ALTERNATIVE MODES FOR TEACHING MATHEMATICAL PROBLEM SOLVING: AN OVERVIEW

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Various modes are proffered as alternatives for teaching mathematical problem solving. Each mode is described briefly, along with general purposes, advantages and disadvantages. Combinations of modes are suggested; general issues identified; recommendations offered; and feedback from teachers summarized.

Introduction

The National Council of Teachers of Mathematics (NCTM) has asserted that "problem solving is an integral part of all mathematics learning" and has recommended increased emphasis on problem solving. While NCTM's principles and standards acknowledge that "there is no one 'right way' to teach," its vision does not specify alternatives to traditional instruction other than that students may learn "alone or in groups." [1]

Mathematics teacher educators can help prospective and in-service teachers by providing them with a repertoire of teaching modes in order to improve students' mathematical problem-solving abilities. A "mode" is a way of structuring students' learning environment for teaching purposes. This term is used here to distinguish modes of teaching from "methods" or "strategies" for problem solving, *per se*.

This article outlines a variety of modes for teaching mathematical problem solving, listed in increasing order of student group size and roughly from student-centered to teacher-centered:

EXPLORATION	PAIRED	COACHING
INDIVIDUAL	THINK ALOUD	BRAINSTORMING
PROBLEM POSING	INTERVIEW	FAMILY
INCUBATION	GAMING	LARGE GROUP
COMPUTER	SMALL GROUPS	PRESENTATION

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Purposes of Modes

Common to all these modes are a dozen general purposes, seen from the teacher's perspective and linked to NCTM's principles and problem-solving process standards [1]. Each purpose references NCTM Principle(s) and/or NCTM Process Standard(s) outlined in the key below:

NCTM Principles	NCTM Process Standards
E Equity	PS Problem Solving
C Curriculum	RP Reasoning and Proof
T Teaching	CM Communication
L Learning	CN Connections
A Assessment	RN Representation
K Technology	

(1) Practical Purpose: To conduct mathematical problem solving in a suitable setting within constraints of time, space, and resources. *Note: Modes are intended primarily for use in the classroom unless indicated otherwise.* PS

(2) Technological Purpose: To use appropriate technology for enhancing teaching and learning. K

(3) Pedagogical Purpose: To engage students in active problem solving and relevant learning. T

(4) Problem-Solving Purpose: To motivate students to apply their problem-solving skills and mathematical knowledge to solve a particular problem. *Note: All modes assume that the problem can be solved by most students within the time frame set by the teacher.* PS

(5) Cognitive Purposes: To stimulate students to think about the problem, and to help them understand related problem-solving concepts and methods. RP

(6) Affective Purposes: To build positive attitudes toward problem solving; to reduce math anxiety; and, to nurture the joy of problem solving. L

(7) Interactive Purpose: To encourage students to communicate about problem solving. CM

(8) Learning Purpose: To develop students' problem-solving skills and related mathematical knowledge. C,L

(9) Metacognitive Purpose: To develop students' ability to monitor, control, and reflect on their problem-solving processes. PS

(10) Cultural Purposes: To respect students' individual differences, heritage, values, and beliefs about mathematics; to include social, economic, and historical perspectives; to uphold college, state, and national standards; and, to promote equity in math education. E

(11) Assessment Purpose: To record students' problem-solving efforts in order to assess their progress. *Note: All modes require individual student reports.* A,RN

(12) Real-World Purposes: To develop students' appreciation for life skills involved in problem solving, and to acknowledge relevant application areas and career opportunities. CN

More specific special purposes are given for each mode in [2].

Modes

In order to help teachers choose a particular mode, each mode is described below in a synopsis, along with salient advantages vs. disadvantages (using the same numbering as the purposes above), along with selected references for follow-up reading. Full descriptions of all modes, including recommended grade/level, time frame, special purposes, and detailed operational guidelines for students and teachers are given in [2].

Exploration Mode

Synopsis: Each student selects a mathematical problem or puzzle to explore, discovers as much as possible about it, and prepares a "map" of what s/he finds. (This map is a guide to the features of the problem.) Students share their maps with each other and then with the teacher, who confirms what is actually needed to solve the problem/puzzle.

Note: This is also known as "Investigation" or "Discovery" mode and may be structured with specific activities for students to follow in stages.

Advantages and Disadvantages:

(1) Students may work in their own chosen space -- library, computer lab, or home. Yet, it's easy to lose focus and to lose track of time. (2) Students can use browsers and search engines via Internet, a

virtually unlimited resource for background information. Yet, they may become distracted by extracurricular information. (3) This mode is good as a warm-up homework assignment; good for introducing a new topic informally; and good for hands-on activity. Yet, without teacher control, students can fool around and might need content scaffolding. (4) It allows field work using real data and is an open-ended opportunity to investigate, experiment, and play—without having to solve the problem. Yet, some students flounder due to lack of structure; some drown in too many possible causes and effects. (5) Students can start thinking naturally and build their own cognitive structures. Yet, it's hard to overcome preconceptions, misconceptions, and mental blocks alone. (6) It's comfortable; there is no overt pressure on students; nobody is watching or demanding results. Yet, some students become frustrated when they can't make progress and may give up too easily. (7) Students can have internal debates about what to do next. Yet, their inner voices may be undeveloped. (8) Natural curiosity is nourished by discovery of potentially endless challenges. Yet, more questions than answers may arise; and there is no guarantee that students will learn underlying mathematics. (9) Students can build a sense of ownership for the problem and use metacognitive skills to guide their own exploration. Yet, these skills may be undeveloped. (10) Students can share cultural aspects of the problem in a "map" for others to use. Yet, they may just represent their own perspective. (11) Students can take pride in self accomplishment. Yet, the teacher can't assess them very well without direct observation. (12) Life skills include: investigation, experimentation, heuristic reasoning, and independent decision-making. Yet, life isn't just a bowl of exploration!

