O(v)$. $\square$

COROLLARY 2.6. If $\alpha = \varepsilon$ then, for any ε-maximizing vertex v, the components of $G[E(v)]$ are K_1s and K_2s and there is a matching which saturates $O(v)$.

Corollary 2.6 was generalized to get the main conjecture (Conjecture 2.1).

CONJECTURE 2.7. For any graph, $\alpha = \varepsilon$ if, and only if, for any ε-maximizing vertex v there exists a matching which saturates $O(v)$.

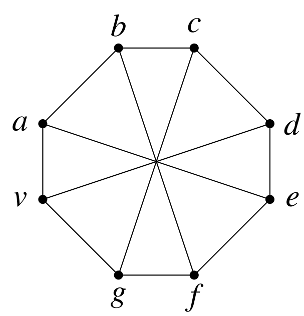


Figure 2.2: This graph is a counterexample since for ε-maximizing vertex v with $O(v) = \{a,d,g\}$ there is a matching $M = \{va,bc,de,fg\}$ which saturates $O(v)$, but $\alpha = 3$ and $\varepsilon = 2$. However, this is not a counterexample to the main conjecture (Conjecture 2.1) since the components of $E(v)$ do not consist of K_1s and K_2s.

THEOREM 2.8. (Hall's Theorem) If G is a bipartite graph with bipartition $G = (A,B)$ then there is a matching which saturates A if and only if, for any set $S \subset A$, $|N(S)| \geq |S|$.

THEOREM 2.9. For any graph G and any maximum independent set I, there is a matching which saturates I^c.

Proof. Let I be a maximum independent set of a graph G. Let M be a maximum matching of I^c. So $I' = I^c \backslash V(M)$ is an independent set. Let $G' = G[I \cup I']$. G' is bipartite with bipartition (I, I'). Let $S \subset I'$. So, $|N_{G'}(S)| \geq |S|$, otherwise $I \backslash N_{G'}(S) \cup S$ is a larger independent set than I. Then by Hall's Theorem, there exists a matching in G' which saturates I^c, and this matching necessarily saturates I^c in G. $\square$

If I is an independent set and there is a matching which saturates I^C is it true that I is a maximum independent set?

No, C_5 is a counterexample as I can be any single vertex, then there is a matching that saturates the remaining four vertices, but $\alpha(C_5) = 2$ and I is not a maximum independent set.

2.0.3 Working With ε

The independence number α is a well-behaved invariant as deleting a vertex can never increase the independence number. That is, for every vertex v, $\alpha(G-v) \leq \alpha(G)$. ε is not well-behaved. It can either increase, decrease, or stay the same when a vertex is deleted. Here we explore ε and other distance related properties as we try to develop tools for attacking conjectures.

The following conjecture was found to be false. We conjectured that there exists a vertex in any graph such that removing that vertex does not affect the distance between the remaining vertices. We hoped this would help to prove the main conjecture 2.1 by completing the attempted proof by induction above.

CONJECTURE 2.10. For any connected graph G, there is a vertex w so that for all $x, y \in V(G-w)$, $d_G(x,y) = d_{G-w}(x,y)$.

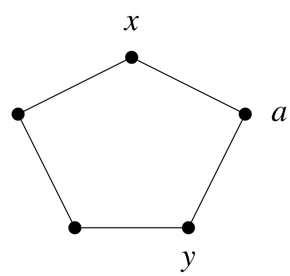


Figure 2.3: A counterexample for Conjecture 2.10 is C_5, since every vertex is on a unique shortest path between some vertices x and y. For this graph, deleting the vertex a increases the distance between x and y.

After working with the ε bound we found that is does not behave very well. By removing a vertex from a graph ε can increase, decrease, or stay the same.

ε can stay the same when removing a vertex w as in C_5 or C_6. ε can increase like the examples from Conjecture 2.11.

ε can also decrease, for example in P_3, by removing a vertex that satisfies the conditions, ε decreases from 2 to 1.

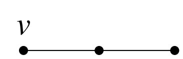


Figure 2.4: Example of ε decreasing

CONJECTURE 2.11. For any connected graph, either $\alpha = \varepsilon$ or there is a vertex v so that $\alpha(G) = \alpha(G-v)$ and $\varepsilon(G) < \varepsilon(G-v)$.

Assume $\alpha \neq \varepsilon$. Then some counterexamples to Conjecture 2.11 include the following two graphs.

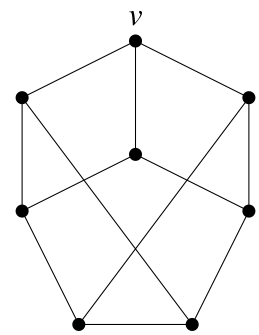


Figure 2.5: G

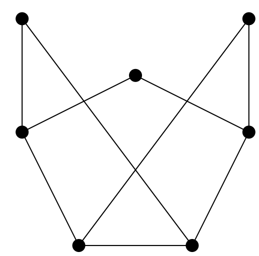


Figure 2.6: $G - v$

For this graph $\alpha(G) = 3$, $\varepsilon(G) = 2$, $\alpha(G - v) = 3$, and $\varepsilon(G - v) = 3$.

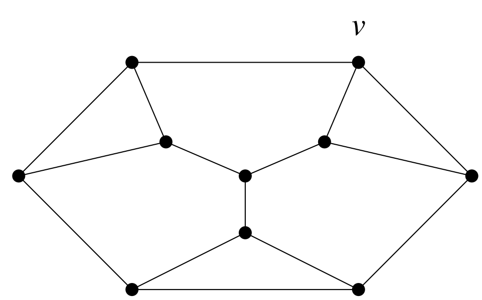


Figure 2.7: G

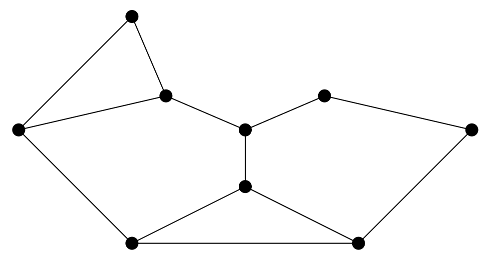


Figure 2.8: $G-v$

For this graph $\alpha(G)=4, \varepsilon(G)=3, \alpha(G-v)=4$, and $\varepsilon(G-v)=4$.

