# Analyzing the Cognitive Demand of Enacted Examples in Precalculus: A Comparative Case Study of Graduate Student Instructors

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**Abstract**

The cognitive demand of mathematical tasks is an important aspect of analyzing the impact of instruction on student learning. The purpose of this study was to examine the instructional examples enacted by graduate student precalculus instructors in order to answer the following questions: What is the cognitive demand of the enacted examples? What does a high cognitive demand example look like when an instructor uses direct instruction? And how are examples drawn from the written curriculum enacted in different ways? Using both random and purposeful sampling of precalculus lessons, I conducted classroom observations as well as pre- and post-observation interviews with the instructors. A modified version of the Task Analysis Guide (Smith & Stein, 1998) was then used to categorize the cognitive demand of the instructional examples. As a result, I found that 25 out of the 93 examples (27%) I observed were enacted at a high level of cognitive demand. I also present vignettes that illustrate how three different instructors chose to enact the same example type at differing levels of cognitive demand.

**Keywords**

cognitive demand, example, precalculus, graduate student instructor

The cognitive demand of mathematical tasks is something that has been widely studied in the literature (Boston & Smith, 2009; Jackson, Shahan, Gibbons, & Cobb, 2012; Kisa & Stein, 2015; Smith & Stein, 1998; Stein, Grover, & Henningsen, 1996). Studies have found that high cognitive demand tasks provide students with more opportunities to learn (Floden, 2002; Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013; Smith & Stein, 1998; Stein, Remillard, & Smith, 2007) but are difficult for instructors to enact (Charalambous, 2010; Henningsen & Stein, 1997; Rogers & Steele, 2016). However, much of the literature on this topic has focused on analyzing the cognitive demand of tasks where students are the primary *doers* of mathematics. This lens makes sense, since reforms in mathematics education have called for more student-centered instruction and engaging students in authentic problem solving (National Council of Teachers of Mathematics, 2000; Mathematical Association of America, 2018). However, this lens makes it difficult to analyze the cognitive demand of instructional examples, which are presented through direct instruction.

## Purpose

The purpose of this collective case study is to examine the instructional examples enacted by graduate student instructors in precalculus courses at a large public university. First, I examine existing literature on the cognitive demand of mathematical tasks and the use of examples. Next, I address some common concerns that often come up from the assertion that instructional examples presented by the instructor can be presented at a high level of cognitive demand. Third, I explain the methods that I used to analyze the cognitive demand of instructional examples used by graduate student instructors teaching a precalculus course. I present a modified version of Smith and Stein's (1998) Task Analysis Guide that disentangles the *who* from the *what*. Finally, I present the results of my analysis and use three vignettes to illustrate what low-level and high-level instructional examples might look like.

## Framework and Research Questions

The framework that I used in this study was Stein, Remillard, and Smith's (2007) temporal phases of curriculum use. Building on the concepts of *formal* or *planned* curriculum, *institutional* or *intended* curriculum, *enacted* curriculum, and *experienced* or *attained* curriculum (Doyle, 1992; Gehrke, Knapp, & Sirotnik, 1992; Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002), Stein et al. identified three temporal stages of curriculum unfolding: *written*, *intended*, and *enacted*. The authors define the *written* curriculum as "the printed page" in textbooks or teacher materials, the *intended* curriculum as "the teachers' plans for instruction," and the *enacted* curriculum as "the actual implementation of curricular-based tasks in the classroom" (p. 321). These three phases are viewed as unfolding in a temporal sequence, and all phases have an impact on student learning. However, studies have shown that the final stage, the *enacted* curriculum, is the phase that has the greatest impact on student learning (Carpenter & Fennema, 1991). At each stage in the process, factors such as teachers' beliefs and knowledge, orientations towards curriculum, professional identity, and professional communities, in addition to the organizational and policy contexts as well as the classroom structures and norms, all impact the unfolding of the curriculum.

The research questions that guided this study were:

- What is the cognitive demand of the *enacted* examples in precalculus courses taught by graduate student instructors?
- What might a high cognitive demand example look like if an instructor chooses to *enact* an example using direct instruction?
- What similarities and differences are there between the examples that graduate student instructors *enact* when they use the same *written* curriculum materials?

## Literature Review

### Cognitive Demand

The framework for analyzing the cognitive demand of mathematical tasks was developed by Stein and Smith (1998). In their framework, they defined lower-level demand tasks as "tasks that ask students to perform a memorized procedure in a routine manner" and higher-level demand tasks as "tasks that require students to think conceptually and that stimulate students to make connections" (p. 269). Each of these categories was then broken down into two subcategories: memorization and procedures without connections (lower-level demands) and procedures with connections and doing mathematics (higher-level demands). Smith and Stein differentiated procedures with and without connections as representing different levels of cognitive demand. They separated these two types of tasks in order to categorize mathematical tasks that "use procedures, but in a way that builds connections to the mathematical meaning" (p. 270) of the underlying concept as a higher-level demand task. Tasks which require doing mathematics are categorized as higher-level demand tasks that require "students to explore and understand the nature of relationships" (Smith & Stein, 1998, p. 347). To aid in differentiating between the different types of tasks, Smith and Stein developed the Task Analysis Guide (p. 348), which lists the characteristics of the four types of mathematical tasks.

Stein et al. (1996) used the Task Analysis Guide to analyze a sample of 144 tasks that were implemented in reform-oriented classrooms. They found "the higher the cognitive demands of tasks at the set-up phase, the lower the percentage of tasks that actually remained that way during implementation" (p. 476). This finding provides confirming evidence for the claim that tasks with high cognitive demand are difficult to enact (National Council of Teachers of Mathematics, 2014, p. 17). To facilitate the design of lesson plans that would support high cognitive demand tasks, Smith, Bill, and Hughes (2008) developed the "Thinking Through a Lesson Protocol" (TTLP). Moving from the intended phase to the enactment phase, Jackson et al. (2012) examined four crucial elements for launching complex tasks: discussing the key contextual features, discussing the key mathematical ideas, developing a common language to describe the key features, and maintaining the cognitive demand. Similar to the TTLP, the authors provide teachers with a set of planning questions to reflect on what to do to launch a complex task effectively. In another paper, Jackson et al. (2013) examined how the launch of tasks correlated with opportunities to learn mathematics during the whole-class discussion. They found that by attending to the crucial elements for developing a common language to describe the key task features and maintaining the cognitive demand of the task during the launch, students had opportunities for higher quality learning during the concluding mathematics discussion.