Reference: See [3] for models of investigation for teaching college science.

Individual Mode

Synopsis: Each student works alone on a problem and annotates his/her own work. The teacher then provides a list of the problem-solving skills and mathematical knowledge involved in solving the problem for the student to identify which s/he actually used and which of them need improvement.

Advantages and Disadvantages:

(1) This mode presumes a quiet and convenient place to work. Yet there is limited private space in the classroom, and intrusions are inevitable at students' homes. (2) It is suitable for use of calculators or personal computers. Yet some students can't afford them. (3) Individual mode is commonly used for homework, drill, and testing. Yet, it is an overused mode with no "teachable

moments" for the teacher. (4) The student has control, can work at his/her own pace, and can focus on the problem. Yet, without help, it is often hard to start, hard to get unstuck, and easy to give up. (5) Mental discipline is exercised, and writing annotations may increase clarity and understanding. Yet: students may just rush to get an answer; writing can interfere with their thinking; there is only one source of ideas, no check against wrong thinking, and no teacher to undo misconceptions. (6) Some students are more comfortable problem solving individually and can build confidence. Yet, lone failures can damage self-esteem. (7) Students can tap their inner voices, and annotations provide a good basis for self communication. Yet, there is no real interaction, no verbalization, and no one to urge them on; so many students prefer collaboration. (8) Students can find out privately what they don't know or can't do. Yet, they can become discouraged if there is too much to be learned. (9) By annotating their own work, students must reflect on it; students can develop a sense of ownership for the problem -- especially if they are successful. Yet it is human nature not to reflect on failure. (10) Students tend to work in their own established sub-cultural context. So, there is little incentive to consider a larger cultural perspective. (11) Students' annotations help the teacher diagnose skills and knowledge they need. Yet, students may believe that it's not their job to assess their own work. (12) Life skills include: test-taking, independent thinking, organization, time/energy management, self-discipline, responsibility, and perseverance. Yet, many people would rather not work alone in the real world.

References: See [4] for a classic book on individual problem solving, and see [5] for a more recent analysis of heuristics, control, and beliefs.

Problem Posing Mode

Synopsis: After problem solving (in another mode), each student is invited to propose new problems and to share them spontaneously with other students. Manipulative materials may be provided.

Note: A "new" problem here means new to the student, not necessarily an original problem.

Note: Problem Posing includes "problem presentation", which concerns how to present a problem—its context, wording, and illustrations—after the essence of it has been *posed*.

Advantages and Disadvantages:

(1) Manipulative materials may be suitable for this mode (depending on choice of problem). Yet, it is a chore to store and retrieve them. (2) An intelligent tutoring system might help students through stages of problem posing. But, such "inspiration" software is not available in schools yet. (3) This is an unusual student-centered activity: the teacher is free to observe or participate; it's particularly good for reinforcing problem-solving skills and knowledge—perhaps on a Friday in review for an impending test; and, it's a good opportunity for creative students to shine. Yet, most students are not used to it and have difficulty getting started; it's hard to detect if students are on task. (4) Students may produce some interesting problems. Yet, many student-posed problems are not very mathematical or not relevant or too silly or too hard or just canned imitations. (5) Posing problems involves both creative and systematic thinking; it can spark insight and solidify understanding. Yet, some students shut down mentally because it seems too challenging. (6) There is no immediate pressure to solve problems, which allows students' confidence in their mathematical creativity to grow. Yet, some students worry that their posed problem isn't good enough. (7) Communication skills are involved in writing and editing problem statements, as well as in explaining a problem to another student. Yet, some students are reluctant to share their posed problems. (8) This mode can motivate students to review related problem-solving skills and knowledge. Yet, even motivated students may find it hard to develop specific problem-posing skills. (9) Since the students clearly own the problems, they can realize that other problems have ownership too. Yet, they may have difficulty judging how hard a problem is; and may inadvertently reinvent problems. (10) This is a good opportunity for students to express their own cultural identity in a problem statement. Yet, they may be conditioned to imitate what they have seen in textbooks. (11) The teacher can select appropriate problems for tests based on collected student-posed problems. Yet, they may not represent what the students actually know. (12) Life skills include: inventing, designing, and teaching. Yet, (math) teachers rarely pose problems themselves; indeed, there are not many opportunities to do actual problem posing in the real world.

Reference: See [6] for an introduction to Problem Posing at about middle school level mathematics.

Incubation Mode

Synopsis: Students consider a problem over an extended time period. They work on it off and on, whenever *their interest* arises or after their ideas have developed.