DEFINITION 2.12. A *connector* is a non-cut-vertex.

CONJECTURE 2.13. For any graph G, if $\alpha(G)=\varepsilon(G)$ and v is an ε-maximizing connector, then $\alpha(G-v)=\varepsilon(G-v)$ and $\varepsilon(G-v)\leq\varepsilon(G)$.

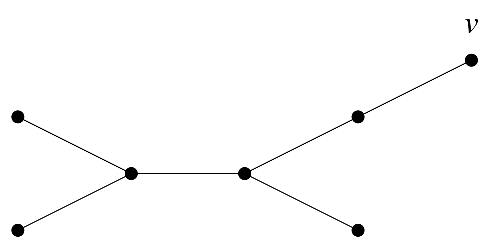


Figure 2.9: This is a counterexample to Conjecture 2.13 since $\alpha=\varepsilon=4$ where v is an ε-maximizing connector but $\alpha(G-v)=4$ and $\varepsilon(G-v)=3$.

CONJECTURE 2.14. For any graph G, if $\alpha(G)=\varepsilon(G)$ and v is an ε-maximizing vertex then $\varepsilon(G-v)\leq\varepsilon(G)$.

This conjecture is still open.

2.1 Vertex Transitive Graphs

The next class of graphs we looked at were vertex transitive graphs. A graph G is vertex-transitive if for any pair of vertices v, w there is an automorphism of the vertices of G that sends v to w. These graphs were very easy to work with since we only had to compute $e(v) - eh(v)$ once to determine ε. This is because for any two vertices v, w in a vertex transitive graph G, we prove that $e(v) - eh(v) = e(w) - eh(w)$. The main conjecture (Conjecture 2.1) was true for all the cubic vertex-transitive graphs up to 22 vertices [11].

DEFINITION 2.15. An *isomorphism* from graph G to H is a bijective function $\theta : V(G) \to V(H)$ such that for all $v, w \in V(G)$, v is adjacent to w if and only if $\theta(v)$ is adjacent to $\theta(w)$.

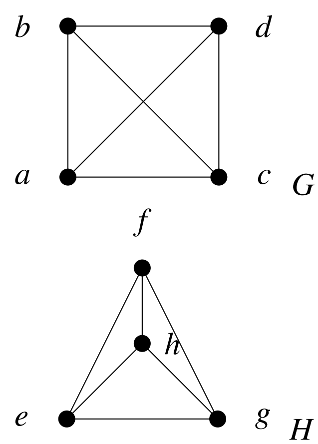


Figure 2.10: Let $\theta : V(G) \to V(H)$ be defined as follows. $\theta(a) = e$, $\theta(b) = f$, $\theta(c) = g$, and $\theta(d) = h$. θ defines an isomorphism between G and H. So, G and H are isomorphic.

DEFINITION 2.16. An *automorphism* is an isomorphism from a set to itself.

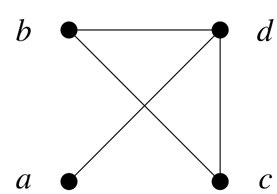


Figure 2.11: An automorphism for the graph is $\phi : V \to V$ defined by $\phi(a) = a$, $\phi(d) = d$, $\phi(b) = c$, $\phi(c) = b$.

DEFINITION 2.17. A graph G is *vertex-transitive* if for every pair of vertices v,w there is an automorphism ϕ: $V(G) \to V(G)$ such that $\phi(v) = w$.

LEMMA 2.18. If G is a vertex-transitive connected graph then for all v,w in $V(G)$ and automorphism $\theta : V(G) \to V(G)$, $d(v,w) = d(\theta(v),\theta(w))$.

Proof. Let $d(v,w)$ be the length of a shortest path from v to w defined by $v = v_0, v_1, \ldots, v_k = w$. So, by the definition of an automorphism, $\theta(v) = \theta(v_0), \theta(v_1), \ldots, \theta(v_k) = \theta(w)$ is a path of length k from $\theta(v)$ to $\theta(w)$. So, $d(\theta(v),\theta(w)) \leq k$. Suppose there is a shorter path from $\theta(v)$ to $\theta(w)$. Then $\theta(v) = u_0, u_1, \ldots, u_l = \theta(w)$, with $l < k$. Then $v = \theta^{-1}(u_0), \theta^{-1}(u_1), \ldots, \theta^{-1}(u_l) = w$ is a path from v to w of length less than k. Which is a contradiction since $d(v,w) = k$. Therefore $d(v,w) = d(\theta(v),\theta(w))$. □

COROLLARY 2.19. Let G be a vertex-transitive connected graph, then $e(v) - eh(v) = e(w) - eh(w)$ for v,w in $V(G)$.

Proof. Assume a graph G is vertex-transitive. Then for every pair of vertices v,w in $V(G)$ there is an automorphism, $\theta : V(G) \to V(G)$ such that $\theta(v) = w$. Then for any vertex y, at even distance from v, θ would send y to a vertex z at the same even distance from w, by Lemma 2.18. So $e(v) = e(w)$. Then any even-horizontal edge to v with endpoints x and y would be sent to an edge with endpoints u and z, by θ that would be an even-horizontal edge to w. Hence $eh(v) = eh(w)$. Therefore, $e(v) - eh(v) = e(w) - eh(w)$. □

THEOREM 2.20. Let G be a vertex-transitive graph, then for any vertex $v \in V(G)$, there is a maximum independent set containing v.

Proof. Let $I = \{v_1, \ldots, v_\alpha\}$ be a maximum independent set in G and $v \in V(G)$. Let $\phi : V(G) \to V(G)$ be an automorphism, so that $\phi(v_1) = v$. Now, let $\phi(I) = \{\phi(v_1), \ldots, \phi(v_\alpha)\}$. By the definition of an automorphism, $\phi(I)$ is a maximum independent set containing v. □

CONJECTURE 2.21. For any vertex-transitive connected graph G, $\alpha = \varepsilon$ if and only if there exists an ε-maximizing vertex v where the components of $E(v)$ consists of K_1s and K_2s and there exists a matching which saturates $O(v)$.

