Math 245: Multivariate Calculus, Linear Algebra, and Differential Equations with Computer I is the first half of a year-long sophomore sequence that emphasizes the subjects' interconnections and grounding in real-world applications. The sequence is aimed primarily at students from physical and mathematical sciences and engineering. In Fall, 1998, as a result of my affiliation with the Science, Technology, Engineering, and Mathematics Teacher Education Collaborative (STEMTEC), I continued and extended previously-introduced reforms in Math 245, including: motivating mathematical ideas with real-world phenomena; student use of computer technology; and, learning by discovery and experimentation. I also introduced additional pedagogical strategies for more actively involving the students in their own learning—a collaborative exam component and in-class problem-solving exercises.

The in-class exercises were well received and usually productive; two were especially effective at revealing normally unarticulated thinking. The collaborative exam component was of questionable benefit and was subsequently abandoned. Overall student performance, as measured by traditional means, was disappointing. Among the plausible reasons for this result is that too much material was covered in too short a time. Experience here suggests that active-learning strategies can be useful, but are unlikely to succeed unless one sets realistic limits to content coverage.

Introduction

The typical full menu of sophomore-level math consists of separate 3-hour, semester-long courses in multivariable calculus, linear algebra, and differential equations. Ordinarily, those courses follow the traditional approach: mathematical concepts and techniques are presented—often in a dogmatic way with little motivation—and illustrated, and only then are they applied; the students solve problems not dissimilar to examples they have already seen in the text or in lecture; and, the problems often require technically complex algebraic manipulations.

Nearly a decade ago, my former colleague, Frank Wattenberg, integrated these three courses into an experimental, year-long sophomore sequence of two 4-hour courses, for which he privately published a new text in three volumes (1999-2000); he also designed parallel web-based
The two courses formed a sequel to Wattenberg's experimental freshman calculus course based upon materials that became his text *Calculus in a Real and Complex World* [2].

These experiments were local instances of a widespread effort to reinvigorate the introductory mathematics curriculum that is generally known as "calculus reform" [3]. That effort extended to linear algebra, thanks in part to the work of the Linear Algebra Curriculum Study Group [4], as well as to differential equations, which had already begun to be influenced by the availability of differential equation solvers and graphers [5].

This article reports on my experience with the first semester of this course in Fall, 1998 when I introduced additional pedagogical strategies into the course. It necessarily discusses my experience with the reforms already then in place, because the totality of innovations in the course affected outcomes of the additional strategies.

**Course Aims**

The principal aims of the local sophomore-level experiment, as originally formulated by Wattenberg, and subsequently extended by me, were as follows:

- To cover nine semester-hours' content from three courses in only eight semester-hours of two courses. Condensing the content was to be accomplished primarily through the efficiencies of integrating the three subjects and avoiding existing overlaps. For example, notions about linear transformations and bases of vector spaces would be covered only once—even though used in disparate contexts—rather than twice, once in linear algebra and again in differential equations. In addition, a few standard topics deemed to be of lesser importance were to be omitted. This was the case, for example, with calculating centroids of solids by means of triple integrals and solving exact first-order ordinary differential equations.

- To render linear algebra more meaningful than the usual, potentially sterile combination of routine calculation and challenging abstraction. The idea was to draw notions about vector spaces and their linear transformations from, and in turn apply them to, situations arising in multivariable calculus and differential equations. For example, eigenvalues and eigenvectors arise from systems of linear differential equations modeling the dynamics of an age-stratified population.
• More generally, to use real-world problems to motivate the key mathematical concepts and much of the mathematical development. For example, in Math 245 linear and affine transformations appear first in the context of two-dimensional computer graphics; vector equations of lines and planes are used to find the shadow of an object illuminated by a point light source; cross-products of 3-vectors are defined so as to measure torque; powers of matrices need to be formed in order to determine the long-term effect of a betting strategy at roulette; various qualitative and quantitative considerations about first-order differential equations are introduced in the context of Newton's law of cooling; linear systems of first-order differential equations are needed to describe oscillations of spring-and-mass systems and RLC electrical circuits.

Note that mathematical modeling was not in itself an aim. Whereas real-world problems were used to motivate mathematical ideas, some of which were, in turn, applied to real-world situations, little attempt was made to teach students to derive mathematical concepts from real-world problems.

• To appeal to different modes of comprehension, by emphasizing visual representations and numerical descriptions, not just symbolic manipulations.

• To enliven the course by more actively involving students in the process of learning. Originally, this aim meant that some key mathematical results would not initially be formulated in full by the lecturer or textbook. Rather, experimentation and exploration in homework problems would lead students to discover such results for themselves. Eventually, when I taught the course in 1998, this aim meant also that some of the exploratory learning would occur in the classroom through in-class exercises.

• To use modern computer technology for teaching and learning in order to facilitate the preceding aims.

