A SIMPLE ANHARMONIC OSCILLATOR

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Introduction

The mathematics curricula in the Upper Schools of St. Catherine's and St. Christopher's Schools are coordinated with students from the two schools sharing classes in the eleventh and twelfth grades. For the past several years, the schools have offered courses in which students take advantage of the power of *Mathematica* to attack difficult problems and to pursue projects in mathematics and science which require extensive computation. In this paper, we shall describe a project undertaken and completed by the first author with the aid of *Mathematica*[1]. The work was performed with the guidance and assistance of her mathematics and physics teachers, the second two authors named above. Katherine presented her results at the annual meeting of the Virginia Association of Independent Schools in November 2000. We shall present our work in narrative form and chronological order without assigning credit for particular results to individual authors.

We set out to develop an idealized mechanical model that would execute simple anharmonic motion, to write an equation of motion for the oscillator, and to solve the equation with the aid of *Mathematica*. We also wished to animate the motion and, finally, we added an experimental component to the project. Throughout our paper, we shall draw boxes about *Mathematica* commands, inputs, and outputs to separate them from the rest of our text.

The Simple Harmonic Oscillator

Let us begin by recalling the simple harmonic oscillator. Suppose that a block of mass m is attached to two identical massless springs as indicated in Figure 1. All motion of the system is confined to a horizontal, coordinate plane. The springs have Hooke's Law force constant k and unstretched length $L >> 1$, and the dimensions of the block are negligible when compared with L and 1 unit. The ends of the springs which are not connected to the block are anchored at the points (0, L) and (0, -L).

The Journal of Mathematics and Science: Collaborative Explorations Volume 3 No 2 (2000) 115-128

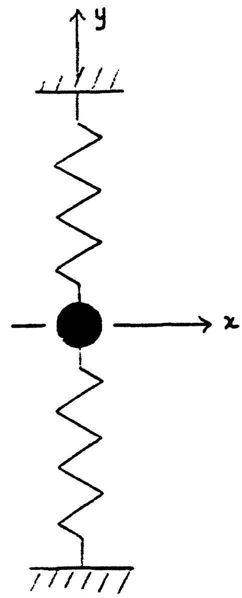

Figure 1. The Simple Harmonic Oscillator

We now suppose that the block is displaced to the point (0, 1) and then released from rest. The block will execute simple harmonic motion on the y-axis. Since there are two springs providing restoring forces upon the block proportional to its displacement, the equation of motion for this simple harmonic oscillator is

$$md^2y/dt^2 = -2ky.$$

The well known solution to this equation under the initial conditions $y = 1$ and $dy/dt = 0$ at $t = 0$ is $y = \cos\left(\sqrt{2k/mt}\right)$. The period of the oscillatory solution is $2\pi\sqrt{2k/m}$. Later, we shall compare these results with those which we obtain for the simple anharmonic oscillator.

The Simple Anharmonic Oscillator

A vibrating mass for which the restoring force is directly proportional to the third power of displacement is known as an anharmonic oscillator. To model such an oscillator, we suppose that we have displaced the block to the point (1, 0) on the x-axis. If we release the block from rest, it will vibrate back and forth along the x-axis with an amplitude of magnitude 1 as indicated in Figure 2.

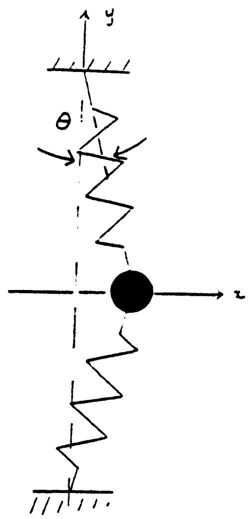

Figure 2. The Simple Anharmonic Oscillator

When the displacement of the block is x, $-1 \le x \le 1$, the stretched lengths of the two springs are $\sqrt{L^2+x^2}$, and the tensions in the spring have magnitude $k\left(\sqrt{L^2+x^2}-L\right)$. The x-components of these tensions provide the unbalanced force which accelerates the block back and forth between (1, 0) and (-1, 0) on the x-axis. Thus, each spring contributes a restoring force of magnitude $k\left(\sqrt{L^2+x^2}-L\right)\sin\theta$. Angle θ is shown in Figure 2, and it should be clear that $\sin\theta = x\Big/\sqrt{L^2+x^2}$.