This conjecture is still open. As mentioned above the statement holds for all vertex-transitive graphs up to 22 vertices.

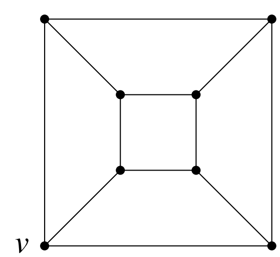


Figure 2.12: For this graph $\varepsilon = \alpha = 4$, the components of $E(v)$ are K_1s and K_2s and there is a matching which saturates $O(v)$.

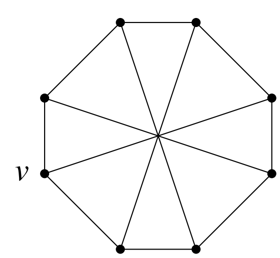


Figure 2.13: For this graph $\alpha = 3$ and $\varepsilon = 2$. There is a matching which saturates $O(v)$ but the components of $E(v)$ do not consist of K_1s and K_2s.

2.2 The Case of Bipartite and KE Graphs

Kónig-Egerváry graphs are graphs where $\alpha(G) + \alpha'(G) = n(G)$. $\alpha'(G)$ is the matching number which is the size of a maximum matching. These graphs will be referred to as KE graphs. KE graphs are a larger class of graphs than bipartite graphs since all bipartite graphs are KE, Figure 2.2. Bipartite graphs are a special case of graphs that could be characterized when $\alpha = \varepsilon$. We worked with bipartite graphs since there are no horizontal edges, as was proved earlier in Proposition 1.23. This made the ε bound even easier to compute since for these graphs $\varepsilon = max_{v \in V}\{e(v)\}$.

THEOREM 2.22. (Kónig-Egerváry Theorem) If a graph G is bipartite then $\alpha(G) + \alpha'(G) = n(G)$. [12]

THEOREM 2.23. If G is bipartite, with bipartition (B,W) then $\alpha = |W| \geq |B|$ if, and only if, there is a matching which saturates B.

Proof. Assume G is bipartite with bipartition (B,W).
Assume $\alpha = |W|$. We know that $\alpha + \alpha' = n$ by the Kónig-Egerváry theorem and $|B| + |W| = n$. So, by substitution we get $\alpha' = |B|$. Let M be a maximum matching. So, $\alpha' = |M|$. Since W and B are independent sets and all edges in M must have one endpoint in B and one in W, it follows that M must saturate B.

Conversely, assume there is a matching that saturates B. Then $\alpha' \geq |B|$. We know $|W| + |B| = n$ and by Theorem 2.22 $n = \alpha + \alpha'$. So, $|W| + |B| \geq \alpha + \alpha' = |W| + |B|$. Now, $\alpha \geq |W|$. Assume $\alpha > |W|$. So, $n = \alpha + \alpha' > |W| + |B| = n$. This says $n > n$ which is a contradiction. Hence, $\alpha = |W|$. □

This statement does not hold in general. The following graph is an example.

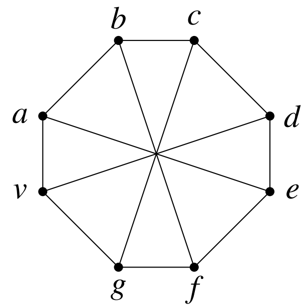


Figure 2.14: This graph is a counterexample since for ε-maximizing vertex v with $O(v) = \{a,d,g\}$ there is a matching $M = \{va, bc, de, fg\}$ which saturates $O(v)$, but $\alpha = 3$ and $\varepsilon = 2$.

COROLLARY 2.24. If G is a bipartite graph, $\alpha = \varepsilon$ if, and only if, there is a vertex v and a matching M so that M saturates $O(v)$.

Proof. Let G be bipartite. Suppose, $\alpha = \varepsilon = \max\{e(v) - eh(v)\}$. Let v be a vertex so that $e(v) = |E(v)|$ is maximized. We know, for every v, $EH(v) = \phi$ and $eh(v) = 0$. So, $e(v) - eh(v) = e(v)$. Since G is bipartite, $(E(v), O(v))$ is a bipartition. By our assumption, we get $\alpha = e(v) - eh(v) = e(v) = |E(v)|$. So, $|E(v)| \geq |O(v)|$. By Theorem 2.23 there is a matching that saturates $O(v)$.

Conversely, assume there is a vertex v and a matching M so that M saturates $O(v)$. We know, $|E(v)| + |O(v)| = n$ and $\alpha + \alpha' = n$ since G is bipartite. Suppose, $\alpha' > |O(v)|$ then $n = \alpha + \alpha' > |E(v)| + |O(v)| = n$. This gives us the contradiction that $n > n$. So, $\alpha' = |O(v)|$ and $\alpha = |E(v)|$. $\square$

After looking at bipartite graphs, we moved onto a larger class of graphs. Since every bipartite graph is KE, the next step was to explore the case of KE graphs.

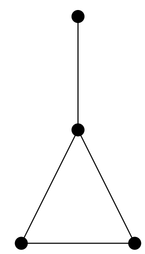


Figure 2.15: An example of a non-bipartite KE graph where $\alpha = \varepsilon$. For this graph $\alpha = 2, \alpha' = 2, n = 4, \varepsilon = 2$.

PROPOSITION 2.25. For any graph G and maximum independent set I, there is a *KE* subgraph G' of G with maximum independent set I. So $\alpha(G') = \alpha(G)$.

Proof. Let $G' = G[I]$. $\square$

It follows from Proposition 2.25 that, for any graph G, any maximum independent set I there is an induced *KE* subgraph G' containing I which is maximum with respect to the number of vertices.

Let G be a connected graph with maximum independent set I. Let G' be a maximum *KE* subgraph containing I. So $I \subset V' \subset V(G)$ with $G' = G[V']$. Let v be an ε-maximizing vertex in G'. Then by Proposition 2.25, $\alpha(G) = \alpha(G')$. Can $\varepsilon(G') > \varepsilon(G)$? The next example shows that the answer is no.