Advantages and Disadvantages:

(1) This mode is easy to accommodate because it puts a problem on a "back burner" and moves problem solving out of the classroom. Yet it's a big change of pace from typical (next-class) deadlines. (2) Students can use resources from the Internet and personal calculators/computers whenever and wherever they are available. Yet some students don't have convenient access to such technology. (3) Incubation is well-suited for an untimed test or large projects which require ample time; the teacher can show trust in students. Yet, undisciplined students may not get mobilized (until the last minute). (4) Students can work in surges, get to know the problem well, and seek multiple solutions. Yet, procrastination is common, and there is no guarantee the problem will be solved. (5) Slow pace allows careful, rigorous thinking: errors die out; students can form mental connections; and subconscious creativity can blossom. Yet, real-world distractions may cause students to forget the problem. (6) With no pressure to solve the problem right away, students can relax and release their negative emotions. Yet, it is frustrating for those who feel they can't do it and aren't making progress. (7) Students may interact with others if they wish. However, if they don't, they won't get help or feedback. (8) Students are given time to develop understanding naturally (like a seed germinating). Yet, students may not be self-motivated to dig in and learn the necessary content. (9) There is plenty of time for reflection here. Yet, students are conditioned to get "the answer" and reach closure; misconceptions deepen with time; and the problem can become haunting. (10) Students can appreciate how valuable other perspectives are -- especially when they are stuck for a long time. Yet, they may not seek help outside of class anyway. (11) After Incubation mode concludes, the teacher can determine which students really can't do the problem and what needs to be taught or reinforced. Yet, this mode doesn't benefit quick problem solvers. (12) Life skills include: patience, perseverance, responsibility, and time management. Yet the real world often imposes short deadlines.

Reference: See [7] for relevant articles on insight in science and creative problem solving.

Computer Mode

Synopsis: Students use (micro)computers or calculators directly to solve a problem. This may involve software tools such as: programming languages; mathematical software packages (e.g., *Mathematica*, *Maple*, *Derive*, *Mathlab*, *Mathcad*); spreadsheets; graphics utilities; simulation and modelling "applets"; courseware; or, intelligent tutoring systems.

Note: If there are enough computers/calculators, each student may work alone—or better yet, in pairs; otherwise, they should be distributed as equally as possible.

Note: Computers/calculators can be used as tools in many other modes as well.

Advantages and Disadvantages:

(1) This mode could be used in a traditional classroom if students have calculators or lap-top computers. Otherwise, use of a computer-equipped classroom or laboratory must be arranged. (2) Modern technology is utilized -- particularly, powerful software. Yet, some students and some teachers oppose such heavy reliance on technology for teaching. (3) The teacher is free to observe, diagnose, and guide. Yet, the teacher may have to teach computer/calculator skills too. (4) Computers offer awesome number-crunching power. Yet, there are many distracting computing issues, such as temptingly easy exhaustive solutions and unexpected roundoff errors. (5) Computer work can be mentally stimulating and endlessly challenging; it can prompt students to elevate their thinking during problem solving by using results of computations which don't require thinking. Yet, this entails large cognitive overhead; and students may not know how computations are actually done. (6) Computers can be exciting, enjoyable, and empowering; this can build students' motivation for using them. Yet some students and teachers are computer phobic; others are unsympathetic computer addicts. (7) Computers offer immediate, accurate, and objective feedback to students; they can serve as a third-party arbiter in discussions of problem solving. Yet, computer interaction is essentially limited to formal language and pre-programmed responses. (8) Computer skills are in demand today; there is much to learn about computers. Yet, maybe there is too much, with little relation to mathematics and problem solving; computing is rarely integrated into curriculum; and computer manuals are notoriously bad for learning. (9) Students take pride in their computer competence. Yet, some prefer playing computer arcade games. (10) A "computer culture" can arise among computerphiles. Yet, other students may ostracize them as "computer geeks." (11) Computer skills can enhance students' math performance. Yet, evaluating these interrelated skills is complicated. (12) Life skills include: computer literacy, computer applications, and computer careers. Yet mathematical programming has become a specialized skill in this technologically developed age.

Reference: See [8] for an early vision of personal computers in education using Logo.

Paired Mode

Synopsis: Pairs of students work together as partners to solve a problem.

Note: This mode is designed without specific roles here in order to allow the students to decide how best to optimize their efforts.

Advantages and Disadvantages:

(1) This is distinctly different from traditional lecturing and individual mode. It's noisy, takes extra time, and may involve some re-arrangement of classroom furniture. (2) Paper/pencil may be adequate, so advanced technology may not be needed. (3) Students can teach each other what they know. Yet, they may not know enough and may convey misconceptions. It's hard for the teacher to monitor all students, and matching partners can be troublesome. (4) Paired mode essentially doubles a student's chance of solving the problem: when one student is stuck, the other can help; they can check each other's work, help keep focused on the problem, and seek better solutions. Yet, individual problem-solving style is compromised; one student slows down the other. (5) "Two heads are better than one" — students exchange ideas, as explaining sharpens the mind. Yet, it's hard to think and talk or listen at the same time. (6) Many students like working together, which reduces their fear of problem solving alone, and may feel more comfortable without the teacher watching. Yet, some students don't know how to collaborate well; and personality conflicts can spoil things. (7) Most students love to talk; this mode gives them "airtime"; they can explain their problem-solving approach and get immediate feedback. Yet, some students' explanatory skills are undeveloped. Some lack social skills or get off task easily while some prefer problem solving independently. (8) Students can use each other as resources; discussion can provoke learning and one can teach the other. Yet, students may unknowingly mislead each other; and it's hard to take notes while cooperating. (9) Students must develop and use a common representation so that the problem becomes "theirs." Yet, ownership of ideas gets blurred quickly. (10) Students have a good opportunity to exchange perspectives and to notice differences in language, customs, etc. Yet, they may be unable to work together because of cultural differences; or, they may be more interested in getting to know each other than in problem solving. (11) It's a lot easier to say "we failed" than "I failed." Yet, a "hitchhiker" is unfair to the better student; it's hard to separate individual contributions. (12) Life skills include: cooperation, communication, willingness to compromise, and

interpersonal sensitivity. Yet, students will often be on their own problem solving in the real world.

Reference: See [9] for an article on fostering basic problem-solving skills, including a protocol of a pair of remedial freshmen mathematics students.