The equation of motion for the transverse vibration of the block along the x-axis is

$$md^2x/dt^2 = -2kx\left(\sqrt{L^2+x^2}-L\right)\Big/\sqrt{L^2+x^2} = -2kx\left(1-1\Big/\sqrt{(x/L)^2+1}\right). \qquad (1)$$

Having written an equation of motion far too formidable for our unaided computational power, we now ask *Mathematica* to give us the third order Maclaurin polynomial in x for the righthand side of the equation.

$$\text{Normal}\left[\text{Series}\left[-2k * x\left(1 - 1\Big/\sqrt{(x-L)^2 + L^2}\right), \{x, 0, 3\}\right]\right]$$
$$-\frac{kx^3}{L^2}$$

Since we have assumed that $L >> 1$, we feel justified in replacing the restoring force with its third order Maclaurin approximation as output above. Thus we turn our attention to the much simpler equation of motion

$$d^2x/dt^2 = -\left(\frac{k}{mL^2}\right)x^3. \tag{2}$$

We have now obtained our equation of simple anharmonic motion and thus have found a model for simple anharmonic motion.

The First Integral of the Equation of Motion

We employ a trick well known to those who study mechanics to discover the velocity $v = dx/dt$ of the oscillating block. The trick which may be termed a "backwards chain rule" is nothing more than the observation that

$$d^2x/dt^2 = dv/dt = (dx/dt)(dv/dx) = v(dv/dx).$$

Equation 2 may now be rewritten as

$$v(dv/dx) = -\left(\frac{k}{mL^2}\right)x^3. \tag{3}$$

We integrate both sides of Equation 3 to obtain another equation which relates velocity and displacement:

$$\frac{v^2}{2} = C - \frac{k}{mL^2}\left(\frac{1}{4}\right)x^4. \tag{4}$$

In this equation, C denotes the constant of integration. The initial conditions that $x = 1$ and $v = dx/dt = 0$ when $t = 0$ imply that $C = k/\left(4mL^2\right)$. Thus,

$$v = dx/dt = \pm\sqrt{\frac{k}{2mL^2}}\sqrt{1-x^4}. \tag{5}$$

During the first quarter period of motion, $0 \le x \le 1$ and $v = dx/dt \le 0$. We are able to fix our attention upon the first quarter period by taking the negative sign for the differential equation above.

The Period of Oscillation

Equation 5 may be rewritten as

$$dt = -\sqrt{\frac{2mL^2}{k}}\frac{dx}{\sqrt{1-x^4}} \tag{6}$$

in the first quarter period. It follows that the period for our simple anharmonic oscillator is given by

$$T = -4\sqrt{\frac{2mL^2}{k}}\int_1^0 \frac{dx}{\sqrt{1-x^4}} = 4\sqrt{\frac{2mL^2}{k}}\int_0^1 \frac{dx}{\sqrt{1-x^4}}. \tag{7}$$

This integration may now be quickly done by *Mathematica* as shown below:

$$4\sqrt{\frac{2mL^2}{k}} * \text{NIntegrate}\left[1/\sqrt{1-x^4}, \{x, 0, 1\}\right]$$

$$7.4163\sqrt{\frac{mL^2}{k}}$$

The period for our oscillator is $7.4163\sqrt{\frac{mL^2}{k}}$ which may also be written as $5.24412\sqrt{\frac{2mL^2}{k}}$.

The Graph of x(t)

Let us define the following function of p, $0 \le p \le 1$, for which p is the lower limit of a definite integral:

$$F(p) = \sqrt{\frac{2mL^2}{k}} \int_p^1 \frac{dx}{\sqrt{1-x^4}}. \tag{8}$$

The function represents the time elapsed in the first quarter period of oscillation between the release of the block and its reaching the point (p, 0). Clearly, F(0) = T/4.

