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# A DOUBLE CHAIN OF COUPLED CIRCUITS IN ANALOGY WITH MECHANICAL LATTICES

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ABSTRACT. A unitary transformation obtained from group theoretical considerations is applied to the problem of finding the resonant frequencies of a system of coupled LC-circuits. This transformation was previously derived to separate the equations of motion for one dimensional mechanical lattices. Computations are performed in matrix notation. The electrical system is an analog of a pair of coupled linear lattices. After the resonant frequencies have been found, comparisons between the electrical and mechanical systems are noted.

KEY WORDS AND PHRASES. Born Cyclic Condition, Unitary Transformation, Symmetry Group, Lagrangin Matrix, Coupled Transmission Lines.
1980 SUBJECT CLASSIFICATION CODES. 20C35, 20G45.

## 1. INTRODUCTION.

This brief note is a by-product of work done in mechanics rather than circuit analysis. But, having worked problems involving coupled mechanical oscillators, simple changes of names have provided us with results for linearly coupled circuits since the underlying mathematics of eigenvalues and eigenfunctions is the same for the mechanical and electrical systems. We claim no specific relevance for our work to matters of immediate practical concern to electrical engineers. However, we do submit our paper in hopes that readers concerned with such topics as coupled transmission lines and translationally invariant circuits will find both the analogy to mechanical lattices and the mathematical exercise to be of interest. [1]

In our note to follow, we shall apply to a system of coupled LC-circuits a unitary transformation which was derived from the symmetries of a mechanical analog of the electrical system. In the mechanical problem, the transformation separated the equations of motion for a one-dimensional lattice of N identical particles having nearest neighbors coupled with harmonic springs. The geometry of the linear array of springs and particles was simplified by using the Born cyclic condition to convert the lattice into a circular ring with the equilibrium positions of the masses at the vertices of a regular, plane N-gon. The symmetry group of the linear array then became the rotation group C(N) for which the rotation by $\frac{2\pi}{N}$ radian serves as a generator, and the irreducible matrix representations of C(N) determine the entries of the unitary transformation matrix

$$U = \frac{1}{\sqrt{N}}(u_{k\ell}) \tag{1.1}$$

where

$$u_{k\ell} = \exp \frac{2\pi k\ell}{N} i\,. \tag{1.2}$$

As we proceed, it should become apparent that U does indeed possess the properties which simplify our calculations. We direct readers interested in the construction of the matrix U to the references cited at the conclusion of this introduction. In our work with mechanical lattices, we wrote a Lagrangian for each system in matrix notation. Then we diagonalized that Lagrangian matrix by performing a similarity transformation with U. From the transformed Lagrangian, the natural frequencies of vibration for the system under consideration were readily obtainable. [2,3,4,5,6,7]

## 2. COUPLED CHAINS OF LC-CIRCUITS.

Let us consider a linear, double array of LC-circuits with 2N circuits in all. By application of the Born condition, we can connect the first and (2N−1)-th circuits and the 2-nd and 2N-th circuits to obtain the circular array with the connection as indicated in Figure 1. We desire to compute the resonant frequencies of this system.

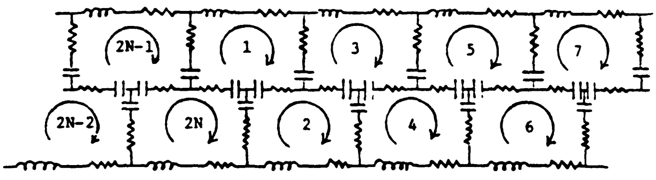


Figure 1.

Circuits 2k and 2k $\pm$ 1 (for k an integer) are coupled by capacitors $C_1$ having resistances $R_1$. (See Figure 2.) Circuits k and k $\pm$ 2 are coupled by capacitors $C_2$ having resistances $R_2$. All inductors have the same value, L, with resistance R.

