

MAKING UPPER DIVISION MATHEMATICS COURSES RELEVANT FOR PRE-SERVICE TEACHERS

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Abstract

This article addresses the disconnect that in-service and pre-service secondary school teachers feel between the material presented in upper division mathematics courses and high school classroom practice. Two examples are given from an abstract algebra course in which this problem is addressed.

The Vertical Disconnect

“How has your classroom practice been affected by the abstract algebra course you took in college?” I’ve asked this question to several groups of high school teachers in Oregon over the past few years, getting responses that range from laughter to groans. I’m not surprised at these responses. Teachers often say that abstract algebra has nothing to do with their teaching because they never talk about groups, rings, fields, and the like. Although many upper division mathematics courses have a large number of prospective teachers, seldom do university mathematics faculty connect the material in advanced mathematics courses to high school level material. In an article addressed to research mathematicians, Al Cuoco of the Center for Mathematics Education at the Education Development Center calls this phenomenon the “vertical disconnect.” He writes:

Most teachers see very little connection between the mathematics they study as undergraduates and the mathematics they teach. This is especially true in algebra, where abstract algebra is seen as a completely different subject from school algebra. As a result, high school algebra has evolved into a subject that is almost indistinguishable from the precalculus study of functions. Another consequence is that, because individual topics are not recognized as things that fit into a larger landscape, the emphasis on a topic may end up being on some low-level application instead of on the mathematically important connections it makes. [1]

It is widely recognized that prospective teachers can benefit from taking advanced mathematics courses such as abstract algebra. In *The Mathematical Education of Teachers*, the Conference Board of the Mathematical Sciences recommends that prospective teachers take courses in abstract algebra and number theory in order to more fully understand the mathematical structures that underlie algebra and number systems [2]. In the NCATE Mathematics Program Standards, we find the recommendation that pre-service teachers “understand and apply the major concepts of abstract algebra.” [3] Zazkis recommends that prospective teachers study unfamiliar number systems and algebraic structures to encourage them to “reconsider their basic mathematical assumptions and analyze their automated responses. [These types of activities] constitute an essential tool for the development of critical thinking in mathematics teacher education.” [4] She claims that, “Working with non-conventional structures helps students in constructing richer and more abstract schemas, in which new knowledge will be assimilated.” Dubinsky claims that “constructing an understanding of even the very beginning of abstract algebra is a major event in the cognitive development of a mathematics student” and that this course is critical in developing prospective teachers' attitudes toward abstraction [5].

Although abstract algebra can be of great benefit to prospective teachers, it does not always fulfill that promise. According to Usiskin, undergraduates do not automatically recognize that the material they study in an abstract algebra course provides underpinnings for high school algebra [6]. These connections are rarely made by university faculty, and students are left to rely on their high school algebra experiences, experiences which are “likely to have been focused on an algorithmic approach to mathematics and unlikely to have contributed to conceptual understanding.” [7] But the problem goes beyond a failure to make these connections as researchers find that “many who are to be ambassadors and salespersons for mathematics at the secondary level develop a negative attitude towards mathematics in general and a fear of abstraction.” [8]

This article presents two case studies of methods I have used, and continue to use, to address the “vertical disconnect” in an abstract algebra course at the University of Portland. This work was supported by the Oregon Collaborative for Excellence in the Preparation of Teachers (OCEPT).

Work Sample Collaboration

In their senior year, our prospective secondary school teachers are required to prepare and implement a “work sample”—a series of lesson plans addressing a coherent unit of study. Usually these work samples are supervised by a faculty member in the School of Education who gives valuable input on lesson design and classroom management issues, but does not have the expertise to give guidance on the mathematical content. This is an excellent opportunity for mathematics faculty to play a role in bridging the “vertical disconnect.”

Angie Mai was a student in my upper division, *Abstract Algebra* course at the University of Portland in 2002. Her student teaching responsibilities prevented her from taking the second semester of the course, so we arranged a “directed study” class. The purpose of this class was for Mai to incorporate the knowledge, methods, and point of view she had been learning in *Abstract Algebra* to her student teaching, specifically to her work sample. Along the way, we looked at many connections abstract algebra has to high school algebra. After considering many topics, we settled on the complex numbers. The complex numbers had come up in the *Abstract Algebra* class in a variety of ways, providing examples of groups, rings, and fields. In chapter nine of *The Mathematical Education of Teachers*, we find the following observation: “It is important for prospective teachers to understand how most extensions of the number system, from natural numbers through complex numbers, are accompanied by new algebraic properties, and why the field axioms are so critical for arithmetic.” [2] In parallel to Mai’s development of her work sample, I guided her through an investigation of Hamilton’s quaternions. Throughout the project, we found many interesting parallels between her learning about this extension of the complex numbers and her high school students’ discovery of the complex numbers as an extension of the real numbers.

