THINKING ABOUT GEOMETRY: LAYING A FOUNDATION FOR FUTURE K-8 TEACHERS

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Introduction

In 1997, the University of Virginia (UVA) joined the NSF-funded Virginia Collaborative for Excellence in the Preparation of Teachers (VCEPT) and led the Collaborative's effort in thinking about geometry courses that would be most appropriate for future K-8 teachers. Working with other VCEPT institutions, we centered our discussion on two basic questions: What parts of elementary geometry are important for K-8 students and why are they important? What types of experiences will future teachers need in preparation to teach K-8 geometry? In this article, I will discuss a geometry course for future K-8 teachers that grew out of this effort. Similar courses were implemented at other VCEPT institutions.

The philosophy of our course is grounded in our conclusion that elementary and middle school geometry is learned primarily by doing and questioning. Our course turns on exploratory hands-on activities; building, cutting, and looking for patterns and structures; activities designed to help the student develop spatial sense, an understanding of spatial structures, and visualization skills. The course's content and format, together with several illustrations of the activities, are discussed here. I have tried to present the material and my views with a minimum of educational terminology, and in a manner that is accessible to all interested parties including, especially, other college mathematics faculty. The discussion begins with a personal look at some specifics separating the history and content of K-8 geometry from the primary strand, arithmetic, in the traditional school mathematics curriculum.

Kenneth Hoffman, after spending years representing mathematics and mathematics education in Washington D.C., often illustrated an essential difference between science education reform and mathematics education reform by drawing attention to the fact that educational change is most difficult when the subject of reform is familiar to a wide audience. He observed that, until recently in the United States, almost all adults had experienced a highly standardized

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K-8 mathematics education focused on arithmetic skills. In contrast, few adults studied substantial amounts of science in the elementary grades. The omission of science is now recognized as having been a mistake, and as a nation we are quite open to innovative teaching and reform in our science classrooms. But, we all studied mathematics and most of us believe that we know what mathematics is and how it should be taught. We tend to be very suspicious of a change that leads toward mathematics instruction philosophically different from that which we experienced in our childhoods. We react in this way in spite of overwhelming evidence that the old methods of mathematics instruction only succeeded with a small minority of students—and they did not succeed with many who oppose reform in mathematics.

The history of K-8 geometry instruction is closer to that of science than to arithmetic. While the subject of geometry is old, except for a few area and perimeter formulas and some instruction on the use of rulers, geometry and measurement were always neglected in the elementary and middle school curricula. As with science, educators generally view this neglect as having been a mistake and an increased emphasis is being placed on geometry. We are, however, starting with very little history in geometry. In K-8 geometry, there are no hard American beliefs or traditions based on previous experiences. At this moment, we are quite free to think through geometry instruction fresh from the ground up. Our generation of mathematics teachers and educators is, in essence, inventing a new subject, or at least a new curriculum. Now and perhaps for a few years while a new norm is precipitating out, we have great latitude to consider geometry's content, rationale, and pedagogy. This is an unusual opportunity that can benefit future generations of students.

As we consider geometry and teachers' needs, two recent documents from the professional communities, the National Council of Teachers of Mathematics (NCTM) 2000 *Principles and Standards* [1] and *The Mathematical Education of Teachers* [2], are especially important. They lay solid groundwork, but they are not complete. Serviceable answers to questions of this type never are. Local circumstances and resources play a large part in determining what is possible. Answers also depend on our changing understanding of the roles of geometry in the broader world, and depend on our knowledge of the concepts children must assemble to construct geometric skills and our understanding of the ways children learn geometry. Regrettably, this knowledge is rarely found where it is needed; especially in college mathematics departments and even among mathematics educators. Over the last five years, I worked to develop my own thinking on these topics. This article and the course it describes provide a snapshot of my present understanding. The next section contains a few personal thoughts on geometry and observations on how we learn it.