## Instructional Examples

Bills et al., (2006) highlighted the importance of studying examples and exemplification in mathematics. First, examples play a central role in the development of mathematics as a discipline and in the teaching and learning of mathematics. Second, "examples offer insight into the nature of mathematics through their use in complex tasks to demonstrate methods, in concept development to indicate relationships, and in explanations and proofs" (pp. 126 – 127). While examples can be presented in a variety of ways, Bills et al. emphasized that "providing worked-out examples with no further explanations or other conceptual support is usually insufficient," as "learners often regard such examples as specific (restricted) patterns which do not seem applicable to them when solving problems that require a slight deviation from the solution presented in the worked-out example" (p. 140). Therefore, the authors emphasize that it is important for worked-out examples to include explanations and reasoning.

By studying the purpose, design, and use of mathematical examples in elementary classrooms, Rowland (2008) found that teachers need to attend to variables, sequencing, representations, and develop learning objectives when choosing which examples to use in the classroom. Similarly, Muir (2007) found that teachers need to attend carefully to the examples that they choose to use when teaching numeracy in order to "avoid the likelihood of students developing common misconceptions about important mathematical concepts" (p. 513). Zodik and Zaslavsky (2008) examined the different characteristics of how teachers choose mathematics examples. They developed a framework that captures the type of examples teachers choose and how the examples are generated. Finally, Mesa et al., (2012) looked at the opportunities to learn through the examples included in college algebra textbooks. In particular, the authors examined the cognitive demand of the examples by coding them according to the categories in Smith and Stein's (1998) framework (i.e., memorization, procedures without connections, procedures with connections, and doing mathematics). They found that of the 488 textbook examples that they analyzed, 445 (91%) of them could be described as procedures without connections. Looking at individual textbooks, 75% – 100% of the examples included fell into this category. Of the remaining examples, 41 (8%) were determined to be procedures with connections, two (<1%) were described as doing mathematics, and none of the examples were coded as memorization tasks.

# Methods

## Overall Approach and Rationale

I chose to use a case study methodology, since I was interested in developing in-depth descriptions of what low-level and high-level cognitive demand enacted examples can look like in precalculus. According to Creswell (2013), "case study research involves the study of a case within a real-life, contemporary context or setting" (p. 97). Case study methodology is rooted in medicine and law, but is also a common methodology in educational research (Yazan, 2015). Because I examined multiple instructors, this project was designed as a *collective* case study (Yin, 2009). Also, since I was interested in examining similarities and differences between the examples that graduate student instructors enacted when they used the same written curriculum materials, I framed this as a *comparative* case study. The main *bounded system* that defined a case in this study was the individual examples. However, I grouped examples by instructor and lesson in order to conduct a cross-case analysis.

## Site Description

### *The Mathematics Department*

This study was conducted at a large, public university in the Midwest. In order to improve student experience and success in lower-level courses, the mathematics department had successfully transformed their precalculus courses by incorporating active learning and course coordination (raising pass rates from 60% to 80%). To oversee this transformation, the department hired a director of first-year mathematics, who was a term faculty member, and formed a faculty committee to help lead a research project to study the department's changes in instruction and to provide formative evaluation to inform and improve the initiative. The department defined active learning as involving teaching methods and classroom norms that promoted student engagement in mathematical reasoning, peer-to-peer interactions, and instructors inquiring into student thinking. Class sizes were capped at 35, and first-time instructors were provided with undergraduate learning assistants who provided additional support during class. In an effort to provide a more uniform experience for students and support for instructors (who were primarily mathematics graduate students), all precalculus courses were heavily coordinated. Each course had an experienced graduate student who served as the primary coordinator and worked closely with a faculty course coordinator. Instructors were expected to use active learning and attended weekly course meetings. There was a common course schedule, homework assignments on WeBWorK, group quizzes (often written by the instructors), individual exams (written by the course coordinators), and shared grading of exams (by the instructors).

### *Precalculus Courses*

Students could choose to take College Algebra (3 credits) and Trigonometry (2 credits) over two semesters or a combined course, College Algebra + Trigonometry (5 credits), during one semester. Graduate students and faculty members were involved in the development of the departmental precalculus curriculum materials, which focused on students making sense of mathematics and developing procedural fluency. Although the structure and content of the materials stayed the same, the coordinators and instructors would collaboratively make changes and improvements throughout the summer and academic year. The curriculum materials included student workbooks and instructor lesson guides, which promoted student engagement and built on student thinking. During class, students were expected to propose questions, communicate their reasoning, and work in groups to complete workbook problems. Instructors were expected to dedicate the majority of class time to group work and student presentations and limit periods of direct instruction to at most 15 – 20 minutes at a time.

### *Instructor Population*

As mentioned previously, the majority of the instructors for precalculus courses were mathematics graduate students in the department's doctoral program. After serving as recitation instructors for Calculus I or II, graduate students were typically assigned to teach College Algebra or Trigonometry in their second year. For many of the graduate students, this was their first experience serving as the instructor of record for a course, so the department ran an intensive three-day workshop before the Fall semester and required graduate student instructors to take a year-long course on teaching and learning mathematics at the post-secondary level. During their third year of doctoral studies, graduate students were typically assigned to teach the combined College Algebra + Trigonometry course for one semester.

## Sampling Techniques

### *Participants*

For my study, I chose to interview and observe instructors who were graduate students in their third year or higher and were teaching a precalculus course for at least the third semester (see Table 1). The reason I chose to study more experienced graduate student instructors instead of novice second-year instructors was twofold. First, while active learning is becoming more common in mathematics education, many of the graduate student instructors had only experienced lecturing in mathematics courses. So not only were they teaching their own course for the first time, but they were also being asked to use classroom practices that were new to them. Second, while the departmental lesson guides were beneficial in that they provided the instructors with suggested sequencing, examples, and timing, the second-year graduate student instructors were still teaching the course for the first time. Therefore, they may have struggled with not understanding some of the content, spotting common student conceptions and misconceptions, or applying different approaches to teaching procedures and concepts. (As a note, instructors chose their own pseudonyms in order to preserve anonymity.)

