# THE PERFECT PERSPECTIVE: A MATHEMATICAL ANALYSIS OF PERSPECTIVE USING TOOLS AVAILABLE TO MIDDLE SCHOOL STUDENTS

W. GRIFFIN
*Liberty Middle School, Hanover County Schools*
*Ashland, VA 23005*
wgriffin@hanover.k12.va.us

## Abstract

This paper examines the basic properties of perspective drawings, the history of perspective drawings, and the basic mathematics of perspective. Using a side view and a top view of a three-dimensional projection, similar triangles can be used to find distances from the axes and vanishing point in a projection. By breaking the three-dimensional projection into two, two-dimensional planes, one can recreate projections based on actual figures, or create placements of figures in real space based on a projection. Using this method, one can change a projection based on the changing position of the vanishing point. This simple approach to perspective makes it accessible to students of different ability levels, as well as creating a strong connection between art and mathematics.

## What is Perspective?

We do not often look at a photograph or a realistic painting and think of it as being a projection of our surroundings. However, this is precisely what a picture is. In a sense, it is flattening our world so that we can carry it with us. Both mathematicians and artists agree that perspective is representing the three- dimensional world in which we live on a two-dimensional plane [1,2].

Linear perspective assumes that the world exists behind a flat rectangular pane of glass. It is a simplification of how we view the world in that it relies on fixed rather than constantly shifting viewpoints and on straight lines to a vanishing point rather than the curvilinear ones that exist [2].



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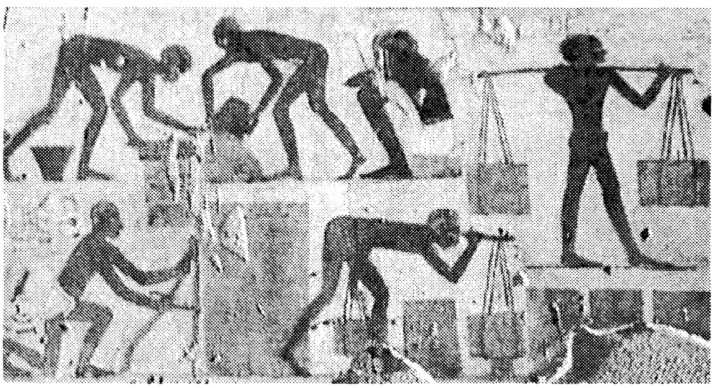
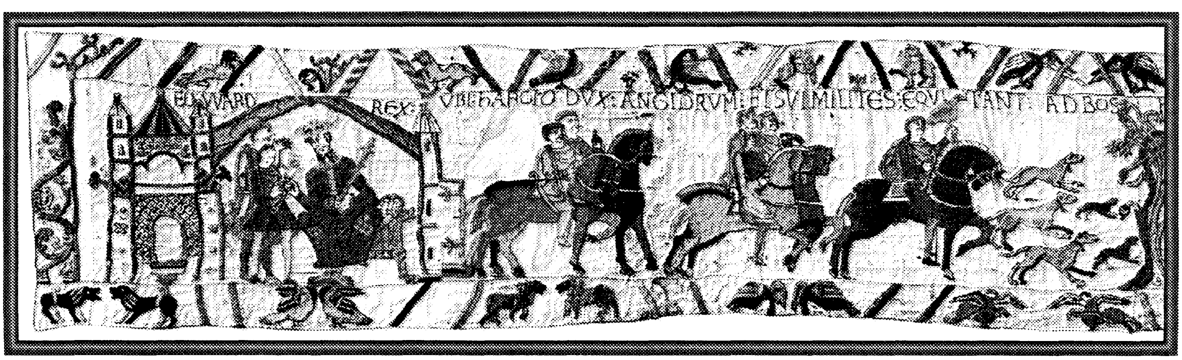
## Brief History of Perspective

Perspective as we know it has evolved over thousands of years. In early civilizations and then for thousands of years following, artists often portrayed all of the objects in a picture as being the same size [2]. This early strategy made paintings very easy to understand and clear. When size was introduced initially, it was used to emphasize importance rather than relation to other objects. For example, the focal point of a painting would be the largest part of the work regardless of its position in respect to the other objects.



This Egyptian record depicting life in the Nile Valley does not use perspective. Notice the men in the background and foreground are the same size. [3]



The Bayeux Tapestry dating back to the 11th century depicts the Battle of Hastings. Notice how the king is much larger than anything else in the tapestry portraying his importance. [4]

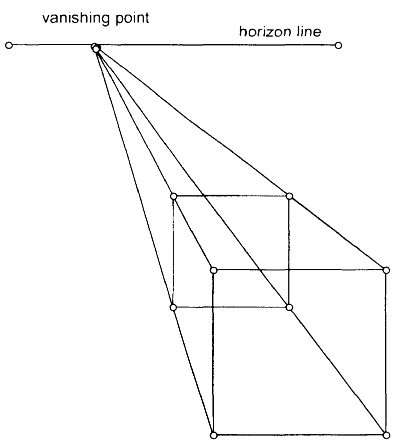
In the late thirteenth century, a mindset of scientific naturalism began to take hold. These ideas of naturalism gave birth to perspective in art [2]. This study of perspective was mastered during the Renaissance period. During this time, artists were not only experimenting with perspective, but rules for the subject were being developed as well. The father of perspective was not an artist or a mathematician for that matter, but rather a goldsmith named Filippo Brunelleschi [2]. He produced a couple of works done with accurate perspective, but he is best known for his design of buildings. None of his writings on perspective exist today and, in fact, it is possible that he may not have written anything in his time on the subject.