With *Mathematica* evaluating F(p) for a very large number of values of p, we obtained an extensive list of ordered pairs (t, x) where t = F(x). We then employed *Mathematica*'s ListPlot command with the PlotJoined option to obtain the following graph of the first quarter period of the displacement x(t).

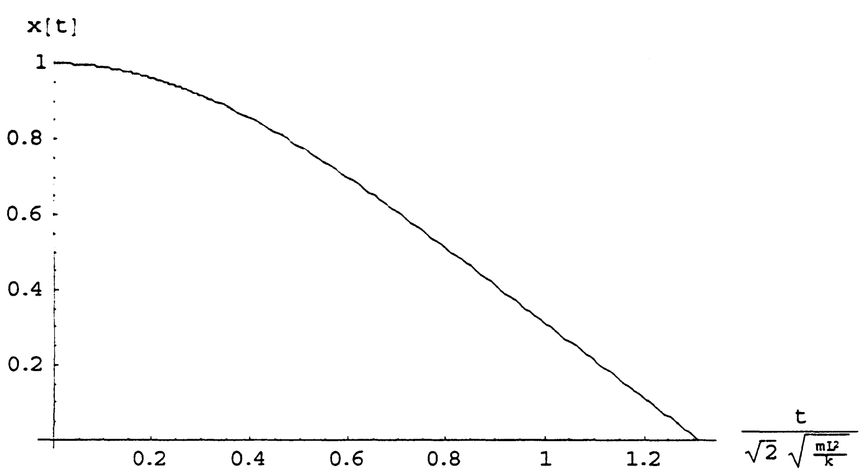

Figure 3. A Graph of Displacement x(t) for $0 \le t \le T/4$

The graph of x(t) for a complete period can be pieced together by simply repeating the graph for the first quarter period with suitable translations and reflections. The result is shown in Figure 4. The dashed curve in that graph represents the cosine solution for simple harmonic motion with the same frequency as our simple anharmonic oscillator.

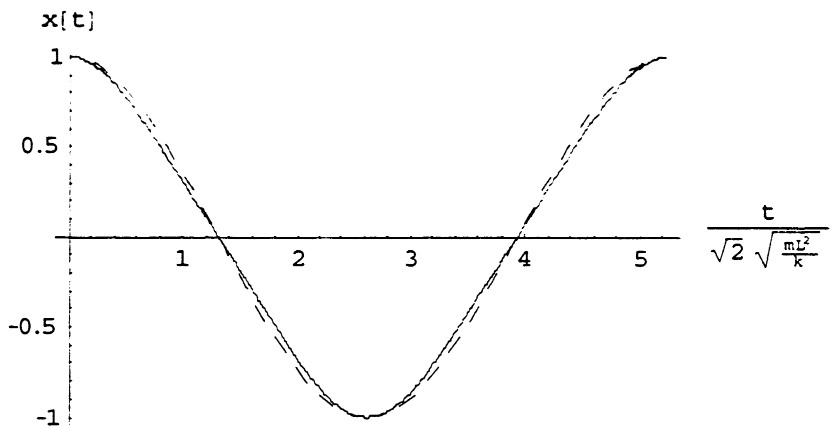

Figure 4. The Graph of x(t) for One Period with the Cosine Curve Shown for Comparison

Displacement as a Function of the Independent Variable Time

Thus far, we have accomplished our first three objectives: to create a model of simple anharmonic oscillator, write its equation of motion, and then solve that equation. The function t = F(x) and the graph of $x(t) = F^{-1}(t)$ extended to a full period as shown in Figure 4 constitute the solution.

Our fourth objective was to animate the motion of the oscillator by using the powerful graphics commands available in *Mathematica*. Here we encountered a difficulty. In our solution t = F(x), displacement plays the role of the independent variable making it relatively easy to obtain ordered pairs (t, x) for uniform increments in x, but not in time t. An animation requires that the variable representing time change at a constant rate. We could have integrated

F(p) using the If function to perform the integrations and record values of $x = p$ when t reached prescribed values separated by integral multiples of some fixed Δt.