**Table 1**
*Instructors' Year in Graduate School and Course Assignment*

| Instructor | Year | Course |
|---|---|---|
| Juno | 3 | Trigonometry |
| Emma | 3 | College Algebra + Trigonometry |
| Kelly | 3 | College Algebra + Trigonometry |
| Alex | 4 | College Algebra + Trigonometry |
| Dan | 4 | College Algebra + Trigonometry |
| Greg | 5 | Trigonometry |
| Selrach | 5 | College Algebra + Trigonometry |

### *Lessons*

During the first semester, I used random sampling for classroom observations. In particular, I asked instructors to pick one date in September, October, and November for me to observe their class. Since I only observed one class, there were times where I only observed part of a lesson because it was spread out over multiple days. During the second semester, I implemented purposeful sampling. First, I analyzed the lesson guides to identify lessons that were more procedural in nature, because I thought that this would provide me with the opportunity to see how a procedural example could be enacted at either a high or low level of cognitive demand. Then I verified that I could observe each instructor teach the lessons I had identified, although I still only observed each instructor teaching three times. Table 2 contains a list of the lessons I observed during the second semester. Several of the lessons were spread out over two days, so I visited the classroom on both days.

**Table 2**
*Purposeful Sampling of Lessons*

| Title | Course |
|---|---|
| The Vertex of a Parabola | College Algebra |
| Function Compositions | College Algebra |
| Logarithms and Their Properties | College Algebra |
| Properties of Inverse Functions | College Algebra |
| Tangent and Reciprocal Trigonometric Functions | Trigonometry |
| Trigonometric Equations and Inverse Functions | Trigonometry |
| End-of-Semester Review | Trigonometry |

## Data Collection Methods

There were three primary forms of data that I collected for the study: curriculum materials, video recordings of my observations, and recordings of my pre/post-observation interviews with the instructors. The curriculum materials that I collected included the departmental lesson guides and student worksheets for each lesson that I observed. Since the lesson guides had undergone multiple revisions throughout the years, instructors tended to use the version that they were given the first time they taught a precalculus course. So, while I observed several instructors teach the same lesson, some of them used different versions of the lesson guides. Also, there were slight differences between the lesson guides for the Trigonometry course and the lesson guides for the trigonometry unit of the College Algebra + Trigonometry course. However, all instructors who taught the same course used the same student workbooks. In addition, I asked instructors to provide me with a copy of their lesson plan, if they made one. Some instructors would create a separate lesson plan that followed the lesson guide, while other instructors used the lesson guide as a lesson plan.

Before each classroom observation, I met with the instructor for approximately 30 minutes to discuss their plan for the lesson. During the interview, I focused on the examples that they planned to present during the class. In some of the lesson guides, exact examples were given, while others provided example parameters or a description of the type of example that should be used. Typically, I conducted the pre-observation interview the morning before I observed the lesson, but occasionally schedules required that we meet the day before. The full semi-structured, pre-observation interview protocol can be found in Miller (2018). During the classroom observations, I videotaped each example the instructor presented and took detailed field notes. After each observation and before the post-observation interview, I analyzed the video recordings to determine the level of cognitive demand of each example. During the post-observation interview, I asked questions about the instructors' decision-making process, which was the focus of another part of the study that is not reported in this paper.

## Data Analysis Procedures

In order to analyze the cognitive demand of the enacted examples, I used Smith and Stein's (1998) Task Analysis Guide. However, the language used in the original framework specified both *who* was doing the mathematics (students) and *what* mathematical work was being done. Since in my study, examples were often presented primarily by the instructor, I found the original framework difficult for me to apply. For example, the original framework includes phrases such as “students need to engage,” “require students to explore and understand,” “require students to access,” and “require students to analyze” (p. 348). While these are all desirable undertakings for students, I realized that many instructors viewed examples as mathematical tasks for the instructor to present through direct instruction and worksheet problems as mathematical tasks to engage students. However, other aspects of the cognitive demand framework focused more on the mathematical work being done, with phrases like “reproducing previously learned facts,” “have no connection to the concepts or meaning,” “represent in multiple ways,” and “analyze the task and actively examine task constraints” (p. 348). Therefore, in order to analyze the cognitive demand of the examples that were presented by the instructor, I felt it was necessary to modify some parts of the original Task Analysis Guide.

Since the Memorization category of the Task Analysis Guide is primarily described in terms of the mathematical work inherent in the task, this first low-level category did not require any modifications. The second low-level category, Procedures without Connections, only required slight modifications. While the descriptors do not reference *who* is completing the task, Smith and Stein (1998) claim that Procedure without Connections tasks will “require limited cognitive demand for successful completion” (p. 348). Since my purpose for using the Task Analysis Guide was to determine the cognitive demand of a task, I found this recursive definition to be problematic. Therefore, I chose to remove this language from both the Procedures without Connections and Procedures with Connections descriptions.

In addition to removing the phrase “require some degree of cognitive effort” (Smith & Stein, 1998, p. 348) from the Procedure with Connections category, I also removed any reference to who was completing the task. In particular, the original description stated that when working on Procedure with Connections tasks, “students need to engage with conceptual ideas that underlie the procedure to complete the task successfully and that develop understanding” (p. 348). While this language could possibly be applied to moments when the teacher is using direct instruction, I thought that the language used earlier in the description for this category (“focus students’ attention on” p. 348) more clearly applied to any situation, so I decided to use this language in two places (see bullets 1 and 4 in Table 3) to describe how students might (directly or indirectly) engage with the task. The final category, Doing Mathematics, required the most modification to remove descriptions for referencing *who* is doing the mathematical work and instead ensure that the statements focus on *what* mathematical work is entailed in the task. In total, I used the phrase “focus students’ attention on” in three places as a replacement for the phrase “require students to.” I also removed the reference to requiring “considerable cognitive effort.” My complete modification of the Task Analysis Guide appears in Table 3.

**Table 3**
*Modification of Smith and Stein’s (1998) Task Analysis Guide*

| **Low-Level Demands** |
|---|
| Memorization<br>• Involve either reproducing previously learned facts, rules, formulas, or definitions or committing facts, rules, formulae, or definitions to memory.<br>• Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure. |

- Are not ambiguous. Such tasks involve exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated.
- Have no connection to the concepts or meanings that underlie the facts, rules, formulas, or definitions being learned or reproduced.