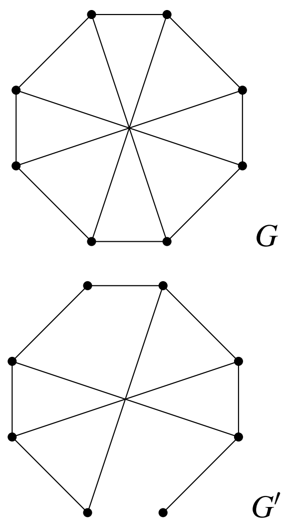


Figure 2.16: For this example $\varepsilon(G') = 4 > \varepsilon(G) = 2$. G' is a maximum KE subgraph since adding any other edge from G would make G' no longer KE.

2.3 Residue

The residue, R, of a graph is an invariant that is found after completing the Havel-Hakimi process. The Havel-Hakimi process is defined in this section. The residue is the number of zeros in the sequence when the Havel-Hakimi process is completed. This invariant is considered to be a good lower bound for the independence number, α, of a graph. The intent of working with this bound was to see how the ε bound and residue compared. We found that ε was as good if not better than residue, for the examples we considered. Some examples of graphs where ε was better than residue were vertex-transitve cubic graphs. See Figure 2.3 for an example.

DEFINITION 2.26. Given a degree sequence $d = (d_1, \ldots, d_n)$ of a graph with $d_1 \geq d_2 \geq \ldots \geq d_n$ the *derived sequence* d' is the sequence formed by deleting d_1 and reducing the next d_1 terms in the sequence by one.

DEFINITION 2.27. A non-negative sequence of integers $d = (d_1, ..., d_n)$ is *graphic* if it is the degree sequence of a graph.

THEOREM 2.28. (Havel-Hakimi Theorem) A sequence d of non-negative integers is graphic if and only if the derived sequence d' is graphic.

COROLLARY 2.29. Given a graphic sequence d, repeated application of the Havel-Hakimi process must end with a sequence of zeros.

DEFINITION 2.30. The *residue*, R, of a graph is the number of zeros in the resulting sequence at the end of the Havel-Hakimi process. (This definition is due to Fajtlowicz [6].)

THEOREM 2.31. The residue R of a graph is a lower bound for the independence number for the graph.

This theorem has been proved multiple times, first by Favaron, Maheo and Sacle [5].

CONJECTURE 2.32. For any graph G, $\varepsilon \geq R$.

This conjecture is true for all vertex-transitive cubic graphs with up to $n = 22$ vertices.

Graphs where ε is a better bound on α than R are vertex-transitive cubic graphs like the graph below.

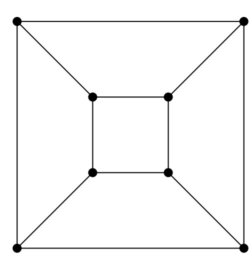


Figure 2.17: For this graph $R = 2$, $\varepsilon = 4$ and $\alpha = 4$.

The derived sequence and residue for Figure 2.3 is found as follows. Since there are two zeros in the final sequence the residue $R = 2$.

$$3,3,3,3,3,3,3,3$$

$$3,3,3,3,2,2,2$$

$$2,2,2,2,2,2$$

$$2,2,2,1,1$$

$$1,1,1,1$$

$$1,1,0$$

$$0,0$$

2.4 Unicyclic and Split Graphs

We looked at unicyclic and split graphs to further classify graphs using the epsilon bound. A unicyclic graph has only one cycle so, for any vertex v of the graph would have at most one even-horizontal edge. Split graphs can be partitioned into a complete graph and an independent set. We were able to classify split graphs completely when $\alpha = \varepsilon$.

Unicyclic graphs are not all KE.

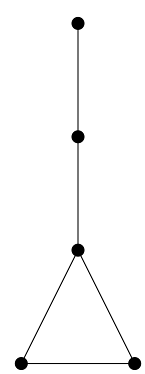


Figure 2.18: For this graph $\alpha = \varepsilon = 2, \alpha' = 2, n = 5$.

CONJECTURE 2.33. For any unicyclic graph, $\alpha = \varepsilon$.

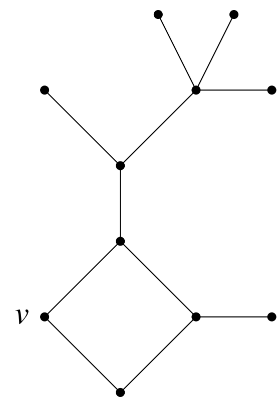


Figure 2.19: This graph is a counterexample to Conjecture 2.33 as $\alpha = 7$ and $\varepsilon = 6$.

After finding a counterexample where $\alpha \neq \varepsilon$ for a unicyclic graph, we worked on finding graphs where $\alpha = \varepsilon$. Since a unicyclic graph has one cycle, we noted that if we remove the cycle the remaining graph would be a tree. Since all trees are bipartite, we know that $\varepsilon = \max_{v \in V} e(v)$. So we decided to use graph minors to accompish this. The graph minor H of a unicyclic graph was formed by contracting the vertices of the cycle to a single vertex. To do this, start with any vertex in the cycle and one of its neighbors. Remove the edge between these and combine the two vertices into one vertex keeping the neighbors intact. Continuing this process until the cycle is one vertex we can obtain the graph minor.

DEFINITION 2.34. A graph H is a *minor* of a graph G if there is a partition $P = \{V_1, ...V_t\}$ of $V(G)$ with $V(H) = P$ and $E(H) = \{V_iV_j | \exists v \in V_i, \exists w \in V_j, and vw \in E(G)\}$

CONJECTURE 2.35. For a unicyclic graph G, with minor H formed by contracting the vertices of the cycle, $\alpha(G) = \varepsilon(G)$ if and only if $\alpha(H) = \varepsilon(H)$.