The amount of programming required to implement this procedure did not appeal to us. It has been our experience that easy-to-formulate, but difficult-to-solve problems often have their solutions already built into *Mathematica*. We knew that the closed-form solution to our differential equation of anharmonic motion (Equation (2)) with our initial conditions involves the elliptic function, cn(t) the Jacobi cosine-amplitude function. It had never occurred to us that the elliptic functions might be available in *Mathematica*; but we investigated, and discovered to our surprise that we could indeed call for cn(t) as JacobiCN$[t,\alpha^2]$ where α is the modulus of the function.

We now had the means to animate the motion of the simple anharmonic oscillator. We could have avoided much of the work described thus far had we known before we started that our software would make it so easy to obtain the elliptic functions of the independent variable t. However, our work was instructive and we would still have needed to integrate for the period of the motion.

The Elliptic Functions

Although long known, the elliptic functions have fallen out of sight in recent years and are rarely studied today[2]. The complex plane is their natural domain; but for our purposes, their domain may be restricted to the set of real numbers. The restricted functions are periodic and differentiable. Unlike the trigonometric functions which they do resemble, the elliptic functions have periods which depend upon their amplitudes. We note that, in our project, we have simplified matters by assuming x(t) to have unit amplitude. The three elliptic functions with which we are concerned are the sine-amplitude, cosine-amplitude, and difference-amplitude functions which we denote by *sn u*, *cn u*, and *dn u*, respectively.

These three functions may be defined by the relationships

$$sn^2u + cn^2u = 1 \tag{9}$$

$$dn^2u + \alpha^2 sn^2u = 1 \tag{10}$$

$$\frac{d}{du}(snu) = (cnu)(dnu) \quad \text{and} \tag{11}$$

$$sn(0) = 0, cn(0) = dn(0) = 1. \tag{12}$$

As previously noted, the constant α is the modulus of the elliptic functions. Functions with different values of α are different functions. Just as the properties of the trigonometric functions may be deduced from the initial statements $\sin^2\theta + \cos^2\theta = 1, \frac{d}{d\theta}(\sin\theta) = \cos\theta, \sin(0) = 0,$ and $\cos(0) = 1$, the properties of the three elliptic functions may be developed from Equations 9 through 12. For example, implicit differentiation of Equation 9 yields $2snu\frac{d}{du}(snu) + 2cnu\frac{d}{du}(cnu) = 0$. By Equation 11. $(snu)(cnu)(dnu) + (cnu)\frac{d}{du}(cnu) = 0$. It follows that

$$\frac{d}{du}(cnu) = -(snu)(dnu). \tag{13}$$

In like manner, we may derive that

$$\frac{d}{du}(dnu) = -\alpha^2(snu)(cnu). \tag{14}$$

The Closed-Form Solution of the Equation of Motion

We return to Equation 2. The animation of simple anharmonic motion will require that we assign a numerical value to the coefficient of x^3. Since the choice is ours to make, we take "the easy way out" and let $\frac{k}{mL^2} = 1$. The simplified equation of motion is

$$d^2x/dt^2 = -x^3. \tag{15}$$

Let us assume that $x(t) = \text{cn}(t)$ satisfies Equation 15. The calculations following upon that assumption will force us to a proper choice for modulus α. Substituting $\text{cn}(t)$ for x in Equation 15 and using the properties given by Equations 9 through 14 lead to the following conclusions:

$$\frac{d^2}{dt^2}(cn\,t) = -\frac{d}{dt}(sn\,t\,dn\,t) = -\left(cn\,t\,dn^2\,t - \alpha^2 cn\,t\,sn^2\,t\right)$$
$$= cn\,t\left(\alpha^2 sn^2\,t - dn^2 t\right) = cn\,t\left(\alpha^2 sn^2\,t - 1 + \alpha^2 sn^2\,t\right)$$
$$= cn\,t\left(2\alpha^2 sn^2\,t - 1\right) = cn\,t\left(2\alpha^2 - 2\alpha^2 cn^2\,t - 1\right).$$

We see that $\frac{d^2}{dt^2}(cn\,t) = -cn^3 t$ if and only if $\alpha^2 = 1/2$. If we let $\alpha = 1/\sqrt{2}$, we have $x(t)$ = cn t as the desired solution of Equation 2. Note that the *Mathematica* input of the Jacobi elliptic functions requires the entry of α^2 rather than α.