Procedures without Connections

- Are algorithmic. Use of the procedure is either specifically called for or is evident from prior instruction, experience, or placement of the task.
- Little ambiguity exists about what needs to be done and how to do it.
- Have no connection to the concepts or meaning that underlie the procedure being used.
- Are focused on producing correct answers instead of on developing mathematical understanding.
- Require no explanations or explanations that focus solely on describing the procedure that was used.

**High-Level Demands**

Procedures with Connections

- Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.
- Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts.
- Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations helps develop meaning.
- Although general procedures may be followed, they cannot be followed mindlessly. They focus students' attention on engaging with conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding.

Doing Mathematics

- Require complex and nonalgorithmic thinking—a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or worked-out example.
- Focus students' attention on exploring and understanding the nature of mathematical concepts, processes, or relationships.
- Demand self-monitoring or self-regulations of one's own cognitive processes.
- Focus students' attention on accessing relevant knowledge and experiences and making appropriate use of them in working through the task.
- Focus students' attention on analyzing the task and actively examining task constraints that may limit possible solution strategies and solutions.
- May involve some level of anxiety for the students because of the unpredictable nature of the solution process required.

## Results

Over the two semesters, I observed a total of 24 lessons presented by the instructors, which spanned 33 days and included 93 different examples (see Table 4). Twenty-five of the examples were presented at a high level of cognitive demand (which I refer to as HCD examples). As a note, Greg was the graduate student course coordinator for Trigonometry, so I was able to observe him during both semesters. All of the other graduate student instructors only taught a precalculus course for one semester.

**Table 4**
*Distribution of Observed Examples*

| Instructor | Lessons | Days | Examples | HCD Examples |
|---|---|---|---|---|
| | | Fall Semester | | |
| Emma | 3 | 3 | 9 | 1 |
| Kelly | 3 | 3 | 7 | 4 |
| Alex | 3 | 3 | 5 | 3 |
| Greg | 3 | 3 | 6 | 1 |
| *Subtotal* | *12* | *12* | *27* | *9* |
| | | Spring Semester | | |
| Juno | 3 | 5 | 14 | 4 |
| Dan | 3 | 6 | 18 | 3 |
| Greg | 3 | 5 | 19 | 9 |
| Selrach | 3 | 5 | 15 | 0 |
| *Subtotal* | *12* | *21* | *66* | *18* |
| Total | 24 | 33 | 93 | 25 |

In the Fall semester when I used random sampling for the observations, I observed an average of 2.25 examples per day, 33% of which were enacted at a high level of cognitive demand. Emma had the lowest percentage of HCD examples (11%) but the highest number of observed examples, and Alex had the highest percentage of HCD examples (60%) but the lowest number of observed examples. In the Spring semester when I used purposeful sampling for the observations, I observed an average of 3.14 examples per day, 24% of which were enacted at a high level of cognitive demand. Selrach had the lowest percentage of HCD examples (0%) and the second lowest number of observed examples, while Greg had the highest percentage of HCD examples (47%) and the highest number of observed examples. Over all of my observations, 27% of the examples were enacted at a high level of cognitive demand.

### Vignettes

In order to gain a better understanding of what low-level versus high-level cognitive demand examples look like, I have included three vignettes below. These three vignettes were selected because they demonstrate how different instructors enacted the same type of example at differing levels of cognitive demand. In the Spring semester, I observed Juno, Greg, and Dan each teach the lesson entitled *Trigonometric Equations and Inverse Functions*. Juno and Greg each taught this lesson in their Trigonometry course in March, while Dan taught this lesson in his College Algebra + Trigonometry course in April. In all three cases the lesson was spread over two consecutive days, and I observed both days. I chose to observe this lesson because through random sampling I had observed Greg teach the second half of this lesson in the Fall semester and realized that it involved a lot of procedures. Within the lesson I chose to focus on a particular example that involved finding all solutions to a trigonometric equation that correspond to a non-standard unit circle angle. I did this because Juno and Greg enacted their examples at a high level of cognitive demand while Dan enacted his example at a low level of cognitive demand.

***Written Lesson Guide Description***

These vignettes feature the final example that was included in the written curriculum for the first day of the lesson entitled Trigonometric Equations and Inverse Functions. The learning objectives for this day were that students should be able to (i) graphically represent solutions to an equation, (ii) understand the process of finding all $\theta$ that satisfy the equation $f(\theta) = a$, where $a$ is fixed and $f$ is a periodic function, and (iii) use the unit circle to solve equations in part (ii) for $f$, a trigonometric function. The lesson guide provided the following outline for the lesson.

I. Demonstrate how solutions to equations of the form $f(x) = a$ can be represented graphically as the intersection points of two equations, $y = f(x)$ and $y = a$.
II. Give students time to work in small groups on a Ferris wheel word problem in the workbook and identify periodic patterns in their solutions.
III. Work through an example of graphing two functions, $y = \cos(\theta)$ and $y = \sqrt{3}/2$, and prompt students to notice the pattern in the occurrence of the intersection points.
IV. Introduce the procedure for solving trigonometric equations by first finding the "initial" or "core" solutions that occur in one period and then adding integer multiples of the period (e.g. $2\pi$).
V. Give students 10 minutes to work in small groups on two similar worksheet problems (i.e. solve $\sin(\theta) = -\sqrt{3}/2$ and $\cos(\theta) = -1/2$)
VI. Bring the class together to discuss the final example (described below).

Even though there were slight differences in the lesson guides used by the instructors (which I describe below), the general lesson outline was the same.

***Lesson Guide Example Descriptions***

Juno and Greg drew from the same version of the lesson guide. Figure 1 contains the example description that Juno and Greg used for this lesson. This example is procedural, explicitly stating that "the strategy for finding $\theta$ is still a two-part process: find initial solutions and then translate them." I categorized this example as Procedures without Connections (see Table 3) because it does not explicitly make connections to the concepts or meaning that underlie the procedure being used (i.e., there is no explanation for why there are two initial solutions or why it is necessary to add the period to an initial solution to determine an infinite family of solutions). In addition, the example, as written, is more focused on students producing correct answers (e.g., "use inverse trigonometric functions on our calculators," "add or subtract copies of $2\pi$ in order to find additional solutions," and "letting $\theta = 0.84$ and calculating $\cos(\theta)$ on the calculator should give a value very close to $2/3$") instead of students developing mathematical understanding of the concepts. However, as I will demonstrate, both Juno and Greg transformed this example in ways that raised the level of cognitive demand.