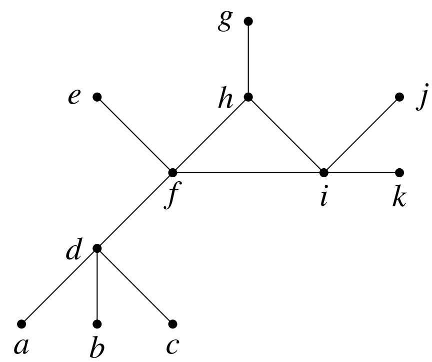


Figure 2.20: For the graph G, $\alpha(G) = \varepsilon(G) = 7$.

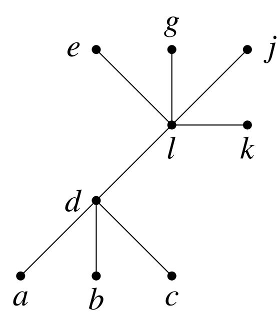


Figure 2.21: The partition used to create the minor H of G is $P = \{\{a\},\{b\},\{c\},\{d\},\{e\},\{g\},\{j\},\{k\},\{f,h,i\}\}$. The vertex l in H is created by contracting the vertices in the set $\{f,h,i\}$ into one vertex. Now, $\alpha(H) = 7$ and $\varepsilon(H) = 5$. This graph is a counterexample to the conjecture.

THEOREM 2.36. For a complete split graph, $\alpha = \varepsilon$.

Proof. Let v be an ε-maximizing vertex for a complete split graph G. If v is in the independent set I then $E(v) = I$ and $EH(v)$ is empty. If v is in the complete part C then $E(v) = \{v\}$ and $EH(v)$ is empty. So, $max\{e(v) - eh(v)\} = |E(v)| - |EH(v)|$ for any v in I. Then $|E(v)| - |EH(v)| = |I| - 0 = \alpha$. $\square$

2.5 Spanning Trees

A spanning tree, T, of a connected graph, G, is a tree on the vertex set of G that uses only edges in G. We worked with spanning trees hoping that knowing about ε for trees would give information about the original graph. To use this we would find an ε maximizing vertex in G and use that vertex to construct a spanning tree in a specific way which is explained in this section.

LEMMA 2.37. Let G be a connected graph and let v be any vertex and u a vertex at maximum distance from v. Then $G' = G - u$ is connected.

Proof. Assume $G' = G - u$ is not connected. We know G is connected so deleting the vertex u must have created components. Let A be the component containing v. Let B be another component. So there must be a w in B with u adjacent to w. This means that there are no edges between vertices of A and B and every path from v to w must contain u. So, u must be on a shortest path from v to a w. So, $d(v,w) = d(v,u) + 1$ and there exists a path of length longer than the path from v to u. This contradicts the fact that u was a vertex at maximum distance from v in G. Hence, $G' = G - u$ is connected. □

LEMMA 2.38. For a connected graph G, and any vertex v, there is a spanning tree T of G so that $d_T(v,w) = d_G(v,w)$ for any w in $V(G)$.

Proof. Let $D_i(v) = \{w | d_G(v,w) = i\}$. Assume the statement is true for connected graphs with fewer than n vertices. Let G be a graph with n vertices and v in $V(G)$ so that no spanning tree T has $d_T(v,w) = d_G(v,w)$. Let u be a vertex so that $d(u,v)$ is maximum. By Lemma 2.37, $G' = G - u$ is connected. By assumption there is a spanning tree T' of G' so $d_{T'}(v,w) = d_{G'}(v,w)$ for every vertex w in G'.

Now, we show $d_T(v,w) = d_G(v,w)$. Let u' be a neighbor of u on a shortest path from v to u in G. Form T with $V(T) = V(G)$ and $E(T) = E(T') \cup vu'$ where u must have degree

1 in T. So, clearly T is a tree. Let w be in G. Case I: Let $w \neq u$. So $d_{T'}(v,w) = d_{G'}(v,w)$ by assumption, and $d_{G'}(v,w) = d_G(v,w)$. Now, there is no shortest path from v to w in G, G' or T' containing u. So the distances must be the same. Case II: Let $w = u$. So $d_T(v,w) = d_T(v,u) = d_T(v,u') + d_T(u',u)$ since u' is on a shortest path from v to u. Then $d_{t'}(v,u') = d_{G'}(v,u') = d_g(v,u') + 1 = d_G(v,u)$. $\square$

The following is the construction for a spanning tree.

Let v be a vertex of G. Separate the remaining vertices of G into levels by distance from $v, D_1(v), D_2(v), \ldots, D_i(v)$. Let $D_m(v)$ be the level at maximum distance from v. Start with a vertex w in $D_m(v)$. Find one neighbor of w in D_{m-1} and draw an edge between them. Repeat for all vertices in $D_m(v)$ then continue for vertices in $D_{m-1}(v)$ and neighbors in $D_{m-2}(v)$. Continue until v is reached. Let the resulting graph be called T. T is a tree since there will be no cycles, otherwise two vertices in the same level would have to be adjacent which would not happen by construction. T is a spanning tree of G since the construction will use all vertices of G only once. Finally, $d_T(v,w) = d_G(v,w)$ for all w in G as the vertices are separated into levels of distance first and paths are created using only one vertex from each level. See Figures 2.5, 2.5, and 2.5.

We tried to use the trees as a proof technique. By their construction we know $e_T(v) - eh_T(v) \geq e_G(v) - eh_G(v)$. We also know Theorem 2.24 which characterizes $\alpha = \varepsilon$ for bipartite graphs. Since all trees are bipartite, the same result holds for trees.

CONJECTURE 2.39. Let v be a vertex so $\varepsilon = e(v) - eh(v)$. Let T be a spanning tree that preserves distances from v. Let T' be formed with $E(T') = E(T) \cup E_G(v)$. If $\alpha(T') = \alpha(G)$ then $\varepsilon(T') = \varepsilon(G)$.

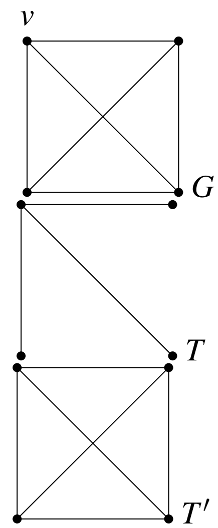


Figure 2.22: For this example, G is not KE and $\alpha(G) = \varepsilon(G) = 1$. T' is also not KE and $\alpha(T') = \varepsilon(T') = 1$.