Mai’s placement was in three sections of “Advanced Algebra” (*Algebra II*) and an introduction to complex numbers was a normal part of the curriculum for that course. The textbook presented a brief and largely unmotivated presentation of addition and multiplication of complex numbers. We felt that with the deeper understanding of the complex numbers we had gained in class and with the perspective on mathematics afforded by the abstract algebra class, we could do a better job.

A reservation I often hear when I talk to teachers or mathematicians about connecting abstract algebra and high school algebra is that such a project could only benefit the top students.

This has most definitely not been my experience in the past and it has not been Mai's experience in this project. By the time she started to teach the complex numbers material, Mai was quite familiar with the skill levels and abilities of her students. Across all three classes, her observation was that even, and sometimes especially, our approach to the complex numbers benefited the “weaker” students.

An Overview of the Lesson

The discussion below gives a brief outline of two weeks of lessons. The questions were presented in class discussions and investigated in small groups working on carefully designed worksheets and through homework assignments we designed. The students also worked their way through the material in their textbook, making the appropriate connection to our approach. One of the main points of this article is that often the point of view of abstract algebra (or any advanced mathematics course), more than the actual theorems and definitions, can make real contributions to teaching high school topics. As you read through this example, think about how generalization is used, how axioms play an important role, and how the students are put in the role of researchers with the question: “How can we build a number system that is even bigger than the real numbers?”

Our approach to the complex number is geometrical and relies on a generalization from the real numbers. The students review the creation of the real numbers in stages, each new number system being motivated by the solution of a new kind of equation (e.g., the rational numbers are needed to solve $3x = 5$). Even with the real numbers in hand there are equations we cannot solve (e.g., $x^2 = -3$), but from a geometric point of view we are at a dead end—the number line is filled up completely! Students were able to suggest a possible solution: move off the line and look at the plane. This gives rise to our basic problem: “How do we turn the points in the plane into a number system?”

To make progress, we are guided by our geometric point of view and the generalization from the real numbers. Using this geometric point of view, a real number (a point on the number line) is specified by its (signed) distance to the origin. When we make our generalization to points in the plane, we need to take into account the distance to the origin, but distance to the origin is not enough information to specify a point. What other information is needed? By responding to a question like, “I live five miles from here—what else do you need to know to get to my house?”

students discover that direction is the missing ingredient. In this way, we develop the polar description of points in the plane, denoting the point at distance r and angle θ by the symbol $r@ \theta$.

We decided to focus on discovering the definition of complex multiplication. Students generalize from their knowledge of multiplication of real numbers, both positive and negative. From our new point of view, we know how to multiply points in the plane if their angles are 0° or 180° . Through worksheets and homework assignments, students conclude that we need to multiply the lengths of our new kind of numbers, but what to do with the angles is a bit of a mystery at this point. Students translate previous knowledge that “negative times negative is positive” to a new situation, finding that when they multiply two numbers with angle 180° we get a number with angle 0° . For example, the real multiplication fact $(-2)(-3) = 6$ becomes the complex number fact that $(2 @ 180^\circ)(3 @ 180^\circ) = (6 @ 0^\circ)$. Students find themselves left with a choice: “Should we add the angles or subtract them?”

We decided to answer this question with a classroom debate. Students chose sides (“Add the angles” vs. “Subtract the angles”) and were given homework assignments to help them prepare for the debate. The class was divided into two teams facing each other: angle addition vs. angle subtraction. Each team took turns presenting an argument and teams were allowed to huddle and offer each other help before presenting an argument. When a team called a huddle on its turn, both teams were allowed to huddle for up to thirty seconds. Once the thirty seconds were up, the team who called the huddle presented an argument or used advice from Mai as a wild card (but only twice). To ensure equal participation, each student had to speak before another student was able to speak again. At the end of the debate, students were asked to choose which operation they believe is the correct operation in a silent secret ballot. Angle addition won the day.

The Role of Abstract Algebra

More than the actual theorems and methods of abstract algebra, we found ourselves using more general ideas and points of view from that course. In this section, we will highlight those contributions.

Investigation: Playing the Researcher — Inquiry- or discovery-based learning is popular now in many fields of study, but many of the examples of this pedagogy tend to lead students along a carefully prepared path with carefully prepared steps. Often lacking is a real sense of investigation of the sort experienced by researchers. This kind of experience is typically

introduced in upper division mathematics courses where students “get a taste of research.” But that kind of investigative activity is available at all levels. In our treatment of the complex numbers, the students definitely don't know what the answer is and instead of laying out the steps needed to get to the “right” answer, we emphasize the same kind of tools and strategies that researchers (in mathematics and many other fields) actually use. The first of these is generalization.