**Figure 1**

*The Example Description in the Version of the Lesson Guide Used by June and Greg*

**Example.** Solve the trigonometric equation $\cos(\theta) = 2/3$.
Notice that $\cos(\theta) = 2/3$ is not satisfied by any standard angle present on the unit circle. However, the strategy for finding $\theta$ is still a two-part process: find initial solutions and then translate them.

(i) Since $\cos(\theta) = \frac{2}{3}$ is not satisfied by any standard angle present on the unit circle, we need to use inverse trigonometric functions on our calculators. This produces one solution of:
$$\theta = \cos^{-1}(2/3) \approx 0.84$$
The second solution we obtain by symmetry, finding $\theta \approx -0.84$.

(ii) Recall now that, since $\cos(\theta)$ is periodic, we can take our two initial solutions to $\theta = 0.84$ and $\theta = -0.84$, and add or subtract copies of $2\pi$ in order to find additional solutions. For example, two additional solutions are given by:
$$\theta = -0.84 + 2\pi \quad \text{and} \quad \theta = 0.84 + 2\pi$$
To account all possible solutions, we write:
$$\theta = 0.84 + 2\pi k \quad \text{and} \quad \theta = -0.84 + 2\pi k$$
Again, $k$ can be **any** integer!

**Important:** We can check our solutions! For example, letting $\theta = -0.84$ and calculating $\cos(\theta)$ on the calculator should produce a value very close to $2/3$ if we have the correct solution. Note that we should have our calculators in radian mode for this task.

**Important:** Solving a trigonometric equation involves finding all of the solutions within a single repeated segment. Once all initial solutions are found, we use the fact that the trigonometric function is periodic to find the other solutions (by adding copies of the period to the solutions). It will not always be the case that we find two initial solutions or that we add $2\pi$. This will be explored in Exercises 4 and 5 of the worksheet and will be seen many times in the future.

**Figure 2**

*The Example Description in the Version of the Lesson Guide Dan Used and the Problem Dan Chose from the Student Workbooks*

**Example.** Discuss the case in which the initial solutions are not angles on their unit circles. You can make up your own example here, or do Problem 3(a) as a class. Remind students that they can check their solutions, and walk them through the process of doing so in this scenario.

**Problem 3(a)** Solve the trigonometric equation $\sin(\theta) = -2/3$.

Dan, however, used a different version of the lesson guide and therefore the description he based his example on was different (see Figure 2). In particular, the version of the lesson guide that Dan used provided much less detail regarding how to present the example and what to emphasize, but the equation used in the example was similar to the one that Juno and Greg used. The lesson guide stated that the instructor could make up their own example or use a problem from the student workbook. Dan decided to do the latter. Because the lesson guide was so vague, it was difficult to determine the cognitive demand of the example. However, I chose to code it as Procedures without Connections (see Table 3) because the problem statement only referenced solving the trigonometric equation and the lesson guide prompted Dan to "remind students that they can check their solution and walk them through the process of doing so in this scenario." So this example also seemed to primarily be focused on students producing correct answers. Dan's presentation did not change the cognitive demand of the task.

**Juno.** During the observation, Juno followed the lesson guide closely (see Figure 1). At the end of the first day, Juno began the final example by directing students' attention to a graph she had drawn on the board at the beginning of class (see Figure 3). In describing how this example was similar to previous problems they had completed (e.g. solve $\sin(\theta) = -\sqrt{3}/2$ and $\cos(\theta) = -1/2$), she emphasized that "the fact that it's not on the unit circle doesn't change anything on the graph…. We still have two solutions in one period, so we still want to find the base solutions, and then add $2\pi k$." Juno proceeded by explaining how to use the inverse cosine function on the calculator to find one base solution, and then she asked her class, "Do you remember how we find the other solution?" Student 1 responded with "add $\pi$," so Juno drew Figure 4.

**Figure 3**
*The Graph Juno Drew on the Board at the Beginning of Her Lesson*

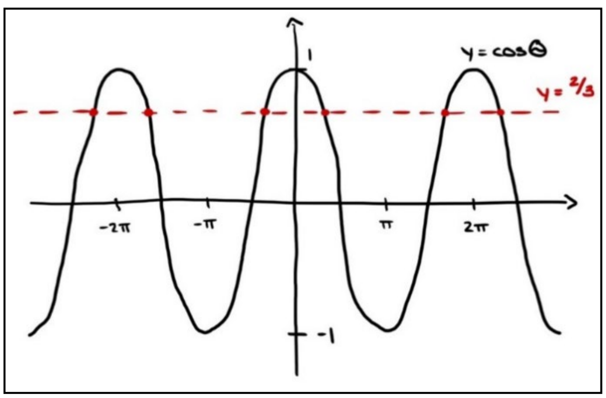
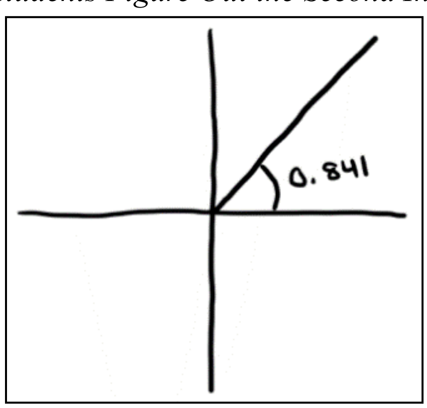



**Figure 4**
*The Graph Juno Drew to Help Students Figure Out the Second Initial Base Solution.*




The discussion that ensued is presented here as an excerpt of the transcript of the observation video.

*Juno:* So, here we got to know what quadrant we are in. So, this is $0.841$ *[draws Figure 4]*. Which other quadrant is cosine positive in?

*Student 2:* Four.

*Juno:* Yeah, fourth quadrant. *[Adds the corresponding fourth quadrant angle to Figure 4.]* So, that's the angle we want. You can either do $2\pi - 0.841$, or you can just do $-0.841$. Ok, so if I do $2\pi - 0.841$ I'm getting this solution *[pointing to the intersection point immediately to the left of $x = 2\pi$ in Figure 3]*. If I do $-0.841$, I'm getting that solution *[pointing to the intersection point immediately to the left of $x = 0$ in Figure 3]*. But it doesn't really matter, because we are just going to be shifting them by multiples of the period anyway. So, then our general solution is going to be

$$\theta = 0.841 + 2\pi k$$
$$\theta = -0.841 + 2\pi k$$

for any integer $k$.