Think Aloud Mode

Synopsis: Two students are given a problem and take on specific roles. One student is the "Solver." The other acts as a "Recorder." The Solver's role is not only to work on the problem, but also to explain his/her thinking—aloud—as s/he goes along. The Recorder's role is not only to record the Solver's work, but also to encourage the Solver to articulate his/her thoughts, to keep him/her focused on the problem, and to review aspects of the problem-solving activity whenever the Solver wishes. However, the Recorder is not permitted to give substantive hints nor to join in actually solving the problem. Then, the same two students switch roles and are given a different problem.

Advantages and Disadvantages:

(1) This is a variant of PAIRED mode. It has not been used much in mathematics education (to date), so students are not used to it. There is not enough private space in a typical classroom; it can be annoyingly noisy or deathly silent. (2) Audio-tape or video-tape can be very helpful here. Yet, playback of tape recordings is awkward during actual problem solving. (3) Formal roles keep students on task. Yet, many students don't like the rigid roles. (4) Both students must focus on problem-solving processes instead of getting the answer. Yet, articulating thoughts interferes with solving the problem; and, there is no substantive help for Solvers. (5) Solvers can check their own thoughts in preparing to speak; Recorders get a 'live' window into Solvers' thinking. Yet, this violates the Solvers' private cognition; and it's hard to explain one's thoughts anyway. (6) The Recorder's presence validates the Solver's efforts and can provide empathy for Solvers performing under pressure. Yet, Solvers may feel self-conscious, intimidated, or worried about being correct; Recorders may feel uncomfortable watching someone struggle. (7) This mode requires 'math talk' as Solvers articulate their problem-solving processes; and well-defined roles prevent one student from dominating. Yet, the roles are unnatural, are not cooperative, and can be overwhelming. (8) Recorders can develop listening skills and learn from Solvers; Solvers can understand their own problem solving better. Yet, Recorders may be too busy monitoring to learn much; and Solvers are focused on solving the problem, not learning *per se*. (9) The Solver must own the problem; both

students must reflect on their thinking and problem solving. Yet, most Solvers are reluctant to air dirty cognitive laundry and may distort things. (10) Students can appreciate how problem solving differs due to language, customs, etc. Yet, they may be deterred by extreme cultural differences. (11) This mode can be used as an alternative assessment tool; a detailed transcript (protocol) of the Solver's problem-solving process is a valuable reference. Yet, it's hard for the Recorder to write everything down accurately and time-consuming to analyze it. (12) Life skills include: explaining, listening, note-taking, carrying out responsibilities, and performing under pressure. Yet, nobody solves math problems like this in the real world.

Reference: See [10] for a close variant of this mode where one student is an "active listener."

Interviewing Mode

Synopsis: Two students interview each other soon after working on a problem (in some other mode). By asking probing questions, the interviewer tries to find out the important and interesting aspects of how the interviewee solved (or attempted to solve) the problem. Then they switch roles.

Advantages and Disadvantages:

(1) This mode is novel, refreshing, and quick. Yet it must be coordinated with a previous mode using the same problem; and classroom space must be re-arranged somewhat to pair students up out of close hearing range. (2) It is suitable for audio- or video-taping, which is valuable to review. Yet, the presence of a microphone or camera can inhibit some students. (3) Interviewing is a good opportunity to review and reinforce problem-solving skills; it gives students significant "authority." Roles keep students on task, and emphasis is on processes, not answers. Yet, if the interview is a rehash, it just wastes time. Students tend to socialize and it's hard to keep them on task with so little teacher control. (4) Students get a second look at the problem—a chance to correct mistakes, get unstuck, improve their solution, and consider related problems. Yet, they may just dwell on what they have already done. (5) Interviewees must describe their thinking, organize their thoughts, and can refine their reasoning, rethink their approach or get new insights; interviewers must focus attention on the problem and get a window into problem-solving processes. Explanations increase understanding for both. Yet interviewees may hide failures or distort the story and forgetting happens! (6) Interviewees can enjoy revisiting familiar work with confidence; interviewers can validate their efforts. The safety of roles allows students to express their feelings about problem

solving openly (and perhaps therapeutically). Yet, interviewers may be insensitive; interviewees may be uncooperative or competitive or worried about sounding stupid. (7) This mode is inherently interactive; students can present their best work to a captive audience, verbalization makes problem solving explicit, and students can talk in their own language. Yet, it may seem unnatural to 'talk math'; some students are reluctant to verbalize while most students lack interviewing skills. (8) Both students can learn skills and knowledge from each other. Yet, they may exchange misinformation: the interviewee may fake an explanation and the interviewer may not learn much. (9) Required reflection can clarify problem solving processes for both students, spawn multiple representations as a basis for comparison, and help them better judge how hard the problem is. Yet, interviewees might become idiosyncratic or obsessed about the problem, and interviewers may not really care. (10) An interview is a good opportunity to draw out the larger context of problem solving. Yet, students may not include cultural aspects unless prompted. (11) The teacher can also interview the students (afterwards). Otherwise, it is hard to assess individual contributions. (12) Life skills include: preparation, accountability, conversing, explaining, listening, interpreting, questioning, and "thinking on your feet." Yet, opportunities for interviewing in the real world are rare.

Reference: See [11] (Chapter 12) for suggestions on where and when to use interviewing in elementary mathematics.

Gaming Mode

Synopsis: Students play a mathematical game with different partners, develop strategies, and discuss them with each other.

