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Article

# Classification of Symmetry Lie Algebras of the Canonical Geodesic Equations of Five-Dimensional Solvable Lie Algebras

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**Abstract:** In this investigation, we present symmetry algebras of the canonical geodesic equations of the indecomposable solvable Lie groups of dimension five, confined to algebras  $A_{5,7}^{abc}$  to  $A_{18}^a$ . For each algebra, the related system of geodesics is provided. Moreover, a basis for the associated Lie algebra of the symmetry vector fields, as well as the corresponding nonzero brackets, are constructed and categorized.

**Keywords:** symmetry algebra; Lie group; canonical connection; system of geodesic equations

## 1. Introduction

The use of Lie symmetry methods has become an increasingly important part of the study of differential equations, ranging from obtaining new solutions from known ones [1,2], reducing the order of a given equation [1–3], deriving conserved quantities [1], to determining whether or not a differential equation can be linearized and construct explicit linearization when one exists [4–6]. Moreover, it can be used to classify equations in accordance with their symmetry algebra [7,8]. This method was devised by Sophus Lie [9–11]. Currently, vast literature exists discussing such techniques, and readers can refer to books by Olver [1], Bluman and Kumei [2], Hydon [3], and Arrigo [12].

Very recently, Lie's symmetry method has been extended to a special case of the inverse problem for geodesic equations of the canonical symmetric connection belonging to any Lie group  $G$ . Ghanam and Thompson initiated an investigation of symmetry algebras of canonical geodesic equations for Lie groups of dimensions two and three [13], as well as four [14]. Furthermore, the first author, together with Ghanam and Thompson, extended the investigation to dimension five [15]. In [15], we concentrated merely on the systems of geodesic equations of indecomposable *nilpotent* Lie groups whose Lie algebras appeared in the article by Patera et al. [16]. We were able to determine the basis of symmetry algebras for each given system of geodesic equations, and then proceeded to classify their corresponding symmetry algebras.

The canonical connection, which we denote by  $\nabla$ , belonging to any Lie group  $G$  was introduced by Cartan and Schouten [17]. Two of the current authors investigated the inverse problem for canonical connection  $\nabla$  in the case of Lie groups of dimension five and less [18–21]. In the following section, we present a brief summary of salient features of the canonical connection on a Lie group.

The current article continues the investigation for symmetry algebras of systems of geodesic equations of five-dimensional indecomposable Lie algebras. In particular, we focus on the geodesics of

*solvable* Lie algebras 7 through 18. In each given case, nontrivial infinitesimal symmetries are detected, and the corresponding Lie algebra of symmetries are identified.

The paper is structured as follows. Section 2 provides a succinct description of the canonical Lie group connection. Section 3 describes the methodology for finding symmetry algebras. Section 4 is the main thrust of the article. In this section, we determine the basis of symmetry algebras for geodesic equations and subsequently calculate their Lie brackets; thereafter, we identify all possible symmetry algebras admitted by the governing systems of equations. We discuss our conclusions and future research in Section 5.

## 2. Canonical Connection of a Lie Group

This section aims to present a brief overview of canonical symmetric connection  $\nabla$  on a Lie group without going into all the details. The background and main properties of such a canonical connection have been well-described in the literature [19–21]. Let  $X$  and  $Y$  be left-invariant vector fields on a Lie group  $G$ ; then, the canonical symmetric connection  $\nabla$  on  $G$  is defined by

$$\nabla_X Y = \frac{1}{2}[X, Y], \quad (1)$$

and then extended to arbitrary vector fields via linearity and the Leibnitz rule. We now quote the following result. For a more detailed presentation, interested readers are referred to Ghanam et al. [19].

**Lemma 1.** *In the definition of  $\nabla$ , we can equally assume that  $X$  and  $Y$  are right-invariant vector fields; hence,  $\nabla$  is also right-invariant and hence bi-invariant.*

Following the above Lemma, connection  $\nabla$  is symmetric, bi-invariant, and the curvature tensor on the left-invariant vector fields is obtained by

$$R(X, Y)Z = \frac{1}{4}[Z, [X, Y]]. \quad (2)$$

Furthermore,  $G$  is a symmetric space in the sense that  $R$  is a parallel tensor field. Ricci tensor  $R_{ij}$  of  $\nabla$  is symmetric and bi-invariant. If  $\{E_i\}$  is the basis of left-invariant vector fields, then

$$[E_i, E_j] = C_{ij}^k E_k, \quad (3)$$

where  $C_{ij}^k$  are the structure constants and relative to this basis; Ricci tensor  $R_{ij}$  is given by

$$R_{ij} = \frac{1}{4}C_{jm}^l C_{il}^m, \quad (4)$$

from which the symmetry of  $R_{ij}$  is apparent. Since  $R_{jkl}^i$  is a parallel tensor field, and  $R_{ij}$  is symmetric, it follows that Ricci gives rise to a quadratic Lagrangian that may, however, not be regular. We assume that  $G$  is indecomposable in the sense that Lie algebra  $\mathfrak{g}$  of  $G$  is not a direct sum of lower-dimensional algebras.

Since our starting point is the Lie algebra  $\mathfrak{g}$  of a Lie group, it was of interest to ask how the ideals of  $\mathfrak{g}$  are related to  $\nabla$ . We quote the following result [22].

**Proposition 1.** *Let  $\nabla$  denote a symmetric connection on a smooth manifold  $M$ . The necessary and sufficient condition that there exist a submersion from  $M$  to a quotient space  $Q$ , such that  $\nabla$  is projectable to  $Q$ , is that there exists an integrable distribution  $D$  on  $M$  that satisfies:*

- (i)  $\nabla_X Y$  belongs to  $D$  whenever  $Y$  belongs to  $D$  and  $X$  is arbitrary.
- (ii)  $R(Z, X)Y$  belongs to  $D$  whenever  $Z$  belongs to  $D$ , and  $X$  and  $Y$  are arbitrary vector fields on  $M$ , where  $R$  denotes the curvature of  $\nabla$ .

In the case of the canonical connection on a Lie group  $G$ , we deduce

**Proposition 2.** *Every ideal  $\mathfrak{h}$  of  $\mathfrak{g}$  gives rise to a quotient space  $Q$  consisting of the leaf space of the integrable distribution determined by  $\mathfrak{h}$  and  $\nabla$  on  $G$  projects to  $Q$ .*

For the sake of completeness, we state the following results, see Ghanam et al. [19].

**Proposition 3.** *Let  $\nabla$  be canonical connection on a Lie group  $G$  and  $R$  denotes the curvature of  $\nabla$ , then the following results hold:*

- (i) *Curvature tensor  $R$  is covariantly constant.*
- (ii) *Connection has torsion zero.*
- (iii) *Curvature tensor  $R$  is zero if and only if the Lie algebra is two-step nilpotent.*
- (iv) *Ricci tensor is symmetric and in fact a multiple of the Killing form.*
- (v) *Ricci tensor is bi-invariant.*
- (vi) *Any left- or right-invariant vector field is a symmetry of the connection.*
- (vii) *Any left- or right-invariant one-form on  $G$  gives rise to a first integral on  $TG$ , i.e., any left- or right-invariant one-form defines a linear first integral of the geodesics.*
- (viii) *Geodesic curves are translations of one-parameter subgroups.*
- (ix) *Any vector field in the center of the Lie algebra is bi-invariant.*

As a way of trying to understand the meaning of symmetry algebras, we note that every left- and right-invariant vector field appears, and they are independent except at identity, and their intersection of course will comprise the bi-invariant vector fields. Any vector field in the center is, as such, a bi-invariant vector field. Thus every symmetry algebra is guaranteed to have a certain number of basic symmetries, that is left and right-invariant vector fields. The more “symmetric” that a certain geodesic system is, the more it will have extra symmetry vector fields, that cannot be so readily interpreted. Closely related to this issue is the fact that many of the Lie algebras in the range 5.7–5.18 in [16] depend on one or more parameters. For certain special values of these parameters, “symmetry is broken”, in the parlance of physicists; one sees this phenomenon particularly in the first example 5.7abc, where there is a variety of cases. In each subcase for each class of Lie algebra, we list at the top the values of the parameters and present the symmetry algebra accordingly. We provide a list of symmetry generators and the non-zero Lie brackets that they engender.

Next, we obtain a formula for connection components  $\Gamma_{jk}^i$  of  $\nabla$  in a coordinate system  $(x^i)$ . Suppose that right-invariant Maurer–Cartan forms of  $G$  are  $\alpha^i$ . Then, there must exist a matrix  $Y_j^i$  of functions such that

$$\alpha^i = Y_j^i dx^j. \quad (5)$$

The fact that such a matrix  $Y_j^i$  exists is the content of Lie’s third theorem (Helgason [23]). We denote the right-invariant vector fields dual to the  $\alpha^i$  by  $E_j$ . It follows that

$$E_i = X_i^k \frac{\partial}{\partial x^k}, \quad (6)$$

where  $X_i^k$  is the inverse of  $Y_j^i$ . We denote the structure constants of  $\mathfrak{g}$  relative to the basis  $E_i$  by  $C_{jk}^i$ . Then, by definition,

$$\nabla_{E_i} E_j = \frac{1}{2} C_{ij}^k E_k. \quad (7)$$

By Equation (7), we find the following condition relating to  $C_{jk}^i$  and  $\Gamma_{jk}^i$ :

$$X_i^k (X_{j,k}^m + X_j^l \Gamma_{kl}^m) = \frac{1}{2} C_{ij}^k X_k^m. \quad (8)$$

Taking the symmetric part of Equation (8), we obtain

$$\Gamma_{pq}^m = -\frac{1}{2}(Y_q^j X_{j,p}^m + Y_p^j X_{j,q}^m). \quad (9)$$

### 3. Formulation of Symmetry Algebra

This section is designed to succinctly discuss Lie's algorithm, adapted to obtain the symmetry algebra of the geodesic system of equations. Consider a system of second-order ordinary differential equations

$$\Delta_i^{(2)} = g_i(t, q, x, y, z, w, \dot{q}, \dot{x}, \dot{y}, \dot{z}, \dot{w}), \quad i = 1, \dots, 5, \quad (10)$$

where  $t$  is the independent variable,  $(q, x, y, z, w)$  are dependent variables, and  $(\dot{q}, \dot{x}, \dot{y}, \dot{z}, \dot{w})$  denote the first-order derivatives of  $(q, x, y, z, w)$  with respect to  $t$ . The Lie algebra  $\mathfrak{g}$  of symmetry algebra of Equation (2) is realized by vector fields

$$\Gamma = T \frac{\partial}{\partial t} + Q \frac{\partial}{\partial q} + X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + Z \frac{\partial}{\partial z} + W \frac{\partial}{\partial w}, \quad (11)$$

with first- and second-order extensions defined as

$$\Gamma^{(1)} = \Gamma + Q_t \frac{\partial}{\partial \dot{q}} + X_t \frac{\partial}{\partial \dot{x}} + Y_t \frac{\partial}{\partial \dot{y}} + Z_t \frac{\partial}{\partial \dot{z}} + W_t \frac{\partial}{\partial \dot{w}}, \quad (12)$$

$$\Gamma^{(2)} = \Gamma^{(1)} + Q_{tt} \frac{\partial}{\partial \ddot{q}} + X_{tt} \frac{\partial}{\partial \ddot{x}} + Y_{tt} \frac{\partial}{\partial \ddot{y}} + Z_{tt} \frac{\partial}{\partial \ddot{z}} + W_{tt} \frac{\partial}{\partial \ddot{w}}, \quad (13)$$

respectively. Expressions  $Q_t, X_t, Y_t, Z_t, W_t, Q_{tt}, X_{tt}, Y_{tt}, Z_{tt}$  and  $W_{tt}$  are given as

$$\begin{aligned} Q_t &= D_t(Q) - \dot{q}D_t(T), & Q_{tt} &= D_t(Q_t) - \ddot{q}D_t(T), \\ X_t &= D_t(X) - \dot{x}D_t(T), & X_{tt} &= D_t(X_t) - \ddot{x}D_t(T), \\ Y_t &= D_t(Y) - \dot{y}D_t(T), & Y_{tt} &= D_t(Y_t) - \ddot{y}D_t(T), \\ Z_t &= D_t(Z) - \dot{z}D_t(T), & Z_{tt} &= D_t(Z_t) - \ddot{z}D_t(T), \\ W_t &= D_t(W) - \dot{w}D_t(T), & W_{tt} &= D_t(W_t) - \ddot{w}D_t(T), \end{aligned} \quad (14)$$

where  $D_t$  is the total  $t$ -derivative defined as

$$D_t = \frac{\partial}{\partial t} + \dot{q} \frac{\partial}{\partial q} + \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z} + \dot{w} \frac{\partial}{\partial w} + \ddot{q} \frac{\partial}{\partial \dot{q}} + \ddot{x} \frac{\partial}{\partial \dot{x}} + \ddot{y} \frac{\partial}{\partial \dot{y}} + \ddot{z} \frac{\partial}{\partial \dot{z}} + \ddot{w} \frac{\partial}{\partial \dot{w}}. \quad (15)$$

Applying second prolongation (13) to (10), we have

$$\Gamma^{(2)}(\Delta_i^{(2)}) \Big|_{\Delta_i^{(2)}=0} = 0. \quad (16)$$

Splitting the resulting expression with respect to the linearly independent derivative terms lead to an overdetermined system of linear PDEs. Such a system is known as a system of *determining equations*, and its solution is the set of all possible infinitesimals  $T, Q, X, Y, Z$ , and  $W$ , from which we obtain the basis for symmetry algebra  $\mathfrak{g}$ .

### 4. Geodesics and Their Symmetry Algebras

The systems of geodesic equations of all indecomposable solvable Lie algebras of dimension five were constructed by Strugar and Thompson [21]. The list of algebras was based on the 1976 list given by Patera et al. [16]. As mentioned in the Introduction, the focus of this article is to construct and classify the symmetry algebras of geodesic equations of solvable Lie algebras. To be more specific,

we consider the geodesic systems of algebras  $A_{5,7}^{abc}$  through  $A_{18}^a$ . Following [16], we denote each of the five-dimensional algebras as  $A_{p,q}^a$ , which means the  $q$ th algebra of dimension  $p$  and the superscripts, if any, represent the continuous parameters upon which the algebra depends. It turns out that, of the twelve solvable Lie algebras that are examined, three involve three parameters, two involve two, and five involve a single parameter. The symmetry algebras may vary, as the parameters take on certain specific values.

In each case, we methodically provide the nonzero brackets of the original Lie algebra, the associated system of geodesic equations, a basis for the symmetry vector fields, and the corresponding nonvanishing Lie brackets. Subsequently, we summarize our findings. An additional point to emphasize is that determining the symmetry algebra basis and identifying its Lie algebraic structure in each of these cases constitutes a major challenge. The intensive computational process was facilitated and verified by the MAPLE symbolic manipulation program. Throughout this section,  $(q, x, y, z, w)$  and their dots represent the position coordinates and the corresponding velocities coordinates as described in [21]; moreover,  $\mathbb{R}^m \rtimes \mathbb{R}^n$  denotes a semidirect product of abelian Lie algebras, in which  $\mathbb{R}^m$  is a subalgebra and  $\mathbb{R}^n$  an ideal. Further,  $H$  and  $N$  are abbreviated to the Heisenberg Lie algebra and nilpotent Lie algebra, respectively. Finally, we use, for example, shorthand  $D_r$  for  $\frac{\partial}{\partial r}$  to denote a coordinate vector field.

#### 4.1. $A_{5,7}^{abc}$ :

$$[e_1, e_5] = e_1, [e_2, e_5] = ae_2, [e_3, e_5] = be_3, [e_4, e_5] = ce_4; \quad (abc \neq 0, -1 \leq c \leq b \leq a \leq 1).$$

System of geodesic equations:

$$\ddot{q} = \dot{q}\dot{w}, \quad \ddot{x} = a\dot{x}\dot{w}, \quad \ddot{y} = b\dot{y}\dot{w}, \quad \ddot{z} = c\dot{z}\dot{w}, \quad \ddot{w} = 0. \quad (17)$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$\begin{aligned} e_1 = D_z, \quad e_2 = e^{bw}D_y, \quad e_3 = e^{aw}D_x, \quad e_4 = D_t, \quad e_5 = D_q, \quad e_6 = D_x, \quad e_7 = D_y, \quad e_8 = e^{cw}D_z, \\ e_9 = e^{w}D_q, \quad e_{10} = wD_t, \quad e_{11} = xD_x, \quad e_{12} = yD_y, \quad e_{13} = zD_z, \quad e_{14} = qD_q, \quad e_{15} = tD_t, \quad e_{16} = D_w. \end{aligned} \quad (18)$$

$$\begin{aligned} [e_1, e_{13}] = e_1, \quad [e_2, e_{12}] = e_2, \quad [e_2, e_{16}] = -be_2, \quad [e_3, e_{11}] = e_3, \quad [e_3, e_{16}] = -ae_3, \quad [e_4, e_{15}] = e_4, \\ [e_5, e_{14}] = e_5, \quad [e_6, e_{11}] = e_6, \quad [e_7, e_{12}] = e_7, \quad [e_8, e_{13}] = e_8, \quad [e_8, e_{16}] = -ce_8, \quad [e_9, e_{14}] = e_9, \\ [e_9, e_{16}] = -e_9, \quad [e_{10}, e_{15}] = e_{10}, \quad [e_{10}, e_{16}] = -e_4. \end{aligned} \quad (19)$$

For a generic case, it is a 16-dimensional indecomposable solvable. It has a 10-dimensional abelian nilradical spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$  and a 6-dimensional abelian complement spanned by  $e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$ . The symmetry algebra as a whole is isomorphic to  $\mathbb{R}^6 \rtimes \mathbb{R}^{10}$ .