My Involvement

Under an Instructional and Laboratory Equipment grant from the National Science Foundation, this experimental sequence became a permanent offering, Math 245–246: Multivariate Calculus, Linear Algebra, and Differential Equations with Computer I–II. Over the past eight years, I have taught this course five times. In Fall, 1998, thanks to my participation in
the STEMTEC Winter 1998 Workshop, I introduced additional pedagogical strategies into the course while continuing to incorporate the reforms that I had already been using in the course in earlier semesters. It is upon my experience in Fall, 1998 that this report focuses.

In Fall, 1998, there was a significant change in Math 245 student demographics that undoubtedly affected both the success of the reforms already in place and the outcome of the additional strategies.

Like the courses that they replace, Math 245 & 246 were taught in lecture sections of at most thirty students. As in many math courses, weekly problem sets were assigned, collected, and graded. Each semester, three exams were given: two mid-semester exams and one final exam, the latter covering just the final third of the course.

Teaching Math 245 was hardly the first occasion on which I had incorporated innovations similar to some of those in Wattenberg's courses. For more than fifteen years in linear algebra courses I had already incorporated student use of computer technology [6]; for several semesters, my students carried out a project on Hill ciphers in which they had to learn the topic entirely on their own [7]. Repeatedly in the sophomore-junior math major course, Fundamental Concepts of Mathematics, and once in junior-senior topology, I minimized lecturing and devoted most class time to students' devising or presenting problem solutions.

Nonetheless, the Fall, 1998, Math 245 was the first lower-division math course aimed primarily at non-majors in which I attempted additional strategies, such as were being demonstrated and advocated at STEMTEC workshops.

Reforms

The overall approach in Math 245 was to motivate the mathematics by means of models of real-world phenomena. The other "reforms" in this course were the students' use of computer technology; innovations in the type of homework, format of exams, and inclusion of in-class problem-solving exercises; and, incorporation of collaborative work.

Computer Technology: The principal technology used in Math 245 was the commercial software package Mathematica from Wolfram Research [8]. This package, documented in [9], is a state-of-the-art "computer algebra system" that is widely used by mathematicians and scientists for research as well as teaching. It provides integrated access to an extensive repertoire of functions through an interactive "live document" interface where input, output (including graphics), and
user- or instructor-supplied text can be intermixed; its functions can be used directly, with user-supplied arguments, or combined into new commands through user-written programs. In *Math 245*, *Mathematica* was used as a computational tool in place of long or complicated paper-and-pencil calculations and, more significantly, as a means of carrying out experimentation and discovery. This package was available for, and sometimes required in, homework problem sets; it was also available for students to use in the second and third of the three exams.

The symbolic, numerical, and graphical capabilities of *Mathematica* were briefly demonstrated at the first class of the semester. Several class sessions during the following week met in a computer lab, where students worked in pairs to learn basics of the software and to apply it to some problems in linear algebra. Resources for learning *Mathematica* included a textbook [10], occasional short lessons during lecture, the program's complete on-line help, and *Mathematica* "notebooks"—interactive documents—prepared by me.

*Mathematica* was also used "live" in lectures to make them more interesting. For example, in the study of oscillatory solutions of second-order differential equations, the phenomenon of beats could be heard—not just represented symbolically by a trigonometric formula or visually in a graph—by evaluating in *Mathematica* the following expressions:

\[
\omega = 263; \\
middleC = \text{DSolve}\{x'[t] + (2\pi \omega)^2 x[t] = 0, x[0] = 0, x'[0] = 1\}, x[t], t\} \\
\text{Play[Evaluate[x[t]/. middleC], \{t,0,(8/5)2\pi\}];} \\
\text{beat = DSolve}\{x'[t] + (2\pi \omega)^2 x[t] = \cos(2\pi(\omega + \delta)t), x[0] = 0, x'[0] = 1\}, x[t], t\}/.\delta \rightarrow 2/10 \\
\text{Play[Evaluate[x[t]/. beat], \{t,0,(8/5)2\pi\}];}
\]

This demonstration aroused student utterances of surprise at the appearance of the plotted oscillations, looks of delight at the sounds produced, and requests to see and hear what would happen with changes to the parameters.

For reviewing or learning more about what was covered in lecture, students were given access to *Mathematica* notebooks that contained examples whose parameters the students could freely modify and where the students could create new examples of their own. These notebooks included several adapted from Wattenberg's materials and others prepared by me [11].

Some of the topics could be learned from a website authored by Wattenberg [1] for the multi-institutional Connected Curriculum Project. This site included some valuable interactive
demonstrations used in class: for example, a Java applet that illustrated the abstract notion of a line segment in a vector space—here, a space of triples of matrices—by dissolving an exterior shot of the Lincoln Memorial into a photograph of the statue of Abraham Lincoln inside.

