STUDENT WORK SECTION

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Editor's Note: *The Journal will regularly feature samples of student work of a nature different from that traditionally expected in college courses. Students frequently become more engaged in the study of mathematics and science when they are given the opportunity and encouragement to place the mathematics or science that they are studying into a broader context, and to bring to bear their individual outside interests and abilities. Such is certainly the case in Contemporary Mathematics, a course designed for non-science majors - particularly prospective K-8 teachers - and currently being developed/refined by the Virginia Collaborative for Excellence in Teacher Preparation. In addition to completing traditional hour exams and quizzes, all 850 students currently enrolled in sections of this course are expected to complete two large projects and complete a substantial number of writing assignments, including two major papers in which they are required to describe the mathematics that they have studied to a non-technical reader.*

The following student paper is an example of such a paper; it accounted for 5% of the grade in the course. The author was a freshman Humanities and Science major who was, in our view, able to combine her interest in fictional writing with a clear exposition of the mathematics that she has learned.

SOLVING PROBLEMS INVOLVING HAMILTON CIRCUITS

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With a disturbing look of confusion, he sits at his desk, contemplating over the trip he must soon take. "Where do I start?" he questions himself. Sipping on a stale cup of black coffee, he looks down his nose through his glasses at the map resting upon the hard oak. Gray smoke rolls from his mouth, dissipating into the late night air after each drag of his cigarette.

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He thinks to himself, "I have four cities to visit in order to deliver my vacuums to my customers, yet I must stay in range of my budget when thinking of air fare. I don't want to pick a wrong route and end up paying far too much than I have to. Plus, I do need to return home so I can show up at the sales room on Monday. Oh, where should I begin? How do I go about solving this problem?"

"Well, my friend," a deep, radio-commercial voice rings from behind the confused salesman's head.

"Who's that?" the salesman shrieks, quickly turning his head to discover a tall, heavenly figure before him. The crisp, yet cheap white suit and peculiar fake gold ring floating above his head gives the salesman the notion that the unexpected guest seems a bit frugal.

"I overheard your complaints," the pale dressed man speaks. "Little did you probably know, I am your guardian salesman. Yes, guys in the sales business have them, too. Anyway, I thought this would be a perfect time for me to introduce myself personally and give you some helpful advice concerning your conflict."

Removing his glasses and rubbing his weary, disbelieving eyes, the salesman takes a closer examination at the character claiming to be his guardian angel. "Of course you can understand that I really don't believe a word you say, but if you have any suggestions, I'm up for them," the salesman shares.

"Alrighty then, make yourself comfortable and pay attention, I'm only gonna explain this once." With one stern look at the dingy wall, a blackboard appears. The tall angel struts to the board and begins his lecture.

"First, we need to address that the route you wish to take is called a Hamilton circuit. These circuits begin and end at the same vertex or point while visiting each exactly once. In order to reduce the amount of strain in finding the best route, you need an organized plan. I plan to show you the different procedures in which to go about solving your problem. Remember, each of these algorithms or in other words, easy processes, will solve your dilemma, yet they differ in efficiency and accuracy, plus you must follow the procedure precisely. It remains your decision to choose the suitable algorithm after you learn them." [1]

"I shall first explain the Brute Force algorithm, which allows you to view each Hamilton circuit and choose the most satisfactory. This trial and error process consists of the following rules:

1) List all of the Hamilton circuits, beginning and ending with the same vertex.
2) Add the variables or weights of each edge in each circuit.
3) Review all of the circuits and choose the optimal circuit, the one with the minimal combined weight. The optimal will result as the best answer.

Now let's apply this to your particular problem as I write it on the board. We first know that you have a total of 24 possible answers, since the first and last points are already designated as A or home. The first place to visit after leaving your hometown has four possibilities-- B, C, D, or E. The second stop only has three choices of cities due to the elimination of the first city (you don't want to backtrack). The third visit only has two selections, while the last city must take the leftover city. By multiplying four, three, two, then one, we end up with 24 possibilities. We must take in consideration duplicates, so we should finally divide by two and end up with 12 non-repeating sequences.

Now that you know the number of circuits you must review, let's list a few of them with their corresponding weights:

ABCDEA	500+305+320+302+205 = 1632 dollars	(dollars represent
ACBEDA	200+305+340+302+185 = 1332 dollars	air fare costs)
ADBCEA	185+360+305+165+205 = 1220 dollars	

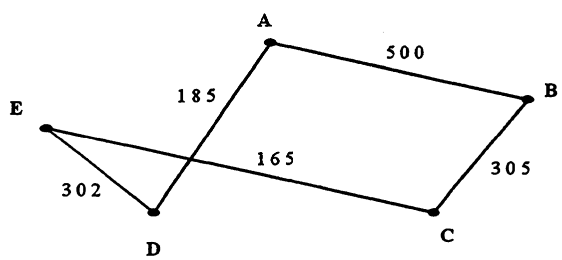
After repeating this process for every Hamilton circuit, find the one with the least numerical value. From the above circuits, and consequently all of the possibilities, the route ADBCEA proves to be the optimal choice.