History of Geometry and Geometry Education

Basic geometry and measurement concern our physical world. The roots of the Greek word "geometry" translate as measurement of the earth: *geo* means earth or world, and *metry* means to measure. This aspect of geometry predates and underlies the more abstract subject of Euclidean geometry. It is, or should be, the essential core of school geometry. The ancient Greeks credited the Egyptians with the origins of the subject. The Nile's yearly floods created an annual need to survey the land; and, Egypt also had a large government whose construction projects, ranging from irrigation canals, temples, and monuments, to office buildings, generated other needs for applied geometry. The early Egyptian geometer's tools included measuring sticks or granite "cubit rods" with chiseled "digit" marks, knotted ropes for measuring longer distances, plumb bobs for establishing vertical directions, squares for constructing perpendicular lines, and compasses for duplicating lengths and constructing circles. Dilke's *Reading the Past* contains an attractive account of this, together with photographs of ancient measuring implements [3].

Egyptian geometers had a strong working knowledge of area and volume calculations and were competent with much of the geometry used today by surveyors, architects, and carpenters. These practical applications gave rise to the development of geometric concepts that are the essential prerequisites for abstract Euclidean geometry. The hands-on work of developing these worldly concepts in children remains the prerequisite for success in their later study of geometry; prerequisites that have been largely ignored for centuries in our schools. How this happened appears largely to be an historical accident.

Geometry entered Europe's historical consciousness after first being filtered through two of history's most influential, successful, and sophisticated philosophical schools: Plato's *Academy* in Athens and the *Library* in Alexandria. These schools were the world's great centers of learning and research. They were unlike today's schools, but it is quite reasonable to view them as ancient analogues of today's research universities. Euclid worked at the *Library* in Alexandria and wrote his great text *The Elements* at the end of the fourth century B.C. It was a scholar's book; an advanced text written for scholars. *The Elements* became Europe's only geometry text for the next two millennia. But a great historical irony occurred as Europe emerged from the Dark Ages and Euclid was rediscovered. *The Elements* became the only source on the subject and all prerequisites disappeared from sight. History's great graduate text became our schoolchildren's text, and the high school geometry book that I used in 1955 was only slightly altered from the first volume of today's standard edition of Euclid [4].

Learning Geometry

The use of Euclid maintained the rigor of geometry, but the origins of the subject with its rich applications to the physical world and all pedagogical considerations disappeared from the schools. The face of geometry became abstract, advanced, and removed from daily life. Children suffered and until very recently, geometry was a subject where only the best and most dedicated students succeeded. Middle school children were not exposed to mathematics that could prepare them for success in high school geometry because we did not recognize that such preparation was necessary or possible. A great gap separated students and the almost sacred text. Only in our times, when research began to illuminate how children learn geometry, did it become possible to glimpse the damage that was being done by traditional geometry instruction [5-7]. Our UVA course attempts to bring the preparation of teachers in line with the type of instruction we now believe is needed for children to learn geometry.

Based on the work of the van Hieles and their successors, we now know that as children and adults learn geometry and measurement, they progress through a developmental sequence in which they piece together their understanding of the elements of geometry: elements like units, dimensions, arrays, angles, area, and congruence. The developmental sequence is largely dependent on the students' experiences; experiences in which they build, measure, and solve problems. Children develop spatial sense and their understanding of geometry by constructing their own (mental) spatial structures that are superimposed on space. All these structures depend on geometric experiences, and the experiences must fit with the students' development. When needed experiences are left out, parts of the developmental conceptual development does not occur. The conceptual gaps become blind spots for students that can remain forever. Typically, these blind spots are not eliminated with formal instruction where students memorize vocabulary, definitions, and theorems, but only with experiences that build the missing concepts and spatial understanding.

It is as if the gaps are missing rungs on a ladder that a student is trying to climb. When too many rungs are missing, the student cannot proceed. This ladder of understanding becomes the true intrinsic prerequisite for success in geometry. This is, in fact, true in a very strong sense.

The subject's logic and language at one level are unintelligible to students at lower levels of development. Using this vocabulary of intrinsic prerequisites, it has been observed that students possessing the prerequisites are typically successful in geometry, and those without the prerequisites typically fail. If our schools have not taught the prerequisites for high school geometry, it is not surprising that few students master the subject. One successful high school teacher of many years' experience recently told me that she has never taught a geometry student that mastered and enjoyed the geometry course in the same manner that these students did with other advanced mathematics courses. Teachers must understand these issues, and they must be able to build activities into their curriculum that allow students to build their own conceptual ladders.