Advantages and Disadvantages:

(1) This mode is a good change of pace; particularly good for a Friday. Yet, it takes time; it requires additional materials and possibly some re-arrangement of the classroom. (2) Students can play against a computer at various levels of difficulty. Yet some would rather play person-vs-person. (3) Gaming is a good pedagogical "ice breaker" for a first class and a possible equalizer for less-mathematically inclined students. Yet, it's unsuitable for immature students. (4) Students must apply their problem-solving skills in a game context; there are interesting problems and numerous challenges to consider. But where's the math? And, how does gaming fit into the curriculum? (5) To think strategically, students must analyze patterns, use if-then logic, and look ahead in a game tree.

Yet, conceptualizing strategies can be hard: some students just guess and hope, some get mental blocks; some misinterpret the rules or don't read the directions. (6) Playing games can be fun. Students can build tolerance for intellectual competition: winning builds confidence while losing can motivate students to improve. Yet, losing can be painful and may lower self esteem while winning can undermine personal relationships. Negative emotions arise from the pressure to make a move—notably fear of failure, performance anxiety, and reluctance to compete. (7) Games are inherently interactive and engaging; they involve hands-on activity and possibly hand-eye coordination; students can express and discuss their strategies. Yet, sometimes this makes students rowdy or tense or bored; competition can discourage talking. Gaming may be unproductive when either player is inexperienced or disinterested, or when opponents are mismatched. (8) Players learn experientially, get immediate feedback in context, and learn from mistakes. Indeed, losing provokes learning, winning reinforces understanding, and practice leads to improvement. Yet, there may be too much to learn in too little time; to play well can be an overwhelming, unrealistic task. (9) Students must consider what worked, what didn't, and what to do next; they can reconsider critical situations, judge levels of difficulty, and invent related games; and there is inevitable comparison with other students. Yet, students are all too often obsessed with winning; such egoistic meta-thoughts can override thinking about game strategy. (10) This mode is a good opportunity for the teacher to introduce the cultural history of the particular game and to recognize expert players from around the world. Yet, students may be more interested in games their close friends play. (11) Gaming is a potentially good diagnostic tool. Yet, it is difficult to discern what mathematical problem-solving skills are embedded in games. (12) Life skills include: competition, quick decision-making, responsibility for decisions, respect for good players, participation in game communities. Yet, gaming is controversial because of its association with gambling, entertainment, competitive sports, and military violence. It's not usually welcome in schools. Many people don't like competition, and few math games represent the real world.

Reference: See the introduction of [12] for a succinct rationale for developing deductive thinking and related problem-solving skills through an instructionally rich game.

<u>Small Group Mode</u>

Synopsis: Groups of three or four students work together cooperatively on a problem.

Note: This mode may also be called "Cooperative Problem Solving" or "Collaborative Problem Solving."

Advantages and Disadvantages:

(1) This mode is a distinct contrast with traditional lecturing and individual mode. It's noisy, takes time, and may involve some re-arrangement of classroom space. (2) Low technology (paper and pencil or chalkboard) may be adequate. Yet, computer technology would bring a competing authority into the group. (3) Students can manage themselves; the teacher can circulate as a "guide on the side" (instead of "sage on the stage"). Yet, it's difficult to form effective groups, hard to monitor students unobtrusively, and awkward to change groups. (4) Small Groups increase each student's chances of solving the problem; everyone has a common goal and can help each other. Yet, students don't work at the same rate; some slack off, and grouping smothers individual problem-solving styles. (5) Students can exchange ideas with each other, build on ideas, go with best idea, check each other, and uncover misconceptions; and re-think their approach. Yet, confusion can arise from conflicting thoughts: it may slow down the better student(s) and it's hard to think and listen at the same time. (6) Many students like working in groups because it reduces pressure; they feel mutual support and have confidence that someone (else) will know what to do. If the group succeeds, all students feel good. Yet, some students don't like groups and just want to solve the problem by themselves; there can be negative tension, strife, competition, pressure to agree, and worry about other students' put-downs. (7) This mode creates "airtime" for students—a chance to talk, be heard, get immediate feedback, and make decisions together. Yet, group dynamics can be explosive due to personality conflicts: two can gang up on one; one can dominate. Furthermore, students' verbal skills may be undeveloped, conversation may get disjointed or too social, students may be uncooperative or lack leadership, and it's not usually easy to reach consensus. (8) Students can learn actively from their peers, different explanations provide more chances for understanding, and it's hard for any one student to hide. Yet, students' abilities differ, students may spread misinformation, and it's awkward to take notes. (9) Students can compare attitudes, thinking, and proposed solutions. Yet, they may be too busy for much metacognition. (10) Students can share their cultural perspectives naturally. Yet, this may be dismissed if it doesn't contribute to solving the problem. (11) The group must produce a mutually understandable representation of the problem and possible solutions; group pride can emerge. Yet, it's hard to separate group and individual accountability; group work may be unfair to better students. (12) Life skills include: cooperation, listening, explaining, non-verbal communication, compromising, consensus decision-making, and interpersonal relationships. Yet, some people don't work well in groups in the real world.

Reference: See [13] for a handbook on cooperative learning in college mathematics education.

Coaching Mode

Synopsis: Students or teacher aids are designated to coach teams of about five students on how to solve a certain type of problem. The coaches try to ensure that everyone knows how to solve such problems. Then teams are given a problem to solve within a time limit, and one student is called on to represent the team in a competition comparing solutions. Afterwards, the coaches review solutions with their teams.