While the Brute Force algorithm may produce the optimal circuit, it remains very time consuming when having to list every possible route. Not only does time pose a problem, it leaves room for error in naming each Hamilton circuit and in the arithmetic; but don't despair, I still have a few more tricks.

My second algorithm I will introduce is the Nearest Neighbor algorithm. Don't worry, Mr. Rogers didn't make this one up, it belongs to some qualified mathematician. This procedure enables you to find a possible Hamilton circuit in less amount of time. Now take a minute to study the procedure.

1) Start the circuit at the home vertex.
2) Pick the next point with the smallest weighted edge connected.
3) If a tie exists between edges, pick one at your discretion. Follow this rule until you've crossed each vertex exactly once.
4) Return to the home vertex.
5) Lastly, add the weights of the edges chosen.

After this process, you conclude with a route and its corresponding weight. This algorithm may seem perfect at first, but it contains a flaw in which to consider. Yes, it shortens the time required to find a route, but it does not guarantee an optimal solution. For example, the solution found through the graph of your routes shown doesn't agree with the optimal we found with the Brute Force algorithm.

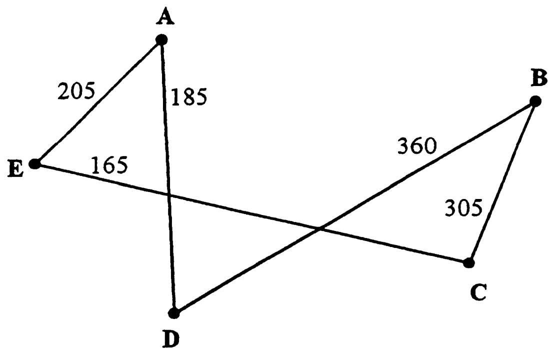


A-D-E-C-B-A 185+302+165+305+500 = 1457 dollars in air fare

Again, I want to describe yet another algorithm which you might find a bit more enticing to your needs-- the Repetitive Nearest Neighbor algorithm. Much like the Nearest Neighbor algorithm, this process allows you to reduce the time required to find a route and area for mistakes. It takes more time than the Nearest Neighbor algorithm, but still remains significantly more efficient than the Brute Force algorithm. Instead, it offers more choices of routes to choose from than the Nearest Neighbor algorithm, which in turn proves more accurate. Now listen close to the guidelines while applying them to your dilemma.

1) Pick a vertex and carry through with the Nearest Neighbor algorithm.
2) Continue this process with each vertex as the start and end.

3) After adding the weights of each circuit, choose the one with the smallest sum.
4) Rewrite the circuit so that the starting point is the necessary vertex.



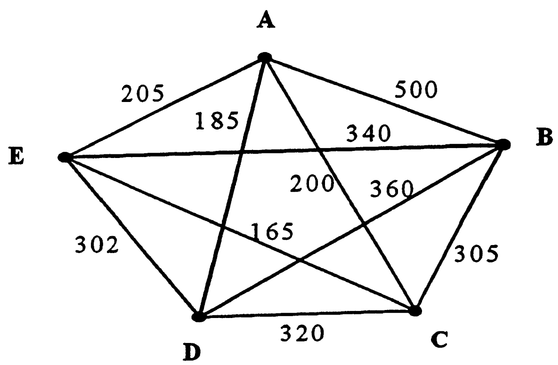
A-D-E-C-B-A = $1457
B-C-E-A-D-B = $1220
C-E-A-D-B-C = $1220
D-A-C-E-B-D = $1250
E-C-A-D-B-E = $1250

We find that the best solutions are the circuits starting with B and C, which consequently matches the value of the optimal found with the Brute Force algorithm. Look closely at the two solutions above, they have the same sequence, except for the beginning and ending cities, therefore they prove as the same circuit. Now we must rewrite the sequence so that it starts and finishes with A or your hometown-- A-D-B-C-E-A. Even if the route chosen does not start with the appropriate vertex, you can always pick up the circuit and follow it through, until you reach home again.

Lastly, I give you another algorithm, the Cheapest Link algorithm. Like the Nearest Neighbor and Repetitive Nearest Neighbor algorithms, it may not always result in an optimal circuit. It saves time in discovering a solution, and lessens the risk of human error. Unlike the Nearest Neighbor and Repetitive Nearest Neighbor algorithms, the Cheapest Link algorithm allows you to pick the edges with the minimal weights, without continuing a circuit from one point chosen to the next. For instance, the first edge picked may rest at one end of the graph while the second lies at the opposite end with no connection between the two.

Before explaining the procedure, I first must state that the Cheapest Link algorithm contains two very crucial rules in which to follow while performing the procedure. You cannot close a circuit while choosing edges until you cross each vertex once, or have three or more edges coming from one vertex. If you do not follow these rules, your resulting route cannot possibly satisfy the definition of a Hamilton circuit, therefore the circuit contains irrelevant edges and weights. If the concluding graph contains a smaller closed graph within itself, it either doesn't cross every vertex or must have three or more edges coming from one point. A Hamilton circuit can only cross each vertex once, if it has edges with the degrees of three or above meeting a single point, then it breaks the rule an a Hamilton circuit. Now, remember these rules while we review the procedure.