##### 4.1.1. $A_{5,7}^{a=1, bc \neq 1}$ :

Symmetries and nonzero Lie brackets are, respectively,

$$\begin{aligned} e_1 = D_x, \quad e_2 = D_t, \quad e_3 = D_y, \quad e_4 = D_z, \quad e_5 = D_q, \quad e_6 = wD_t, \quad e_7 = e^{w}D_q, \quad e_8 = e^{w}D_x, \\ e_9 = e^{bw}D_y, \quad e_{10} = e^{cw}D_z, \quad e_{11} = D_w, \quad e_{12} = tD_t, \quad e_{13} = yD_y, \quad e_{14} = qD_q + xD_x, \quad e_{15} = zD_z, \\ e_{16} = qD_x, \quad e_{17} = -qD_q + xD_x, \quad e_{18} = xD_q. \end{aligned} \quad (20)$$

$$\begin{aligned}
[e_1, e_{14}] &= e_1, & [e_1, e_{17}] &= e_1, & [e_1, e_{18}] &= e_5, & [e_2, e_{12}] &= e_2, & [e_3, e_{13}] &= e_3, \\
[e_4, e_{15}] &= e_4, & [e_5, e_{14}] &= e_5, & [e_5, e_{16}] &= e_1, & [e_5, e_{17}] &= -e_5, & [e_6, e_{11}] &= -e_2, \\
[e_6, e_{12}] &= e_6, & [e_7, e_{11}] &= -e_7, & [e_7, e_{14}] &= e_7, & [e_7, e_{16}] &= e_8, & [e_7, e_{17}] &= -e_7, \\
[e_8, e_{11}] &= -e_8, & [e_8, e_{14}] &= e_8, & [e_8, e_{17}] &= e_8, & [e_8, e_{18}] &= e_7, & [e_9, e_{11}] &= -be_9, \\
[e_9, e_{13}] &= e_9, & [e_{10}, e_{11}] &= -ce_{10}, & [e_{10}, e_{15}] &= e_{10}, & [e_{16}, e_{17}] &= 2e_{16}, & [e_{16}, e_{18}] &= -e_{17}, \\
[e_{17}, e_{18}] &= 2e_{18}.
\end{aligned} \tag{21}$$

4.1.2.  $A_{5,7}^{a=b, a \neq c}$ :

Symmetries and nonzero Lie brackets are, respectively,

$$\begin{aligned}
e_1 &= D_y, & e_2 &= D_t, & e_3 &= D_q, & e_4 &= D_z, & e_5 &= D_x, & e_6 &= wD_t, & e_7 &= e^{wv}D_q, & e_8 &= e^{bw}D_x, \\
e_9 &= e^{bw}D_y, & e_{10} &= e^{cw}D_z, & e_{11} &= zD_z, & e_{12} &= D_w, & e_{13} &= tD_t, & e_{14} &= qD_q, & e_{15} &= xD_x + yD_y, \\
e_{16} &= xD_y, & e_{17} &= -xD_x + yD_y, & e_{18} &= yD_x.
\end{aligned} \tag{22}$$

$$\begin{aligned}
[e_1, e_{15}] &= e_1, & [e_1, e_{17}] &= e_1, & [e_1, e_{18}] &= e_5, & [e_2, e_{13}] &= e_2, & [e_3, e_{14}] &= e_3, \\
[e_4, e_{11}] &= e_4, & [e_5, e_{15}] &= e_5, & [e_5, e_{16}] &= e_1, & [e_5, e_{17}] &= -e_5, & [e_6, e_{12}] &= -e_2, \\
[e_6, e_{13}] &= e_6, & [e_7, e_{12}] &= -e_7, & [e_7, e_{14}] &= e_7, & [e_8, e_{12}] &= -be_8, & [e_8, e_{15}] &= e_8, \\
[e_8, e_{16}] &= e_9, & [e_8, e_{17}] &= -e_8, & [e_9, e_{12}] &= -be_9, & [e_9, e_{15}] &= e_9, & [e_9, e_{17}] &= e_9, \\
[e_9, e_{18}] &= e_8, & [e_{10}, e_{11}] &= e_{10}, & [e_{10}, e_{12}] &= -ce_{10}, & [e_{16}, e_{17}] &= 2e_{16}, & [e_{16}, e_{18}] &= -e_{17}, \\
[e_{17}, e_{18}] &= 2e_{18}.
\end{aligned} \tag{23}$$

4.1.3.  $A_{5,7}^{b=c, a \neq bc}$ :

Symmetries and nonzero Lie brackets are, respectively,

$$\begin{aligned}
e_1 &= D_z, & e_2 &= D_t, & e_3 &= D_q, & e_4 &= D_x, & e_5 &= D_y, & e_6 &= e^{aw}D_x, & e_7 &= wD_t, & e_8 &= e^{cw}D_z, \\
e_9 &= e^{wv}D_q, & e_{10} &= e^{cw}D_y, & e_{11} &= tD_t, & e_{12} &= xD_x, & e_{13} &= qD_q, & e_{14} &= D_w, & e_{15} &= yD_y + zD_z, \\
e_{16} &= yD_z, & e_{17} &= -yD_y + zD_z, & e_{18} &= zD_y.
\end{aligned} \tag{24}$$

$$\begin{aligned}
[e_1, e_{15}] &= e_1, & [e_1, e_{17}] &= e_1, & [e_1, e_{18}] &= e_5, & [e_2, e_{11}] &= e_2, & [e_3, e_{13}] &= e_3, \\
[e_4, e_{12}] &= e_4, & [e_5, e_{15}] &= e_5, & [e_5, e_{16}] &= e_1, & [e_5, e_{17}] &= -e_5, & [e_6, e_{12}] &= e_6, \\
[e_6, e_{14}] &= -ae_6, & [e_7, e_{11}] &= e_7, & [e_7, e_{14}] &= -e_2, & [e_8, e_{14}] &= -ce_8, & [e_8, e_{15}] &= e_8, \\
[e_8, e_{17}] &= e_8, & [e_8, e_{18}] &= e_{10}, & [e_9, e_{13}] &= e_9, & [e_9, e_{14}] &= -e_9, & [e_{10}, e_{14}] &= -ce_{10}, \\
[e_{10}, e_{15}] &= e_{10}, & [e_{10}, e_{16}] &= e_8, & [e_{10}, e_{17}] &= -e_{10}, & [e_{16}, e_{17}] &= 2e_{16}, & [e_{16}, e_{18}] &= -e_{17}, \\
[e_{17}, e_{18}] &= 2e_{18}.
\end{aligned} \tag{25}$$

For all subcases, the Lie symmetry algebra for each subcase is indecomposable Levi decomposition  $\mathfrak{sl}(2, \mathbb{R}) \times (\mathbb{R}^5 \times \mathbb{R}^{10})$ , where the semisimple part is spanned by  $e_{16}, e_{17}, e_{18}$ . The radical consists of a 10-dimensional indecomposable nilradical spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$  and a 5-dimensional abelian complement spanned by  $e_{11}, e_{12}, e_{13}, e_{14}, e_{15}$ .

4.1.4.  $A_{5,7}^{a=1,b=1,c \neq 1}$ :

Symmetries and nonzero Lie brackets are, respectively,

$$\begin{aligned} e_1 &= D_x, & e_2 &= D_y, & e_3 &= D_z, & e_4 &= D_t, & e_5 &= D_q, & e_6 &= e^w D_y, & e_7 &= e^w D_q, & e_8 &= e^{cw} D_z, \\ e_9 &= w D_t, & e_{10} &= e^w D_x, & e_{11} &= D_w, & e_{12} &= t D_t, & e_{13} &= z D_z, & e_{14} &= q D_q + x D_x + y D_y, \\ e_{15} &= y D_x, & e_{16} &= q D_y, & e_{17} &= x D_y, & e_{18} &= -q D_q + x D_x, & e_{19} &= -q D_q + y D_y, & e_{20} &= x D_q, \\ e_{21} &= y D_q, & e_{22} &= q D_x. \end{aligned} \quad (26)$$

$$\begin{aligned} [e_1, e_{14}] &= e_1, & [e_1, e_{17}] &= e_2, & [e_1, e_{18}] &= e_1, & [e_1, e_{20}] &= e_5, & [e_2, e_{14}] &= e_2, \\ [e_2, e_{15}] &= e_1, & [e_2, e_{19}] &= e_2, & [e_2, e_{21}] &= e_5, & [e_3, e_{13}] &= e_3, & [e_4, e_{12}] &= e_4, \\ [e_5, e_{14}] &= e_5, & [e_5, e_{16}] &= e_2, & [e_5, e_{18}] &= -e_5, & [e_5, e_{19}] &= -e_5, & [e_5, e_{22}] &= e_1, \\ [e_6, e_{11}] &= -e_6, & [e_6, e_{14}] &= e_6, & [e_6, e_{15}] &= e_{10}, & [e_6, e_{19}] &= e_6, & [e_6, e_{21}] &= e_7, \\ [e_7, e_{11}] &= -e_7, & [e_7, e_{14}] &= e_7, & [e_7, e_{16}] &= e_6, & [e_7, e_{18}] &= -e_7, & [e_7, e_{19}] &= -e_7, \\ [e_7, e_{22}] &= e_{10}, & [e_8, e_{11}] &= -ce_8, & [e_8, e_{13}] &= e_8, & [e_9, e_{11}] &= -e_4, & [e_9, e_{12}] &= e_9, \\ [e_{10}, e_{11}] &= -e_{10}, & [e_{10}, e_{14}] &= e_{10}, & [e_{10}, e_{17}] &= e_6, & [e_{10}, e_{18}] &= e_{10}, & [e_{10}, e_{20}] &= e_7, \\ [e_{15}, e_{16}] &= -e_{22}, & [e_{15}, e_{17}] &= -e_{18} + e_{19}, & [e_{15}, e_{18}] &= e_{15}, & [e_{15}, e_{19}] &= -e_{15}, & [e_{15}, e_{20}] &= e_{21}, \\ [e_{16}, e_{18}] &= e_{16}, & [e_{16}, e_{19}] &= 2e_{16}, & [e_{16}, e_{20}] &= -e_{17}, & [e_{16}, e_{21}] &= -e_{19}, & [e_{17}, e_{18}] &= -e_{17}, \\ [e_{17}, e_{19}] &= e_{17}, & [e_{17}, e_{21}] &= e_{20}, & [e_{17}, e_{22}] &= -e_{16}, & [e_{18}, e_{20}] &= 2e_{20}, & [e_{18}, e_{21}] &= e_{21}, \\ [e_{18}, e_{22}] &= -2e_{22}, & [e_{19}, e_{20}] &= e_{20}, & [e_{19}, e_{21}] &= 2e_{21}, & [e_{19}, e_{22}] &= -e_{22}, & [e_{20}, e_{22}] &= e_{18}, \\ [e_{21}, e_{22}] &= e_{15}. \end{aligned} \quad (27)$$

4.1.5.  $A_{5,7}^{a=b=c}$ :

Symmetries and nonzero Lie brackets are, respectively,

$$\begin{aligned} e_1 &= D_y, & e_2 &= D_z, & e_3 &= D_q, & e_4 &= D_t, & e_5 &= D_x, & e_6 &= e^{cw} D_y, & e_7 &= e^{cw} D_z, & e_8 &= e^{cw} D_x, \\ e_9 &= w D_t, & e_{10} &= e^w D_q, & e_{11} &= t D_t, & e_{12} &= q D_q, & e_{13} &= D_w, & e_{14} &= x D_x + y D_y + z D_z, \\ e_{15} &= z D_y, & e_{16} &= x D_z, & e_{17} &= y D_z, & e_{18} &= -x D_x + y D_y, & e_{19} &= -x D_x + z D_z, & e_{20} &= y D_x, \\ e_{21} &= z D_x, & e_{22} &= x D_y \end{aligned} \quad (28)$$

$$\begin{aligned} [e_1, e_{14}] &= e_1, & [e_1, e_{17}] &= e_2, & [e_1, e_{18}] &= e_1, & [e_1, e_{20}] &= e_5, & [e_2, e_{14}] &= e_2, \\ [e_2, e_{15}] &= e_1, & [e_2, e_{19}] &= e_2, & [e_2, e_{21}] &= e_5, & [e_3, e_{12}] &= e_3, & [e_4, e_{11}] &= e_4, \\ [e_5, e_{14}] &= e_5, & [e_5, e_{16}] &= e_2, & [e_5, e_{18}] &= -e_5, & [e_5, e_{19}] &= -e_5, & [e_5, e_{22}] &= e_1, \\ [e_6, e_{13}] &= -ce_6, & [e_6, e_{14}] &= e_6, & [e_6, e_{17}] &= e_7, & [e_6, e_{18}] &= e_6, & [e_6, e_{20}] &= e_8, \\ [e_7, e_{13}] &= -ce_7, & [e_7, e_{14}] &= e_7, & [e_7, e_{15}] &= e_6, & [e_7, e_{19}] &= e_7, & [e_7, e_{21}] &= e_8, \\ [e_8, e_{13}] &= -ce_8, & [e_8, e_{14}] &= e_8, & [e_8, e_{16}] &= e_7, & [e_8, e_{18}] &= -e_8, & [e_8, e_{19}] &= -e_8, \\ [e_8, e_{22}] &= e_6, & [e_9, e_{11}] &= e_9, & [e_9, e_{13}] &= -e_4, & [e_{10}, e_{12}] &= e_{10}, & [e_{10}, e_{13}] &= -e_{10}, \\ [e_{15}, e_{16}] &= -e_{22}, & [e_{15}, e_{17}] &= -e_{18} + e_{19}, & [e_{15}, e_{18}] &= e_{15}, & [e_{15}, e_{19}] &= -e_{15}, & [e_{15}, e_{20}] &= e_{21}, \\ [e_{16}, e_{18}] &= e_{16}, & [e_{16}, e_{19}] &= 2e_{16}, & [e_{16}, e_{20}] &= -e_{17}, & [e_{16}, e_{21}] &= -e_{19}, & [e_{17}, e_{18}] &= -e_{17}, \\ [e_{17}, e_{19}] &= e_{17}, & [e_{17}, e_{21}] &= e_{20}, & [e_{17}, e_{22}] &= -e_{16}, & [e_{18}, e_{20}] &= 2e_{20}, & [e_{18}, e_{21}] &= e_{21}, \\ [e_{18}, e_{22}] &= -2e_{22}, & [e_{19}, e_{20}] &= e_{20}, & [e_{19}, e_{21}] &= 2e_{21}, & [e_{19}, e_{22}] &= -e_{22}, & [e_{20}, e_{22}] &= e_{18}, \\ [e_{21}, e_{22}] &= e_{15}. \end{aligned} \quad (29)$$



For both subcases, the symmetry algebra is a 22-dimensional indecomposable Levi decomposition with an 8-dimensional semisimple  $\mathfrak{sl}(3, \mathbb{R})$  spanned by  $e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{22}$  as well as a 14-dimensional solvable consisting of a 10-dimensional abelian nilradical spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$  and a 4-dimensional abelian complement spanned by  $e_{11}, e_{12}, e_{13}, e_{14}$ .

4.1.6.  $A_{5,7}^{a=1,b=1,c=1}$ :

Symmetries and nonzero Lie brackets are, respectively,

$$\begin{aligned}
 e_1 &= D_x, & e_2 &= D_y, & e_3 &= D_z, & e_4 &= D_t, & e_5 &= D_q, & e_6 &= e^w D_x, & e_7 &= e^w D_y, & e_8 &= e^w D_z, \\
 e_9 &= e^w D_q, & e_{10} &= w D_t, & e_{11} &= t D_t, & e_{12} &= q D_q + x D_x + y D_y + z D_z, & e_{13} &= D_w, & e_{14} &= z D_x, \\
 e_{15} &= q D_y, & e_{16} &= x D_y, & e_{17} &= z D_y, & e_{18} &= q D_q - x D_x, & e_{19} &= -x D_x + y D_y, & e_{20} &= q D_z, \\
 e_{21} &= x D_q, & e_{22} &= x D_z, & e_{23} &= y D_z, & e_{24} &= -x D_x + z D_z, & e_{25} &= y D_q, & e_{26} &= z D_q, & e_{27} &= q D_x, \\
 e_{28} &= y D_x.
 \end{aligned}
 \tag{30}$$

$$\begin{aligned}
 [e_1, e_{12}] &= e_1, & [e_1, e_{16}] &= e_2, & [e_1, e_{18}] &= -e_1, & [e_1, e_{19}] &= -e_1, & [e_1, e_{21}] &= e_5, \\
 [e_1, e_{22}] &= e_3, & [e_1, e_{24}] &= -e_1, & [e_2, e_{12}] &= e_2, & [e_2, e_{19}] &= e_2, & [e_2, e_{23}] &= e_3, \\
 [e_2, e_{25}] &= e_5, & [e_2, e_{28}] &= e_1, & [e_3, e_{12}] &= e_3, & [e_3, e_{14}] &= e_1, & [e_3, e_{17}] &= e_2, \\
 [e_3, e_{24}] &= e_3, & [e_3, e_{26}] &= e_5, & [e_4, e_{11}] &= e_4, & [e_5, e_{12}] &= e_5, & [e_5, e_{15}] &= e_2, \\
 [e_5, e_{18}] &= e_5, & [e_5, e_{20}] &= e_3, & [e_5, e_{27}] &= e_1, & [e_6, e_{12}] &= e_6, & [e_6, e_{13}] &= -e_6, \\
 [e_6, e_{16}] &= e_7, & [e_6, e_{18}] &= -e_6, & [e_6, e_{19}] &= -e_6, & [e_6, e_{21}] &= e_9, & [e_6, e_{22}] &= e_8, \\
 [e_6, e_{24}] &= -e_6, & [e_7, e_{12}] &= e_7, & [e_7, e_{13}] &= -e_7, & [e_7, e_{19}] &= e_7, & [e_7, e_{23}] &= e_8, \\
 [e_7, e_{25}] &= e_9, & [e_7, e_{28}] &= e_6, & [e_8, e_{12}] &= e_8, & [e_8, e_{13}] &= -e_8, & [e_8, e_{14}] &= e_6, \\
 [e_8, e_{17}] &= e_7, & [e_8, e_{24}] &= e_8, & [e_8, e_{26}] &= e_9, & [e_9, e_{12}] &= e_9, & [e_9, e_{13}] &= -e_9, \\
 [e_9, e_{15}] &= e_7, & [e_9, e_{18}] &= e_9, & [e_9, e_{20}] &= e_8, & [e_9, e_{27}] &= e_6, & [e_{10}, e_{11}] &= e_{10}, \\
 [e_{10}, e_{13}] &= -e_4, & [e_{14}, e_{16}] &= e_{17}, & [e_{14}, e_{18}] &= -e_{14}, & [e_{14}, e_{19}] &= -e_{14}, & [e_{14}, e_{20}] &= -e_{27}, \\
 [e_{14}, e_{21}] &= e_{26}, & [e_{14}, e_{22}] &= e_{24}, & [e_{14}, e_{23}] &= -e_{28}, & [e_{14}, e_{24}] &= -2e_{14}, & [e_{15}, e_{18}] &= -e_{15}, \\
 [e_{15}, e_{19}] &= e_{15}, & [e_{15}, e_{21}] &= -e_{16}, & [e_{15}, e_{23}] &= e_{20}, & [e_{15}, e_{25}] &= e_{18} - e_{19}, & [e_{15}, e_{26}] &= -e_{17}, \\
 [e_{15}, e_{28}] &= e_{27}, & [e_{16}, e_{18}] &= e_{16}, & [e_{16}, e_{19}] &= 2e_{16}, & [e_{16}, e_{23}] &= e_{22}, & [e_{16}, e_{24}] &= e_{16}, \\
 [e_{16}, e_{25}] &= e_{21}, & [e_{16}, e_{27}] &= -e_{15}, & [e_{16}, e_{28}] &= -e_{19}, & [e_{17}, e_{19}] &= e_{17}, & [e_{17}, e_{20}] &= -e_{15}, \\
 [e_{17}, e_{22}] &= -e_{16}, & [e_{17}, e_{23}] &= -e_{19} + e_{24}, & [e_{17}, e_{24}] &= -e_{17}, & [e_{17}, e_{25}] &= e_{26}, & [e_{17}, e_{28}] &= e_{14}, \\
 [e_{18}, e_{20}] &= e_{20}, & [e_{18}, e_{21}] &= -2e_{21}, & [e_{18}, e_{22}] &= -e_{22}, & [e_{18}, e_{25}] &= -e_{25}, & [e_{18}, e_{26}] &= -e_{26}, \\
 [e_{18}, e_{27}] &= 2e_{27}, & [e_{18}, e_{28}] &= e_{28}, & [e_{19}, e_{21}] &= -e_{21}, & [e_{19}, e_{22}] &= -e_{22}, & [e_{19}, e_{23}] &= e_{23}, \\
 [e_{19}, e_{25}] &= e_{25}, & [e_{19}, e_{27}] &= e_{27}, & [e_{19}, e_{28}] &= 2e_{28}, & [e_{20}, e_{21}] &= -e_{22}, & [e_{20}, e_{24}] &= e_{20}, \\
 [e_{20}, e_{25}] &= -e_{23}, & [e_{20}, e_{26}] &= e_{18} - e_{24}, & [e_{21}, e_{24}] &= e_{21}, & [e_{21}, e_{27}] &= -e_{18}, & [e_{21}, e_{28}] &= -e_{25}, \\
 [e_{22}, e_{24}] &= 2e_{22}, & [e_{22}, e_{26}] &= e_{21}, & [e_{22}, e_{27}] &= -e_{20}, & [e_{22}, e_{28}] &= -e_{23}, & [e_{23}, e_{24}] &= e_{23}, \\
 [e_{23}, e_{26}] &= e_{25}, & [e_{24}, e_{26}] &= e_{26}, & [e_{24}, e_{27}] &= e_{27}, & [e_{24}, e_{28}] &= e_{28}, & [e_{25}, e_{27}] &= e_{28}, \\
 [e_{26}, e_{27}] &= e_{14}.
 \end{aligned}
 \tag{31}$$