Advantages and Disadvantages:

(1) This mode utilizes selected students or teaching aids as resources; it has a manageable group size. Yet it is noisy, difficult to monitor, time consuming, and may need some classroom rearrangement. (2) An overhead projector, chalkboard, or computer can be used for display of instructional information. Yet, paper handouts may be sufficient. (3) Intensive peer tutoring is good for review; it gets students' attention and encourages everyone to work. Students are in control, and the special role of "coach" allows students to be "expert for the day." Yet, extra effort is required to train coaches: a coach may not have good teaching skills; the team may not accept the coach; and, there is no direct instruction by the teacher. (4) The coach can help motivate students who are stuck. Yet, there is no help during problem-solving competition, and no guarantee that the team will solve the problem. (5) It helps students get into a good frame of mind and be mentally focused. Yet, thinking gets narrowed: there may be too much information at once and students tend to cram instead of understand. (6) Peers can offer moral support for each other as team members; there may be more willingness to take risks with peers; and competitive instincts can be channeled into team spirit. Yet, it's inherently competitive and stressful. There is a lot of pressure on the coach whose weaknesses may get exposed: the coach may be insensitive, uncommunicative, overwhelmed, or otherwise bad. (7) Students get immediate feedback and help from peers; explanations are in student language. Yet, not all students may actually contribute to the team interactively. (8) Students learn best when they have to teach. As a coach, one can freshen up skills. Indeed, a coach does extra work and must know the content well; actually, it's a knowledge-rich experience for all. Yet, the coach may convey misinformation; so the teacher must rectify things later. (9) In order to prepare, the coach must reflect on mathematical content, consider students' different ways of thinking, recognize others' strengths, and may uncover misconceptions. Yet, the teacher may be

biased in selecting coaches and team comparisons may be embarrassing. (10) This mode is a good equalizer if all students get a turn to be a coach. Yet, students may reveal prejudices about who is qualified. (11) The teacher can observe some students' performances publicly and review coaches' notes afterwards. Yet, it's easy to blame failure on the coach. (12) Life skills include: teamwork, responsibility, leadership, competition, study skills, coaching, teaching. Yet, beware prolonged dependency on a coach.

Reference: See [14] for a discussion of an apprenticeship system for coaching problem solving. Also see www.mathcounts.org for a program for training school teachers to coach "mathletes" in mathematics competitions.

Brainstorming Mode

Synopsis: In groups of about seven, students generate ideas for solving a problem (or designing a math project). Specific roles are assigned: Moderator, Recorder, Purger, Timekeeper; and, everyone is an Idea Generator. Emphasis is on creative imagination and no criticism is permitted. Ideas are then categorized, evaluated, selected, and implemented.

Note: Groups may pool ideas and may implement them using a different mode.

Advantages and Disadvantages:

(1) This mode is quick and does not require access to resources (other than students' brains). Yet, it is very noisy; and a large board is needed to display ideas. (2) Audio- or video-tape can relieve the burden of taking notes. Yet, it's awkward to replay tape during brainstorming. (3) It is refreshing, energizing, and a good 'starter'; roles help control student behavior. The teacher may participate. Yet, Moderator and Recorder roles are very demanding; students may burn out quickly. Flights of fancy are not usually welcomed in the classroom. (4) This productive idea fest yields many diverse options which may lead to multiple solutions; students can build on each other's ideas. Yet, some ideas may be outrageous and unusable. Good students may get distracted. It's tempting to try solving the problem before generating more ideas. (5) Brainstorming opens the mind and encourages creative thinking "outside the box"; it challenges assumptions and counters the typical tendency to go with the first idea. Yet it can be chaotic and confusing: too much, too fast; and, possibly misleading as a license to think irresponsibly. (6) It's low risk and fun, students can feel

positive about their contributions, there are no wrong answers, everything is accepted, and "disenfranchised" students may rise to the occasion. Yet, it may be intimidating for shy students; some worry that their crazy ideas may get ridiculed; others may feel hurt when theirs get ignored. (7) This mode sparks spontaneous and intensive 'math talk'; everyone can participate freely. Yet, it's a verbal frenzy, with interruptive style and extroverts dominate. (8) Students can hear a variety of approaches to problem solving. Yet, they can't get explanations right away, and there is no guidance for learning how to be creative here. (9) Students collectively own the results; connections can be emphasized in a "knowledge web." Yet, it's hard to detect good ideas; indeed, there is no check against bad ideas, no way to correct misconceptions, and no time for metacognition while generating ideas. (10) This is a great opportunity for students to express diverse perspectives safely. Yet, there is no guarantee that students will honor cultural values. (11) The Recorder produces a record of the group's ideas. Yet guidelines are needed for assessing individual students' contributions. (12) Life skills include: "lateral thinking," fast thinking, valuing others' ideas, workplace applications (e.g., product design, advertising, management). Yet, brainstorming is not used productively often enough in the real world.

References: See [15] for an early introduction to "lateral thinking" and write Perfection Learning, 10520 New York Ave., Des Moines, IA 50322 for deBono's CoRT Thinking lessons which are available to schools. Also, see www.mindtools.com/brainstm.html for an intoduction to brainstorming, and other leads.

Family Mode

Synopsis: Each student is given a problem to work on and share with parent(s), caretaker(s), sibling(s), or other family member(s) or friend(s) at home. The student observes their efforts, records their work, and compares it with his/hers.