Generalization as Guide to Discovery — We treat the complex numbers as a generalization of the real numbers. Starting from the perspective that we want to multiply points in the plane, we discuss what form would be most appropriate. In this decision, we are guided by generalization. How do we multiply points on the real line? We just find out how far they are from the origin and then multiply those numbers. So when we represent points in the plane, distance is important. If you know the distance from a point to the origin, what else do you need to know to locate that point? If someone tells you they live five miles from here, what else do you ask them to find out where they live? Direction! In this way, we arrive at our $r@θ$ notation for points in the plane. This part of the lesson is more strongly guided by questioning than the next phase in which we decide how to multiply complex numbers.

Where should we look for guidance when we ask, “How should we multiply two points in the plane?” The principle of generalization says that whatever definition we come up with had better agree with our method of multiplication of real numbers, so we can use that information to guide us in our exploration. As we saw above, this gives us some information, but doesn't quite decide the matter. We need another perspective from abstract algebra.

Axioms Decide the Debate — In abstract algebra, we study many structures defined by axioms: groups, rings, fields and the like. The perspective is that these axioms are a guide to useful mathematics and should not be easily sacrificed. Many of the arguments in the “great debate” were based on preservation of the axioms we have depended on in our study of integers, rational numbers, and real numbers. In deciding for adding angles over subtracting angles, the crucial arguments were, “Do you really want to give up a property like $ab = ba$ or $(ab)c = a(bc)$?”

Payoff — After we come up with our definition of complex multiplication and check that our definition corresponds to the algebraic definition, it's time to reap some benefits. As I say many times in our abstract algebra course, whenever there is more than one way to look at a given

question good things can happen. The first benefit of the geometric definition is that the complex number i appears naturally—in fact, the students discover it on their own. What else can we do with our new perspective on complex multiplication? In their textbook, the students learn the $a + bi$ definition of complex numbers and their operations. Are there questions that are easier to answer with our different (yet equivalent) definition? The students explore these questions mainly in homework, where they discover more general roots as well as the various “roots of unity” that our abstract algebra class studied as well. Could we have done this using the algebraic definition? Possibly, but if you try you will find yourself quickly in difficulty. The connection between these two perspectives is the addition formulas for sine and cosine, but that connection will have to wait for a later class.

In-Service Workshops

A wonderful opportunity to connect upper division mathematics material to high school teaching practice comes in the form of in-service workshops for high school teachers. I design workshops that build connections to abstract algebra for in-service teachers and I use students in my upper division classes as teaching assistants. The workshop is very much activity-based with the high school teachers working in small groups. My abstract algebra students are expert in the topic and use the questioning strategies of the Peer-Led Team Learning model as they assist the teachers [9].

An Outline of the Workshop

The workshop I will outline takes material my students were learning about symmetry groups, specifically dihedral groups, and connects this material to standard topics in the high school curriculum: properties of functions, matrix multiplication, and trigonometric identities.

The workshop begins with each teacher learning to use a “dihedral calculator” (Figure 1). This is simply a regular hexagon made of posterboard with labeled vertices on the front and back. We investigate and list all the possible symmetries of this object, finding that there are twelve - six counterclockwise rotations of $0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ$ which we call R_0, R_1, \dots, R_5 , respectively (the subscript representing the number of “clicks” we rotate counterclockwise) and six “flips” over the six lines of symmetry, which we call L_1, \dots, L_6 as shown below.

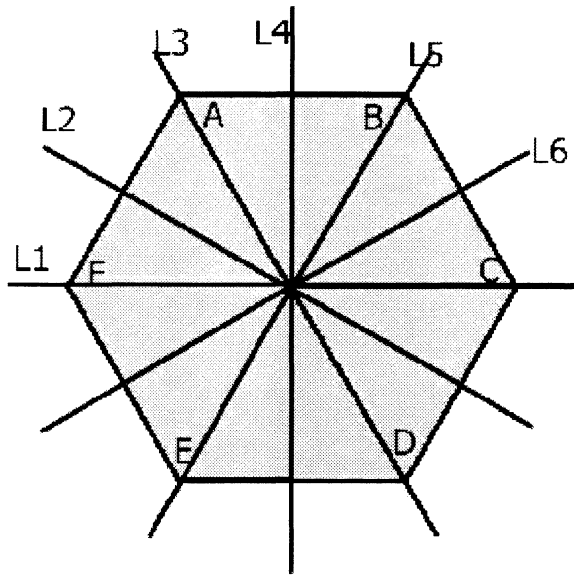


Figure 1

The workshop begins with the high school teachers working their way through some exploratory exercises with composition of symmetries. For example, in Figure 2 below, the hexagon begins in “standard position,” then the symmetry L_4 is performed interchanging left and right. This is followed by the symmetry L_5 and the teachers observe the composition, $L_5 \circ L_4$ is equivalent to a 300° counterclockwise rotation. After some discussion about which order to write this in, we agree on