This is a 28-dimensional indecomposable with nontrivial Levi decomposition  $\mathfrak{sl}(4, \mathbb{R}) \times (\mathbb{R}^3 \times \mathbb{R}^{10})$ . The semisimple is spanned by  $e_{14}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{22}, e_{23}, e_{24}, e_{25}, e_{26}, e_{27}, e_{28}$ . The radical is a semidirect product of a 10-dimensional indecomposable nilradical spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$  and a 3-dimensional abelian complement spanned by  $e_{11}, e_{12}, e_{13}$ .

4.2.  $A_{5,8}^c$ :

$$[e_2, e_5] = e_1, [e_3, e_5] = e_3, [e_4, e_5] = ce_4; \quad (0 \leq |c| \leq 1).$$

System of geodesic equations:

$$\ddot{q} = \dot{x}\dot{w}, \quad \ddot{x} = 0, \quad \ddot{y} = \dot{y}\dot{w}, \quad \ddot{z} = c\dot{z}\dot{w}, \quad \ddot{w} = 0. \tag{32}$$

The symmetry algebra basis and nonvanishing brackets are, respectively,

$$\begin{aligned} e_1 &= D_x, & e_2 &= D_t, & e_3 &= D_y, & e_4 &= D_q, & e_5 &= D_z, & e_6 &= wD_t, & e_7 &= wD_q, & e_8 &= e^{w}D_y, \\ e_9 &= e^{cw}D_t, & e_{10} &= \frac{1}{2}w^2D_q + wD_x, & e_{11} &= \frac{x D_q}{2} + D_w, & e_{12} &= qD_q + tD_t + xD_x, & e_{13} &= yD_y, \\ e_{14} &= zD_z, & e_{15} &= \frac{(wx - 2q)D_q}{2} + tD_t, & e_{16} &= tD_q, & e_{17} &= xD_t, & e_{18} &= xD_q, \\ e_{19} &= qD_q + xD_x + (xw - 2q)D_q, & e_{20} &= tD_x + \frac{twD_q}{2}, & e_{21} &= (xw - 2q)D_t, \\ e_{22} &= \left(\frac{xw^2}{2} - qw\right)D_q + (xw - 2q)D_x. \end{aligned} \tag{33}$$

$$\begin{aligned} [e_1, e_{11}] &= \frac{e_4}{2}, & [e_1, e_{12}] &= e_1, & [e_1, e_{15}] &= \frac{e_7}{2}, & [e_1, e_{17}] &= e_2, \\ [e_1, e_{18}] &= e_4, & [e_1, e_{19}] &= e_1 + e_7, & [e_1, e_{21}] &= e_6, & [e_1, e_{22}] &= e_{10}, \\ [e_2, e_{12}] &= e_2, & [e_2, e_{15}] &= e_2, & [e_2, e_{16}] &= e_4, & [e_2, e_{20}] &= e_1 + \frac{e_7}{2}, \\ [e_3, e_{13}] &= e_3, & [e_4, e_{12}] &= e_4, & [e_4, e_{15}] &= -e_4, & [e_4, e_{19}] &= -e_4, \\ [e_4, e_{21}] &= -2e_2, & [e_4, e_{22}] &= -2e_1 - e_7, & [e_5, e_{14}] &= e_5, & [e_6, e_{11}] &= -e_2, \\ [e_6, e_{12}] &= e_6, & [e_6, e_{15}] &= e_6, & [e_6, e_{16}] &= e_7, & [e_6, e_{20}] &= -e_{10}, \\ [e_7, e_{11}] &= -e_4, & [e_7, e_{12}] &= e_7, & [e_7, e_{15}] &= -e_7, & [e_7, e_{19}] &= -e_7, \\ [e_7, e_{21}] &= -2e_6, & [e_7, e_{22}] &= -2e_{10}, & [e_8, e_{11}] &= -e_8, & [e_8, e_{13}] &= e_8, \\ [e_9, e_{11}] &= -ce_9, & [e_9, e_{14}] &= e_9, & [e_{10}, e_{11}] &= -e_1 - \frac{e_7}{2}, & [e_{10}, e_{12}] &= e_{10}, \\ [e_{10}, e_{17}] &= e_6, & [e_{10}, e_{18}] &= e_7, & [e_{10}, e_{19}] &= e_{10}, & [e_{15}, e_{16}] &= 2e_{16}, \\ [e_{15}, e_{17}] &= -e_{17}, & [e_{15}, e_{18}] &= e_{18}, & [e_{15}, e_{20}] &= e_{20}, & [e_{15}, e_{21}] &= -2e_{21}, \\ [e_{15}, e_{22}] &= -e_{22}, & [e_{16}, e_{17}] &= -e_{18}, & [e_{16}, e_{19}] &= -e_{16}, & [e_{16}, e_{21}] &= -2e_{15}, \\ [e_{16}, e_{22}] &= -2e_{20}, & [e_{17}, e_{19}] &= -e_{17}, & [e_{17}, e_{20}] &= -e_{15} + e_{19}, & [e_{17}, e_{22}] &= -e_{21}, \\ [e_{18}, e_{19}] &= -2e_{18}, & [e_{18}, e_{20}] &= -e_{16}, & [e_{18}, e_{21}] &= -2e_{17}, & [e_{18}, e_{22}] &= -2e_{19}, \\ [e_{19}, e_{20}] &= -e_{20}, & [e_{19}, e_{21}] &= -e_{21}, & [e_{19}, e_{22}] &= -2e_{22}, & [e_{20}, e_{21}] &= -e_{22}. \end{aligned} \tag{34}$$

For the generic case, the symmetry algebra is  $\mathfrak{sl}(3, \mathbb{R}) \times (\mathbb{R}^4 \times \mathbb{R}^{10})$ . The semisimple part  $\mathfrak{sl}(3, \mathbb{R})$  is spanned by  $e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{22}$ ; the abelian nilradical  $\mathbb{R}^{10}$  is spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$ , and the abelian complement to  $\mathbb{R}^{10}$  is spanned by  $e_{11}, e_{12}, e_{13}, e_{14}$ .

$A_{5,8}^{c=1}$ :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned}
e_1 &= D_x, & e_2 &= D_t, & e_3 &= D_q, & e_4 &= D_y, & e_5 &= D_z, & e_6 &= wD_t, & e_7 &= wD_q, & e_8 &= e^{tw}D_y, \\
e_9 &= e^{tw}D_z, & e_{10} &= \frac{w^2}{2}D_q + wD_x, & e_{11} &= D_w + \frac{x}{2}D_q, & e_{12} &= qD_q + tD_t + xD_x, & e_{13} &= yD_y + zD_z, \\
e_{14} &= yD_y - zD_z, & e_{15} &= zD_y, & e_{16} &= yD_z, & e_{17} &= tD_t + \frac{(wx-2q)}{2}D_q, & e_{18} &= xD_q, & e_{19} &= tD_q, \\
e_{20} &= qD_q + xD_x + (wx-2q)D_q, & e_{21} &= xD_t, & e_{22} &= \frac{tw}{2}D_q + tD_x, & e_{23} &= (wx-2q)D_t, \\
e_{24} &= \frac{(xw^2 - qw)}{2}D_q + (wx-2q)D_x.
\end{aligned} \tag{35}$$

$$\begin{aligned}
[e_1, e_{11}] &= \frac{e_3}{2}, & [e_1, e_{12}] &= e_1, & [e_1, e_{17}] &= \frac{e_7}{2}, & [e_1, e_{18}] &= e_3, \\
[e_1, e_{20}] &= e_1 + e_7, & [e_1, e_{21}] &= e_2, & [e_1, e_{23}] &= e_6, & [e_1, e_{24}] &= e_{10}, \\
[e_2, e_{12}] &= e_2, & [e_2, e_{17}] &= e_2, & [e_2, e_{19}] &= e_3, & [e_2, e_{22}] &= e_1 + \frac{e_7}{2}, \\
[e_3, e_{12}] &= e_3, & [e_3, e_{17}] &= -e_3, & [e_3, e_{20}] &= -e_3, & [e_3, e_{23}] &= -2e_2, \\
[e_3, e_{24}] &= -2e_1 - e_7, & [e_4, e_{13}] &= e_4, & [e_4, e_{14}] &= e_4, & [e_4, e_{16}] &= e_5, \\
[e_5, e_{13}] &= e_5, & [e_5, e_{14}] &= -e_5, & [e_5, e_{15}] &= e_4, & [e_6, e_{11}] &= -e_2, \\
[e_6, e_{12}] &= e_6, & [e_6, e_{17}] &= e_6, & [e_6, e_{19}] &= e_7, & [e_6, e_{22}] &= e_{10}, \\
[e_7, e_{11}] &= -e_3, & [e_7, e_{12}] &= e_7, & [e_7, e_{17}] &= -e_7, & [e_7, e_{20}] &= -e_7, \\
[e_7, e_{23}] &= -2e_6, & [e_7, e_{24}] &= -2e_{10}, & [e_8, e_{11}] &= -e_8, & [e_8, e_{13}] &= e_8, \\
[e_8, e_{14}] &= e_8, & [e_8, e_{16}] &= e_9, & [e_9, e_{11}] &= -e_9, & [e_9, e_{13}] &= e_9, \\
[e_9, e_{14}] &= -e_9, & [e_9, e_{15}] &= e_8, & [e_{10}, e_{11}] &= -e_1 - \frac{e_7}{2}, & [e_{10}, e_{12}] &= e_{10}, \\
[e_{10}, e_{18}] &= e_7, & [e_{10}, e_{20}] &= e_{10}, & [e_{10}, e_{21}] &= e_6, & [e_{14}, e_{15}] &= -2e_{15}, \\
[e_{14}, e_{16}] &= 2e_{16}, & [e_{15}, e_{16}] &= -e_{14}, & [e_{17}, e_{18}] &= e_{18}, & [e_{17}, e_{19}] &= 2e_{19}, \\
[e_{17}, e_{21}] &= -e_{21}, & [e_{17}, e_{22}] &= e_{22}, & [e_{17}, e_{23}] &= -2e_{23}, & [e_{17}, e_{24}] &= -e_{24}, \\
[e_{18}, e_{20}] &= -2e_{18}, & [e_{18}, e_{22}] &= -e_{19}, & [e_{18}, e_{23}] &= -2e_{21}, & [e_{18}, e_{24}] &= -2e_{20}, \\
[e_{19}, e_{20}] &= -e_{19}, & [e_{19}, e_{21}] &= -e_{18}, & [e_{19}, e_{23}] &= -2e_{17}, & [e_{19}, e_{24}] &= -2e_{22}, \\
[e_{20}, e_{21}] &= e_{21}, & [e_{20}, e_{22}] &= -e_{22}, & [e_{20}, e_{23}] &= -e_{23}, & [e_{20}, e_{24}] &= -2e_{24}, \\
[e_{21}, e_{22}] &= -e_{17} + e_{20}, & [e_{21}, e_{24}] &= -e_{23}, & [e_{22}, e_{23}] &= -e_{24}.
\end{aligned} \tag{36}$$

The algebra is a  $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R}) \times (\mathbb{R}^3 \times \mathbb{R}^{10})$  Levi decomposition algebra. The semisimple factor is direct sum of  $\mathfrak{sl}(2, \mathbb{R})$  spanned by  $e_{14}, e_{15}, e_{16}$  and  $\mathfrak{sl}(3, \mathbb{R})$  spanned by  $e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{22}, e_{23}, e_{24}$ . The radical comprises a 10-dimensional indecomposable nilradical spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$  and a 3-dimensional abelian complement spanned by  $e_{11}, e_{12}, e_{13}$ .

#### 4.3. $A_{5,9}^{bc}$ :

$$[e_1, e_5] = e_1, [e_2, e_5] = e_1 + e_2, [e_3, e_5] = be_3, [e_4, e_5] = ce_4; \quad (bc \neq 0).$$

System of geodesic equations:

$$\ddot{q} = \dot{q}\dot{w} + \dot{x}\dot{w}, \quad \ddot{x} = \dot{x}\dot{w}, \quad \ddot{y} = b\dot{y}\dot{w}, \quad \ddot{z} = c\dot{z}\dot{w}, \quad \ddot{w} = 0. \tag{37}$$

The symmetry basis and nonzero brackets are, respectively,

$$\begin{aligned}
e_1 &= D_x, & e_2 &= D_q, & e_3 &= xD_q, & e_4 &= e^{tw}D_q, & e_5 &= (w-1)e^{tw}D_q + e^{tw}D_x, & e_6 &= wD_t, & e_7 &= D_y, \\
e_8 &= D_z, & e_9 &= e^{bw}D_y, & e_{10} &= e^{cw}D_z, & e_{11} &= D_t, & e_{12} &= tD_t, & e_{13} &= D_w, & e_{14} &= yD_y, \\
e_{15} &= zD_z, & e_{16} &= qD_q + xD_x.
\end{aligned} \tag{38}$$

$$\begin{aligned}
 [e_1, e_3] &= e_2, & [e_1, e_{16}] &= e_1, & [e_2, e_{16}] &= e_2, & [e_3, e_5] &= -e_4, & [e_4, e_{13}] &= -e_4, \\
 [e_4, e_{16}] &= e_4, & [e_5, e_{13}] &= -e_4 - e_5, & [e_5, e_{16}] &= e_5, & [e_6, e_{12}] &= e_6, & [e_6, e_{13}] &= -e_{11}, \\
 [e_7, e_{14}] &= e_7, & [e_8, e_{15}] &= e_8, & [e_9, e_{13}] &= -be_9, & [e_9, e_{14}] &= e_9, & [e_{10}, e_{13}] &= -ce_{10}, \\
 [e_{10}, e_{15}] &= e_{10}, & [e_{11}, e_{12}] &= e_{11}.
 \end{aligned}
 \tag{39}$$

For the generic case, the symmetry algebra is a 16-dimensional indecomposable solvable  $\mathbb{R}^5 \rtimes (H_5 \oplus \mathbb{R}^6)$ . The nonabelian nilradical is  $H_5 \oplus \mathbb{R}^6$ . Here,  $H$  denotes the 5-dimensional Heisenberg algebra spanned by  $e_1, e_2, e_3, e_4, e_5$ , and the  $\mathbb{R}^6$  summand is spanned by  $e_6, e_7, e_8, e_9, e_{10}, e_{11}$ . The complement to the nilradical is abelian spanned by  $e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$ .