Since we have let $\frac{k}{mL^2} = 1$, the period of cn(t) must be 7.4163. In Figure 5, we display the graph of $x(t)$ = cn(t) for one period. We again show as a dashed curve the graph of the cosine with the same period. We are happy to note that Figures 4 and 5 appear to be identical.

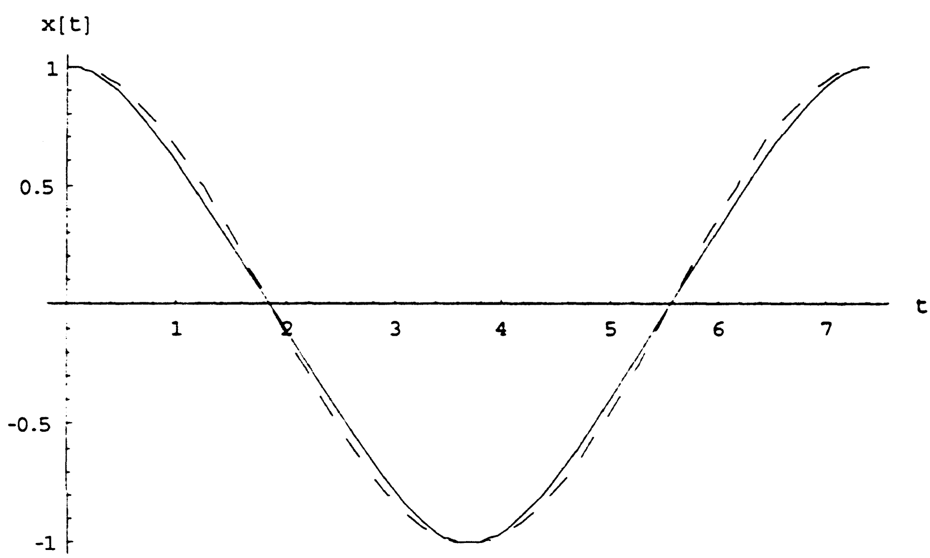

Figure 5. The Graph of $x(t)$ = cn(t)

The Animation of the Simple Anharmonic Motion

The procedure for animating a motion with *Mathematica* is to create a sequence of n still images of the changing system. The images must depict the system at times $t_1, t_1 + \Delta t, t_1 + 2\Delta t, \ldots, t_1 + (n-1)\Delta t$ for a fixed value of Δt. Upon command, *Mathematica* will show the images in rapid succession, thereby creating the sensation of viewing the motion. Now that we have $x(t) = \text{cn}(t)$ with independent variable t, the animation becomes easy to achieve.

The following program created 21 images uniformly separated in time over one period of oscillation. Note that, in the first step below, the constant a is assigned its value so that $4\sqrt{2}\text{a} = 7.4163$ which corresponds to one period.

```
a = 1.31103
anharpts =
        Table [{4√2 a * n/20,   JacobiCN[4√2 a * n/20,1/2]},{n,0,20}];

set1[n_] := {{0,3}, {anharpts[[n,2]], 0}, {0, -3}}

Do[plot1 = ListPlot[set1[n], PlotRange → {{-1.5, 1.5}, {-3, 3}},
        PlotJoined → True, PlotStyle → Dashing[{0.02}],
        AspectRatio → Automatic,Ticks → False,
        DisplayFunction → Identity];
plot2 = Graphics[{PointSize[0.1], Point[{anharpts[[n, 2]], 0}]},
        AspectRatio → Automatic, Axes → True,
        PlotRange → {{-1.5, 1.5}, {-3, 3}}]; Show[plot2, plot1], {n, 1, 21}]
```

In the next figure, one frame from the sequence of images is shown. The springs are represented by the dashed line segments.