*Homework Problems:* The weekly homework problems often involved experimentation, exploration, and discovery. Here are two examples:

1. Given functions \( \vec{f} : \mathbb{R} \to \mathbb{R}^3 \) and \( \vec{g} : \mathbb{R} \to \mathbb{R}^3 \), discover and derive a nice formula for the derivative of \( \vec{f} \times \vec{g} \), the cross-product of the two functions. Although your formula should *not* involve coordinates, your derivation may.

   This problem is already more sophisticated than what is ordinarily assigned in a traditional multivariable calculus course: it asks the student not to derive a formula already supplied by the instructor or text, but to discover a formula and to derive it. Moreover, the symbolic powers of *Mathematica* can be used to do the work of manipulating coordinate functions, beginning with the following input:

   \[
   f[t_] = \{x[t], y[t], z[t]\}; \quad g[t_] = \{u[t], v[t], w[t]\}; \quad h[t_] = f[t] \times g[t]; \quad h'[t].
   \]

   Evaluating this compound expression in *Mathematica* returns as its result a vector of functions of \( t \) each of whose terms is of a familiar Leibnizian “first times derivative of second plus second times derivative of first” form. The student can then conjecture that the correct answer is \( f \times g' + f' \times g \); use *Mathematica* to evaluate

   \[
   f[t] \times g'[t] + f'[t] \times g[t]
   \]

   and see by inspection that the result is the same as before; and even let *Mathematica* check that the two results are in fact identical by obtaining True as the result of evaluating:

   \[
   h'[t] == f[t] \times g'[t] + f'[t] \times g[t].
   \]

2. Every Friday, each student at State U. goes to either Sue’s Subs or Paulo’s Pizzeria for a late-night snack. This Friday, 70% of the State U. students went to Sue’s and the rest went to Paulo’s. In each case below, what fractions of State U. students go to the two restaurants on Friday of weeks 2, 3, 4, ..., 20? What happens after many, many weeks go by: Is there an equilibrium state toward which the weekly vectors of fractions tend in the limit? If so, what is it?
(a) Suppose that each week, 60% of the students who go to Sue's on a Friday return to Sue's the following Friday, whereas 40% go instead to Paulo's; and 70% of those who go to Sue's on that Friday return to Sue's the following Friday, whereas 30% go instead to Paulo's.

(b) Suppose that each week, half the students who go to Sue's return to Sue's the following Friday, and the other half go instead to Paulo's; and half those who go to Sue's return to Sue's, and the other half go instead to Paulo's.

(c) Suppose that each week, all the students who go to Sue's that Friday go instead to Paulo's the following Friday; and all the students who go to Paulo's that Friday go instead to Sue's the following Friday.

This problem involves an unrealistically simple situation; other problems about Markov chains involved more realistic ones.

Homework problems often were rich—although hardly ever open-ended—and of a higher order of difficulty than is typical in sophomore math classes. For example:

3. An image of a cat's face has its nose at (2, 0). What sequence of transformations will create a movie in which the face moves counterclockwise around the origin with its nose always on the circle of radius 2? The face has to remain upright so the cat doesn't get dizzy!

Problems about composing transformations most commonly ask for synthesizing given transformations into more complex ones. Problem 3, by contrast, calls for analyzing transformations into composites of simpler ones. To solve it requires geometric insight and visualization skills and explicit formulation of a good strategy, not just a sound grasp of function composition and inversion. The crux of the cat-spinning problem is how to keep the face upright when, for a single frame of the animation, the nose is rotated through an angle $\theta$ around the origin. Some students could not figure out how to do this; the effect of their solution is as shown in Fig 1 (a). A correct solution should produce an animation whose superimposed frames appear as in Fig. 1 (b). And several different strategies led to correct solutions, including an elegantly simple one not anticipated by me.
Although they were not asked to do so, many students chose not only to describe how to create the movie, but also to realize the requisite sequence of transformations in Mathematica and apply them to produce an actual animation. In fact, many used Mathematica to experiment with their ideas and to verify whether their solutions were correct. Except for the few who made little or no progress on a solution, students were particularly engaged with, and seemed to enjoy, the cat-spinning problem. This reaction occurred despite the problem's difficulty and apparently, according to what several students said, because of the satisfaction of seeing the solution realized in action.

4. Spotted owls and flying squirrels inhabit California old-growth Douglas fir forests. The owls prey upon the squirrels. The numbers $o_k$ of owls and $s_k$ of squirrels in year $k$ are related to the corresponding numbers in year $k + 1$ by

$$
\begin{align*}
    o_{k+1} &= 0.4\, o_k + 0.2\, s_k, \\
    s_{k+1} &= -0.325\, o_k + 1.2\, s_k.
\end{align*}
$$

Suppose initially there are 100 owls and 175 squirrels. For each $k$, let $\bar{x}_k = (o_k, s_k)$.

(a) Explain what the sizes and signs of the coefficients indicate about the ecological relationship between the owls and squirrels.