We recognize three, interwoven linear arrays of circuits, each array being analogous to a linear lattice of particles and springs:

$$1, 2, 3, \ldots, 2N-1, 2N;$$
$$1, 3, 5, \ldots, 2N-3, 2N-1; \quad \text{and}$$
$$2, 4, 6, \ldots, 2N-2, 2N.$$

We also recognize that the permutation P which sends circuit k to circuit k + 1(Mod 2N) is a symmetry operation for the system and that the group $G = \{P, P^2, P^3, \ldots, P^{2N}\}$ is a symmetry group of the double array once we have connected the first and second circuits to the (2N − 1)-th and 2N-th circuits. Furthermore, G is isomorphic to the rotation group C(2N).

In Figure 1, there are 2N current loops indicated. In the k-th circuit, $\dot{q}_k$ denotes the current in its loop while $q_k = \int \dot{q}_k dt$ gives the charge associated with that current on each capacitor. Figure 2 shows the currents in all parts of the k-th circuit.

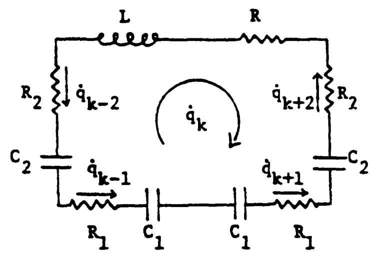


Figure 2.

Since there is no impressed E M F, we can write the k-th circuit equation as

$$L\ddot{q}_k + (R + 2R_1 + 2R_2)\,\dot{q}_k - R_1(\dot{q}_{k-1} + \dot{q}_{k+1})$$
$$- R_2(\dot{q}_{k-2} + \dot{q}_{k+2}) + \left(\frac{2}{C_1} + \frac{2}{C_2}\right)q_k$$
$$- \frac{1}{C_1}(q_{k-1} + q_{k+1}) - \frac{1}{C_2}(q_{k-2} + q_{k+2}) = 0.$$

The equations for all the circuits can then be condensed into the single matrix equation

$$L\ddot{Q} + R\dot{Q} + R_1A\dot{Q} + R_2B\dot{Q} + \frac{1}{C_1}AQ + \frac{1}{C_2}BQ = 0. \tag{2}$$

In Equation 2, Q is the column matrix giving the 2N components of the charge: $Q = \mathrm{col}(q_1 q_2 \ldots q_{2N})$. Similarly, $\dot{Q} = \mathrm{col}(\dot{q}_1\dot{q}_2 \ldots \dot{q}_{2N})$ and $\ddot{Q} = \mathrm{col}(\ddot{q}_1\ddot{q}_2 \ldots \ddot{q}_{2N})$. The matrices A and B are symmetric and have dimensions 2N × 2N:

$$A=\begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ -1 & 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix},$$

$$B=\begin{bmatrix} 2 & 0 & -1 & 0 & \dots & -1 & 0 \\ 0 & 2 & 0 & -1 & \dots & 0 & -1 \\ -1 & 0 & 2 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & -1 & 0 & 0 & \dots & 0 & 2 \end{bmatrix}.$$

We now perform a similarity transformation upon this matrix equation. The unitary transformation matrix is obtained from Equations 1.1 and 1.2 by replacing N with 2N. Thus U becomes

$$\begin{bmatrix} \exp\frac{2\pi i}{2N} & \exp\frac{4\pi i}{2N} & \exp\frac{6\pi i}{2N} & \dots & 1 \\ \exp\frac{4\pi i}{2n} & \exp\frac{8\pi i}{2N} & \exp\frac{12\pi i}{2N} & \dots & 1 \\ \exp\frac{6\pi i}{2N} & \exp\frac{12\pi i}{2N} & \exp\frac{18\pi i}{2N} & \dots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}.$$