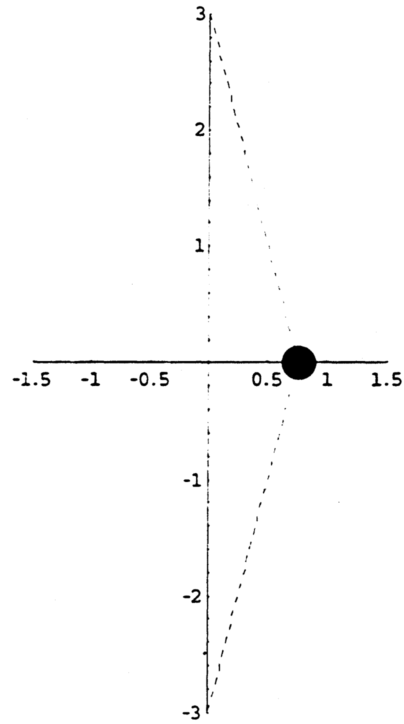

Figure 6. One Frame from the Animating Sequence

An Experiment

In our advanced physics laboratory, we constructed an oscillator from two springs and a light mass (0.004 kg.) to approximate the ideal oscillator depicted in Figure 1. However, the springs and mass were aligned vertically. The undistorted length of each of the springs was 3 cm. However, the springs were stretched by about 50% of their undistorted length when they were attached to the mass in its equilibrium position. This was done so that the springs would operate in their optimum linear range.

Any additional distortion in the springs produced by the weight of the mass was assumed to be negligible. In any event, no additional stretching of the upper spring and compression of the lower were observed. The stretched length of the springs was large with respect to the dimensions of the mass. The mass (to which we had taped a small section cut from a stiff index card) was pulled aside to a horizontal displacement of approximately 1 cm. and then released from rest.

The motion of the spring-mass system took place in a vertical plane with the mass vibrating on a horizontal line. The length of the springs was always much greater than the horizontal displacement of the mass. We employed a VERNIER™ Ultrasonic Motion Detector and interfaced it with the computer via a U L I in order to record the position of the vibrating mass at fifty equally spaced instants of time per second. The section of index card served to reflect the ultrasonic waves produced by the device. The software program which we used to treat the data was MacMotion (version 4). The next figure displays the displacement of the mass for two consecutive periods chosen well after the motion had settled into a steady pattern. We make no claims that our experimental work was closely controlled or quantitative. It seems to us that the most meaningful use to make of the data would be statistical in nature.