4.3.1.  $A_{5,9}^{b=1,c \neq 1}$ :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned}
 e_1 &= xD_q, & e_2 &= D_x, & e_3 &= D_q, & e_4 &= xD_z, & e_5 &= (w-1)e^{wq}D_q + e^{wq}D_x, & e_6 &= e^{wq}D_q, \\
 e_7 &= zD_q, & e_8 &= e^{wq}D_z, & e_9 &= D_z, & e_{10} &= D_y, & e_{11} &= D_t, & e_{12} &= e^{cw}D_y, & e_{13} &= wD_t, \\
 e_{14} &= tD_t, & e_{15} &= D_w, & e_{16} &= yD_y, & e_{17} &= zD_z, & e_{18} &= qD_q + xD_x.
 \end{aligned}
 \tag{40}$$

$$\begin{aligned}
 [e_1, e_2] &= -e_3, & [e_1, e_5] &= -e_6, & [e_2, e_{14}] &= e_9, & [e_2, e_{18}] &= e_2, & [e_3, e_{18}] &= e_3, \\
 [e_4, e_5] &= -e_8, & [e_4, e_7] &= e_1, & [e_4, e_{17}] &= e_4, & [e_4, e_{18}] &= -e_4, & [e_5, e_{15}] &= -e_5 - e_6, \\
 [e_5, e_{18}] &= e_5, & [e_6, e_{15}] &= -e_6, & [e_6, e_{18}] &= e_6, & [e_7, e_8] &= -e_6, & [e_7, e_9] &= -e_3, \\
 [e_7, e_{17}] &= -e_7, & [e_7, e_{18}] &= e_7, & [e_8, e_{15}] &= -e_8, & [e_8, e_{17}] &= e_8, & [e_9, e_{17}] &= e_9, \\
 [e_{10}, e_{16}] &= e_{10}, & [e_{11}, e_{14}] &= e_{11}, & [e_{12}, e_{15}] &= -ce_{12}, & [e_{12}, e_{16}] &= e_{12}, & [e_{13}, e_{14}] &= e_{13}, \\
 [e_{13}, e_{15}] &= -e_{11}.
 \end{aligned}
 \tag{41}$$

4.3.2.  $A_{5,9}^{b \neq 1,c=1}$ :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned}
 e_1 &= xD_q, & e_2 &= D_x, & e_3 &= D_q, & e_4 &= xD_z, & e_5 &= (w-1)e^{wq}D_q + e^{wq}D_x, & e_6 &= e^{wq}D_q, \\
 e_7 &= zD_q, & e_8 &= e^{wq}D_z, & e_9 &= D_z, & e_{10} &= D_y, & e_{11} &= D_t, & e_{12} &= e^{bw}D_y, & e_{13} &= wD_t, \\
 e_{14} &= tD_t, & e_{15} &= D_w, & e_{16} &= yD_y, & e_{17} &= zD_z, & e_{18} &= qD_q + xD_x.
 \end{aligned}
 \tag{42}$$

$$\begin{aligned}
 [e_1, e_2] &= -e_3, & [e_1, e_5] &= -e_6, & [e_2, e_{14}] &= e_9, & [e_2, e_{18}] &= e_2, & [e_3, e_{18}] &= e_3, \\
 [e_4, e_5] &= -e_8, & [e_4, e_7] &= e_1, & [e_4, e_{17}] &= e_4, & [e_4, e_{18}] &= -e_4, & [e_5, e_{15}] &= -e_5 - e_6, \\
 [e_5, e_{18}] &= e_5, & [e_6, e_{15}] &= -e_6, & [e_6, e_{18}] &= e_6, & [e_7, e_8] &= -e_6, & [e_7, e_9] &= -e_3, \\
 [e_7, e_{17}] &= -e_7, & [e_7, e_{18}] &= e_7, & [e_8, e_{15}] &= -e_8, & [e_8, e_{17}] &= e_8, & [e_9, e_{17}] &= e_9, \\
 [e_{10}, e_{16}] &= e_{10}, & [e_{11}, e_{14}] &= e_{11}, & [e_{12}, e_{15}] &= -be_{12}, & [e_{12}, e_{16}] &= e_{12}, & [e_{13}, e_{14}] &= e_{13}, \\
 [e_{13}, e_{15}] &= -e_{11}.
 \end{aligned}
 \tag{43}$$

For both subcases, the symmetry algebra is an 18-dimensional indecomposable solvable algebra. The nilradical is a nonabelian Lie algebra,  $N_9 \oplus \mathbb{R}^4$ , where  $N_9$  is a 9-dimensional indecomposable nilpotent spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$ , and  $\mathbb{R}^4$  is spanned by  $e_{10}, e_{11}, e_{12}, e_{13}$ . The complement to the nilradical is a 5-dimensional abelian spanned by  $e_{14}, e_{15}, e_{16}, e_{17}, e_{18}$ .

4.3.3.  $A_{5,9}^{b=c}$  :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned} e_1 = D_x, \quad e_2 = e^w D_q, \quad e_3 = x D_q, \quad e_4 = (w-1)e^w D_q + e^w D_x, \quad e_5 = D_q, \quad e_6 = D_z, \quad e_7 = D_y, \\ e_8 = D_t, \quad e_9 = e^{cw} D_z, \quad e_{10} = e^{cw} D_y, \quad e_{11} = w D_t, \quad e_{12} = q D_q + x D_x, \quad e_{13} = t D_t, \\ e_{14} = D_w, \quad e_{15} = y D_y + z D_z, \quad e_{16} = y D_y - z D_z, \quad e_{17} = z D_y, \quad e_{18} = y D_z. \end{aligned} \quad (44)$$

$$\begin{aligned} [e_1, e_3] = e_5, \quad [e_1, e_{12}] = e_1, \quad [e_2, e_{12}] = e_2, \quad [e_2, e_{14}] = -e_2, \quad [e_3, e_4] = -e_2, \\ [e_4, e_{12}] = e_4, \quad [e_4, e_{14}] = -e_2 - e_4, \quad [e_5, e_{12}] = e_5, \quad [e_6, e_{15}] = e_6, \quad [e_6, e_{16}] = -e_6, \\ [e_6, e_{17}] = e_7, \quad [e_7, e_{15}] = e_7, \quad [e_7, e_{16}] = e_7, \quad [e_7, e_{18}] = e_6, \quad [e_8, e_{13}] = e_8, \\ [e_9, e_{14}] = -ce_9, \quad [e_9, e_{15}] = e_9, \quad [e_9, e_{16}] = -e_9, \quad [e_9, e_{17}] = e_{10}, \quad [e_{10}, e_{14}] = -ce_{10}, \\ [e_{10}, e_{15}] = e_{10}, \quad [e_{10}, e_{16}] = e_{10}, \quad [e_{10}, e_{18}] = e_9, \quad [e_{11}, e_{13}] = e_{11}, \quad [e_{11}, e_{14}] = -e_8, \\ [e_{16}, e_{17}] = -2e_{17}, \quad [e_{16}, e_{18}] = 2e_{18}, \quad [e_{17}, e_{18}] = -e_{16}. \end{aligned} \quad (45)$$

The symmetry algebra is a  $\mathfrak{sl}(2, \mathbb{R}) \times (\mathbb{R}^4 \times \mathbb{R}^6 \oplus H_5)$  Levi decomposition, where the radical consists of a decomposable nilradical  $\mathbb{R}^6 \oplus H_5$  spanned by  $e_6, e_7, e_8, e_9, e_{10}, e_{11}$  and  $e_1, e_2, e_3, e_4, e_5$ , respectively, as well as an abelian complement spanned by  $e_{12}, e_{13}, e_{14}, e_{15}$ . The semisimple part is  $\mathfrak{sl}(2, \mathbb{R})$  spanned by  $e_{16}, e_{17}, e_{18}$ .

4.3.4.  $A_{5,9}^{b=1, c=1}$  :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned} e_1 = D_x, \quad e_2 = x D_q, \quad e_3 = x D_y, \quad e_4 = e^w D_q, \quad e_5 = (w-1)e^w D_q + e^w D_x, \quad e_6 = D_t, \\ e_7 = D_y, \quad e_8 = e^w D_y, \quad e_9 = w D_t, \quad e_{10} = z D_q, \quad e_{11} = D_q, \quad e_{12} = y D_q, \quad e_{13} = D_z, \\ e_{14} = e^w D_z, \quad e_{15} = x D_z, \quad e_{16} = D_w, \quad e_{17} = t D_t, \quad e_{18} = q D_q + x D_x, \quad e_{19} = y D_y + z D_z, \\ e_{20} = y D_y - z D_z, \quad e_{21} = z D_y, \quad e_{22} = y D_z. \end{aligned} \quad (46)$$

$$\begin{aligned} [e_1, e_2] = e_{11}, \quad [e_1, e_3] = e_7, \quad [e_1, e_{15}] = e_{13}, \quad [e_1, e_{18}] = e_1, \quad [e_2, e_5] = -e_4, \\ [e_3, e_5] = -e_8, \quad [e_3, e_{12}] = e_2, \quad [e_3, e_{18}] = -e_3, \quad [e_3, e_{19}] = e_3, \quad [e_3, e_{20}] = e_3, \\ [e_3, e_{22}] = e_{15}, \quad [e_4, e_{16}] = -e_4, \quad [e_4, e_{18}] = e_4, \quad [e_5, e_{15}] = e_{14}, \quad [e_5, e_{16}] = -e_4 - e_5, \\ [e_5, e_{18}] = e_5, \quad [e_6, e_{17}] = e_6, \quad [e_7, e_{12}] = e_{11}, \quad [e_7, e_{19}] = e_7, \quad [e_7, e_{20}] = e_7, \\ [e_7, e_{22}] = e_{13}, \quad [e_8, e_{12}] = e_4, \quad [e_8, e_{16}] = -e_8, \quad [e_8, e_{19}] = e_8, \quad [e_8, e_{20}] = e_8, \\ [e_8, e_{22}] = e_{14}, \quad [e_9, e_{16}] = -e_6, \quad [e_9, e_{17}] = e_9, \quad [e_{10}, e_{13}] = -e_{11}, \quad [e_{10}, e_{14}] = -e_4, \\ [e_{10}, e_{15}] = -e_2, \quad [e_{10}, e_{18}] = e_{10}, \quad [e_{10}, e_{19}] = -e_{10}, \quad [e_{10}, e_{20}] = e_{10}, \quad [e_{10}, e_{22}] = -e_{12}, \\ [e_{11}, e_{18}] = e_{11}, \quad [e_{12}, e_{18}] = e_{12}, \quad [e_{12}, e_{19}] = -e_{12}, \quad [e_{12}, e_{20}] = -e_{12}, \quad [e_{12}, e_{21}] = -e_{10}, \\ [e_{13}, e_{19}] = e_{13}, \quad [e_{13}, e_{20}] = -e_{13}, \quad [e_{13}, e_{21}] = e_7, \quad [e_{14}, e_{16}] = -e_{14}, \quad [e_{14}, e_{19}] = e_{14}, \\ [e_{14}, e_{20}] = -e_{14}, \quad [e_{14}, e_{21}] = e_8, \quad [e_{15}, e_{18}] = -e_{15}, \quad [e_{15}, e_{19}] = e_{15}, \quad [e_{15}, e_{20}] = -e_{15}, \\ [e_{15}, e_{21}] = e_3, \quad [e_{20}, e_{21}] = -2e_{21}, \quad [e_{20}, e_{22}] = 2e_{22}, \quad [e_{21}, e_{22}] = -e_{20}. \end{aligned} \quad (47)$$

The symmetry algebra is  $\mathfrak{sl}(2, \mathbb{R}) \times (\mathbb{R}^4 \times \mathbb{R}^{15})$  indecomposable Levi decomposition with a 22-dimensional. It has a 19-dimensional solvable consisting of a 15-dimensional nonabelian nilradical spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}$  and a 4-dimensional abelian complement spanned by  $e_{16}, e_{17}, e_{18}, e_{19}$ . The  $\mathfrak{sl}(2, \mathbb{R})$  part is semisimple spanned by  $e_{20}, e_{21}, e_{22}$ .

4.4.  $A_{5,10}$ :

$$[e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3.$$

System of geodesic equations:

$$\ddot{q} = \dot{x}\dot{w}, \quad \ddot{x} = \dot{y}\dot{w}, \quad \dot{y} = 0, \quad \ddot{z} = \dot{z}\dot{w}, \quad \ddot{w} = 0. \quad (48)$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$\begin{aligned} e_1 &= D_t, & e_2 &= tD_q, & e_3 &= D_z, & e_4 &= D_q, & e_5 &= D_x, & e_6 &= D_y, & e_7 &= wD_t, & e_8 &= yD_t, & e_9 &= yD_q, \\ e_{10} &= wD_q, & e_{11} &= xD_q + yD_x, & e_{12} &= e^w D_z, & e_{13} &= \frac{1}{2}w^2 D_q + wD_x, & e_{14} &= wyD_q + 2yD_x, \\ e_{15} &= \frac{1}{6}w^3 D_q + \frac{1}{2}w^2 D_x + wD_y, & e_{16} &= D_w, & e_{17} &= zD_z, \\ e_{18} &= qD_q + xD_x + yD_y - \frac{1}{2}(wx - w^2y)D_q - \frac{1}{2}(-wy + 2x)D_x, \\ e_{19} &= tD_t + \frac{1}{2}(wx - w^2y)D_q + \frac{1}{2}(-wy + 2x)D_x, \\ e_{20} &= tD_t - \frac{1}{2}(wx - w^2y)D_q - \frac{1}{2}(-wy + 2x)D_x, & e_{21} &= twD_q + 2tD_x, & e_{22} &= (wy - 2x)D_t. \end{aligned} \quad (49)$$

$$\begin{aligned} [e_1, e_2] &= e_4, & [e_1, e_{19}] &= e_1, & [e_1, e_{20}] &= e_1, & [e_1, e_{21}] &= e_{10} + 2e_5, \\ [e_2, e_7] &= -e_{10}, & [e_2, e_8] &= -e_9, & [e_2, e_{18}] &= e_2, & [e_2, e_{19}] &= -e_2, \\ [e_2, e_{20}] &= -e_2, & [e_2, e_{22}] &= 2e_{11} - e_{14}, & [e_3, e_{17}] &= e_3, & [e_4, e_{18}] &= e_4, \\ [e_5, e_{11}] &= e_4, & [e_5, e_{18}] &= -\frac{1}{2}e_{10}, & [e_5, e_{19}] &= e_5 + \frac{1}{2}e_{10}, & [e_5, e_{20}] &= -\frac{1}{2}e_{10} - e_5, \\ [e_5, e_{22}] &= -2e_1, & [e_6, e_8] &= e_1, & [e_6, e_9] &= e_4, & [e_6, e_{11}] &= e_5, \\ [e_6, e_{14}] &= e_{10} + 2e_5, & [e_6, e_{18}] &= e_6 + \frac{1}{2}e_{13}, & [e_6, e_{19}] &= -\frac{1}{2}e_{13}, & [e_6, e_{20}] &= \frac{1}{2}e_{13}, \\ [e_6, e_{22}] &= e_7, & [e_7, e_{16}] &= -e_1, & [e_7, e_{19}] &= e_7, & [e_7, e_{20}] &= e_7, \\ [e_7, e_{21}] &= 2e_{13}, & [e_8, e_{15}] &= -e_7, & [e_8, e_{18}] &= -e_8, & [e_8, e_{19}] &= e_8, \\ [e_8, e_{20}] &= e_8, & [e_8, e_{21}] &= e_{14}, & [e_9, e_{15}] &= -e_{10}, & [e_{10}, e_{16}] &= -e_4, \\ [e_{10}, e_{18}] &= e_{10}, & [e_{11}, e_{13}] &= -e_{10}, & [e_{11}, e_{14}] &= -2e_9, & [e_{11}, e_{15}] &= -e_{13}, \\ [e_{11}, e_{18}] &= e_{11} - e_{14}, & [e_{11}, e_{19}] &= -e_{11} + e_{14}, & [e_{11}, e_{20}] &= e_{11} - e_{14}, & [e_{11}, e_{21}] &= -2e_2, \\ [e_{11}, e_{22}] &= -2e_8, & [e_{12}, e_{16}] &= -e_{12}, & [e_{12}, e_{17}] &= e_{12}, & [e_{13}, e_{16}] &= -e_{10} - e_5, \\ [e_{13}, e_{19}] &= e_{13}, & [e_{13}, e_{20}] &= -e_{13}, & [e_{13}, e_{22}] &= -2e_7, & [e_{14}, e_{15}] &= -2e_{13}, \\ [e_{14}, e_{16}] &= -e_9, & [e_{14}, e_{18}] &= -e_{14}, & [e_{14}, e_{19}] &= e_{14}, & [e_{14}, e_{20}] &= -e_{14}, \\ [e_{14}, e_{22}] &= -4e_8, & [e_{15}, e_{16}] &= -e_{13} - e_6, & [e_{15}, e_{18}] &= e_{15}, & [e_{16}, e_{18}] &= \frac{1}{2}(-e_{11} + e_{14}), \\ [e_{16}, e_{19}] &= \frac{1}{2}(e_{11} - e_{14}), & [e_{16}, e_{20}] &= \frac{1}{2}(-e_{11} + e_{14}), & [e_{16}, e_{21}] &= e_2, & [e_{16}, e_{22}] &= e_8, \\ [e_{20}, e_{21}] &= 2e_{21}, & [e_{20}, e_{22}] &= -2e_{22}, & [e_{21}, e_{22}] &= -4e_{20}. \end{aligned} \quad (50)$$

The symmetry algebra is a 22-dimensional indecomposable Levi decomposition, where the semisimple is  $\mathfrak{sl}(2, \mathbb{R})$  spanned by  $e_{20}, e_{21}, e_{22}$  and the nilradical is nonabelian spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}$ .

4.5.  $A_{5,11}^c$ :

$$[e_1, e_5] = e_1, [e_2, e_5] = e_1 + e_2, [e_3, e_5] = e_2 + e_3, [e_4, e_5] = ce_4; \quad (c \neq 0).$$

System of geodesic equations:

$$\ddot{q} = \dot{q}\dot{w} + \dot{x}\dot{w}, \quad \dot{x} = \dot{x}\dot{w} + \dot{y}\dot{w}, \quad \dot{y} = \dot{y}\dot{w}, \quad \dot{z} = \dot{z}\dot{w}, \quad \ddot{w} = 0. \quad (51)$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$\begin{aligned} e_1 &= D_q, & e_2 &= D_x, & e_3 &= xD_q + yD_x, & e_4 &= D_y, & e_5 &= (w-1)e^w D_q + e^w D_x, & e_6 &= e^w D_q, \\ e_7 &= yD_q, & e_8 &= \left(\frac{w^2}{2} - w + 1\right)e^w D_q + (w-1)e^w D_x + e^w D_y, & e_9 &= D_z, & e_{10} &= e^{cw} D_z, & e_{11} &= D_t, \\ e_{12} &= wD_t, & e_{13} &= D_w, & e_{14} &= tD_t, & e_{15} &= zD_z, & e_{16} &= qD_q + xD_x + yD_y. \end{aligned} \quad (52)$$

$$\begin{aligned} [e_1, e_{16}] &= e_1, & [e_2, e_3] &= e_1, & [e_2, e_{16}] &= e_2, & [e_3, e_4] &= -e_2, & [e_3, e_5] &= -e_6, \\ [e_3, e_8] &= -e_5, & [e_4, e_7] &= e_1, & [e_4, e_{16}] &= e_4, & [e_5, e_{13}] &= -e_5 - e_6, & [e_5, e_{16}] &= e_5, \\ [e_6, e_{13}] &= -e_6, & [e_6, e_{16}] &= e_6, & [e_7, e_8] &= -e_6, & [e_8, e_{13}] &= -e_5 - e_8, & [e_8, e_{16}] &= e_8, \\ [e_9, e_{15}] &= e_9, & [e_{10}, e_{13}] &= -ce_{10}, & [e_{10}, e_{15}] &= e_{10}, & [e_{11}, e_{14}] &= e_{11}, & [e_{12}, e_{13}] &= -e_{11}, \\ [e_{12}, e_{14}] &= e_{12}. \end{aligned} \quad (53)$$

For the generic case, the symmetry algebra is a 16-dimensional indecomposable solvable Lie algebra  $\mathbb{R}^4 \rtimes (N_9 \oplus \mathbb{R}^3)$ . The nilradical is composed of a 12-dimensional decomposable, a direct sum of 9-dimensional nilpotent spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$  and  $\mathbb{R}^3$  spanned by  $e_{10}, e_{11}, e_{12}$ . The complement to the nilradical is a 4-dimensional abelian spanned by  $e_{13}, e_{14}, e_{15}, e_{16}$ .