Examples of Missing Rungs

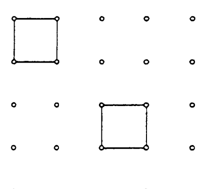
The above remarks are difficult to understand in the abstract without concrete examples. The following examples are taken from my experiences teaching informal geometry to undergraduates at UVA. The examples illustrate both typical activities in the course, and problems that students must cope with when they are missing rungs in their conceptual ladder. The students that I refer to here are all talented and dedicated, and have successfully completed a high school geometry course. Their academic achievements were sufficient to earn admission to an elite, highly selective university, but they often did not understand the simplest geometric concepts: concepts like squares and rectangles. Before I taught the informal geometry course at UVA, neither I nor others in our mathematics department knew these education gaps existed. The examples provide compelling evidence that for many, successful completion of a high school geometry course has been a farcical experience. They have been swindled.

To help students learn of the many conceptual pieces that go into a mature understanding of concepts like volume and area, we present a relatively complete developmental development of area [8]. The missing ingredient in our development is time; in part because of college instructional schedules, and in part because we originally assumed that our students have a solid foundation in school mathematics, and that their primary need is for them to experience how the pieces fit together. The original area sequence included activities where:

- areas of simple, differently shaped regions were compared;
- students were asked to construct a rectangle with the same area as that of an irregularly shaped region;
- students found areas of simple and complex polygons on Geoboards and drawn on square dot paper;
- students dissected and recomposed regions to develop standard area formulas and the Pythagorean Theorem; and,
- students both compared and measured the areas of irregularly shaped regions.

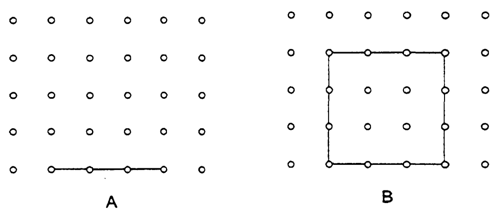
Example One — My first surprise discovery was that many students have never engaged in meaningful activities with squares and rectangles. More precisely, their experiences with geometry were so shallow that they did not lead to a usable understanding of squares and rectangles. These students possessed no precise information about these shapes that can reliably be called upon when measuring or drawing a square or computing its area. Some had never used a ruler.

One particular activity starts with a few line segments drawn on square dot paper. For each given segment, the students are asked to construct a square with the segment as one edge. They are then asked to find the area of the square. The purpose is to develop an understanding of area, but visualization and elementary construction skills are the essential tools. I assumed, without consciously expressing the thought, that all students would be able to see that the dots on the paper outlined small squares,



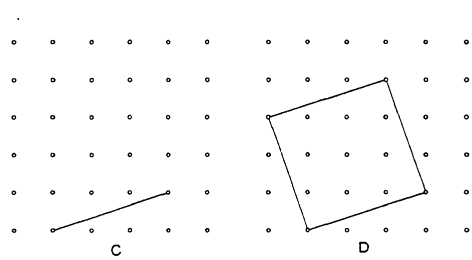
that these squares cover the plane in a regular array, and that the squares could be used as units for measuring area. My assumptions were naïve. Most students owned most of this picture, but their knowledge was not always structured in a usable way, and in some there were very surprising gaps.

When the line segment on the sheet looked like that in figure A, then the students quickly produced the drawing in B.



In this case, there was a high probability that they would give the correct answer for the area of the square, but a few would consistently count the dots rather than the spaces and reach the conclusion that this is a 4 by 4 square rather than a 3 by 3 square.

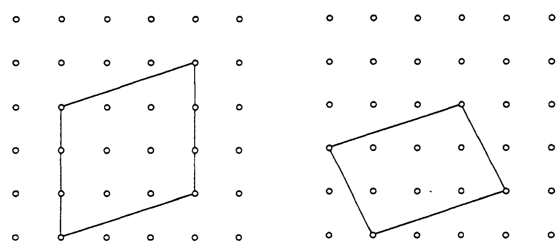
The next step raised the level of difficulty more than I imagined. This time there was a segment like that in figure C, for which I expected the answer drawn in D.