In this exchange, Juno referenced graphical representations from earlier in the lesson in order to help her students understand how to find the second initial solution based on the value given by calculating with inverse cosine. She also made connections to the original graph she had drawn with intersection points for the two functions (see Figure 3) in order to help students understand which intersection point corresponded with $\theta = -0.841$ and which intersection point corresponded with $\theta = 2\pi - 0.841$. Juno wrapped up the example by asking if any students had questions. One student asked if they could just write $\theta = 0.841 + \pi k$ to capture all the solutions to the equation. Juno explained that shifting one intersection point by only half a period ($\pi$) would not result in landing (graphically) on another intersection point. Another student asked if it would be possible to turn $2\pi - 0.841$ into a decimal. Juno explained that the number $0.841$ was already a rounded approximation, so converting $2\pi - 0.841$ to a decimal would also be an approximation and not an exact solution. Finally, Juno wrapped up the example by explaining how students could check their work by plugging in a few values from their solution families for $\theta$ into the expression $\cos(\theta)$ to make sure the results were close to $2/3$.

Even though Juno was doing much of the mathematical work, the mathematical focus of the example justified it being coded as Procedures with Connections (see Table 3). While Juno used a procedure for finding initial base solutions and then translated them, she focused students' attention on the use of the procedure for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. In particular, Juno emphasized that, regardless of whether or not the trigonometric equation corresponds with a standard unit circle angle, the process still involves finding intersection points. Also, she consistently referenced the graphical representation of the problem (Figure 3) in order to help her students understand that, while there are an infinite number of intersection points, they actually correspond with two initial solutions that repeat in a periodic fashion. Second, Juno did not represent the procedure for solving the problem in an algorithmic way. [1]

Juno also used multiple representations to help her students understand the example. Throughout her presentation, she consistently referenced the graphical representation of the problem (Figure 3) to help her students develop an understanding of why there were infinitely

[1] In contrast, I observed Selrach present a similar example where he listed a five-step algorithm on the board and instructed students to follow the algorithm in order to solve trigonometric equations.

many solutions. When her students struggled to identify how to find the second initial base solution, Juno referenced another representation (Figure 4) to help her students reason with the symmetry of the unit circle to find the solution instead of memorizing and implementing a formula like $2\pi - \cos^{-1}(x)$. Finally, although Juno followed the general procedure for finding initial base solutions and then translating them, she challenged her students to engage with the conceptual ideas that underlie the procedure. In particular, Juno focused students' attention on the periodic nature of the cosine function, the symmetry of corresponding angles on the unit circle, and the graphical consequence of choosing different equivalent representations for the initial base angles (e.g. $-0.841$ vs. $2\pi - 0.841$).

**Greg.** For the beginning of class, Greg followed the lesson guide directly. However, Greg decided to use a different final example than the one provided. Instead of illustrating how to solve $\cos(\theta) = 2/3$, Greg chose to use the equation $1 + 2\sin(\theta) = 4/3$.[2] During the pre-observation interview, Greg explained that he changed the example because he wanted his students to understand that even when there is a lot of other "stuff" going on (i.e., transformations of the trigonometric function), they should still follow the same procedure. He also wanted to connect students' prior experiences with solving linear equations with what they were learning about solving trigonometric equations. Moving forward, Greg knew that his students would need to continue to combine ideas (i.e., solving linear and trigonometric equations) when working through problems. In selecting his example, he knew that he wanted to incorporate a constant multiple of the sine function plus a constant, and he didn't want the solution to involve a unit circle angle because the prior example did so. He wanted to introduce the students to the general procedure for solving trigonometric equations regardless of whether or not the problem required analyzing standard unit circle angles.

When Greg first introduced the final example, he acknowledged that they were "stepping it up a little bit." However, he emphasized that, although the example may look intimidating, "this is really just combining two ideas that you already know." In order to help his students recognize that part of the problem involved solving a linear equation, Greg decided to substitute $X = \sin(\theta)$ into the equation. Greg explained that he used a capital $X$, in order to "remind myself that I'm not solving for $X$, I'm solving for $\theta$." However, making this temporary substitution would make it clear what the next step should be to work through the problem. Greg then rewrote the original equation as $1 + 2X = 4/3$ and worked through the steps for solving the linear equation to get $X = 1/6$. Greg labeled this as Step 1 and said "for Step 2, remember we didn't want to solve our equation initially for $X$, we wanted to solve it for $\theta$." So, he substituted $X = \sin(\theta)$ which resulted in the equation $\sin(\theta) = 1/6$.

With the trigonometric equation isolated, Greg explained that they couldn't use the unit circle, because $1/6$ is not a standard unit circle angle. So instead, he asked his class "What do we want to use to move the sine to the other side?" The discussion that ensued is presented here as an excerpt of the transcript of the observation video.

*Student 1:* Arcsine.
*Greg:* Yeah, we'll use an arcsine. Ok? I'll write it as sine inverse. So, I get $\theta = \sin^{-1}(1/6)$. So that's one solution. Where does the second solution come from? I gave a chart.
*Student 2:* I just have a quick question. How did we know to use arcsine?

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[2] As a note, solving more complicated trigonometric equations was the first learning objective for the second day of the lesson, so Greg was leading into what they would be doing next.

*Greg:* [*To Student 1*] How did you know to do that?
*Student 1:* Because you're trying to find $\theta$, and to get rid of sine you have to move it to the other side using sine inverse.
*Student 2:* Oh, yeah.

As a quick aside, Greg mentions that it is not acceptable to divide both sides by sin to isolate $\theta$ and arrive at $\theta = \frac{1}{6\sin}$. Greg identifies $\theta = \sin^{-1}(1/6)$ as an initial solution and asks, "Someone, remind me where the other solution comes from whenever you use arcsine? The other solution in $[0, 2\pi]$." One student responds with, "$2\pi - \sin^{-1}(1/6)$," and another asks, "Isn't it just $\pi -$?"

*Greg:* Why do you think it's $\pi -$? I mean that's the chart, right? But…the $2\pi -$ has the effect of flipping over the $x$-axis, and that works for cosine. But for sine values, we want to flip over the $y$-axis. And to do that, we use $\pi -$.