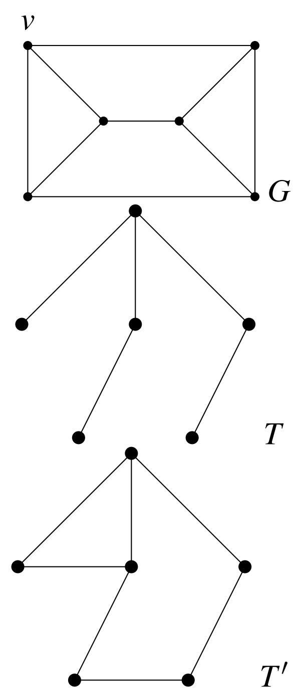


Figure 2.23: For this example, G is not KE and $\alpha(G) = \varepsilon(G) = 2$. T' is KE and $\alpha(T') = \varepsilon(T') = 3$.

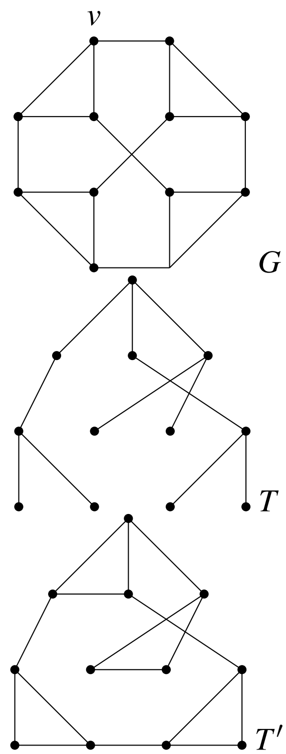


Figure 2.24: For this example, G is not KE and $\alpha(G) = \varepsilon(G) = 4$. T' is not KE and $\alpha(T') = \varepsilon(T') = 4$.

Hence, T' is not always KE and $\alpha(G) \neq \alpha(T')$ or $\varepsilon(G) \neq \varepsilon(T')$. Though, $\alpha(T') = \varepsilon(T')$ for all examples.

2.6 Summary of Results and Open Problems

Important results obtained from working on the main conjecture include:

1. For any graph G, if $\alpha(G) = \varepsilon(G)$ then for every ε-maximizing vertex v, the graph $G[E(v)]$ consists of components which are K_1s and K_2s.

2. If $\alpha = \varepsilon$ then, for any ε-maximizing vertex v, there is a matching which saturates $O(v)$.

3. The main conjecture (Conjecture 2.1) is true for vertex-transitive graphs up to 22 vertices.

4. The main conjecture (Conjecture 2.1) is proven to be true for bipartite graphs.

5. $\alpha = \varepsilon$ for complete split graphs.

Ideas that were disproven include:

1. For any graph G, $\alpha = \varepsilon$ if and only if, for every ε-maximizing vertex v, the components of $G[E(v)]$ are K_1s and K_2s.

2. For any graph G, $\alpha(G) = \varepsilon(G)$ if and only if for all ε-maximizing vertices v there exists a matching which saturates $O(v)$.

3. For unicyclic graphs, $\alpha(G) = \varepsilon(G)$.

A few open problems include:

1. The main conjecture (Conjecture 2.1). For any graph G, $\alpha(G) = \varepsilon(G)$ if and only if there is an ε-maximizing vertex v such that the components of $G[E(v)]$ consists of K_1s and K_2s and there exists a matching which saturates $O(v)$.

2. Is the main conjecture (Conjecture 2.1) true for KE graphs?

3. Is the main conjecture (Conjecture 2.1) true for unicyclic graphs?

4. Is ε a better bound than residue for all graphs?

5. Is the main conjecture (Conjecture 2.1) true for vertex-transitive graphs?

Related Graffiti Conjectures

This chapter states and explores other conjectures of Graffiti from Fajtlowicz's "Written on the Wall" [4]. These are related to the original conjecture that was modified to give us the ε bound for α.

DEFINITION 3.1.

- The number of vertices at even distance from v is $e(v)$.
- The number of vertices at odd distance from v is $o(v)$.
- The number of edges that are horizontal to v is $h(v)$.
- The number of edges that are even horizontal to v is $eh(v)$.
- The number of edges that are odd horizontal to v is $oh(v)$.
- $\overline{e(v)}$ is the average of $e(v)$ for all vertices v.

DEFINITION 3.2. A *cubic graph* is one in which every vertex has degree three.

CONJECTURE 3.3. (WoW # 842) For a fullerene and a vertex v such that $h(v)$ is maximized, $\alpha \leq e(v) - 2$.

CONJECTURE 3.4. (WoW # 750) For any graph, $\alpha \geq \max_{v \in V} o(v) - \min_{v \in V} h(v)$.

As stated in "Written on the Wall" a counterexample to this conjecture was found by Peter Puget [4].

The remaining conjectures do not apply to the majority of the standard graph classes C_n, P_n, S_n, K_n except K_4, and $K_{m,n}$ except $K_{3,3}$ since they are not cubic graphs. Noticing that both K_4 and $K_{3,3}$ are vertex-transitive graphs, we decided to explore these conjectures for cubic vertex-transitive graphs.

CONJECTURE 3.5. (WoW # 766) If G is a cubic graph then $\alpha \geq \min o(v) - \min oh(v)$ and $\alpha \geq \min e(v) - \min eh(v)$.

We don't know the truth for cubic graphs, but a parallel statement is true for vertex-transitive graphs.

THEOREM 3.6. If G is a vertex-transitive graph then $\alpha \geq \min o(v) - \min oh(v)$ and $\alpha \geq \min e(v) - \min eh(v)$.

Proof. If G is vertex-transitive with ε-maximizing vertex v and w any vertex in $V(G)$, $\min_w e(w) = e(v)$ and $\min_w eh(w) = eh(v)$. So $\min_w(e(w) - eh(w)) = e(v) - eh(v) = \min_w e(w) - \min_w eh(w)$. So $\alpha \geq \min_w e(w) - \min_w eh(w)$. □

The proof that $\alpha \geq \min o(v) - \min oh(v)$ is similar.