$A_{5,11}^{c=1}$ :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned} e_1 &= D_t, & e_2 &= D_q, & e_3 &= D_x, & e_4 &= D_z, & e_5 &= D_y, & e_6 &= yD_z, & e_7 &= yD_q, & e_8 &= zD_q, \\ e_9 &= wD_t, & e_{10} &= xD_q + yD_x, & e_{11} &= e^w D_q, & e_{12} &= e^w D_z, & e_{13} &= (w-1)e^w D_q + e^w D_x, \\ e_{14} &= \left(\frac{w^2}{2} - w + 1\right)e^w D_q + (w-1)e^w D_x + e^w D_y, & e_{15} &= D_w, & e_{16} &= tD_t, & e_{17} &= zD_z, \\ e_{18} &= qD_q + xD_x + yD_y. \end{aligned} \quad (54)$$

$$\begin{aligned} [e_1, e_{16}] &= e_1, & [e_2, e_{18}] &= e_2, & [e_3, e_{10}] &= e_2, & [e_3, e_{18}] &= e_3, \\ [e_4, e_8] &= e_2, & [e_4, e_{17}] &= e_4, & [e_5, e_6] &= e_4, & [e_5, e_7] &= e_2, \\ [e_5, e_{10}] &= e_3, & [e_5, e_{18}] &= e_5, & [e_6, e_8] &= e_7, & [e_6, e_{14}] &= -e_{12}, \\ [e_6, e_{17}] &= e_6, & [e_6, e_{18}] &= -e_6, & [e_7, e_{14}] &= -e_{11}, & [e_8, e_{12}] &= -e_{11}, \\ [e_8, e_{17}] &= -e_8, & [e_8, e_{18}] &= e_8, & [e_9, e_{15}] &= -e_1, & [e_9, e_{16}] &= e_9, \\ [e_{10}, e_{13}] &= -e_{11}, & [e_{10}, e_{14}] &= -e_{13}, & [e_{11}, e_{15}] &= -e_{11}, & [e_{11}, e_{18}] &= e_{11}, \\ [e_{12}, e_{15}] &= -e_{12}, & [e_{12}, e_{17}] &= e_{12}, & [e_{13}, e_{15}] &= -e_{11} - e_{13}, & [e_{13}, e_{18}] &= e_{13}, \\ [e_{14}, e_{15}] &= -e_{13} - e_{14}, & [e_{14}, e_{18}] &= e_{14}. \end{aligned} \quad (55)$$

The symmetry algebra is  $\mathbb{R}^4 \rtimes \mathbb{R}^{14}$  indecomposable solvable. It has a 14-dimensional nonabelian nilradical spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}$  and a 4-dimensional abelian complement spanned by  $e_{15}, e_{16}, e_{17}, e_{18}$ .

4.6.  $A_{5,12}$ :

$$[e_1, e_5] = e_1, \quad [e_2, e_5] = e_1 + e_2, \quad [e_3, e_5] = e_2 + e_3, \quad [e_4, e_5] = e_3 + e_4.$$

System of geodesic equations:

$$\ddot{q} = \dot{q}\dot{w} + \dot{x}\dot{w}, \quad \ddot{x} = \dot{x}\dot{w} + \dot{y}\dot{w}, \quad \ddot{y} = \dot{y}\dot{w} + \dot{z}\dot{w}, \quad \ddot{z} = \dot{z}\dot{w}, \quad \ddot{w} = 0. \tag{56}$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$\begin{aligned} e_1 &= D_q, & e_2 &= D_y, & e_3 &= D_x, & e_4 &= D_z, & e_5 &= D_t, & e_6 &= zD_q, & e_7 &= wD_t, & e_8 &= yD_q + zD_x, \\ e_9 &= e^{tw}D_q, & e_{10} &= xD_q + yD_x + zD_y, & e_{11} &= (w-1)e^{tw}D_q + e^{tw}D_x, \\ e_{12} &= (w^2 - 2w + 2)e^{tw}D_q + (2w - 2)e^{tw}D_x + 2e^{tw}D_y, \\ e_{13} &= \left(\frac{w^3}{6} - \frac{w^2}{2} + w - 1\right)e^{tw}D_q + \left(\frac{w^2}{2} - w + 1\right)e^{tw}D_x + (w-1)e^{tw}D_y + e^{tw}D_z, & e_{14} &= tD_t, & e_{15} &= D_w, \\ e_{16} &= qD_q + xD_x + yD_y + zD_z. \end{aligned} \tag{57}$$

$$\begin{aligned} [e_1, e_{16}] &= e_1, & [e_2, e_8] &= e_1, & [e_2, e_{10}] &= e_3, & [e_2, e_{16}] &= e_2, \\ [e_3, e_{10}] &= e_1, & [e_3, e_{16}] &= e_3, & [e_4, e_6] &= e_1, & [e_4, e_8] &= e_3, \\ [e_4, e_{10}] &= e_2, & [e_4, e_{16}] &= e_4, & [e_5, e_{14}] &= e_5, & [e_6, e_{13}] &= -e_9, \\ [e_7, e_{14}] &= e_7, & [e_7, e_{15}] &= -e_5, & [e_8, e_{12}] &= -2e_9, & [e_8, e_{13}] &= -e_{11}, \\ [e_9, e_{15}] &= -e_9, & [e_9, e_{16}] &= e_9, & [e_{10}, e_{11}] &= -e_9, & [e_{10}, e_{12}] &= -2e_{11}, \\ [e_{10}, e_{13}] &= -\frac{e_{12}}{2}, & [e_{11}, e_{15}] &= -e_{11} - e_9, & [e_{11}, e_{16}] &= e_{11}, & [e_{12}, e_{15}] &= -2e_{11} - e_{12}, \\ [e_{12}, e_{16}] &= e_{12}, & [e_{13}, e_{15}] &= -\frac{e_{12}}{2} - e_{13}, & [e_{13}, e_{16}] &= e_{13}. \end{aligned} \tag{58}$$

It is a 16-dimensional indecomposable solvable algebra. The nilradical is a 19-dimensional nonabelian spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}$ , where its complement  $\mathbb{R}^3$  is abelian, spanned by  $e_{14}, e_{15}, e_{16}$ .

4.7.  $A_{5,13}^{abc}$ :

$$[e_1, e_5] = e_1, \quad [e_2, e_5] = ae_2, \quad [e_3, e_5] = be_3 - ce_4, \quad [e_4, e_5] = ce_3 + be_4; \quad (ac \neq 0, |a| \leq 1).$$

System of geodesic equations:

$$\ddot{q} = \dot{q}\dot{w}, \quad \ddot{x} = a\dot{x}\dot{w}, \quad \ddot{y} = b\dot{y}\dot{w} + c\dot{z}\dot{w}, \quad \ddot{z} = -c\dot{y}\dot{w} + b\dot{z}\dot{w}, \quad \ddot{w} = 0. \tag{59}$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$\begin{aligned} e_1 &= D_y, & e_2 &= D_z, & e_3 &= D_q, & e_4 &= D_x, & e_5 &= D_t, & e_6 &= wD_t, & e_7 &= e^{tw}D_q, & e_8 &= e^{aw}D_x, \\ e_9 &= e^{bw}(\sin cwD_y + \cos cwD_z), & e_{10} &= e^{bw}(\cos cwD_y - \sin cwD_z), & e_{11} &= D_w, \\ e_{12} &= tD_t, & e_{13} &= qD_q, & e_{14} &= xD_x, & e_{15} &= yD_y + zD_z, & e_{16} &= zD_y - yD_z. \end{aligned} \tag{60}$$

$$\begin{aligned} [e_1, e_{15}] &= e_1, & [e_1, e_{16}] &= -e_2, & [e_2, e_{15}] &= e_2, & [e_2, e_{16}] &= e_1, & [e_3, e_{13}] &= e_3, \\ [e_4, e_{14}] &= e_4, & [e_5, e_{12}] &= e_5, & [e_6, e_{11}] &= -e_5, & [e_6, e_{12}] &= e_6, & [e_7, e_{11}] &= -e_7, \\ [e_7, e_{13}] &= e_7, & [e_8, e_{11}] &= -ae_8, & [e_8, e_{14}] &= e_8, & [e_9, e_{11}] &= -be_9 - ce_{10}, & [e_9, e_{15}] &= e_9, \\ [e_9, e_{16}] &= e_{10}, & [e_{10}, e_{11}] &= -be_{10} + ce_9, & [e_{10}, e_{15}] &= e_{10}, & [e_{10}, e_{16}] &= -e_9. \end{aligned} \tag{61}$$

For the generic case, it is  $\mathbb{R}^6 \times \mathbb{R}^{10}$  indecomposable solvable Lie algebra. The nilradical and its complement are abelian, spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$  and  $e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$ , respectively.



4.7.1.  $A_{5,13}^{a=1, bc \neq 1}$ :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned} e_1 &= D_t, & e_2 &= D_y, & e_3 &= D_z, & e_4 &= D_q, & e_5 &= D_x, & e_6 &= wD_t, & e_7 &= e^w D_q, & e_8 &= e^w D_x, \\ e_9 &= e^{bw}(\sin cwD_y + \cos cwD_z), & e_{10} &= e^{bw}(\cos cwD_y - \sin cwD_z), & e_{11} &= D_w, & e_{12} &= tD_t, \\ e_{13} &= yD_y + zD_z, & e_{14} &= zD_y - yD_z, & e_{15} &= qD_q + xD_x, & e_{16} &= qD_q - xD_x, & e_{17} &= xD_q, \\ e_{18} &= qD_x. \end{aligned} \quad (62)$$

$$\begin{aligned} [e_1, e_{12}] &= e_1, & [e_2, e_{13}] &= e_2, & [e_2, e_{14}] &= -e_3, & [e_3, e_{13}] &= e_3, \\ [e_3, e_{14}] &= e_2, & [e_4, e_{15}] &= e_4, & [e_4, e_{16}] &= e_4, & [e_4, e_{18}] &= e_5, \\ [e_5, e_{15}] &= e_5, & [e_5, e_{16}] &= -e_5, & [e_5, e_{17}] &= e_4, & [e_6, e_{11}] &= -e_1, \\ [e_6, e_{12}] &= e_6, & [e_7, e_{11}] &= -e_7, & [e_7, e_{15}] &= e_7, & [e_7, e_{16}] &= e_7, \\ [e_7, e_{18}] &= e_8, & [e_8, e_{11}] &= -e_8, & [e_8, e_{15}] &= e_8, & [e_8, e_{16}] &= -e_8, \\ [e_8, e_{17}] &= e_7, & [e_9, e_{11}] &= -be_9 - ce_{10}, & [e_9, e_{13}] &= e_9, & [e_9, e_{14}] &= e_{10}, \\ [e_{10}, e_{11}] &= -be_{10} + ce_9, & [e_{10}, e_{13}] &= e_{10}, & [e_{10}, e_{14}] &= -e_9, & [e_{16}, e_{17}] &= -2e_{17}, \\ [e_{16}, e_{18}] &= 2e_{18}, & [e_{17}, e_{18}] &= -e_{16}. \end{aligned} \quad (63)$$

4.7.2.  $A_{5,13}^{b=0, ac \neq 0}$ :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned} e_1 &= D_t, & e_2 &= D_y, & e_3 &= D_z, & e_4 &= D_q, & e_5 &= D_x, & e_6 &= wD_t, & e_7 &= e^w D_q, & e_8 &= e^{aw} D_x, \\ e_9 &= \sin cwD_y + \cos cwD_z, & e_{10} &= \cos cwD_y - \sin cwD_z, & e_{11} &= D_w + \frac{c}{2}(zD_y - yD_z), \\ e_{12} &= tD_t, & e_{13} &= qD_q, & e_{14} &= xD_x, & e_{15} &= yD_y + zD_z, & e_{16} &= zD_y - yD_z, \\ e_{17} &= (z \cos cw + y \sin cw)D_y + (y \cos cw - z \sin cw)D_z, \\ e_{18} &= (y \cos cw - z \sin cw)D_y - (z \cos cw + y \sin cw)D_z. \end{aligned} \quad (64)$$

$$\begin{aligned} [e_1, e_{12}] &= e_1, & [e_2, e_{11}] &= \frac{-c}{2}e_3, & [e_2, e_{15}] &= e_2, & [e_2, e_{16}] &= -e_3, & [e_2, e_{17}] &= e_9, \\ [e_2, e_{18}] &= e_{10}, & [e_3, e_{11}] &= \frac{c}{2}e_2, & [e_3, e_{15}] &= e_3, & [e_3, e_{16}] &= e_2, & [e_3, e_{17}] &= e_{10}, \\ [e_3, e_{18}] &= -e_9, & [e_4, e_{13}] &= e_4, & [e_5, e_{14}] &= e_5, & [e_6, e_{11}] &= -e_1, & [e_6, e_{12}] &= e_6, \\ [e_7, e_{11}] &= -e_7, & [e_7, e_{13}] &= e_7, & [e_8, e_{11}] &= -ae_8, & [e_8, e_{14}] &= e_8, & [e_9, e_{11}] &= \frac{-c}{2}e_{10}, \\ [e_9, e_{15}] &= e_9, & [e_9, e_{16}] &= e_{10}, & [e_9, e_{17}] &= e_2, & [e_9, e_{18}] &= -e_3, & [e_{10}, e_{11}] &= \frac{c}{2}e_9, \\ [e_{10}, e_{15}] &= e_{10}, & [e_{10}, e_{16}] &= -e_9, & [e_{10}, e_{17}] &= e_3, & [e_{10}, e_{18}] &= e_2, & [e_{16}, e_{17}] &= -2e_{18}, \\ [e_{16}, e_{18}] &= 2e_{17}, & [e_{17}, e_{18}] &= 2e_{16}. \end{aligned} \quad (65)$$

For both subcases, the symmetry algebra is  $\mathfrak{sl}(2, \mathbb{R}) \rtimes (\mathbb{R}^5 \rtimes \mathbb{R}^{10})$ , where  $\mathfrak{sl}(2, \mathbb{R})$  is spanned by  $e_{16}, e_{17}, e_{18}$ . The  $\mathbb{R}^5$  factor is spanned by  $e_{11}, e_{12}, e_{13}, e_{14}, e_{15}$  and the nilradical  $\mathbb{R}^{10}$  is spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$ .

4.7.3.  $A_{5,13}^{b=1ac, \neq 1}$ :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned} e_1 = D_y, \quad e_2 = D_z, \quad e_3 = D_q, \quad e_4 = D_x, \quad e_5 = D_t, \quad e_6 = wD_t, \quad e_7 = e^{aw}D_q, \quad e_8 = e^{aw}D_x, \\ e_9 = e^{aw}(\sin cwD_y + \cos cwD_z), \quad e_{10} = e^{aw}(\cos cwD_y - \sin cwD_z), \quad e_{11} = D_w, \quad e_{12} = tD_t, \quad (66) \\ e_{13} = qD_q, \quad e_{14} = xD_x, \quad e_{15} = yD_y + zD_z, \quad e_{16} = zD_y - yD_z. \end{aligned}$$

$$\begin{aligned} [e_1, e_{15}] = e_1, \quad [e_1, e_{16}] = -e_2, \quad [e_2, e_{15}] = e_2, \quad [e_2, e_{16}] = e_1, \quad [e_3, e_{13}] = e_3, \\ [e_4, e_{14}] = e_4, \quad [e_5, e_{12}] = e_5, \quad [e_6, e_{11}] = -e_5, \quad [e_6, e_{12}] = e_6, \quad [e_7, e_{11}] = -e_7, \quad (67) \\ [e_7, e_{13}] = e_7, \quad [e_8, e_{11}] = -ae_8, \quad [e_8, e_{14}] = e_8, \quad [e_9, e_{11}] = -ce_{10} - e_9, \quad [e_9, e_{15}] = e_9, \\ [e_9, e_{16}] = e_{10}, \quad [e_{10}, e_{11}] = ce_9 - e_{10}, \quad [e_{10}, e_{15}] = e_{10}, \quad [e_{10}, e_{16}] = -e_9. \end{aligned}$$

The symmetry algebra is  $\mathbb{R}^6 \times \mathbb{R}^{10}$  indecomposable solvable where the nilradical and its complement are abelian, spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$  and  $e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$ , respectively.