$$L_4 * L_5 = R_5$$

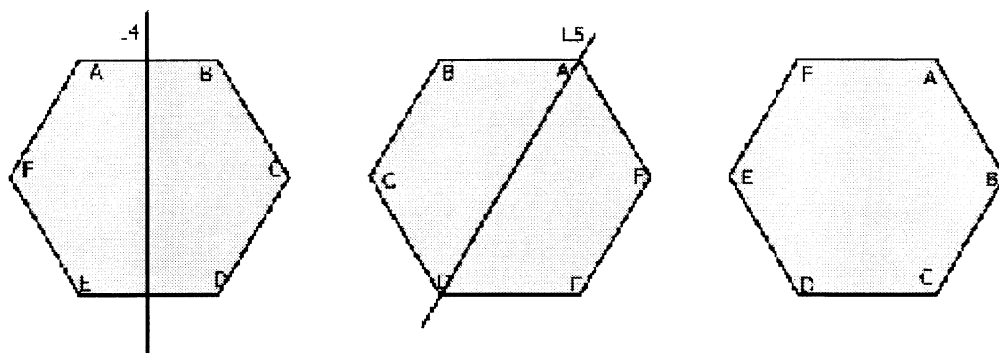


Figure 2

From this point, we begin to investigate different ways to represent our “dihedral calculator” symbolically. For example, using permutations, we can represent the symmetry R_5 as the permutation

$$\begin{pmatrix} A & B & C & D & E & F \\ B & C & D & E & F & A \end{pmatrix}$$

Through a guided set of activities, investigating this way of looking at symmetry pushes on the teachers’ understanding of basic concepts such as function, onto, one-to-one, inverse function, and associative and commutative laws. Usually, one of the teachers will recall the definition of an abstract group, and there will be some discussion about how this concept was taught in their abstract algebra course.

Finally, we look at matrix representations of these functions. Many, but by no means all, of the teachers are able to represent the transformation R_1 as a matrix. After first introducing a coordinate system, the teachers use trigonometry, to find that the rotation R_1 can be represented as:

$$R_1 = \begin{pmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{pmatrix}$$

and more generally, a counterclockwise rotation through an angle α is given by

$$\begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$$

This way of representing a rotation is familiar to some, but new to many of the teachers and we do several examples of applying these transformations to vectors until everyone is comfortable with the concepts. What is new to most of the teachers is the connection to sum of angle formulae for trigonometric functions. We begin by investigating the composition $R_1 \circ R_1 = R_2$ and comparing it to the matrix calculation

$$\begin{pmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{pmatrix} \begin{pmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{pmatrix} = \begin{pmatrix} \cos 120^\circ & -\sin 120^\circ \\ \sin 120^\circ & \cos 120^\circ \end{pmatrix}$$

The equality of this matrix multiplication is equivalent to the double angle formulas for sin and cos. Further experiments with other R_i and rotations through general angles reveal the connection to the general sum formulas for sin and cos.

Benefits for In-Service and Pre-Service Teachers

Although the workshop is ostensibly for the in-service high school teachers, I see the greater benefits accruing to the pre-service abstract algebra teachers. The in-service teachers are usually very happy to see these connections between material they currently teach and a class from college they had supposed was irrelevant to their teaching. However, in a one-day workshop, there is little chance this experience will actually make a long-lasting difference in their classroom teaching practice. For my abstract algebra students, there are more substantial benefits. They begin to build the expectation that the material they are exposed to in upper division courses should connect to high school mathematics topics. They begin to ask questions of me and of themselves about these connections. As they help the high school teachers through the material, many useful conversations occur in which the teachers relate actual classroom situations in which the concepts addressed in the workshop arise.

It is interesting to note the connection between the material on complex numbers that Mai presented to her *Algebra II* classes and the material presented above that could be part of any high school trigonometry class. If we had thought of the hexagon as being part of the complex plane, we could have represented the rotations as multiplications by a complex number. For example, the rotation R_1 could be represented as multiplication by the complex number $1 @ 60^\circ$. The connection to the trigonometric addition formulas comes about by writing the multiplication of complex numbers in polar form and in rectangular form, and comparing the answers. Although none of the high school teachers thought to do this, they might well have if their abstract algebra or complex analysis professors at college had bridged the “vertical divide.” ■

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