I anticipated that the students' responses would be governed by knowledge that squares have equal angles and equal sides. If they knew this, then even if they could not see the pattern for constructing the square, I felt that trial and error would lead to the correct solution. Eventually the pattern, "*When the bottom edge goes over three spaces and up one space, then the right hand side edge will go up three spaces and over one*" would emerge. In any case, since algebra students have been taught the rule for the slopes of perpendicular lines, they could resort to using this fact as a last resort. I thought they would be able to use this to find the square algebraically, even if they did not see the geometric pattern. Finally, I expected that the most visually talented students would see the square as having been rotated slightly about its center, and for them the rule should follow. This was not the case. Not at all.

Typical students would have some initial trouble, but would discover how to construct the triangle in a few minutes. They used the anticipated trial and error approach with little or no analysis, but they became visually adept at finding the square's missing sides. In discussions, none mentioned perpendicular lines or slopes. One highly exceptional student mentioned the rotated square idea.

The real surprise came with students, often as many as 15% at UVA, who drew figures like those in E and F.

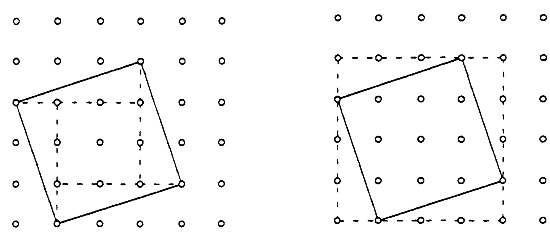


E F

These students turned out to be members of a large segment in our society for whom the sentence, "This is a square," means approximately that, "This figure looks much like other figures that we call squares." For these students, squares do not have sharp mathematical definitions and properties, but "square" is a fuzzy concept. Squares are "squarish" with more or less equal angles and more or less equal sides that are more or less straight. Some adult students do not recognize squares when the bases are not aligned with the edges of the paper. The students had passed geometry courses, but there was a giant gap between what they were thinking and what we thought we were teaching! For them, after years of instruction, squares had not entered the domain of precise mathematical argument.

As the course progressed, observation showed that these individuals were not lacking mathematical talent, but they lacked the habits of precise mathematical thought and the rich experiences from informal geometry that build precise geometric concepts. In personal interviews, some students stated they had never built things with blocks or tiles, or hammer and nails. Others had never used a ruler or a protractor. They were truly geometrically deprived, and the other students, the successful students, were not far ahead! They had not had experiences that taught things like the practical importance of right angles and perpendicular lines. Without this background, no understanding of right angles and the squares of mathematics was possible beyond that of an abstract definition to try memorizing for the test. In fact, they did not understand that the mathematics they had been taught depended on the properties of squares. In my square problem, they looked for the square that I desired, and when it proved to be difficult to find, they settled for something that looked sort of like a square. They never saw a reason why it might matter.

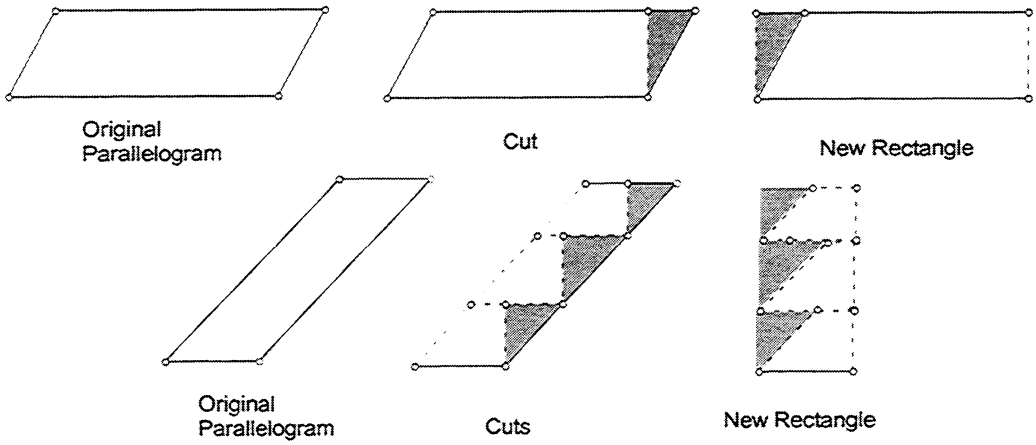
Other gaps were disclosed in the students' area calculations. By this time, the students had repeated experiences with dissection problems, and most found the correct area by using one of the dissections indicated in figures G and H.