(b) Calculate enough entries in the $\bar{x}_k$ sequence to guess what the limit is, as $k \to \infty$, of the ratio of corresponding coordinates in successive values $\bar{x}_k, \bar{x}_{k+1}$; and to guess what the
limits are of the ratios of each coordinate of $x_k$. Draw informative plots of: (I) both components of $x_k$ vs. time $k$; and (ii) one component of $x_k$ vs. the other.

(c) Use eigenvalue-eigenvector analysis to help explain what you guessed to be the various limiting ratios.

Problem 4 goes well beyond what is commonly asked about eigenvalue-eigenvector analysis of a matrix: calculate the eigenvalues and associated eigenvectors, then decompose a given vector into its components in the eigenspaces. This problem has a real-world connection, and it asks for: interpreting the parameters of a discrete dynamical model; observing the model's dynamics experimentally, with Mathematica; and, explaining the observations in terms of eigenvalue-eigenvector analysis.

Exams: All three exams lasted at least two hours, including the mid-semester exams that in sophomore courses ordinarily take just one 50- to 75-minute period. The exams were not multiple choice or short-answer but rather, as is the case in nearly all our sophomore math courses, consisted of problems whose solutions were to be written out.

The first mid-semester exam was given in an ordinary classroom, and students could use their calculators. The following question is representative in its level of difficulty, its mix of the conceptual with the computational, and its combination of geometric and algebraic reasoning.

1. (a) Express reflection $F$ across the line $x_2 = x_1 + 3$ in $\mathbb{R}^2$ as a composition of translations, rotations, and reflections across the coordinate axes. (No proof is required.)

(b) Let $L$ be the line in $\mathbb{R}^3$ passing through distinct points $\vec{a}$ and $\vec{b}$ and let $T$ be translation by the vector $\vec{v}$. Use vector algebra to show that the image $T(L)$ lies on the line passing through $T(\vec{a})$ and $T(\vec{b})$.

Part (a) of this question was essentially the same as one most had solved for homework; part (b) had been a homework problem, only solved completely by some of the students. Solutions to both homework problems had been provided.

This first exam included a small-group collaborative component. During the first ninety minutes, students worked individually; their papers were then collected. By a counting-off procedure, students were then randomly assigned to groups of three or four, and during the next twenty minutes, each group gathered at a reserved section of blackboards in order to work collaboratively on the same problems. Students were then given a new, blank copy of the exam,
and during the final thirty minutes, they again worked individually on any problems or parts of problems they wished. To each student's score on the original work, 50% of any improved score from the group work was added.

Whereas the other two exams involved individual work only, they were administered in a computer lab where the students could freely use *Mathematica* for symbolic and numeric calculations as well as for visualization. The first two of the following three questions are from the second mid-term exam, and the third is from the final exam.

2. Let $T: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the linear transformation whose standard matrix representation is

$$A = \begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & 3 & 18 \\ 3 & 6 & 3 & 21 \\ 4 & 8 & 4 & 28 \end{bmatrix}$$

(a) Find a basis of the kernel of $T$.

(b) What is the dimension of the image of $T$? Why? Is $T$ one-to-one? Why or why not?

3. In the following two-species population model, $p(t)$ and $q(t)$ are the two populations' sizes, in hundreds, after $t$ years:

$$\begin{cases} p'(t) = (2 - 1.2q)p, \\ q'(t) = (-1 + 0.9p)q. \end{cases}$$

(a) What, according to this model, is the relationship between the two species—competitive, aggressive, predator/prey, or something else—and why?

(b) Estimate the sizes of the two populations after ten years if initially the two populations' sizes are 1 and 0.5 (in hundreds), respectively.

4. Without using *Mathematica*'s built-in DSolve, find all solutions of the differential equation $y'' - 6y' + 9y = e^{3t}$.

The preceding three questions are not unlike what might be asked on a traditional paper-and-pencil-only exam—except that many traditional differential equations courses would not even reach the topic of nonlinear systems, since so much time would already have been devoted to obtaining exact solutions through symbolic manipulations. In questions 2 and 3, however, the
examinees were able to avoid tedious or error-prone numerical calculations through use of Mathematica; in question 4 they were able to use Mathematica to do most of the symbolic calculations as well as to check their answers.

In-Class Problem-Solving Exercises: On average at least once a week—and most often during one or both of the 75-minute class meetings rather than during the third, 50-minute class meeting—students worked during class on problem-solving exercises. Many of these were straightforward exercises to check, on the spot, understanding of principles and methods that had just been covered in lecture or, less commonly, in the previous class or reading assignment. For example:

1. Prove that the reflection $refl_L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ across a line $L$ through the origin in $\mathbb{R}^n$ is linear. You may use the fact that the projection $proj_L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ onto $L$ is linear as well as the formula $refl_L (\bar{v}) = 2 \text{proj}_{L}(\bar{v}) - \bar{v}$ derived in class.

The purpose of Exercise 1 was to establish linearity by verifying in this instance its abstract characterizing property $T(a\bar{v} + b\bar{w}) = aT(\bar{v}) + bT(\bar{w})$ rather than through concrete representation by a matrix.