CONJECTURE 3.7. (WoW # 767) If G is a cubic graph then $\alpha \geq \overline{e(v)} - \overline{eh(v)} = \overline{e(v) - eh(v)}$.

THEOREM 3.8. If G is a vertex-transitive graph then $\alpha \geq \overline{e(v)} - \overline{eh(v)} = \overline{e(v) - eh(v)}$.

Proof. If G is vertex-transitive, $\overline{e(v)} = e(v)$ and $\overline{eh(v)} = eh(v)$. Also, $\overline{e(v) - eh(v)} = e(v) - eh(v)$. Therefore, $\alpha \geq e(v) - eh(v)$. Since G is vertex-transitve, $e(v) - eh(v) = max\{e(v) - eh(v)\}$. Hence, $\alpha \geq max\{e(v) - eh(v)\}$. □

CONJECTURE 3.9. (WoW # 768) If G is a cubic graph then $\alpha \geq \overline{e(v)} - \min_{v \in V} eh(v)$.

THEOREM 3.10. If G is a vertex-transitive graph then $\alpha \geq \overline{e(v)} - \min_{v \in V} eh(v)$.

Proof. If G is vertex-transitive, $\overline{e(v)} = e(v)$ and $min_{v \in V} eh(v) = eh(v)$. Then $\alpha \geq e(v) - eh(v) = max\{e(v) - eh(v)\}$. □

CONJECTURE 3.11. (WoW # 769) If G is a cubic graph then $\alpha \leq \max_{v \in V} e(v)$.

As stated in "Written on the Wall" a counterexample was found to this conjecture by Gilles Caporossi, Pierre Hansen, and Florian Pujol with an 18-vertex graph [4].

CONJECTURE 3.12. (WoW # 770) If G is a cubic graph then $\alpha \geq \frac{1+\max_{v \in V} e(v)}{2}$.

ε-critical Graphs and the Core

An ε-critical graph is defined in this section as a graph where removing any vertex either disconnects the graph or increases ε. The goal of identifying this class of graphs was to determine what conditions apply to these.

DEFINITION 4.1. A graph is *ε-critical* if removing any vertex either disconnects the graph or increases ε.

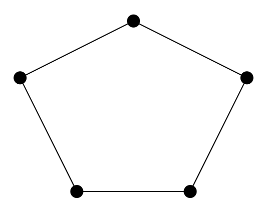


Figure 4.1: C_5 is an example.

PROPOSITION 4.2. If G is ε-critical, then $\alpha > \varepsilon$.

Proof. Suppose $\alpha = \varepsilon$. Let v be a vertex that is not in every maximum independent set. Then $\alpha(G-v) = \alpha(G)$ and $\varepsilon(G-v) > \varepsilon(G)$. This gives, $\varepsilon(G-v) > \alpha(G-v)$ which is a contradiction. □

DEFINITION 4.3. A *connector* is a non-cut-vertex.

DEFINITION 4.4. The *core* of a graph is the set of vertices which are in every maximum independent set.

For example, P_3 has two vertices in its core, and the core of P_4 is empty.

CONJECTURE 4.5. If G is a graph where every connector is in the core, then $\alpha(G) = \varepsilon(G)$.

CONJECTURE 4.6. Every connected graph G has a connected induced subgraph G' where $\alpha(G) = \alpha(G') = \varepsilon(G')$.

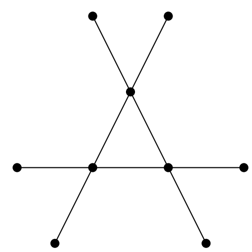


Figure 4.2: This graph is a counterexample to both conjectures because the pendant set is the unique maximum independent set. So, for any connected subgraph G' with $\alpha(G) = \alpha(G')$, G' must include all pendants. Since G' is connected it must include all the neighbors of the pendants. Hence $G' = G$. This graph is not a counterexample if the statement does not require the subgraph to be an induced subgraph.

CONJECTURE 4.7. Every connected graph G has a connected subgraph G' where $\alpha(G) = \alpha(G') = \varepsilon(G')$.

Generalized Petersen Graphs

A Generalized Petersen graph, $G(n,k)$, is a graph on the vertex set $\{u_0,u_1,...,u_{n-1},v_0,v_1,...,v_{n-1}\}$ with the edges $\{u_iu_{i+1},u_iv_i,v_iv_{i+k} : i=0,...,n-1\}$ with the subscripts reduced modulo n. We expected ε to be a good bound for these graphs but it was not. Examples of this are shown in this section.

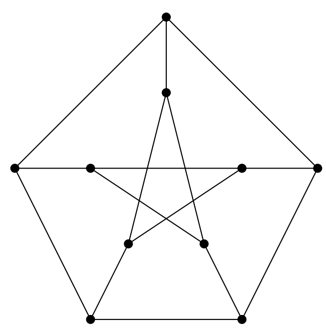


Figure 5.1: This graph is the Petersen graph, $G(5,2)$ where $\alpha = 4$ and $\varepsilon = 1$. The independence number was calculated for many Generalized Petersen graphs by Azadi, Besharati, and Ebrahimi. [1]

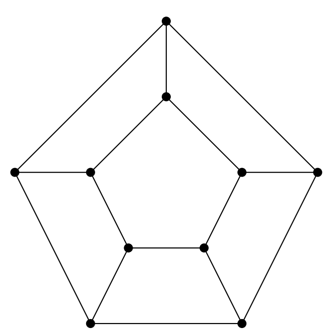


Figure 5.2: This graph is the Generalized Petersen graph, $G(5,1)$ where $\alpha = 4$ and $\varepsilon = 4$.

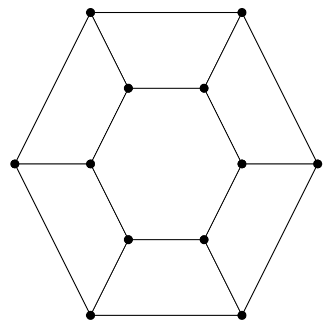


Figure 5.3: This graph is the Generalized Petersen graph, $G(6,1)$ where $\alpha = 6$ and $\varepsilon = 6$.