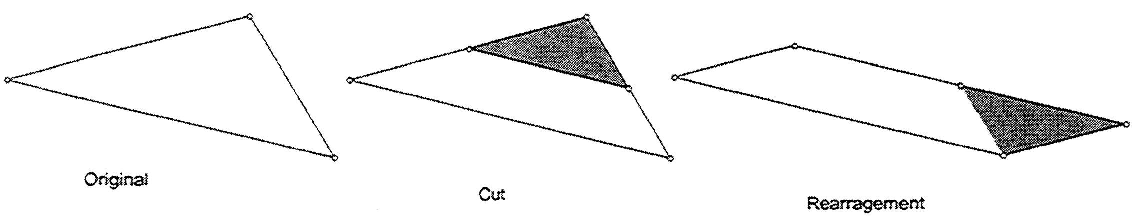
G H

In figure G, the area is shown as the area of the small inscribed square of area 4 squares together with the areas of 4 triangles, each of area 3/2 squares, while H shows the area to be from the large circumscribed square of area 16 squares by removing 4 triangles of area 3/2 squares each. These students basically understood the structure of the grid paper, but those that did not produced a variety of mistakes, including several in which the area was reported to be 9 squares because the *length* of the segment was perceived to be 3 units.

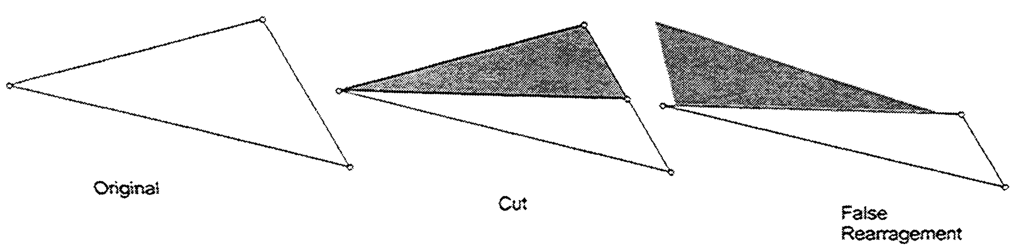
Example Two — A second set of area dissection problems asked students to cut simple regions into pieces and reassemble them to make other simple shapes. For example, making a rectangle out of a parallelogram leads to the area formula for a parallelogram. Here are two appropriate cuts for this purpose.



A similar, but slightly more advanced problem is to cut a triangle into two pieces and reassemble it into a parallelogram. The standard cut and rearrangement for this is illustrated in the next three figures. Midpoints of two opposite sides are joined and the tip cut off and joined to either end of the resulting trapezoid.

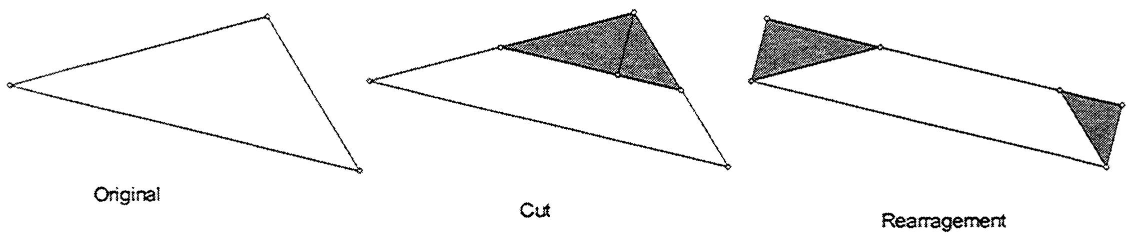


Students, and the reader, are asked to justify that the resulting figure is a parallelogram. This problem also has its own standard false solution:



The point to observe in this figure is that, while the resulting figure resembles a parallelogram, upon close examination it is clearly not one. The example again exhibits that students who have not had sufficient experience to construct an appropriate mathematical understanding of a parallelogram, are ready to accept as a solution anything that looks close.