Greg labels $\theta = \pi - \sin^{-1}(1/6)$ as the second initial solution and then reminds the class that in the earlier graphed examples, they saw that sine and cosine usually have two initial solutions. He then wrote the following on the board: $(\text{init})+(\text{per})k$ where init stands for initial solution and per represents the period. Greg first introduced this notation for the general form of an infinite solution family of a trigonometric equation during the second example when they solved the equation $\cos(\theta) = \sqrt{3}/2$. When I observed Greg teach the second half of this lesson the prior Fall, he did not use this notation. However, Greg consistently used this notation during both lessons in the second semester as a way to help students see that the structure of the solutions was consistent regardless of the differences in the trigonometric equations.

Next, Greg asked, "What's the period that I'm looking for in this case?" He paused for approximately 6 seconds, but no one responded, so he answered his own question and explained that it was possible to find the period by looking at the equation $\sin(\theta) = 1/6$. Greg wrote the two solution families on the board, using the general form $(\text{init})+(\text{per})k$, and summarized that "It's really the same pattern that we were using before. Now it's just two steps." At the end, Greg asked if any students had questions. One student asked what type of equation would require a change in the period in the general form. Greg explained that an equation like $1 + 2\sin(3\theta) = 4/3$ would have period $2\pi/3$. In fact, the change from $\theta$ to $3\theta$ in the equation would mean that both the initial solutions and the period would have different values in the general form. Another student asked if the final answer was simply the two initial solutions, or the "longer equations." Greg explained that the final answer must include all solutions, not just the initial solutions. Finally, Greg wrapped up the example by explaining how students could check their work by plugging in some of the solutions to the original equation into make sure they get $4/3$.

In this example, Greg involved students in the process of finding the solutions more than Juno did, but he still worked through most of the mathematics himself. Greg listed some procedures and referred to Step 1 (i.e., isolate the trigonometric function using algebraic manipulations) and Step 2 (i.e., solve the trigonometric equation), but he mainly focused on the broad, general procedures and on developing student understanding. So I coded this example as Procedures with Connections (see Table 3). In particular, Greg consistently focused students' attention on the general procedure for finding initial solutions and then adding on integer multiples of the period. He introduced the general form $(\text{init})+(\text{per})k$ in the previous example, brought it up again in this example, made connections between the general form and the graphical representation of the trigonometric equation, and emphasized repeatedly that the general procedures for finding initial solutions and identifying the period were core features of the problem. Like Juno, he did not present the procedure algorithmically but rather emphasized how the procedure was connected to the periodic nature of trigonometric functions. Although he did not graph $1 + 2\sin(\theta) = 4/3$, he did make references to the graphs he had drawn previously when explaining why there are two initial solutions. He also referenced the graph when explaining why the second initial solution was $\pi - \sin^{-1}(1/6)$ and not $2\pi - \sin^{-1}(1/6)$. Finally, Greg often paused and asked his students questions, which helped them do more than just mindlessly follow the procedure. Rather, he consistently focused their attention on engaging with the concept that solution families of trigonometric equations have the general form $(\text{init})+(\text{per})k$.

**Dan.** While the example in the lesson guide that Dan used was not as detailed as the one followed by Greg and Juno, he ended up using an example similar to Juno's example: Solve the trigonometric equation $\sin(\theta) = -2/3$. In the pre-observation interview, Dan explained that he picked this equation because he wanted an example that didn't have a lot of clutter so that the students could focus on how to solve trigonometric equations with a "not nice" (i.e., non-standard) unit circle coordinate. He planned to walk his students through five steps: (1) draw the unit circle, (2) draw $y = -2/3$, (3) note which quadrant(s) the two solutions are in, (4) find the actual angles using arcsine, and (5) given that the period is $2\pi$, write the equations representing infinitely many solutions. Although he didn't list these steps explicitly on the board, he did reference them verbally in the example presented here and his previous example (solve $\cos(\theta) = \sqrt{3}/2$).

Dan started the example by mentioning that although the equation $\sin(\theta) = -2/3$ does not correspond to a "nice" unit circle coordinate, "we shouldn't throw the baby out with the bath water." In particular, Dan emphasized that all of the work was going to be almost exactly the same as with the previous example. But instead of using the unit circle to find the initial solutions, they would have to use inverse trigonometric functions. Here is an excerpt from the observation video transcript where Dan begins to work through the example:

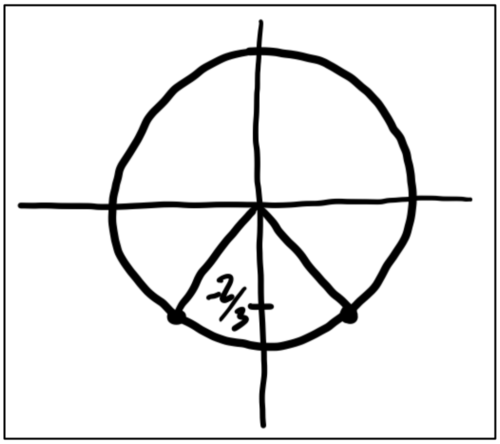
> *Dan:* Again, the overall analysis is exactly the same. We draw our picture [*draws Figure 5 while talking*], we turn our equation into a label. Sine tells me I'm looking at $y$-coordinates. $-2/3$ tells me I want $y = -2/3$. That gives me two points, one in the third quadrant and one in the fourth quadrant.

Dan reminded the class that they should use arcsine on their calculators but that the value provided, $\theta \approx -0.73$, is an angle in either the first or fourth quadrant. He explained that the fourth quadrant angle represents starting at 0 and subtracting 0.73 to get $-0.73$, while the third quadrant angle is found by starting at $\pi$ and adding 0.73 to get $\pi + 0.73$. Once Dan had the two initial solutions, he skipped immediately to writing the solution families (shown below) and then asked if anyone had questions.

$$\theta = (-0.73) + 2\pi k$$
$$\theta = (\pi + 0.73) + 2\pi k$$
$$k \text{ any whole number}$$

One student asked, "So you know how if you take $\arcsin(-2/3)$ it's $-0.73$? So, for the second one, you did $\pi + 0.73$. What about the negative?" Dan explained that the two points in the third and fourth quadrant are based on the symmetry of the unit circle, and the $-0.73$ is the result of moving clockwise while the $+0.73$ is the result of moving counter-clockwise. Another student asked if there are four ways to write the solution, since each initial solution could start at 0 and move either clockwise or counter-clockwise. Dan explained that while it is possible to come up with different initial solutions, all solutions to the equation are equivalent and “it’s sort of all taken care of in this $+2\pi k$ business.”