4.8.  $A_{5,14}^a$ :

$$[e_2, e_5] = e_1, \quad [e_3, e_5] = ae_3 - e_4, \quad [e_4, e_5] = e_3 + ae_4.$$

System of geodesic equations:

$$\ddot{q} = \dot{x}\dot{w}, \quad \ddot{x} = 0, \quad \ddot{y} = a\dot{y}\dot{w} + \dot{z}\dot{w}, \quad \ddot{z} = -\dot{y}\dot{w} + a\dot{z}\dot{w}, \quad \ddot{w} = 0. \quad (68)$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$\begin{aligned} e_1 = D_t, \quad e_2 = e^{aw}(\sin wD_z - \cos wD_y), \quad e_3 = D_q, \quad e_4 = D_z, \quad e_5 = D_y, \quad e_6 = D_x, \\ e_7 = e^{aw}(\sin wD_y + \cos wD_z), \quad e_8 = wD_q, \quad e_9 = wD_t, \quad e_{10} = \frac{1}{2}w^2D_q + wD_x, \quad e_{11} = -zD_y + yD_z, \\ e_{12} = yD_y + zD_z, \quad e_{13} = qD_q + tD_t + xD_x, \quad e_{14} = D_w + \frac{1}{2}xD_q, \quad e_{15} = tD_q, \quad (69) \\ e_{16} = tD_t + \frac{1}{2}(wx - 2q)D_q, \quad e_{17} = xD_q, \quad e_{18} = xD_t, \quad e_{19} = qD_q + xD_x + (wx - 2q)D_q, \\ e_{20} = \frac{1}{2}twD_q + tD_x, \quad e_{21} = (wx - 2q)D_t, \quad e_{22} = (\frac{1}{2}xw^2 - qw)D_q + (wx - 2q)D_x. \end{aligned}$$

$$\begin{aligned}
[e_1, e_{13}] &= e_1, & [e_1, e_{15}] &= e_3, & [e_1, e_{16}] &= e_1, & [e_1, e_{20}] &= e_6 + \frac{1}{2}e_8, \\
[e_2, e_{11}] &= -e_7, & [e_2, e_{12}] &= e_2, & [e_2, e_{14}] &= -ae_2 - e_7, & [e_3, e_{13}] &= e_3, \\
[e_3, e_{16}] &= -e_3, & [e_3, e_{19}] &= -e_3, & [e_3, e_{21}] &= -2e_1, & [e_3, e_{22}] &= -2e_6 - e_8, \\
[e_4, e_{11}] &= -e_5, & [e_4, e_{12}] &= e_4, & [e_5, e_{11}] &= e_4, & [e_5, e_{12}] &= e_5, \\
[e_6, e_{13}] &= e_6, & [e_6, e_{14}] &= \frac{1}{2}e_3, & [e_6, e_{16}] &= \frac{1}{2}e_8, & [e_6, e_{17}] &= e_3, \\
[e_6, e_{18}] &= e_1, & [e_6, e_{19}] &= e_6 + e_8, & [e_6, e_{21}] &= e_9, & [e_6, e_{22}] &= e_{10}, \\
[e_7, e_{11}] &= e_2, & [e_7, e_{12}] &= e_7, & [e_7, e_{14}] &= -ae_7 + e_2, & [e_8, e_{13}] &= e_8, \\
[e_8, e_{14}] &= -e_3, & [e_8, e_{16}] &= -e_8, & [e_8, e_{19}] &= -e_8, & [e_8, e_{21}] &= -2e_9, \\
[e_8, e_{22}] &= -2e_{10}, & [e_9, e_{13}] &= e_9, & [e_9, e_{14}] &= -e_1, & [e_9, e_{15}] &= e_8, \\
[e_9, e_{16}] &= e_9, & [e_9, e_{20}] &= e_{10}, & [e_{10}, e_{13}] &= e_{10}, & [e_{10}, e_{14}] &= -e_6 - \frac{1}{2}e_8, \quad (70) \\
[e_{10}, e_{17}] &= e_8, & [e_{10}, e_{18}] &= e_9, & [e_{10}, e_{19}] &= e_{10}, & [e_{15}, e_{16}] &= -2e_{15}, \\
[e_{15}, e_{18}] &= -e_{17}, & [e_{15}, e_{19}] &= -e_{15}, & [e_{15}, e_{21}] &= -2e_{16}, & [e_{15}, e_{22}] &= -2e_{20}, \\
[e_{16}, e_{17}] &= e_{17}, & [e_{16}, e_{18}] &= -e_{18}, & [e_{16}, e_{20}] &= e_{20}, & [e_{16}, e_{21}] &= -2e_{21}, \\
[e_{16}, e_{22}] &= -e_{22}, & [e_{17}, e_{19}] &= -2e_{17}, & [e_{17}, e_{20}] &= -e_{15}, & [e_{17}, e_{21}] &= -2e_{18}, \\
[e_{17}, e_{22}] &= -2e_{19}, & [e_{18}, e_{19}] &= -e_{18}, & [e_{18}, e_{20}] &= -e_{16} + e_{19}, & [e_{18}, e_{22}] &= -e_{21}, \\
[e_{19}, e_{20}] &= -e_{20}, & [e_{19}, e_{21}] &= -e_{21}, & [e_{19}, e_{22}] &= -2e_{22}, & [e_{20}, e_{21}] &= -e_{22}.
\end{aligned}$$

For the generic case, the symmetry algebra has a  $\mathfrak{sl}(3, \mathbb{R}) \times (\mathbb{R}^4 \times \mathbb{R}^{10})$  Levi decomposition, where the semisimple part is  $\mathfrak{sl}(3, \mathbb{R})$  spanned by  $e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{22}$ . The radical  $\mathbb{R}^4 \times \mathbb{R}^{10}$  is spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$  and  $e_{11}, e_{12}, e_{13}, e_{14}$ .

$A_{5,14}^{a=0}$  :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned}
e_1 &= D_t, & e_2 &= D_q, & e_3 &= D_z, & e_4 &= D_y, & e_5 &= D_x, & e_6 &= \sin w D_z - \cos w D_y, \\
e_7 &= \sin w D_y + \cos w D_z, & e_8 &= w D_q, & e_9 &= w D_t, & e_{10} &= \frac{1}{2}w^2 D_q + w D_x, & e_{11} &= y D_y + z D_z, \\
e_{12} &= q D_q + t D_t + x D_x, & e_{13} &= D_w + \frac{1}{2}(x D_q + z D_y - y D_z), & e_{14} &= t D_q, \\
e_{15} &= t D_t + \frac{1}{2}(wx - 2q) D_q, & e_{16} &= x D_t, & e_{17} &= x D_q, & e_{18} &= \left(\frac{1}{2}xw^2 - qw\right) D_q + (wx - 2q) D_x, \quad (71) \\
e_{19} &= \frac{1}{2}tw D_q + t D_x, & e_{20} &= q D_q + x D_x + (wx - 2q) D_q, & e_{21} &= (wx - 2q) D_t, & e_{22} &= -z D_y + y D_z, \\
e_{23} &= (z \cos w + y \sin w) D_y + (y \cos w - z \sin w) D_z, \\
e_{24} &= (-y \cos w + z \sin w) D_y + (z \cos w + y \sin w) D_z.
\end{aligned}$$

$$\begin{aligned}
 [e_1, e_{12}] &= e_1, & [e_1, e_{14}] &= e_2, & [e_1, e_{15}] &= e_1, & [e_1, e_{19}] &= e_5 + \frac{1}{2}e_8, \\
 [e_2, e_{12}] &= e_2, & [e_2, e_{15}] &= -e_2, & [e_2, e_{18}] &= -2e_5 - e_8, & [e_2, e_{20}] &= -e_2, \\
 [e_2, e_{21}] &= -2e_1, & [e_3, e_{11}] &= e_3, & [e_3, e_{13}] &= \frac{1}{2}e_4, & [e_3, e_{22}] &= -e_4, \\
 [e_3, e_{23}] &= -e_6, & [e_3, e_{24}] &= e_7, & [e_4, e_{11}] &= e_4, & [e_4, e_{13}] &= -\frac{1}{2}e_3, \\
 [e_4, e_{22}] &= e_3, & [e_4, e_{23}] &= e_7, & [e_4, e_{24}] &= e_6, & [e_5, e_{12}] &= e_5, \\
 [e_5, e_{13}] &= \frac{1}{2}e_2, & [e_5, e_{15}] &= \frac{1}{2}e_8, & [e_5, e_{16}] &= e_1, & [e_5, e_{17}] &= e_2, \\
 [e_5, e_{18}] &= e_{10}, & [e_5, e_{20}] &= e_5 + e_8, & [e_5, e_{21}] &= e_9, & [e_6, e_{11}] &= e_6, \\
 [e_6, e_{13}] &= -\frac{1}{2}e_7, & [e_6, e_{22}] &= -e_7, & [e_6, e_{23}] &= -e_3, & [e_6, e_{24}] &= e_4, \\
 [e_7, e_{11}] &= e_7, & [e_7, e_{13}] &= \frac{1}{2}e_6, & [e_7, e_{22}] &= e_6, & [e_7, e_{23}] &= e_4, \\
 [e_7, e_{24}] &= e_3, & [e_8, e_{12}] &= e_8, & [e_8, e_{13}] &= -e_2, & [e_8, e_{15}] &= -e_8, \\
 [e_8, e_{18}] &= -2e_{10}, & [e_8, e_{20}] &= -e_8, & [e_8, e_{21}] &= -2e_9, & [e_9, e_{12}] &= e_9, \\
 [e_9, e_{13}] &= -e_1, & [e_9, e_{14}] &= e_8, & [e_9, e_{15}] &= e_9, & [e_9, e_{19}] &= e_{10}, \\
 [e_{10}, e_{12}] &= e_{10}, & [e_{10}, e_{13}] &= -e_5 - \frac{1}{2}e_8, & [e_{10}, e_{16}] &= e_9, & [e_{10}, e_{17}] &= e_8, \\
 [e_{10}, e_{20}] &= e_{10}, & [e_{14}, e_{15}] &= -2e_{14}, & [e_{14}, e_{16}] &= -e_{17}, & [e_{14}, e_{18}] &= -2e_{19}, \\
 [e_{14}, e_{20}] &= -e_{14}, & [e_{14}, e_{21}] &= -2e_{15}, & [e_{15}, e_{16}] &= -e_{16}, & [e_{15}, e_{17}] &= e_{17}, \\
 [e_{15}, e_{18}] &= -e_{18}, & [e_{15}, e_{19}] &= e_{19}, & [e_{15}, e_{21}] &= -2e_{21}, & [e_{16}, e_{18}] &= -e_{21}, \\
 [e_{16}, e_{19}] &= e_{20} - e_{15}, & [e_{16}, e_{20}] &= -e_{16}, & [e_{17}, e_{18}] &= -2e_{20}, & [e_{17}, e_{19}] &= -e_{14}, \\
 [e_{17}, e_{20}] &= -2e_{17}, & [e_{17}, e_{21}] &= -2e_{16}, & [e_{18}, e_{20}] &= 2e_{18}, & [e_{19}, e_{20}] &= e_{19}, \\
 [e_{19}, e_{21}] &= -e_{18}, & [e_{20}, e_{21}] &= -e_{21}, & [e_{22}, e_{23}] &= -2e_{24}, & [e_{22}, e_{24}] &= 2e_{23}, \\
 [e_{23}, e_{24}] &= 2e_{22}.
 \end{aligned} \tag{72}$$

The symmetry algebra has a  $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \times (\mathbb{R}^3 \times \mathbb{R}^{10})$  Levi decomposition algebra. The semisimple part is  $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$  spanned by  $e_{14}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}$  and  $e_{22}, e_{23}, e_{24}$ , respectively. The solvable comprises a 3-dimensional abelian complement spanned by  $e_{11}, e_{12}, e_{13}$  and 10-dimensional abelian nilradical spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_9, e_{10}$ .

4.9.  $A_{5,15}^a$ :

$$[e_1, e_5] = e_1, [e_2, e_5] = e_1 + e_2, [e_3, e_5] = ae_3, [e_4, e_5] = e_3 + ae_4; \quad (|a| \leq 1).$$

System of geodesic equations:

$$\dot{q} = \dot{q}\dot{w} + \dot{x}\dot{w}, \quad \dot{x} = \dot{x}\dot{w}, \quad \dot{y} = a\dot{y}\dot{w} + \dot{z}\dot{y}, \quad \dot{z} = a\dot{z}\dot{w}, \quad \dot{w} = 0. \tag{73}$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$\begin{aligned}
 e_1 &= D_x, \quad e_2 = D_q, \quad e_3 = xD_q, \quad e_4 = (w - 1)e^{aw}D_q + e^{aw}D_x, \quad e_5 = e^{aw}D_q, \quad e_6 = \frac{e^{aw}}{a}D_y, \\
 e_7 &= \frac{(aw - 1)e^{aw}}{a}D_y + e^{aw}D_z, \quad e_8 = zD_y, \quad e_9 = D_z, \quad e_{10} = D_y, \quad e_{11} = D_t, \quad e_{12} = wD_t, \\
 e_{13} &= D_w, \quad e_{14} = tD_t, \quad e_{15} = qD_q + xD_x, \quad e_{16} = yD_y + zD_z.
 \end{aligned} \tag{74}$$

$$\begin{aligned}
 [e_1, e_3] &= e_2, & [e_1, e_{15}] &= e_1, & [e_2, e_{15}] &= e_2, & [e_3, e_4] &= -e_5, \\
 [e_4, e_{13}] &= -e_4 - e_5, & [e_4, e_{15}] &= e_4, & [e_5, e_{13}] &= -e_5, & [e_5, e_{15}] &= e_5, \\
 [e_6, e_{13}] &= -ae_6, & [e_6, e_{16}] &= e_6, & [e_7, e_8] &= ae_6, & [e_7, e_{13}] &= -ae_6 - ae_7, \\
 [e_7, e_{16}] &= e_7, & [e_8, e_9] &= -e_{10}, & [e_9, e_{16}] &= e_9, & [e_{10}, e_{16}] &= e_{10}, \\
 [e_{11}, e_{14}] &= e_{11}, & [e_{12}, e_{13}] &= -e_{11}, & [e_{12}, e_{14}] &= e_{12}.
 \end{aligned} \tag{75}$$

For the generic case, the symmetry algebra is  $\mathbb{R}^4 \rtimes (H_5 \oplus N_7)$  indecomposable solvable, where the decomposable nilradical comprises a five-dimensional Heisenberg spanned by  $e_1, e_2, e_3, e_4, e_5$  and a 7-dimensional nilpotent spanned by  $e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}$ . The  $\mathbb{R}^4$  factor is abelian, spanned by  $e_{13}, e_{14}, e_{15}, e_{16}$ .

4.9.1.  $A_{5,15}^{a=0}$  :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned}
 e_1 &= e^w D_q, & e_2 &= D_q, & e_3 &= D_x, & e_4 &= xD_q, & e_5 &= (w-1)e^w D_q + e^w D_x, & e_6 &= wD_t, & e_7 &= wD_y, \\
 e_8 &= D_y, & e_9 &= D_z, & e_{10} &= \frac{1}{2}w^2 D_y + wD_z, & e_{11} &= D_t, & e_{12} &= tD_t + \frac{1}{2}wz D_y + zD_z + (y - \frac{wz}{2})D_y, \\
 e_{13} &= qD_q + xD_x, & e_{14} &= D_w + \frac{1}{2}zD_y, & e_{15} &= tD_y, & e_{16} &= tD_t - (y - \frac{1}{2}wz)D_y, & e_{17} &= zD_t, \\
 e_{18} &= zD_y, & e_{19} &= \frac{1}{2}twD_y + tD_z, & e_{20} &= (wz - 2y)D_t, & e_{21} &= \frac{1}{2}wz D_y + zD_z - (y - \frac{wz}{2})D_y, \\
 e_{22} &= (\frac{1}{2}zw^2 - wy)D_y + (wz - 2y)D_z.
 \end{aligned} \tag{76}$$

$$\begin{aligned}
 [e_1, e_{13}] &= e_1, & [e_1, e_{14}] &= -e_1, & [e_2, e_{13}] &= e_2, & [e_3, e_4] &= e_2, \\
 [e_3, e_{13}] &= e_3, & [e_4, e_5] &= -e_1, & [e_5, e_{13}] &= e_5, & [e_5, e_{14}] &= -e_1 - e_5, \\
 [e_6, e_{12}] &= e_6, & [e_6, e_{14}] &= -e_{11}, & [e_6, e_{15}] &= e_7, & [e_6, e_{16}] &= e_6, \\
 [e_6, e_{19}] &= e_{10}, & [e_7, e_{12}] &= e_7, & [e_7, e_{14}] &= -e_8, & [e_7, e_{16}] &= -e_7, \\
 [e_7, e_{20}] &= -2e_6, & [e_7, e_{21}] &= -e_7, & [e_7, e_{22}] &= -2e_{10}, & [e_8, e_{12}] &= e_8, \\
 [e_8, e_{16}] &= -e_8, & [e_8, e_{20}] &= -2e_{11}, & [e_8, e_{21}] &= -e_8, & [e_8, e_{22}] &= -e_7 - 2e_9, \\
 [e_9, e_{12}] &= e_9, & [e_9, e_{14}] &= \frac{1}{2}e_8, & [e_9, e_{16}] &= \frac{1}{2}e_7, & [e_9, e_{17}] &= e_{11}, \\
 [e_9, e_{18}] &= e_8, & [e_9, e_{20}] &= e_6, & [e_9, e_{21}] &= e_7 + e_9, & [e_9, e_{22}] &= e_{10}, \\
 [e_{10}, e_{12}] &= e_{10}, & [e_{10}, e_{14}] &= -\frac{1}{2}e_7 - e_9, & [e_{10}, e_{17}] &= e_6, & [e_{10}, e_{18}] &= e_7, \\
 [e_{10}, e_{21}] &= e_{10}, & [e_{11}, e_{12}] &= e_{11}, & [e_{11}, e_{15}] &= e_8, & [e_{11}, e_{16}] &= e_{11}, \\
 [e_{11}, e_{19}] &= \frac{1}{2}e_7 + e_9, & [e_{15}, e_{16}] &= -2e_{15}, & [e_{15}, e_{17}] &= -e_{18}, & [e_{15}, e_{20}] &= -2e_{16}, \\
 [e_{15}, e_{21}] &= -e_{15}, & [e_{15}, e_{22}] &= -2e_{19}, & [e_{16}, e_{17}] &= -e_{17}, & [e_{16}, e_{18}] &= e_{18}, \\
 [e_{16}, e_{19}] &= e_{19}, & [e_{16}, e_{20}] &= -2e_{20}, & [e_{16}, e_{22}] &= -e_{22}, & [e_{17}, e_{19}] &= -e_{16} + e_{21}, \\
 [e_{17}, e_{21}] &= -e_{17}, & [e_{17}, e_{22}] &= -e_{20}, & [e_{18}, e_{19}] &= -e_{15}, & [e_{18}, e_{20}] &= -2e_{17}, \\
 [e_{18}, e_{21}] &= -2e_{18}, & [e_{18}, e_{22}] &= -2e_{21}, & [e_{19}, e_{20}] &= -e_{22}, & [e_{19}, e_{21}] &= e_{19}, \\
 [e_{20}, e_{21}] &= e_{20}, & [e_{21}, e_{22}] &= -2e_{22}.
 \end{aligned} \tag{77}$$

The symmetry algebra is 22-dimensional indecomposable,  $\mathfrak{sl}(3, \mathbb{R}) \rtimes (\mathbb{R}^3 \rtimes H_5 \oplus \mathbb{R}^6)$ . The radical is a semidirect product  $\mathbb{R}^3 \rtimes (H_5 \oplus \mathbb{R}^6)$  with an 11-dimensional nonabelian nilradical spanned by  $e_1, e_2, e_3, e_4, e_5$ , and  $e_6, e_7, e_8, e_9, e_{10}, e_{11}$ , respectively, as well as a 3-dimensional abelian complement spanned by  $e_{12}, e_{13}, e_{14}$ . The semisimple part is  $\mathfrak{sl}(3, \mathbb{R})$ , spanned by  $e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{22}$ .