Example Three — Finally, I will mention the problem of cutting a triangle to make a rectangle. Here is one solution.



In this problem, I have observed UVA undergraduates using a completely non-mathematical approach by repeatedly cutting off bits and pieces until something roughly rectangular is achieved. In the most extreme case of this that I have seen, the figure that was submitted as a rectangle had, in fact, nine sides.

Lessons Learned

Geometry's roots are in the physical world, and for most students its importance must be understood in these terms. Elementary geometry is the focus of our work when we are learning to make sense out of the physical/geometric world of shapes, solids, and space. These activities are dominated by a few fundamental ideas and the relationships between them: length, area, volume, angle, congruence, similarity, and symmetry. The importance of an informal mastery of these topics can hardly be overemphasized. They give the student a set of powerful skills and tools that are referred to as spatial sense, but might also be termed the geometer's eye. With them, students see spatial structures that are invisible to those without these skills. These skills are of great utility in many technical occupations ranging from heating technician to engineering. The same skills are important to artists, movie animating technicians, craftsmen, and furniture movers. They are also the foundation for the insights that geometry students need to understand definitions, theorems, and proofs.

Through the observations discussed in the last section and the research on how children learn geometry, it becomes clear that most of the future teachers that I have taught will not be prepared to teach their students to see spatial structures and relationships without considerable work. For them to become effective, a special geometry course is needed that will teach them the content, but more importantly, the course must have the power to transform future teachers into mathematical thinkers. Within their minds, the topics of elementary geometry, such as squares and area, need to be transformed into subjects where mathematical precision and analysis are applicable.

With a coherent set of geometric activities, future teachers can develop their own geometric concepts. But teachers require more than a superficial understanding. Their experiences must prepare them to create rich, geometrical classroom environments that can serve their students' geometric needs. In earlier times, rural environments and physical activities both at home and in the crafts, provided many children with opportunities to learn basic geometry outside the classroom, but very few children have comparable experiences today. Generally, the needed experience can only be found in the classroom, and for this reason there is a special need for geometry-rich environments in our schools. Geometry is a natural subject to integrate with other parts of the school curriculum, and a geometry course for future teachers must prepare teachers for this task.

Our UVA course is our first attempt to address this issue. The course has been popular with education students. These students learn considerable amounts of informal geometry. Through their experiences, they gain knowledge of the developmental sequence for learning geometry, and the use of hands-on activities with physical objects in geometry instruction. The unanswered question is whether the course is sufficient. The course moves students in the right direction, but does it effectively fill the geometry gaps that our students bring to college? ■

References

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Appendix A
(Syllabus Outline)

- Experiences and learning geometry: the van Hiele model;
- Length: comparisons and measurement;
- Angles: comparisons and measurement;
- Area: comparisons and measurement, explorations; comparisons, dot paper, Geoboards, cutting and recomposing, formulas, informal units, Pythagorean Theorem, similar figures, irregular figures, perimeter and area;
- Volume: comparisons and measurement, explorations; cutting and recomposing, informal units, surface area and volume of familiar solids and irregular solids, similar figures;
- Shapes: sorts, and properties;
- Analysis: angles, and parallel lines;
- Analysis: triangles, decomposing and recomposing, angle sums;
- Analysis: geometry on the surface of a balloon;
- Regular polygons;
- Constructions: with paper folding, with compass and straightedge;
- Symmetries: reflections with paper folding, mirrors, and MIRA's;
- Transformations and symmetries in 2 dimensions;
- Symmetries in 3 dimensions;
- Tessellations.