Advantages and Disadvantages:

(1) Students can work on their own schedule outside the classroom. Yet it's hard to find time; and they may impose on family/friends' routines. (2) Students can use technology available at home. Yet, low-income families can't afford computers. (3) This "homework" is a unique opportunity to do math with family/friends; they can help teach the student and appreciate the need for mathematics education. Yet, complex psychological factors are involved; some families/friends are

dysfunctional and activities are beyond the teacher's control. (4) Students can see how someone else tackles the problem; this helps sustain their interest; and they get several chances to solve it. Yet, there may be very different levels of math ability: the family/friend may not be able to solve the problem at all; or, a mathphile family/friend may spoil it for the student. (5) There are different sources of ideas. Yet, family/friends' minds may be too similar. (6) A familiar emotional environment allows more risk-taking; bonding may occur. Yet, past emotional baggage may inhibit risk-taking: the family/friend may be uninterested or mathphobic, or may ridicule the student; and, arguments, stress, and sibling rivalry may erupt. (7) This mode draws on established relationships, with no language/cultural barriers, and allows intimate discussion; it calls for "quality math time"; students get individual attention, help, honest feedback, patient explanation, and a natural way to follow up. Yet, it may be hard to get family/friends to participate. The family/friend may know math but can't explain it well or might dominate interaction; and, feedback may be too painfully honest. (8) This is a special opportunity to learn skills and knowledge from family/friends and vice versa; it might help build a learning community. Yet, the family/friend may be a bad model; and beware distractions, excuses, and an anti-math home learning environment. (9) The student and the family/friend can contrast their solutions, discuss how hard the problem is, and take pride in mutual accomplishment. Yet, a competitive comparison may be counterproductive. (10) It is certainly a good opportunity to bridge college and home cultures. Yet, the students' family/friends may prefer keeping them separate. (11) The teacher gains another basis for assessment. Yet, it is not objective: How much help did the student get? (12) Life skills include: family-teacher-student relationships, listening, explaining, and valuing lifelong learning. Yet, this is a very controversial mode; old wounds can be opened which might provoke need for therapy.

Reference: See [16] for a university-based project to involve families in math and science education.

Large Group Mode

Synopsis: A large number of students work on a problem at the same time, with leadership from the teacher who calls on students to contribute ideas and assists them toward solutions.

Note: "A large number" may be twenty to thirty students in a stereotypical classroom, to hundreds in a lecture hall, or even thousands in a virtual classroom via the Internet "distance education."

Note: This mode is "problem-based" whereas in traditional lecture mode background, information is usually presented first.

Advantages and Disadvantages:

(1) This mode is efficient for teaching many students problem solving at the same time; if the classroom is already arranged for a large group, no changes are needed. (2) It is well-suited for a chalkboard, overhead projector, or computer displays for demonstrations, simulations, websites, etc. Yet, such technology must be set up in advance. (3) It's a teacher-centered format, engaging the whole class in collective problem solving; it's good for observing students, collecting data, and modeling effective problem solving; and, there is equal opportunity for all students to contribute. Yet, student participation is bottlenecked through the teacher; it's impossible to bring all students up to the same level; discipline problems can arise. It is an overwhelming job. (4) This mode maximizes chances of solving the problem in minimal time: there is a common goal, many possible approaches, and potential synergy. The teacher can guide students toward multiple solutions. Yet, a mob approach can compromise individual problem-solving style, and students may mislead each other. (5) The teacher can stimulate students' thinking, pool their thoughts, build on their ideas or create "cognitive dissonance." Everyone has the right to check each other; argumentation can be beneficial, and the teacher can correct misconceptions immediately. Yet, there may be too many different ideas, different rates of thinking, or mass confusion; it's hard to hold thoughts in mind and some students tune out. (6) A large group is common for lecturing where students can listen and observe safely. Yet, it's risky to speak up. Students may fear ridicule by other students or the teacher; or, they're quickly frustrated when they can't get a word in. (7) Interaction relies on conventional hand-raising. Confident, verbal students can excel; some develop public speaking skills. Yet, there is limited "air time": a few students can dominate while passive students get shut out; side conversations are disruptive; and there may be too many voices—or a deadly silence. (8) Students can watch, listen, and decide what to learn from the teacher and other students. But it's easy to rely on others to do the work and not learn much. The teacher must correct students' misinformation, which slows down better students. (9) Students tend to compare themselves with others, which may motivate them. Yet, some may be reluctant to reconsider their problem-solving approach; the teacher may misinterpret or misphrase a student's idea publicly. (10) Students get to hear many other students' perspectives; and the teacher can add cultural background about the problem. Yet, some students may not appreciate more information which does not directly help solve it. (11) The teacher has a good vantage point to spot strong students. Yet, it is difficult to assess individuals' contributions which piggyback on others and to detect if those who don't

contribute at all had good understanding (and, therefore, teachers may have to rely on individual reports anyway). (12) Life skills include: public speaking, listening, patience, consensus-building, communal problem solving. Yet, real-world problem solving is not usually done in a large group.

Reference: See [17] for how the process of group problem solving through discussion is essential to democracy.

Presentation Mode

Synopsis: The teacher presents relevant problem-solving skills and mathematical knowledge along with an illustrative problem while students listen, take notes, and ask questions. Then the teacher demonstrates how to solve the problem. Later, students practice solving similar problems (for homework) and present their solutions (in the next class).