4.9.2.  $A_{5,15}^{a=1}$ :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned} e_1 &= D_t, & e_2 &= D_q, & e_3 &= D_y, & e_4 &= D_x, & e_5 &= D_z, & e_6 &= wD_t, & e_7 &= xD_q, & e_8 &= zD_q, \\ e_9 &= xD_y, & e_{10} &= zD_y, & e_{11} &= e^w D_q, & e_{12} &= e^w D_y, & e_{13} &= (w-1)e^w D_q + e^w D_x, \\ e_{14} &= (w-1)e^w D_y + e^w D_z, & e_{15} &= D_w, & e_{16} &= tD_t, & e_{17} &= qD_q + xD_x + yD_y + zD_z, \\ e_{18} &= qD_q + xD_x - yD_y - zD_z, & e_{19} &= yD_q + zD_x, & e_{20} &= qD_y + xD_z. \end{aligned} \quad (78)$$

$$\begin{aligned} [e_1, e_{16}] &= e_1, & [e_2, e_{17}] &= e_2, & [e_2, e_{18}] &= e_2, & [e_2, e_{20}] &= e_3, \\ [e_3, e_{17}] &= e_3, & [e_3, e_{18}] &= -e_3, & [e_3, e_{19}] &= e_2, & [e_4, e_7] &= e_2, \\ [e_4, e_9] &= e_3, & [e_4, e_{17}] &= e_4, & [e_4, e_{18}] &= e_4, & [e_4, e_{20}] &= e_5, \\ [e_5, e_8] &= e_2, & [e_5, e_{10}] &= e_3, & [e_5, e_{17}] &= e_5, & [e_5, e_{18}] &= -e_5, \\ [e_5, e_{19}] &= e_4, & [e_6, e_{15}] &= -e_1, & [e_6, e_{16}] &= e_6, & [e_7, e_{13}] &= -e_{11}, \\ [e_7, e_{19}] &= -e_8, & [e_7, e_{20}] &= e_9, & [e_8, e_{14}] &= -e_{11}, & [e_8, e_{18}] &= 2e_8, \\ [e_8, e_{20}] &= e_{10} - e_7, & [e_9, e_{13}] &= -e_{12}, & [e_9, e_{18}] &= -2e_9, & [e_9, e_{19}] &= -e_{10} + e_7, \\ [e_{10}, e_{14}] &= -e_{12}, & [e_{10}, e_{19}] &= e_8, & [e_{10}, e_{20}] &= -e_9, & [e_{11}, e_{15}] &= -e_{11}, \\ [e_{11}, e_{17}] &= e_{11}, & [e_{11}, e_{18}] &= e_{11}, & [e_{11}, e_{20}] &= e_{12}, & [e_{12}, e_{15}] &= -e_{12}, \\ [e_{12}, e_{17}] &= e_{12}, & [e_{12}, e_{18}] &= -e_{12}, & [e_{12}, e_{19}] &= e_{11}, & [e_{13}, e_{15}] &= -e_{11} - e_{13}, \\ [e_{13}, e_{17}] &= e_{13}, & [e_{13}, e_{18}] &= e_{13}, & [e_{13}, e_{20}] &= e_{14}, & [e_{14}, e_{15}] &= -e_{12} - e_{14}, \\ [e_{14}, e_{17}] &= e_{14}, & [e_{14}, e_{18}] &= -e_{14}, & [e_{14}, e_{19}] &= e_{13}, & [e_{18}, e_{19}] &= -2e_{19}, \\ [e_{18}, e_{20}] &= 2e_{20}, & [e_{19}, e_{20}] &= -e_{18}. \end{aligned} \quad (79)$$

It is a 20-dimensional indecomposable. The semisimple part is  $\mathfrak{sl}(2, \mathbb{R})$  spanned by  $e_{18}, e_{19}, e_{20}$ . The radical constitutes a 3-dimensional abelian complement and a 14-dimensional nonabelian nilradical spanned by  $e_{15}, e_{16}, e_{17}$  and  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}$ , respectively.

4.10.  $A_{5,16}^{ab}$ :

$$[e_1, e_5] = e_1, [e_2, e_5] = e_1 + e_2, [e_3, e_5] = ae_3 - be_4, [e_4, e_5] = be_3 + ae_4; \quad (b \neq 0).$$

System of geodesic equations:

$$\ddot{q} = \dot{q}\dot{w} + \dot{x}\dot{w}, \quad \dot{x} = \dot{x}\dot{w}, \quad \dot{y} = a\dot{y}\dot{w} + b\dot{z}\dot{w}, \quad \dot{z} = -b\dot{y}\dot{w} + a\dot{z}\dot{w}, \quad \dot{w} = 0. \quad (80)$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$\begin{aligned} e_1 &= D_q, & e_2 &= D_z, & e_3 &= D_y, & e_4 &= D_x, & e_5 &= e^w D_q, & e_6 &= xD_q, \\ e_7 &= (w-1)e^w D_q + e^w D_x, & e_8 &= e^{aw} (\cos bw D_y - \sin bw D_z), & e_9 &= e^{aw} (\sin bw D_y + \cos bw D_z), \\ e_{10} &= D_t, & e_{11} &= wD_t, & e_{12} &= qD_q + xD_x, & e_{13} &= zD_y - yD_z, & e_{14} &= yD_y + zD_z, & e_{15} &= tD_t, \\ e_{16} &= D_w. \end{aligned} \quad (81)$$

$$\begin{aligned}
[e_1, e_{12}] &= e_1, & [e_2, e_{13}] &= e_3, & [e_2, e_{14}] &= e_2, & [e_3, e_{13}] &= -e_2, \\
[e_3, e_{14}] &= e_3, & [e_4, e_6] &= e_1, & [e_4, e_{12}] &= e_4, & [e_5, e_{12}] &= e_5, \\
[e_5, e_{16}] &= -e_5, & [e_6, e_7] &= -e_5, & [e_7, e_{12}] &= e_7, & [e_7, e_{16}] &= -e_5 - e_7, \\
[e_8, e_{13}] &= -e_9, & [e_8, e_{14}] &= e_8, & [e_8, e_{16}] &= -ae_8 + be_9, & [e_9, e_{13}] &= e_8, \\
[e_9, e_{14}] &= e_9, & [e_9, e_{16}] &= -ae_9 - be_8, & [e_{10}, e_{15}] &= e_{10}, & [e_{11}, e_{15}] &= e_{11}, \\
[e_{11}, e_{16}] &= -e_{10}.
\end{aligned} \tag{82}$$

For the generic case, the symmetry algebra is a 16-dimensional indecomposable solvable. The nilradical is an 11-dimensional decomposable,  $N_7 \oplus \mathbb{R}^4$ , spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7$  and  $e_8, e_9, e_{10}, e_{11}$ . The complement to the nilradical is a five-dimensional abelian spanned by  $e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$ .

#### 4.10.1. $A_{5,16}^{a=0, b \neq 0}$ :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned}
e_1 &= D_q, & e_2 &= (w-1)e^w D_q + e^w D_x, & e_3 &= xD_q, & e_4 &= D_x, & e_5 &= e^w D_q, & e_6 &= wD_t, & e_7 &= D_z, \\
e_8 &= D_t, & e_9 &= \sin bw D_y + \cos bw D_z, & e_{10} &= \cos bw D_y - \sin bw D_z, & e_{11} &= D_y, & e_{12} &= tD_t, \\
e_{13} &= Dw + \frac{b}{2}(zD_y - yD_z), & e_{14} &= qD_q + xD_x, & e_{15} &= yD_y + zD_z, & e_{16} &= zD_y - yD_z, \\
e_{17} &= (z \cos bw + y \sin bw)D_y + (y \cos bw - z \sin bw)D_z, \\
e_{18} &= (y \cos bw - z \sin bw)D_y + (-z \cos bw - y \sin bw)D_z.
\end{aligned} \tag{83}$$

$$\begin{aligned}
[e_1, e_{14}] &= e_1, & [e_2, e_3] &= e_5, & [e_2, e_{13}] &= -e_2 - e_5, & [e_2, e_{14}] &= e_2, & [e_3, e_4] &= -e_1, \\
[e_4, e_{14}] &= e_4, & [e_5, e_{13}] &= -e_5, & [e_5, e_{14}] &= e_5, & [e_6, e_{12}] &= e_6, & [e_6, e_{13}] &= -e_8, \\
[e_7, e_{13}] &= \frac{b}{2}e_{11}, & [e_7, e_{15}] &= e_7, & [e_7, e_{16}] &= e_{11}, & [e_7, e_{17}] &= e_{10}, & [e_7, e_{18}] &= -e_9, \\
[e_8, e_{12}] &= e_8, & [e_9, e_{13}] &= \frac{-b}{2}e_{10}, & [e_9, e_{15}] &= e_9, & [e_9, e_{16}] &= e_{10}, & [e_9, e_{17}] &= e_{11}, \\
[e_9, e_{18}] &= -e_7, & [e_{10}, e_{13}] &= \frac{b}{2}e_9, & [e_{10}, e_{15}] &= e_{10}, & [e_{10}, e_{16}] &= -e_9, & [e_{10}, e_{17}] &= e_7, \\
[e_{10}, e_{18}] &= e_{11}, & [e_{11}, e_{13}] &= -\frac{b}{2}e_7, & [e_{11}, e_{15}] &= e_{11}, & [e_{11}, e_{16}] &= -e_7, & [e_{11}, e_{17}] &= e_9, \\
[e_{11}, e_{18}] &= e_{10}, & [e_{16}, e_{17}] &= -2e_{18}, & [e_{16}, e_{18}] &= 2e_{17}, & [e_{17}, e_{18}] &= 2e_{16}.
\end{aligned} \tag{84}$$

The symmetry algebra is an 18-dimensional indecomposable Levi decomposition. The semisimple is  $\mathfrak{sl}(2, \mathbb{R})$  spanned by  $e_{16}, e_{17}, e_{18}$ , whereas the radical comprises an 11-dimensional decomposable nilradical  $H_5 \oplus \mathbb{R}^6$  spanned by  $e_1, e_2, e_3, e_4, e_5$  and  $e_7, e_8, e_9, e_{10}, e_{11}$ , respectively, as well as a 4-dimensional abelian complement spanned by  $e_{12}, e_{13}, e_{14}, e_{15}$ .

#### 4.10.2. $A_{5,16}^{a=1, b \neq 1}$ :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned}
e_1 &= D_q, & e_2 &= (w-1)e^w D_q + e^w D_x, & e_3 &= xD_q, & e_4 &= D_x, & e_5 &= e^w D_q, & e_6 &= wD_t, & e_7 &= D_z, \\
e_8 &= D_t, & e_9 &= D_y, & e_{10} &= e^w(\sin bw D_y + \cos bw D_z), & e_{11} &= e^w(\cos bw D_y - \sin bw D_z), \\
e_{12} &= tD_t, & e_{13} &= Dw, & e_{14} &= qD_q + xD_x, & e_{15} &= yD_y + zD_z, & e_{16} &= zD_y - yD_z.
\end{aligned} \tag{85}$$

$$\begin{aligned}
[e_1, e_{14}] &= e_1, & [e_2, e_3] &= e_5, & [e_2, e_{13}] &= -e_2 - e_5, & [e_2, e_{14}] &= e_2, \\
[e_3, e_4] &= -e_1, & [e_4, e_{14}] &= e_4, & [e_5, e_{13}] &= -e_5, & [e_5, e_{14}] &= e_5, \\
[e_6, e_{12}] &= e_6, & [e_6, e_{13}] &= -e_8, & [e_7, e_{15}] &= e_7, & [e_7, e_{16}] &= e_9, \\
[e_8, e_{12}] &= e_8, & [e_9, e_{15}] &= e_9, & [e_9, e_{16}] &= -e_7, & [e_{10}, e_{13}] &= -be_{11} - e_{10}, \\
[e_{10}, e_{15}] &= e_{10}, & [e_{10}, e_{16}] &= e_{11}, & [e_{11}, e_{13}] &= be_{10} - e_{11}, & [e_{11}, e_{15}] &= e_{11}, \\
[e_{11}, e_{16}] &= -e_{10}.
\end{aligned} \tag{86}$$

4.10.3.  $A_{5,16}^{a=1,b=1}$ :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned}
e_1 &= D_q, & e_2 &= (w-1)e^{w}(D_q + D_x), & e_3 &= xD_q, & e_4 &= D_x, & e_5 &= e^{w}D_q, & e_6 &= wD_t, & e_7 &= D_y, \\
e_8 &= D_t, & e_9 &= D_z, & e_{10} &= e^{w}(\sin wD_y + \cos wD_z), & e_{11} &= e^{w}(\sin wD_z - \cos wD_y), & e_{12} &= tD_t, \\
e_{13} &= D_w, & e_{14} &= qD_q + xD_x, & e_{15} &= yD_y + zD_z, & e_{16} &= yD_z - zD_y.
\end{aligned} \tag{87}$$

$$\begin{aligned}
[e_1, e_{14}] &= e_1, & [e_2, e_3] &= e_5, & [e_2, e_{13}] &= -e_2 - e_5, & [e_2, e_{14}] &= e_2, \\
[e_3, e_4] &= -e_1, & [e_4, e_{14}] &= e_4, & [e_5, e_{13}] &= -e_5, & [e_5, e_{14}] &= e_5, \\
[e_6, e_{12}] &= e_6, & [e_6, e_{13}] &= -e_8, & [e_7, e_{15}] &= e_7, & [e_7, e_{16}] &= e_9, \\
[e_8, e_{12}] &= e_8, & [e_9, e_{15}] &= e_9, & [e_9, e_{16}] &= -e_7, & [e_{10}, e_{13}] &= -e_{10} + e_{11}, \\
[e_{10}, e_{15}] &= e_{10}, & [e_{10}, e_{16}] &= e_{11}, & [e_{11}, e_{13}] &= -e_{10} - e_{11}, & [e_{11}, e_{15}] &= e_{11}, \\
[e_{11}, e_{16}] &= -e_{10}.
\end{aligned} \tag{88}$$

For both subcases, the symmetry algebra is  $\mathbb{R}^5 \times (\mathbb{R}^6 \oplus H_5)$  indecomposable solvable. Its nilradical is an 11-dimensional nonabelian spanned by  $e_1, e_2, e_3, e_4, e_5$  and  $e_6, e_7, e_8, e_9, e_{10}, e_{11}$ . The complement to the nilradical is abelian spanned by  $e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$ .

4.11.  $A_{5,17}^{abc}$ :

$$[e_1, e_5] = ae_1 - e_2, \quad [e_2, e_5] = e_1 + ae_2, \quad [e_3, e_5] = be_3 - ce_4, \quad [e_4, e_5] = ce_3 + be_4; \quad (c \neq 0).$$

System of geodesic equations:

$$\ddot{q} = a\dot{q}\dot{w} + \dot{x}\dot{w}, \quad \ddot{x} = -\dot{q}\dot{w} + a\dot{x}\dot{w}, \quad \ddot{y} = b\dot{y}\dot{w} + c\dot{z}\dot{w}, \quad \ddot{z} = -c\dot{y}\dot{w} + b\dot{z}\dot{w}, \quad \ddot{w} = 0. \tag{89}$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$\begin{aligned}
e_1 &= e^{bw}(\cos cwD_y - \sin cwD_z), & e_2 &= D_y, & e_3 &= D_z, & e_4 &= D_q, & e_5 &= D_t, \\
e_6 &= e^{bw}(\sin cwD_y + \cos cwD_z), & e_7 &= D_x, & e_8 &= wD_t, & e_9 &= e^{aw}(-\cos wD_q + \sin wD_x), \\
e_{10} &= e^{aw}(\sin wD_q + \cos wD_x), & e_{11} &= yD_y + zD_z, & e_{12} &= qD_q + xD_x, & e_{13} &= -xD_q + qD_x, \\
e_{14} &= zD_y - yD_z, & e_{15} &= D_w, & e_{16} &= tD_t.
\end{aligned} \tag{90}$$

$$\begin{aligned}
[e_1, e_{11}] &= e_1, & [e_1, e_{14}] &= -e_6, & [e_1, e_{15}] &= -be_1 + ce_6, & [e_2, e_{11}] &= e_2, & [e_2, e_{14}] &= -e_3, \\
[e_3, e_{11}] &= e_3, & [e_3, e_{14}] &= e_2, & [e_4, e_{12}] &= e_4, & [e_4, e_{13}] &= e_7, & [e_5, e_{16}] &= e_5, \\
[e_6, e_{11}] &= e_6, & [e_6, e_{14}] &= e_1, & [e_6, e_{15}] &= -be_6 - ce_1, & [e_7, e_{12}] &= e_7, & [e_7, e_{13}] &= -e_4, \\
[e_8, e_{15}] &= -e_5, & [e_8, e_{16}] &= e_8, & [e_9, e_{12}] &= e_9, & [e_9, e_{13}] &= -e_{10}, & [e_9, e_{15}] &= -ae_9 - e_{10}, \\
[e_{10}, e_{12}] &= e_{10}, & [e_{10}, e_{13}] &= e_9, & [e_{10}, e_{15}] &= -ae_{10} + e_9
\end{aligned} \tag{91}$$



For the generic case, the symmetry algebra comprises a 16-dimensional semidirect product,  $\mathbb{R}^6 \rtimes \mathbb{R}^{10}$ . The nilradical and its complement are abelian, spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$  and  $e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$ , respectively.