2. Find all solutions of the ODE $y' = y/t$. Then find those satisfying initial conditions $y(1) = 2$ and $y(1) = 0$, respectively.

Exercise 2 was intended to reinforce the idea that equilibrium solutions of a separable differential equation had to be obtained separately, before the method of separation of variables could be legitimately applied.

Some in-class exercises were intended to guide the students to discover a principle. For example:

3. Let $A$ be an $m \times n$ matrix. Determine $A\bar{e}_1, A\bar{e}_2$, and, in general, $A\bar{e}_j$ for $j = 1, 2, ..., n$. Here $\bar{e}_j$ is the $n$-vector having 1 as its $j$th entry and 0s elsewhere.

4. Let $a$, $b$, $c$ be constants. For what values of the scalar $r$ is $x = e^{rt}$ a solution of the ODE $ax'' + bx' + cx = 0$?
Results such as those in Exercises 3 and 4 are key facts that are usually derived in lecture by the instructor and in the textbook. The reason for giving such exercises was that the students might better remember these facts, and understand why they are true, by having to obtain the results for themselves. There was a trade-off, however, in that far more class time had to be consumed when the students, instead of the instructor, did the work.

One extended in-class exercise involved a series of problems, with intervening short expository passages, devoted to: defining the complex number field as consisting of ordered pairs \((a, b)\) of real numbers—with suitably defined operations of addition and multiplication; forming the correspondence between pairs of the form \((a, 0)\) and real numbers \(a\); and, seeing that these special complex numbers and corresponding real numbers behave the same way with respect to addition and multiplication. One purpose of this exercise was to tell the “truth” about complex numbers that had been glossed over by the textbook which, like most, treated complex numbers naively as “expressions of the form \(a + bi\)”. A more fundamental purpose was to provide an opportunity in class for guided learning through active reading. Unfortunately, the concepts involved are subtle; students’ puzzled looks, slow progress with the exercise, and questions to me indicated that the students did not, in fact, get the mathematical point of the exercise.

Two of the in-class exercises were effective in unexpected ways. The first was done on the first day of class, and the second at a point two-thirds into the semester.

5. In a coordinate plane, draw a triangle that is not isosceles and is not in any “special” position—for example, does not have any sides or vertices on the coordinate axes. Consider the translation \(T\) of the plane by the vector \((4, -3)\) and the reflection \(S\) of the plane across the y-axis. Construct the images of your triangle under the compositions \(T \circ S\) and \(S \circ T\). Are the two images the same?

Exercise 5 was posed verbally rather than—as was the case with all subsequent exercises—on printed handouts. Following Wattenberg’s practice, I provided each student with a marker along with a sheet of paper and a sheet of plastic printed with coordinate axes. The instructions were to draw the original triangle on the paper and a copy on the plastic; to effect the transformations by suitably manipulating the plastic; and to copy onto the paper the resultant triangle’s position on the plastic. The technical vocabulary of ‘translation’, ‘reflection’, and ‘vector’ was not actually used, nor was the notation \(\circ\) for composition—indeed, one of the purposes of the exercise was to introduce all these notions and corresponding notations. Rather,
the translation was described as “move 4 units rightward and 3 units downward” and the reflection as “flip around the vertical axis”; composition was described in terms of doing first one operation and then the other. The term “image” was described in terms of the set of all the points you get as result.

![Image of triangle under composition of rigid motions](image)

Fig. 2: Images of triangle under composition of rigid motions

The sort of configuration the students obtained is illustrated in Fig. 2, where for clarification I have drawn the original triangle with solid lines but the images with dashed lines. One of the purposes of representing the composites graphically was to allow students to realize more readily that the images are *not* the same—in other words, that composing rigid motions in the plane need not be commutative—without having first to calculate images of the vertices. Much to my surprise and seemingly despite the evidence of their eyes, many students claimed, “The two images are the same.” When asked why, the students volunteered explanations such as, “The images are the same triangle as the original one, but just in different places.” And, in fact, they considered each of the images in this sense as being the “same triangle” as the original one. This reaction revealed a fundamental misconception about points in the plane $\mathbb{R}^2$ and their relationship to cartesian coordinates.

What was the source of the misconception? Perhaps the notion of image was not sufficiently clear from the couple of examples, with single transformations and no compositions, that I had already presented on the board. Or perhaps the students were confused about the
definition of 'same' in this context as meaning 'equal'—if not confused about the very meaning of 'equal'. Students may have been misled by their prior experience with vectors in physics or high school math, where one deliberately regards two directed line segments as being "the same" when they have equal magnitudes and directions, no matter what their locations. Maybe, too, they were misled by the practice in high school geometry of calling line segments or angles "equal" that merely have identical measure. Whatever the source of the misconception, this in-class exercise serendipitously had the effect of informing me immediately about the misunderstanding, so that I could attempt to correct it through explicit explanations and further illustrations of the concepts at issue.