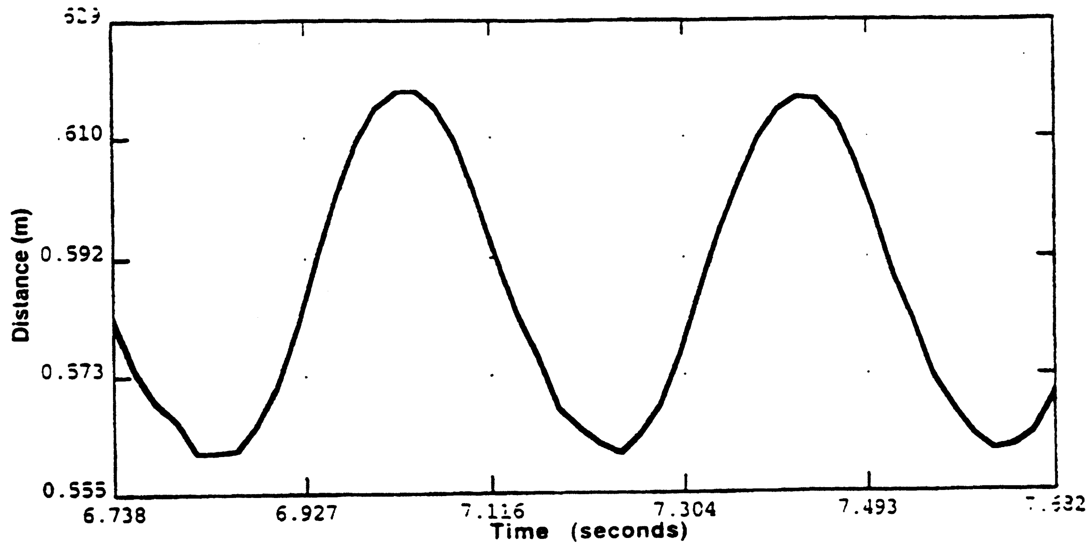

Figure 7. Displacement (Distance) as a Function of Time for the Real Oscillator

In Figure 8, we display nine points (x, t) within a single period with respect to the dashed cosine graph for the simple harmonic motion of the same period, phase, and amplitude as those of the real motion. The points marked with the symbol "#" are those which we judge to depart from the cosine graph in the manner predicted by Figure 4, in the case that our real oscillator was actually a simple anharmonic oscillator. It had occurred to us that we might look at a large number of such periods and accumulate a count of points "better explained" by an anharmonic model than by the harmonic model of oscillation. We then might do a bit of statistics by developing a nonparametric sign test of the null hypothesis that the data were explained by the harmonic model. We would hope to reject that hypothesis. We decided not to pursue these ideas, but would be happy to share the data with the joint Advanced Placement Statistics class in our two schools.

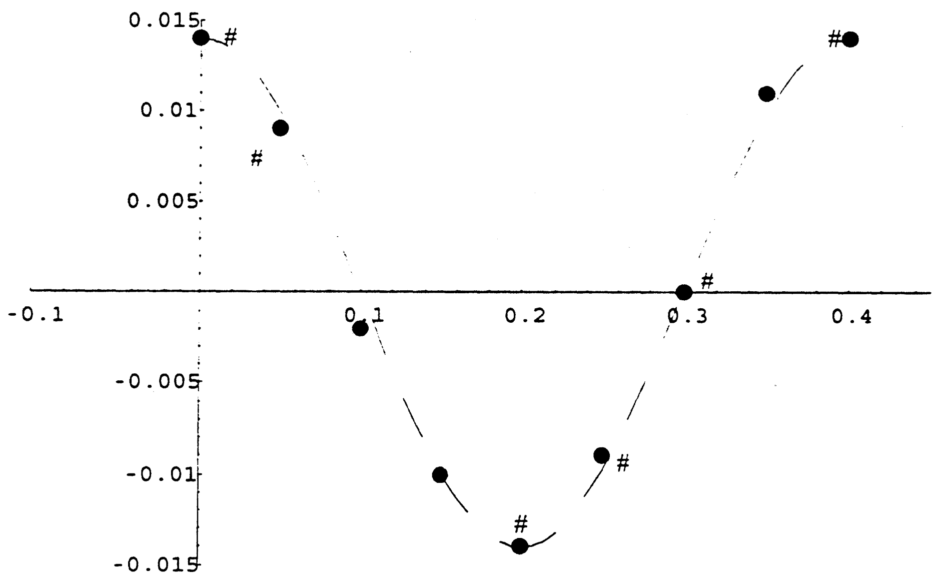

Figure 8. Nine Points (x, t) from One Period

Conclusions

All three authors, regardless of the levels of their mathematical and technological sophistication, benefited from participation in this work. Each is able to point out new ideas and combinations of ideas with which he or she had to grapple in order to bring the project to its conclusion. We believe that others teaching and studying in secondary schools at the Advanced Placement Level can with profit and success attempt interdisciplinary projects of the sort which we have described. ■

References

[1] B. F. Torrence and E. A. Torrence, *The Student's Introduction to Mathematica*, Cambridge Press, New York, 1999.

[2] A. C. Dixon, *The Elementary Properties of the Elliptic Functions*, MacMillan and Company, 1894.