1) Choose the edge with the smallest weight, if a tie, decide at random.
2) Find the next smallest weighted edge and continue until you complete a Hamilton circuit.
3) Calculate the weights chosen to find the total value of the circuit.

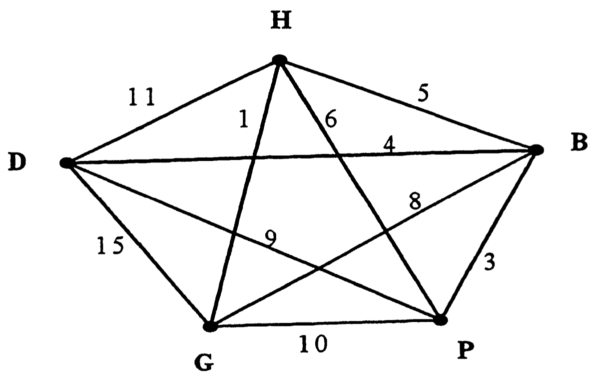


We can write the solution two ways: A-C-E-B-D-A or A-D-B-E-C-A. No matter what the order, the sum of the weights remain the same: 1250. The result of the Cheapest Link algorithm upon this problem varied from the optimal circuit. This proves that this procedure does not consistently offer the optimal solution, yet its efficiency in time and simplicity makes this algorithm a definite consideration when choosing a plan to find a Hamilton Circuit.

Not only do salesmen need the assistance of algorithms, other people such as families, students, and the bus driver may benefit from these procedures in their everyday lives. For example, a mother needs to make several errands in one day while traveling the least possible distance in order to save gas and time. After leaving home, she needs to visit the bank, post office, grocery store, K-mart, and drug store in no particular order before returning home. Let's explore the four algorithms to find solutions to this problem and determine which procedure proves most suitable.

Brute Force Algorithm:	H-B-P-G-D-H	5+3+10+15+11 = 44
	H-B-P-D-G-H	5+3+9+15+1 = 33
	H-B-G-P-D-H	5+8+10+9+11 = 43
	H-B-G-D-P-H	5+8+15+9+6 = 43
	H-B-D-P-G-H	5+4+9+10+1 = 29
	H-B-D-G-P-H	5+4+15+10+6 = 40
	H-P-B-G-D-H	6+3+8+15+11 = 43
	H-P-B-D-G-H	6+3+4+15+1 = 29
	H-P-G-B-D-H	6+10+8+4+11 = 39
	H-P-D-B-G-H	6+9+4+8+1 = 28
	H-G-B-P-D-H	1+8+3+9+11 = 32
	H-G-P-B-D-H	1+10+3+4+11 = 29

Optimal solution: H-P-D-B-G-H



Nearest Neighbor Algorithm: H-G-B-P-D-H 1+8+3+9+11 = 32

Repetitive Nearest Neighbor Algorithm:

H-G-B-P-D-H	1+8+3+9+11 = 32
B-P-H-G-D-H	3+6+1+15+11 = 36
P-B-D-H-G-P	3+4+11+1+10 = 29
G-H-B-P-D-G	1+5+3+9+15 = 33
D-B-P-H-G-D	4+3+6+1+15 = 29

Final Circuit: H-G-P-B-D-H

Cheapest Link Algorithm:	H-P-B-D-G-H	6+3+4+15+1 = 29
	H-G-D-B-P-H	1+15+4+3+6 = 29

With the Brute Force algorithm, we find the optimal circuit, yet it does not appear as the result of the other three algorithms. This alone shows that the Nearest Neighbor, Repetitive Nearest Neighbor, and Cheapest Link algorithms do not guarantee the optimal solution. Each procedure constitutes a Hamilton circuit, yet they vary in sequence and weight. The Brute Force's inefficient, bothersome process offers the optimal solution, yet some people prefer a faster procedure and sacrifice the difference in the weight of a non-optimal solution to the optimal. If the problem can produce a realistic number of possibilities, the Brute Force Algorithm could be used, but not every dilemma contains few enough vertices to make the application of this algorithm possible. When more involved graphs are in play, the Nearest Neighbor, Repetitive Nearest Neighbor, and Cheapest Link algorithms are needed."

"Now that I've thoroughly explained the algorithms available to you in order to solve your travel problem, which will you choose?" the angel questions as he places his chalk in the tray and glances at the salesman. With a deep sigh, the heavenly figure regains his breath from the tiresome lecture and walks over to the salesman. A slumped figure sits in the wooden chair with his forehead resting on his arms, asleep on his desk. With a caring hand and thoughtful mind, the angel takes a piece of paper and rewrites the algorithms on the paper for the salesman to refer upon. On another sheet of paper, he writes the optimal solution to the "Traveling Salesman Problem", then disappears, leaving a small cloud of sparkling dust suspended in the air and helpful tips for the salesman. ■

References

[1] P. Tannenbaum and R. Arnold, *Excursions in Modern Mathematics*, 2nd Ed, Prentice Hall, Englewood Cliffs, NJ, 1995.