6. When the coroner arrived at midnight, a corpse on the floor had a temperature of in the 85°F in the 65°F room. The corpse was left where it was found as the investigators went about their work. Three hours later, the temperature of the corpse had dropped to 70°F, while the room temperature still remained a steady 65°F. When did the death occur?

Problems like Exercise 6 are "old chestnuts" in differential equations. When the exercise was posed, the class had already learned about Newton's law of cooling from the textbook and in a previous lecture, but only as a model for the decay in time of a freshly poured cup of coffee's temperature. During the exercise, I circulated around the classroom, eavesdropping on group discussions and observing what was being written down. I could tell that most students recognized the relevance of Newton's law, but many were having difficulty deciding how to identify the relevant variables and constants from the given data. The students had been, as usual, encouraged to work in small groups. In one group, a member expressed confusion about whether the corpse's clothes made any difference in temperature, how the corpse's temperature was taken, etc. Another member quickly replied, "But you don't have to worry about all those details. The model takes care of all that...." That insight quickly spread to other groups and helped their members, too, make a substantial start at solving the problem.

The student-to-student interchange that occurred with this exercise represents the kind of thinking one hopes students learn to internalize for their own problem solving. That Newton's model simplifies the complexities of the situation is something that an experienced instructor might explain while presenting the solution as part of a lecture; but, it is something that his students might well miss while busy copying from the board all the symbolic and numerical manipulations of the solution. Having the insight uttered by one student and understood
immediately by others as they concentrated upon solving the problem themselves was a fortuitous instance of catching problem solving in action.

**Collaboration:** The several class sessions held in the computer lab near the semester's start were especially productive instances of cooperative learning. Discussion within pairs was generally very lively, and it was evident that students were helping one another, on matters that ranged from navigating windows on the Macintosh to understanding subtleties of *Mathematica* and *Mathematica*’s representation of motions of the plane. This experience is something I have often seen before with paired work at computers.

On the homework problem sets, students were explicitly permitted and even encouraged to work together—provided they acknowledged their collaboration. Whereas no organization into groups was imposed, the students did work together in a variety of modes—in pairs, small groups, and large groups.

For the collaborative component of the first exam, no group score was derived; rather, each student wrote his own new or revised solutions after the in-group consultations. This scoring strategy was used because the purposes of the collaborative component were: (1) to provide at the exam, to some degree, an opportunity for help from others similar to what was available for homework problems, and (2) to evaluate students' ability to learn from consultation, but not their ability to contribute toward a collaborative effort.

The typical format for the in-class problem-solving exercises was a printed problem, possibly accompanied by introductory text, which was distributed to the class. Students were instructed to read the problem and then to individually attempt to start solving it. After everybody had a chance to think and work alone for some five to ten minutes, students were asked to work in a group with one or two neighbors. At first during the semester, nearly all students required repeated prompting to work in these informal groups; later, entering group mode usually required only minimal prompting.

**Outcomes**

In my Fall, 1998 offering of *Math 245*, the various reforms in general, and the innovations that semester in particular, had outcomes ranging from productive to disappointing.

**Computer technology:** Nearly all students did learn to use *Mathematica* productively, as was evident from homework papers and office-hour interactions. For unknown reasons, some failed
to exploit it sufficiently during the exams where it was available. On homework, most students did not hesitate to use Mathematica or to seek out situations where they could do so. In fact, for homework many students employed Mathematica's mathematical word-processing capabilities to create a product with a polished appearance, despite no requirement that they do so nor evidence that a better grade would result.

Students did not report much use of Wattenberg's website outside class on their own. Why that was so was not ascertained. In any case, what was available there often duplicated what was in the text.

Homework problems: As measured by average homework scores, performance on the ten weekly problem sets was good to excellent for nearly all students (see Table 1).

Students were allowed and encouraged to collaborate on homework problems, but all too often groups of students submitted identical, or nearly identical solutions. This meant that a student who could not solve a problem, or who did not have the time to work on it, copied the solution wholesale from a classmate. Math 245 students' common practice of preparing homework solutions in the form of Mathematica notebooks evidently facilitated plagiarism. Unfortunately, no mechanism was in place—aside from the independent check by means of exams—to ensure that students collaborated responsibly and accounted for their own contributions.