**Figure 5**
*Figure Dan Drew to Demonstrate which Quadrants Contain the Angles Corresponding with the Equation* $\sin(\theta) = -2/3$.




In this vignette, Dan worked through the same procedure as Juno and Greg. However, he emphasized the procedure itself instead of using the procedure to highlight the underlying conceptual ideas. So, I categorized this example as Procedures without Connections (see Table 3). While Dan included a graphical representation in his explanation, he mainly used it to determine which quadrant the solutions are in. So, his representation was used primarily to produce correct answers instead of to develop mathematical meaning or understanding. He did reference the conceptual structure of the infinite families of solutions for trigonometric equations, but his references were brief and mainly focused on writing down the solution (i.e., he skipped immediately from finding the initial solutions to writing the family of solutions without discussing why it is necessary to add constant multiples of the period or why the period is $2\pi$). When the student asked if there were four different ways to write the initial solutions (based on working clockwise or counter-clockwise around the unit circle), Dan could have explained how different initial solutions correspond with different intersection points of the graphs of $y = \sin(\theta)$ and $y = -2/3$ (like Juno did), but instead he said it did not matter which approach was used and “it’s sort of all taken care of in this $+2\pi k$ business.”

## Discussion

In this study, I examined the cognitive demand of the examples that graduate student instructors chose to enact in precalculus courses. At first, I attempted to use the original Task Analysis Guide developed by Smith and Stein (1998) to code the cognitive demand of the enacted examples. However, the original version of the Guide includes language that specifies that students are the ones doing the mathematical work (e.g., “require students to explore” and “students need to engage”). While some instructors did involve students explicitly in working out examples, others chose to present examples using direct instruction. Therefore, I created a

Modified Task Analysis Guide for analyzing the cognitive demand of examples (Table 3) that removed any language concerning *who* is doing the mathematical work. Using this Modified Task Analysis Guide, I found that of the 93 examples that I observed over two semesters, 25 (27%) of them were enacted at a high level of cognitive demand.

Next, I conducted a cross-case analysis in order to illustrate what high cognitive demand precalculus examples might look like when instructors use direct instruction and to identify similarities and differences between the examples that different instructors enacted when using the same written curriculum materials. In the vignettes of Juno, Greg, and Dan, I illustrated how instructors can enact the same type of example at a high level of cognitive demand but emphasize different concepts (i.e., connections between algebraic manipulations and graphical representations versus the underlying structure of solution families). In addition, I found that while high and low cognitive demand examples might use similar representations (i.e., algebraic and graphical), focusing on finding the answer instead of on developing student understanding can lower the cognitive demand.

## Implications

One implication of this study is that even through moments of direct instruction (i.e., during the enactment of an example), instructors can present mathematical tasks with higher levels of cognitive demand. A precalculus instructor who was not a participant in this study but taught in the department where the study was conducted told me that all of the content he covered was easy and did not require deep thinking. However, Juno, Greg, and several of the other graduate student instructors demonstrated that it is possible to teach concepts in precalculus that focus on developing deeper understandings of the underlying mathematics and make connections between multiple representations.

The Modified Task Analysis Guide that I developed for analyzing the cognitive demand of examples is useful for both researchers and practitioners. First, this framework gives researchers a way to analyze the cognitive demand of tasks independent of *who* is doing the mathematical work. This is especially important for examples, since instructors can present them in a variety of ways. While it is similar in many ways to the original Task Analysis Guide (Smith & Stein, 1998), I modified the categories by removing any reference to *who* is doing the mathematical work. The Modified Task Analysis Guide is also useful for practitioners as a planning and reflection tool. As teachers plan and reflect on their teaching, they can use this framework to assess the cognitive demand of the examples they use.

## Limitations

One limitation of this study is that while the instructors were using the department lesson guides, the different versions they were using had some clear and distinct differences. In the example that I highlighted in my vignettes, the structure of the two lesson guides was the same; however, the amount of descriptive text that accompanied the example varied greatly. The version with more descriptive text still focused mostly on producing the correct answer, so it was still categorized as a Procedures with Connections task. However, it is difficult to determine what effect the different versions may have had on the cognitive demand of the enacted examples.

Another limitation of this study is that not only did the instructors know when I was observing them teach, but they also knew I was there to examine the cognitive demand of the examples they presented. Therefore, they may have spent more time thinking about the cognitive demand of the examples that they enacted on the days that I observed, which may have influenced the results. In conversations with Greg, who I observed both semesters, he mentioned that our interviews made him think more deeply about the content that he was teaching. So, while these conversations may have impacted the results, if the outcome was that he presented more examples at a high level of cognitive demand, then I view that as a positive impact of this work.

Finally, another limitation of my study is that I cannot make any claims related to student learning. Since the focus of this study was the graduate student instructors and the choices that they made when planning and enacting examples in their precalculus classrooms, I did not collect any student data. Therefore, I cannot make any claims about whether high cognitive demand examples presented through direct instruction have a positive impact on student learning or understanding. Rather, enacting these types of examples provides students with *opportunities to engage* with mathematical tasks that require higher levels of cognitive demand. I was not able to determine whether or not students actually took advantage of these opportunities.

## Future Directions

There are some aspects of my modifications to the Task Analysis Guide that may need further attention. I chose to use the phrase “focus students’ attention on” instead of “require students to” in order to remove references to *who* is doing the mathematical work of the task and to allow the framework to be used to analyze instructional examples that are presented using direct instruction. However, it is possible that a teacher could lecture for 50 minutes and claim their aim is to “focus students’ attention on” high-cognitive demand tasks, but the students may never truly engage with the mathematics. Therefore, I think it would be beneficial to analyze student engagement, or perhaps students’ opportunity to struggle, in addition to analyzing cognitive demand. This would provide a clearer picture of the type of mathematical work that students actually engage in during class. Finally, a follow up study that I could complete from my data set is to analyze the cognitive demand of the written and planned examples and then examine whether the cognitive demand of the examples increased, decreased, or stayed constant as the lesson unfolded.

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