#### 4.11.1. $A_{5,17}^{a=0, bc \neq 0}$ :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned} e_1 &= D_z, & e_2 &= D_x, & e_3 &= D_q, & e_4 &= D_t, & e_5 &= D_y, & e_6 &= wD_t, & e_7 &= \sin wD_q + \cos wD_x, \\ e_8 &= \sin wD_x - \cos wD_q, & e_9 &= e^{bw}(\sin cwD_y + \cos cwD_z), & e_{10} &= e^{bw}(\cos cwD_y - \sin cwD_z), \\ e_{11} &= tD_t, & e_{12} &= qD_q + xD_x, & e_{13} &= yD_y + zD_z, & e_{14} &= zD_y - yD_z, & e_{15} &= D_w + \frac{1}{2}(xD_q - qD_x), & (92) \\ e_{16} &= qD_x - xD_q, & e_{17} &= (x \sin w - q \cos w)D_q + (x \cos w + q \sin w)D_x, \\ e_{18} &= (x \cos w + q \sin w)D_q + (q \cos w - x \sin w)D_x. \end{aligned}$$

$$\begin{aligned} [e_1, e_{13}] &= e_1, & [e_1, e_{14}] &= e_5, & [e_2, e_{12}] &= e_2, & [e_2, e_{15}] &= \frac{1}{2}e_3, \\ [e_2, e_{16}] &= -e_3, & [e_2, e_{17}] &= e_7, & [e_2, e_{18}] &= -e_8, & [e_3, e_{12}] &= e_3, \\ [e_3, e_{15}] &= -\frac{1}{2}e_2, & [e_3, e_{16}] &= e_2, & [e_3, e_{17}] &= e_8, & [e_3, e_{18}] &= e_7, \\ [e_4, e_{11}] &= e_4, & [e_5, e_{13}] &= e_5, & [e_5, e_{14}] &= -e_1, & [e_6, e_{11}] &= e_6, \\ [e_6, e_{15}] &= -e_4, & [e_7, e_{12}] &= e_7, & [e_7, e_{15}] &= \frac{1}{2}e_8, & [e_7, e_{16}] &= e_8, & (93) \\ [e_7, e_{17}] &= e_2, & [e_7, e_{18}] &= e_3, & [e_8, e_{12}] &= e_8, & [e_8, e_{15}] &= -\frac{1}{2}e_7, \\ [e_8, e_{16}] &= -e_7, & [e_8, e_{17}] &= e_3, & [e_8, e_{18}] &= -e_2, & [e_9, e_{13}] &= e_9, \\ [e_9, e_{14}] &= e_{10}, & [e_9, e_{15}] &= -be_9 - ce_{10}, & [e_{10}, e_{13}] &= e_{10}, & [e_{10}, e_{14}] &= -e_9, \\ [e_{10}, e_{15}] &= -be_{10} + ce_9, & [e_{16}, e_{17}] &= 2e_{18}, & [e_{16}, e_{18}] &= -2e_{17}, & [e_{17}, e_{18}] &= -2e_{16}. \end{aligned}$$

#### 4.11.2. $A_{5,17}^{b=0, ac \neq 0}$ :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned} e_1 &= D_q, & e_2 &= D_y, & e_3 &= D_z, & e_4 &= D_t, & e_5 &= D_x, & e_6 &= wD_t, & e_7 &= \sin cwD_y + \cos cwD_z, \\ e_8 &= \cos cwD_y - \sin cwD_z, & e_9 &= e^{aw}(\sin wD_q + \cos wD_x), & e_{10} &= e^{aw}(\sin wD_x - \cos wD_q), \\ e_{11} &= tD_t, & e_{12} &= yD_y + zD_z, & e_{13} &= qD_q + xD_x, & e_{14} &= qD_x - xD_q, & e_{15} &= D_w + \frac{c}{2}(zD_y - yD_z), & (94) \\ e_{16} &= zD_y - yD_z, & e_{17} &= (z \cos cw + y \sin cw)D_y + (y \cos cw - z \sin cw)D_z, \\ e_{18} &= (y \cos cw - z \sin cw)D_y - (z \cos cw + y \sin cw)D_z. \end{aligned}$$

$$\begin{aligned}
[e_1, e_{13}] &= e_1, & [e_1, e_{14}] &= e_5, & [e_2, e_{12}] &= e_2, & [e_2, e_{15}] &= -\frac{c}{2}e_3, \\
[e_2, e_{16}] &= -e_3, & [e_2, e_{17}] &= e_7, & [e_2, e_{18}] &= e_8, & [e_3, e_{12}] &= e_3, \\
[e_3, e_{15}] &= \frac{c}{2}e_2, & [e_3, e_{16}] &= e_2, & [e_3, e_{17}] &= e_8, & [e_3, e_{18}] &= -e_7, \\
[e_4, e_{11}] &= e_4, & [e_5, e_{13}] &= e_5, & [e_5, e_{14}] &= -e_1, & [e_6, e_{11}] &= e_6, \\
[e_6, e_{15}] &= -e_4, & [e_7, e_{12}] &= e_7, & [e_7, e_{15}] &= -\frac{c}{2}e_8, & [e_7, e_{16}] &= e_8, \\
[e_7, e_{17}] &= e_2, & [e_7, e_{18}] &= -e_3, & [e_8, e_{12}] &= e_8, & [e_8, e_{15}] &= \frac{c}{2}e_7, \\
[e_8, e_{16}] &= -e_7, & [e_8, e_{17}] &= e_3, & [e_8, e_{18}] &= e_2, & [e_9, e_{13}] &= e_9, \\
[e_9, e_{14}] &= e_{10}, & [e_9, e_{15}] &= -ae_9 + e_{10}, & [e_{10}, e_{13}] &= e_{10}, & [e_{10}, e_{14}] &= -e_9, \\
[e_{10}, e_{15}] &= -ae_{10} - e_9, & [e_{16}, e_{17}] &= -2e_{18}, & [e_{16}, e_{18}] &= 2e_{17}, & [e_{17}, e_{18}] &= 2e_{16}
\end{aligned} \tag{95}$$

For both subcases, the symmetry algebra is  $\mathfrak{sl}(2, \mathbb{R}) \times (\mathbb{R}^5 \times \mathbb{R}^{10})$  Levi decomposition algebra. The radical comprises a five-dimensional abelian complement and a 10-dimensional abelian nilradical spanned by  $e_{11}, e_{12}, e_{13}, e_{14}, e_{15}$  and  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$ , respectively. The  $\mathfrak{sl}(2, \mathbb{R})$  is spanned by  $e_{16}, e_{17}, e_{18}$ .

#### 4.11.3. $A_{5,17}^{a=0, b=0, c \neq 0}$ :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned}
e_1 &= D_z, & e_2 &= D_x, & e_3 &= D_q, & e_4 &= D_t, & e_5 &= D_y, & e_6 &= wD_t, & e_7 &= \sin wD_q + \cos wD_x, \\
e_8 &= -\cos wD_q + \sin wD_x, & e_9 &= \sin cwqD_y + \cos cwD_z, & e_{10} &= \cos cwD_y - \sin cwD_z, & e_{11} &= tD_t, \\
e_{12} &= qD_q + xD_x, & e_{13} &= yD_y + zD_z, & e_{14} &= D_w + \frac{1}{2}xD_q - \frac{1}{2}qD_x + \frac{c}{2}(zD_y - yD_z), \\
e_{15} &= qD_x - xD_q, & e_{16} &= (x \cos w + q \sin w)D_q + (q \cos w - x \sin w)D_x, \\
e_{17} &= (x \sin w - q \cos w)D_q + (x \cos w + q \sin w)D_x, & e_{18} &= zD_y - yD_z, \\
e_{19} &= (z \cos cw + y \sin cw)D_y + (y \cos cw - z \sin cw)D_z, \\
e_{20} &= (y \cos cw - z \sin cw)D_y - (z \cos cw + y \sin cw)D_z.
\end{aligned} \tag{96}$$

$$\begin{aligned}
[e_1, e_{13}] &= e_1, & [e_1, e_{14}] &= \frac{c}{2}e_5, & [e_1, e_{18}] &= e_5, & [e_1, e_{19}] &= e_{10}, & [e_1, e_{20}] &= -e_9, \\
[e_2, e_{12}] &= e_2, & [e_2, e_{14}] &= \frac{1}{2}e_3, & [e_2, e_{15}] &= -e_3, & [e_2, e_{16}] &= -e_8, & [e_2, e_{17}] &= e_7, \\
[e_3, e_{12}] &= e_3, & [e_3, e_{14}] &= -\frac{1}{2}e_2, & [e_3, e_{15}] &= e_2, & [e_3, e_{16}] &= e_7, & [e_3, e_{17}] &= e_8, \\
[e_4, e_{11}] &= e_4, & [e_5, e_{13}] &= e_5, & [e_5, e_{14}] &= -\frac{c}{2}e_1, & [e_5, e_{18}] &= -e_1, & [e_5, e_{19}] &= e_9, \\
[e_5, e_{20}] &= e_{10}, & [e_6, e_{11}] &= e_6, & [e_6, e_{14}] &= -e_4, & [e_7, e_{12}] &= e_7, & [e_7, e_{14}] &= \frac{1}{2}e_8, \\
[e_7, e_{15}] &= e_8, & [e_7, e_{16}] &= e_3, & [e_7, e_{17}] &= e_2, & [e_8, e_{12}] &= e_8, & [e_8, e_{14}] &= -\frac{1}{2}e_7, \\
[e_8, e_{15}] &= -e_7, & [e_8, e_{16}] &= -e_2, & [e_8, e_{17}] &= e_3, & [e_9, e_{13}] &= e_9, & [e_9, e_{14}] &= -\frac{c}{2}e_{10}, \\
[e_9, e_{18}] &= e_{10}, & [e_9, e_{19}] &= e_5, & [e_9, e_{20}] &= -e_1, & [e_{10}, e_{13}] &= e_{10}, & [e_{10}, e_{14}] &= \frac{c}{2}e_9, \\
[e_{10}, e_{18}] &= -e_9, & [e_{10}, e_{19}] &= e_1, & [e_{10}, e_{20}] &= e_5, & [e_{15}, e_{16}] &= -2e_{17}, & [e_{15}, e_{17}] &= 2e_{16}, \\
[e_{16}, e_{17}] &= 2e_{15}, & [e_{18}, e_{19}] &= -2e_{20}, & [e_{18}, e_{20}] &= 2e_{19}, & [e_{19}, e_{20}] &= 2e_{18}.
\end{aligned} \tag{97}$$

The symmetry algebra is indecomposable with Levi factor  $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \times (\mathbb{R}^4 \times \mathbb{R}^{10})$ , where the semisimple has two copies of  $\mathfrak{sl}(2, \mathbb{R})$  spanned by  $e_{15}, e_{16}, e_{17}$  and  $e_{18}, e_{19}, e_{20}$ .



$$\begin{aligned}
[e_1, e_7] &= e_3, & [e_1, e_8] &= e_2, & [e_1, e_{15}] &= e_1, \\
[e_1, e_{16}] &= e_5, & [e_2, e_{15}] &= e_2, & [e_2, e_{16}] &= -e_3, \\
[e_3, e_{15}] &= e_3, & [e_3, e_{16}] &= e_2, & [e_4, e_{13}] &= e_4, \\
[e_5, e_7] &= e_2, & [e_5, e_8] &= -e_3, & [e_5, e_{15}] &= e_5, \\
[e_5, e_{16}] &= -e_1, & [e_6, e_{13}] &= e_6, & [e_6, e_{14}] &= -e_4, \\
[e_7, e_{11}] &= -ae_{10} - e_9, & [e_7, e_{12}] &= -ae_9 + e_{10}, & [e_8, e_{11}] &= ae_9 - e_{10}, \\
[e_8, e_{12}] &= -ae_{10} - e_9, & [e_9, e_{14}] &= -ae_9 + e_{10}, & [e_9, e_{15}] &= e_9, \\
[e_9, e_{16}] &= e_{10}, & [e_{10}, e_{14}] &= -ae_{10} - e_9, & [e_{10}, e_{15}] &= e_{10}, \\
[e_{10}, e_{16}] &= -e_9, & [e_{11}, e_{14}] &= -ae_{11} - e_{12} - 2e_9, & [e_{11}, e_{15}] &= e_{11}, \\
[e_{11}, e_{16}] &= ae_{10} - e_{12} - e_9, & [e_{12}, e_{14}] &= -ae_{12} - 2ae_9 + e_{11}, & [e_{12}, e_{15}] &= e_{12}, \\
[e_{12}, e_{16}] &= -ae_9 - e_{10} + e_{11}.
\end{aligned} \tag{102}$$

For the generic case, it is a 16-dimensional indecomposable solvable Lie algebra with a 12-dimensional nonabelian nilradical spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}$  and a 4-dimensional abelian complement spanned by  $e_{13}, e_{14}, e_{15}, e_{16}$ .

$A_{5,18}^{a=0}$ :

Symmetries and nonzero brackets are, respectively,

$$\begin{aligned}
e_1 &= D_y, & e_2 &= D_x, & e_3 &= D_q, & e_4 &= D_t, & e_5 &= D_z, & e_6 &= wD_t, & e_7 &= yD_q + zD_x, \\
e_8 &= yD_x - zD_q, & e_9 &= \sin wD_x - \cos wD_q, & e_{10} &= -\sin wD_q - \cos wD_x, \\
e_{11} &= (z \sin w - y \cos w)D_q + (z \cos w + y \sin w)D_x, \\
e_{12} &= -(z \cos w + y \sin w)D_q + (z \sin w - y \cos w)D_x, \\
e_{13} &= (2 \cos w + w \sin w)D_q + (w \cos w - 2 \sin w)D_x + \sin wD_y + \cos wD_z, \\
e_{14} &= (\sin w - w \cos w)D_q + (\cos w + w \sin w)D_x - \cos wD_y + \sin wD_z, \\
e_{15} &= qD_q + xD_x + yD_y + zD_z, & e_{16} &= tD_t, & e_{17} &= D_w + \frac{1}{2}(xD_q - qD_x + zD_y - yD_z), \\
e_{18} &= qD_x - zD_y + yD_z - xD_q, \\
e_{19} &= (z \sin w - y \cos w)D_q + (z \cos w + y \sin w)D_x + \frac{1}{2}((wz - x + 2y) \cos w - \\
&\sin w(q + 2z - wy))D_q + \frac{1}{2}((wy - q - 2z) \cos w - \sin w(wz - x + 2y))D_x + \\
&\frac{1}{2}(z \cos w + y \sin w)D_y + \frac{1}{2}(y \cos w - z \sin w)D_z, \\
e_{20} &= -(z \cos w + y \sin w)D_q + (z \sin w - y \cos w)D_x + ((q + z - wy) \cos w + (wz - x + y) \sin w)D_q + \\
&((wz - x + y) \cos w - \sin w(q + z - wy))D_x + (z \sin w - y \cos w)D_y + (z \cos w + y \sin w)D_z
\end{aligned} \tag{103}$$

$$\begin{aligned}
[e_1, e_7] &= e_3, & [e_1, e_8] &= e_2, & [e_1, e_{11}] &= e_9, \\
[e_1, e_{12}] &= e_{10}, & [e_1, e_{15}] &= e_1, & [e_1, e_{17}] &= -\frac{1}{2}e_5, \\
[e_1, e_{18}] &= e_5, & [e_1, e_{19}] &= e_9 + \frac{1}{2}e_{13}, & [e_1, e_{20}] &= e_{10} + e_{14}, \\
[e_2, e_{15}] &= e_2, & [e_2, e_{17}] &= \frac{1}{2}e_3, & [e_2, e_{18}] &= -e_3, \\
[e_2, e_{19}] &= \frac{1}{2}e_9, & [e_2, e_{20}] &= e_{10}, & [e_3, e_{15}] &= e_3, \\
[e_3, e_{17}] &= -\frac{1}{2}e_2, & [e_3, e_{18}] &= e_2, & [e_3, e_{19}] &= \frac{1}{2}e_{10}, \\
[e_3, e_{20}] &= -e_9, & [e_4, e_{16}] &= e_4, & [e_5, e_7] &= e_2, \\
[e_5, e_8] &= -e_3, & [e_5, e_{11}] &= -e_{10}, & [e_5, e_{12}] &= e_9, \\
[e_5, e_{15}] &= e_5, & [e_5, e_{17}] &= \frac{1}{2}e_1, & [e_5, e_{18}] &= -e_1, \\
[e_5, e_{19}] &= -\frac{1}{2}(e_{10} + e_{14}), & [e_5, e_{20}] &= e_{13} + 2e_9, & [e_6, e_{16}] &= e_6, \\
[e_6, e_{17}] &= -e_4, & [e_7, e_{13}] &= e_{10}, & [e_7, e_{14}] &= -e_9, \\
[e_7, e_{19}] &= e_{12}, & [e_7, e_{20}] &= -2e_{11}, & [e_8, e_{13}] &= -e_9, \\
[e_8, e_{14}] &= -e_{10}, & [e_9, e_{15}] &= e_9, & [e_9, e_{17}] &= \frac{1}{2}e_{10}, \\
[e_9, e_{18}] &= e_{10}, & [e_9, e_{19}] &= \frac{1}{2}e_2, & [e_9, e_{20}] &= -e_3, \\
[e_{10}, e_{15}] &= e_{10}, & [e_{10}, e_{17}] &= -\frac{1}{2}e_9, & [e_{10}, e_{18}] &= -e_9, \\
[e_{10}, e_{19}] &= \frac{1}{2}e_3, & [e_{10}, e_{20}] &= e_2, & [e_{11}, e_{13}] &= -e_2, \\
[e_{11}, e_{14}] &= -e_3, & [e_{11}, e_{18}] &= 2e_{12}, & [e_{11}, e_{20}] &= -2e_7, \\
[e_{12}, e_{13}] &= e_3, & [e_{12}, e_{14}] &= -e_2, & [e_{12}, e_{18}] &= -2e_{11}, \\
[e_{12}, e_{19}] &= e_7, & [e_{13}, e_{15}] &= e_{13}, & [e_{13}, e_{17}] &= \frac{1}{2}(e_{10} + e_{14}), \\
[e_{13}, e_{18}] &= -e_{10} + e_{14}, & [e_{13}, e_{19}] &= \frac{1}{2}e_1 - e_2, & [e_{13}, e_{20}] &= 2e_3 + e_5, \\
[e_{14}, e_{15}] &= e_{14}, & [e_{14}, e_{17}] &= -\frac{1}{2}(e_{13} + 3e_9), & [e_{14}, e_{18}] &= -e_{13} - e_9, \\
[e_{14}, e_{19}] &= -\frac{1}{2}(e_3 + e_5), & [e_{14}, e_{20}] &= e_1 - e_2, & [e_{17}, e_{19}] &= -\frac{1}{2}e_{12}, \\
[e_{17}, e_{20}] &= e_{11}, & [e_{18}, e_{19}] &= -e_{20}, & [e_{18}, e_{20}] &= 4e_{19}, \\
[e_{19}, e_{20}] &= e_{18}.
\end{aligned} \tag{104}$$

The symmetry algebra is  $\mathfrak{sl}(2, \mathbb{R}) \times (\mathbb{R}^3 \times \mathbb{R}^{14})$  with nontrivial Levi decomposition. The semisimple part is  $\mathfrak{sl}(2, \mathbb{R})$  spanned by  $e_{18}, e_{19}, e_{20}$ , whereas nilradical  $\mathbb{R}^{14}$  and its complement  $\mathbb{R}^3$  are abelian, spanned by  $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}$  and  $e_{15}, e_{16}, e_{17}$ , respectively.

## 5. Conclusions and Future Work

In this work, we investigated the Lie symmetry algebras properties of second-order systems of geodesic equations of the five-dimensional Lie groups and elaborated on the methods of obtaining them. We specifically considered the geodesic systems of solvable Lie algebras 5.7–5.18. For each system, we found a basis for the symmetry algebra and calculated the corresponding Lie brackets. We then proceeded to identify each symmetry Lie algebra, noting whether it was solvable, semisimple, or neither, and, in the latter case, gave the semidirect sum of semisimple and solvable algebras.

Algebras  $A_{5,7}$  through  $A_{5,18}$  have a four-dimensional abelian nilradical, as described by [16], are mutually nonisomorphic, are unlike many of the low-dimensional nilpotent Lie algebras, and belong to continuous families that include parameters.

We remarked that, for exceptional values of the parameters, the dimension of symmetry algebras appeared to be larger than that of other comparable algebras. There may be two underlying causes that help to describe this phenomenon. First, solvable algebras, at least in dimension five, depend

on parameters. Furthermore, it seems as though geodesic systems for some solvable Lie algebras do not always contain more than one trivial geodesic equation, that is, when the right-hand side is zero. Finally, we observed that the complement of the radical of case ten was not a Lie algebra, and this may be because its geodesics were trivial and did not contain parameters. In the future, we plan to apply our procedures to the geodesics of algebras  $A_{5,19}$  to  $A_{5,40}$ , and the six and higher dimensional algebras, though new methods are required.

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