<table>
<thead>
<tr>
<th>Table 1: Mean homework scores (n = 19)</th>
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<tr>
<td>Letter</td>
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<tr>
<td>A</td>
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<tr>
<td>B</td>
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<tr>
<td>C</td>
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<td>D</td>
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<td>F</td>
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Exams: The scores on the first exam, including both the individual and collaborative components, were distributed as shown in Table 2.
Table 2: Scores on first mid-term exam, with collaborative component \((n = 20)\)

<table>
<thead>
<tr>
<th>Letter</th>
<th>Range</th>
<th>Percent</th>
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<tbody>
<tr>
<td>A</td>
<td>85-100</td>
<td>5%</td>
</tr>
<tr>
<td>B</td>
<td>75-84</td>
<td>15%</td>
</tr>
<tr>
<td>C</td>
<td>60-74</td>
<td>20%</td>
</tr>
<tr>
<td>D</td>
<td>50-60</td>
<td>15%</td>
</tr>
<tr>
<td>F</td>
<td>0-49</td>
<td>49%</td>
</tr>
</tbody>
</table>

In view of the results, it would not be surprising that students were not uniformly enthusiastic about the format. In fact, according to a mid-semester questionnaire, the students were about equally divided as to whether they favored such collaboration for their remaining exams. Informally, individual students said that they preferred to have the entire time to work by themselves. Consequently, the collaborative part was abandoned for the subsequent two exams, although now the use of Mathematica was allowed. The results, shown in Table 3, were better.

Table 3: Mean of second mid-term and final exam scores \((n = 18)\)

<table>
<thead>
<tr>
<th>Letter</th>
<th>Range</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>85-100</td>
<td>22.2%</td>
</tr>
<tr>
<td>B</td>
<td>75-84</td>
<td>27.8%</td>
</tr>
<tr>
<td>C</td>
<td>60-74</td>
<td>33.3%</td>
</tr>
<tr>
<td>D</td>
<td>50-60</td>
<td>5.5%</td>
</tr>
<tr>
<td>F</td>
<td>0-49</td>
<td>11.1%</td>
</tr>
</tbody>
</table>

In-Class Problem-Solving Exercises: Overall, the in-class exercises were productive and well-received. Most students worked energetically on them, and many students informally told me they liked doing them. The in-class Exercise 5 concerning rigid motions of the plane and Exercise 6 about modeling a cooling corpse with Newton’s law were especially efficacious. However, taking time for all the in-class exercises interfered with adequately covering in class all topics on the syllabus, and vice versa.

Overall course results: Among the students who enrolled in Math 245 in Fall, 1998, many dropped the course during the two-week no-penalty add-drop period. Of those who remained, only 50% completed the course successfully, that is, with a grade of D or higher, and 42% officially withdrew after the add-drop period. On the whole, this situation—and the results on the first mid-semester exam that in part gave rise to it—was demoralizing to the students and disheartening to me.
Overall course scores were computed with weights of 54% for the average of all three exam scores, 36% for the best 8 of 11 problem set scores, and 10% for class participation. The distribution of course scores and corresponding grades is shown in Table 4. Whereas fully 40% of students still enrolled at the semester's end did earn grades of B or higher, that number represents only 22.2% of the students who remained after the add-drop period and a smaller percentage yet of all who originally enrolled.

<table>
<thead>
<tr>
<th>Letter</th>
<th>Range</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>85-100</td>
<td>20%</td>
</tr>
<tr>
<td>B</td>
<td>75-84</td>
<td>20%</td>
</tr>
<tr>
<td>C</td>
<td>60-74</td>
<td>40%</td>
</tr>
<tr>
<td>D</td>
<td>50-60</td>
<td>10%</td>
</tr>
<tr>
<td>F</td>
<td>0-49</td>
<td>10%</td>
</tr>
</tbody>
</table>

Several Math 245 students did truly outstanding work—and they became teaching assistants in the course the next year. One expects a few students in a typical math class to stand out above the rest; what is notable in the case of Math 245 is that these few did so while enjoying a significantly richer and more challenging experience than they would otherwise have had in traditional courses.

**Explanation of the Outcomes**

There is no evidence that the disappointing results in the Fall, 1998, version of Math 245 can be attributed to the specific innovations introduced that semester. Rather, in view of the actual situation and what students individually said to me about it, the following factors are plausible explanations:

1. *Math 245* included far too much material, especially difficult abstract material, covered at far too fast a pace. By treating all three subjects together and thereby avoiding repetition of topics common to them, *Math 245 & 246* together were designed to cover the material more efficiently. But in Fall, 1998, at the behest of a new, major client department for the course, the year-long syllabus was rearranged so as to cover in the first semester most of the linear algebra and all the ordinary differential equations content.

2. In *Math 245*, students experienced dissonance with their view of what a math course should be. They typically had done quite well in freshman calculus and may have had a fairly easy time...
there, especially since so many had already taken calculus in high school. They were overtly hostile to being required to write prose explaining what they were doing, and they were unaccustomed to working problems with the aim of discovering a principle rather than, as in calculus, demonstrating they understood a method already taught to them.

In this regard, many students had a discouraging experience with their first problem set papers. Interpreting my grading directions too strictly, the graduate assistant who read homework papers was a stickler for complete, coherent write-ups; he gave little or no credit for correct answers that included minimally necessary calculations, but lacked adequate prose explanations.

3. Some students enrolled in Math 245 because, they said, they thought that use of computing would make learning mathematics easier for them. They may have been under the illusion that because Math 245 involved computing, less mathematical thought would be required than in the traditional courses.