If we denote the transformed charge vector by $P=UQ=\mathrm{col}(p_1p_2 \dots p_{2N})$, Equation 2 then becomes

$$L U\ddot{Q} + RU\dot{Q} + R_1UAU^{-1}U\dot{Q} + R_2UBU^{-1}U\dot{Q} + \frac{1}{C_1}\, UAU^{-1}UQ + \frac{1}{C_2}\, UBU^{-1}UQ = 0$$

or

$$L\ddot{P} + R\dot{P} + R_1UAU^{-1}\dot{P} + R_2UBU^{-1}\dot{P} + \frac{1}{C_1}\, UAU^{-1}P + \frac{1}{C_2}\, UBU^{-1}P = 0. \tag{3}$$

By straightforward computation previously performed in simplifying the equations of motion for mechanical lattices [2], we know that

$$UAU^{-1}=\begin{bmatrix} 4\sin^2\frac{\pi}{2N} & 0 & 0 & \dots & 0 \\ 0 & 4\sin^2\frac{2\pi}{2N} & 0 & \dots & 0 \\ 0 & 0 & 4\sin^2\frac{3\pi}{2N} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

and

$$UBU^{-1}=\begin{bmatrix} 4\sin^2\frac{\pi}{N} & 0 & 0 & \dots & 0 \\ 0 & 4\sin^2\frac{2\pi}{N} & 0 & \dots & 0 \\ 0 & 0 & 4\sin^2\frac{3\pi}{N} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then matrix Equation 3 implies that the equation for the k-th transformed coordinate is

$$L\ddot{p}_k + \left(R + 4R_1 \sin^2 \frac{k\pi}{2N} + 4R_2 \sin \frac{k\pi}{N}\right)\dot{p}_k + \left(\frac{4}{C_1} \sin^2 \frac{k\pi}{2N} + \frac{4}{C_2} \sin^2 \frac{k\pi}{N}\right)p_k = 0.$$

The technique for solution of this differential equation by letting $p_k = e^{\alpha(k)t}$ is well known. We find that, if $\left(R + 4R_1 \sin^2 \frac{k\pi}{2N} + 4R_2 \sin^2 \frac{k\pi}{N}\right)^2 < 4L\left(\frac{4}{C_1} \sin^2 \frac{k\pi}{2N} + \frac{4}{C_2} \sin^2 \frac{k\pi}{N}\right)$, we obtain the resonant frequencies $f(k) = \frac{1}{4\pi L}\left[4L\left(\frac{4}{C_1} \sin^2 \frac{k\pi}{2N} + \frac{4}{C_2} \sin^2 \frac{k\pi}{N}\right) - \left(R + 4R_1 \sin^2 \frac{k\pi}{2N} + 4R_2 \sin^2 \frac{k\pi}{N}\right)^2\right]^{\frac{1}{2}}$.

## 3. OBSERVATIONS.

In the event that $R = R_1 = R_2 = 0$, the frequency distribution for $k \epsilon \{1,2,3,...,2N\}$ reduces to that for coupled linear lattices for which there is no energy loss during oscillation.

The array of circuits corresponds to a double chain of identical particles as shown in Figure 3. Line segments between vertices indicate connecting ideal springs. The unit cell for the double chain is a parallelogram, and the triangles with vertices k, k+1, k+2 are isosceles. The frequencies computed correspond to longitudinal vibrations which are parallel to the center line (CL) of the chain.

We observe that the chains, 1,3,5,...,2N−1 and 2,4,6,...,2N, are uncoupled by letting $C_1 \to \infty$. If the resistances are all taken to be zero, the resulting frequency distribution is just that for longitudinal vibrations in a linear lattice of N particles with mass numerically equal to L which are connected by harmonic springs of force constant $\frac{1}{C_2}$ [2].

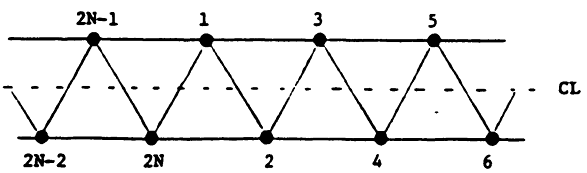


Figure 3.

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