Advantages and Disadvantages:

(1) This mode is efficient for disseminating a lot of information to all students in the same place at the same time. Yet, the classroom (furniture) must be arranged accordingly for an audience. (2) It is well-suited for using technology to enhance teaching with traditional chalkboard, overhead projector, film, video, or computer display system (with presentation software) and Internet connection. Yet, it involves setting up equipment in advance, and some students resent technology which allegedly de-personalizes teaching. (3) It is a strongly teacher-controlled mode for introduction or review of specific skills and knowledge; there is equal opportunity for all students to learn the same information. Yet, it is usually paced for the "average student." (4) Students are given a solution template for solving a certain type of problem and can practice solving similar problems. Yet, this may be a crutch which doesn't help them solve other (non-routine) problems. (5) Students get to hear the requisite vocabulary before attempting to solve a problem; a dynamic presentation stimulates thinking. Yet, few students can think during a presentation dense with information; boredom is likely. (6) Students are familiar with lecturing. It's easy for them to hide; public put-downs by the teacher or peers can damage self-esteem. (7) Students can listen, ask appropriate questions, and develop public speaking ability. Yet, there is little "air time" for each student. (8) Students are told what skills/knowledge are needed for problem solving. Yet, rote practice does not guarantee conceptual understanding. (9) Students own their presentations. But, they don't own presentations by the teacher. (10) The teacher can take the opportunity to point out cultural

perspectives on problem solving. Yet, it may be difficult to integrate different perspectives. (11) This mode is convenient and fair for testing students who have been presented the same information. Yet, assessing their presentations is complicated by stage presence. (12) Life skills include: listening, note-taking, questioning, public speaking. Yet learning "on the job" is not like this.

Combinations of Modes

These modes may well be combined; indeed, there are many potential combinations of only two or three modes. For example: Small Groups reporting back to the Large Group; Individual work then Interviewing; Exploration then Paired problem solving; Small Groups then "jigsaw" Small Groups (re-formed with one student from each); Gaming with Think Aloud about strategies; Paired students in Computer mode; Brainstorming ideas, then implementation in Small Groups; Family sharing, then Presentation; Incubation, then Coaching; Exploration, then Problem Posing, then Individual problem solving. See [2] for details.

Other Modes

Other possible modes include: EXPERT mode, in which students observe an expert solving problems in a "cognitive apprenticeship" [18]; SILENT mode, in which the teacher shows students how to solve a problem without saying a word -- using only diagrams, body language, and writing in response to questions; PERSUADE mode, where a small group works on a problem, then presents their solution to another small group, trying to persuade them that they have solved the problem [19]; PYRAMID mode, in which students work on a problem individually, then pair up, then join in two pairs, then brainstorm in a group of eight, then combine results in a large group with the teacher; INTERNET mode, in which the teacher posts problems on an on-line bulletin board and/or e-mails problems to students who work on them collaboratively via e-mail distribution lists and/or chat rooms, and then post their results for critique by the teacher and other students. See [2] for details.

General Advantages and Disadvantages

Overall, the modes presented here offer some general advantages, balanced by intrinsic disadvantages: (1) Teachers can choose alternative modes (or combinations of modes) which fit best within the practical constraints of their particular situations. Yet, choices are limited by the classroom space, time available, and existing resources. (2) Technology provides powerful tools for enhancing teaching and learning. Yet, it incurs extra costs, technical difficulties occur, and training and additional effort are required. Some teachers distrust technology and it may also distract students away from mathematics. (3) Teachers can choose modes which are compatible with curricular content, suit their philosophy or teaching style, and motivate students. Yet, a new mode may cause a "Hawthorne effect" (getting their attention just due to a change). (4) The mode should increase the students' chances of solving the problem. Yet, it does not guarantee they will actually solve it. (5) The mode should enable students to do more productive thinking. Yet, their minds may become overloaded or distracted by new activities. (6) Making students emotionally comfortable should reduce math anxiety. Yet, anxiety may increase when they have to change their usual routines. (7) Interactive student communication can help problem solving and learning. Yet, it may increase confusion; and, there are always language barriers. (8) Some students learn problem-solving skills and knowledge better in certain modes. Yet, they may not learn well in all modes; and there is no guarantee of transfer from one mode to another. (9) Students can become more aware of their own problem-solving processes manifest in different modes, judge how well they are progressing, and compare their performance with their own expectations or with other students'. Yet, they may be reluctant to confront what they know vs. what they need; and, it is hard to do problem solving and metacognition at the same time. (10) Problem solving in different modes can help students respect individual differences and appreciate multicultural perspectives. Yet, it may conflict with practices within their family or local community; and it may raise complex issues about gender, race, special needs, etc. (11) Students' reports provide a record for both formative and summative assessment. Yet, it may be hard to get students to write reports; and some students dislike peers influencing their grades. (12) These modes allow development of life skills. Yet, many students have short-term interests.

General Issues and Recommendations

A number of general issues pervade all these modes, including: how to evaluate the mode; changes in the teacher's role; how to choose a mode to match a given problem; how to choose an appropriate problem to match the mode; how to group students or accommodate their differing abilities; how to assess students' work; how to deal with students' non-participation, avoidance of learning, cheating, and other discipline problems. Since these issues are beyond the scope of this article, they will not be addressed here.

A general recommendation for mathematics teacher educators is to arrange for prospective and/or in-service teachers to try out these modes themselves (acting as students) in a course, before using them in school classrooms.

Recommendations for teachers:

- Choose a mode which maximizes your purposes (but don't rely on it alone to fulfill all purposes).
- Explain the mode and model it for students before using it.
- Use a mode by itself before combining it with another mode.
- Practice the mode several times before evaluating it.

Feedback

These modes have been introduced to math teachers at elementary, middle-school, secondary, and higher education levels as part of a graduate course offered at University of Massachusetts' School of Education for the past ten years. About 100 teachers personally tried out each mode, informally field-tested most modes in their own classrooms, then critiqued all modes and discussed issues related to mathematics education reform.

Although these modes were not formally evaluated, feedback from teachers generally indicated that they welcomed alternative modes (and combinations) for vitalizing their teaching, that different modes enabled them to meet different students' needs, and that, while most modes are time-consuming, the actual advantages usually outweighed any disadvantages.

Hopefully, mathematics teacher educators will encourage teachers to consider adopting and adapting these and other effective modes in order to help students become better mathematical problem solvers. ■

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Bio

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