4. Math 245 students’ initial hands-on experience with Mathematica at the class lab sessions was discouraging. When multiple users simultaneously accessed the same Mathematica notebook from the lab server, performance of the client-based Mathematica systems slowed to a crawl; saving files to a PC-formatted diskette—which in theory was possible with the Macintosh computers—caused erratic system freezes and, consequently, lost work. (A work-around and, later, a fix, were eventually found.) These frustrations persuaded some students to drop the course.

Concluding Remarks

The overall disappointing results in my Fall, 1998, offering of Math 245 can be attributed in part to a changed client department profile. Previously, the Math 245–246 sequence was merely one option available to all students who would take some sophomore mathematics. For 1998–99, the Electrical and Computer Engineering (ECE) Department—which had previously enrolled few students in the course—asked the Mathematics and Statistics Department to revise Math 245–246 to better serve their own students. The plan was that all ECE majors would take the first semester, but only the electrical engineers would take the second semester. Because my department wanted to prop up enrollment in what had been a dangerously low-enrollment course, I accordingly revised Math 245–246. Most of the more difficult material from linear algebra and everything about ordinary differential equations would now be covered in Math 245, and the less
abstract topics of surfaces and multivariable integration would be shifted to Math 246. This accommodation was made beginning in Fall, 1998.

For Fall, 1998, ECE strongly advised its majors to take Math 245, and many did enroll. But as soon as they faced a rapid pace, demanding workload, difficult and harshly-graded homework assignments, and a discouraging initial experience at the lab, a disproportionately large number of ECE students bailed out.

Evaluating the reforms and innovations in the Fall, 1998, version of Math 245 ought to include comparing the experience there with what happened in preceding and succeeding years. Unfortunately, such comparisons are problematic. Prior to Fall, 1998, enrollment was smaller by roughly a factor of five, and the material was more reasonably distributed in amount and difficulty between the two semesters of the year-long course. Afterwards, in Fall, 1999, ECE majors no longer had the option to defect early. That semester, three lecture sections were offered, of which I taught only one. The other two instructors conducted their classes in a traditional lecture format. While the task of preparing homework problem sets rotated among the three instructors, the syllabus became more rigid and uniform across all three sections.

A premise of Math 245–246 has been that students would be more motivated to learn, and better able to understand, the mathematical ideas if they realized how these ideas arise from solving problems about real-world situations. In point of fact, some of the “real-world” problems were more fanciful than real. And perhaps those, as well as some of the genuinely real problems, were simply not compelling for the clientele in this course—not things the students really wanted to know. Indeed, finding realistic problems in mathematics—especially ones that do not require unduly extended excursions into the field of applicability—is challenging, at least compared with finding such problems in a physical or biological science whose very subject is the real world.

In-class problem-solving exercises are a strategy that I not only would use in other courses, but have—predicated on the condition that the syllabus be more modest in coverage than is typically the case for such courses. Using such exercises in separate calculus, linear algebra, and differential equations courses has been successful, as suggested by positive student reactions and apparently positive effects upon student learning.

To get students to collaborate willingly on in-class exercises, and not just work individually, took repeated reminders and encouragement. By contrast, when students were
paired in front of computers at the start-of-semester lab sessions, they immediately and freely communicated with each other and helped each other learn. For a few in-class exercises, students were allowed and encouraged to use the two computers available in the room; given the size of the class and the classroom layout—with the computers deployed in a front corner of the room—that arrangement was awkward. These considerations suggest that an ideal way to teach a mathematics course exploiting computer technology would be in a “workshop-classroom” where students work in small groups each near its own computer, but where the entire class can readily assemble for student or instructor presentations to the entire class.

This tale of Math 245 is cautionary. My experience there indicates that active-learning strategies can be quite useful, but that they are unlikely to fulfill their promise unless the instructor is realistic as to how much can be covered. In the end, less may be more!

Acknowledgments
Frank Wattenberg conceived and originally implemented the integrated sophomore course described here. To him I am indebted not only for demonstrating the possibility of application-motivated mathematics instruction, but also for specific guidance in using his written and electronic materials. Kenneth Levasseur suggested the adage “Less is more.” Howard A. Peelle generously gave an earlier draft of this paper a critical reading; STEMTEC reviewers James A. McDonald and Richard Yuretich made useful suggestions for improvement.

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Bio
Since 1965, Murray Eisenberg has taught at the University of Massachusetts Amherst, where he is Professor of Mathematics and Statistics. His Ph.D. is from Wesleyan University. His main mathematical interest is the topology of dynamical systems. In recent years, he has studied cognitive and pedagogical issues of using computer programming to teach mathematics. He has published articles on topological dynamics, topology, the APL and J programming languages, and the use of computing in teaching math; he is the author of three advanced undergraduate texts. For more than twenty years, he has directed his department’s undergraduate computer lab and used computers to help students learn mathematics.
References