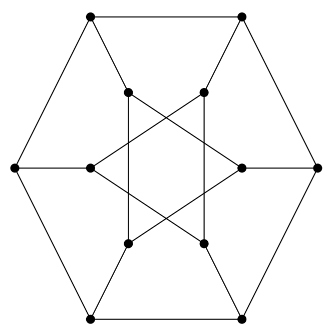


Figure 5.4: This graph is the Generalized Petersen graph, $G(6,2)$ where $\alpha = 4$ and $\varepsilon = 3$.

CONJECTURE 5.1. For a Generalized Petersen graph G, $\alpha = \varepsilon$ if, and only if, there exists an ε-maximizing vertex v so that the components of $G[E(v)]$ are K_1s and K_2s.

This conjecture is still open. The necessary condition follows from Theorem 2.2.

Difficult Graphs

Difficult and Super Difficult graphs are those whose independence numbers are very hard to compute or even estimate using known upper and lower bounds for the independence number. William Willis computed many known bounds for α on these Difficult and Super Difficult graphs in his Master's Thesis [10]. We computed ε for these graphs to see how well the ε bound compared to known bounds. For all of the graphs in this section, either ε was equal to α or ε was one less than α.

Difficult Graphs where ε is a better bound than the lower bounds investigated by Willis in his thesis.

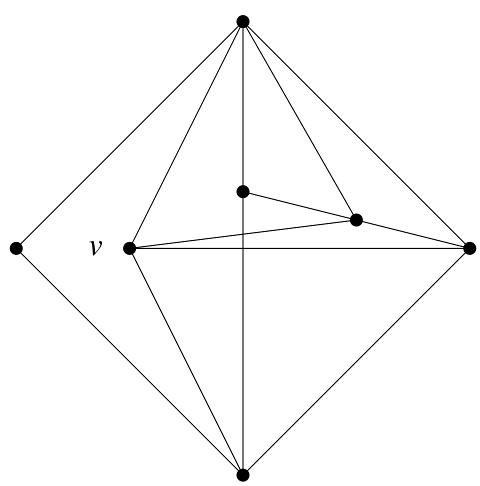


Figure 6.1: For this graph $\varepsilon = \alpha = 3$.

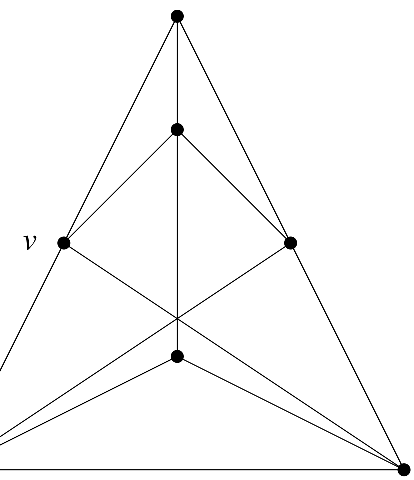


Figure 6.2: For this graph $\varepsilon = \alpha = 3$.

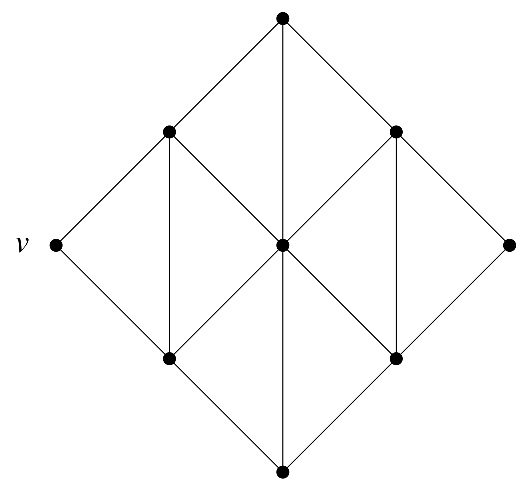


Figure 6.3: For this graph $\varepsilon = 3$ but $\alpha = 4$.

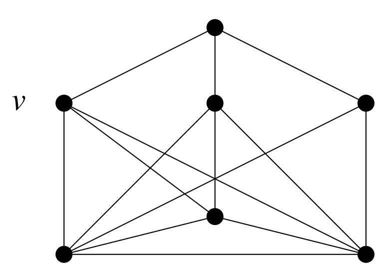


Figure 6.4: For this graph $\varepsilon = \alpha = 3$.

Super Difficult graphs where ε is a better bound than known lower bounds.

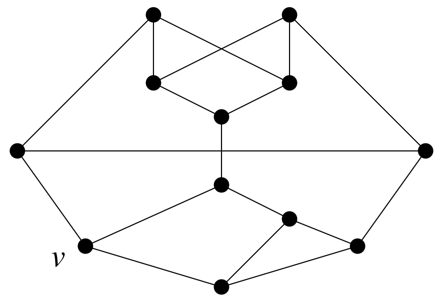


Figure 6.5: For this graph $\varepsilon = \alpha = 5$

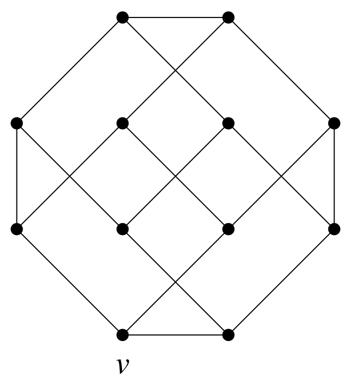


Figure 6.6: For this graph $\varepsilon = 4$ but $\alpha = 5$.

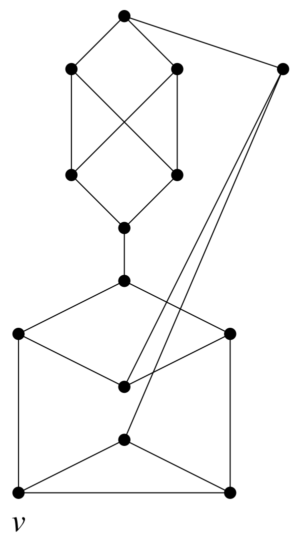


Figure 6.7: For this graph $\varepsilon = \alpha = 6$.

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