The first actual writings that we have today were done by a learned humanist named Leon Battista Alberti (1404-1472) [2]. As artists perfected this method on canvas, many artists and mathematicians alike created and perfected the method of finding true perspective.

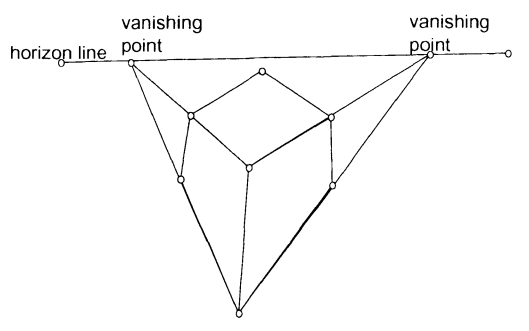
**Different Scenarios of Perspective**

The horizon line or eye level line in a perspective drawing is the line where the sky meets the ground or the line that the eye falls on naturally when looking straight at the drawing [1]. Somewhere on that line, you will usually find the vanishing point or points in the painting.

The vanishing point is the point in which any two or more lines of the painting converge in the plane [1]. As illustrated in the simple diagrams below, you can see that there can be one vanishing point. When there is only one vanishing point, the drawing has one-point perspective. Another scenario illustrated below shows two vanishing points. When a painting has two or more vanishing points, it is said to have two-point perspective.




One-Point Perspective

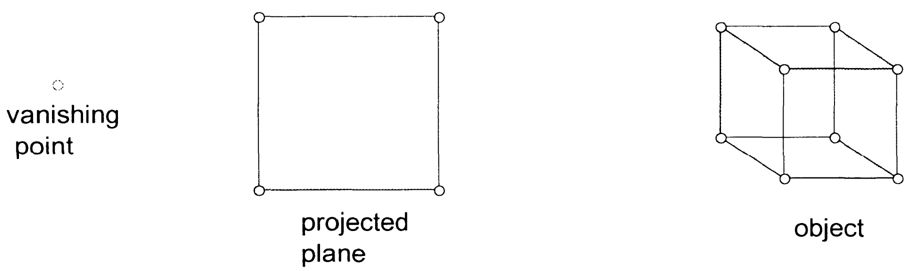



Two-Point Perspective

**Finding the Projected Coordinates**

In my research, I have chosen to focus on one-point perspective paintings. Although this problem can be solved using the equation of lines in three-space and where they intersect different planes, I have chosen instead to focus on how to solve the problem using mathematics accessible to middle school students [5]. Although the process has many steps, curriculum could be developed from this research that could lead students through the method. The majority of the mathematics used in this project includes visualizing in three-space, and using proportions to find the lengths of corresponding sides of similar triangles.

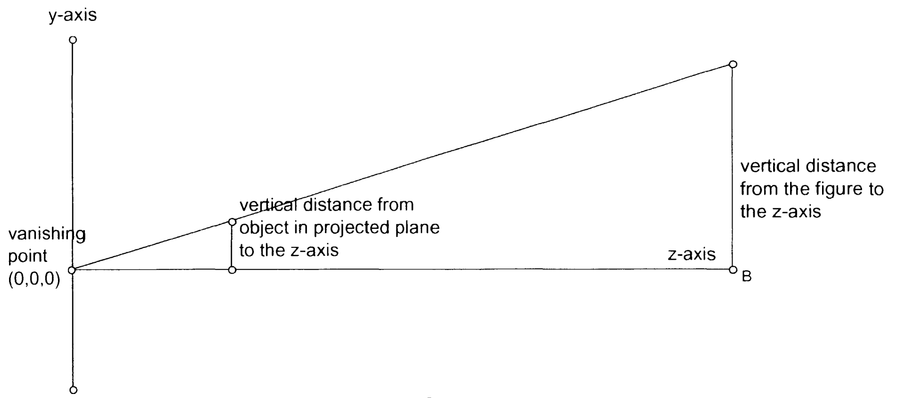
In my initial problem, I had a rectangular prism that I wanted to recreate using one-point perspective. I knew the size and position of the prism; I chose the position of the vanishing point, and the position of the projection plane. My task was to find the coordinates of the cube in the projection plane. I first placed my prism on graph paper and found the three-dimensional coordinates of each of the six vertices. Since I could set the vanishing point to be anywhere, I chose to place it (0,0,0). I then chose to place my projection plane between the object and the vanishing point.




Since I was thinking and graphing points in three-dimensions, I needed to find a system to simplify the process. I chose to look at the setup from two separate two-dimensional planes and then combine them to get the three-dimensional coordinates. In order to do this, I followed a few steps:

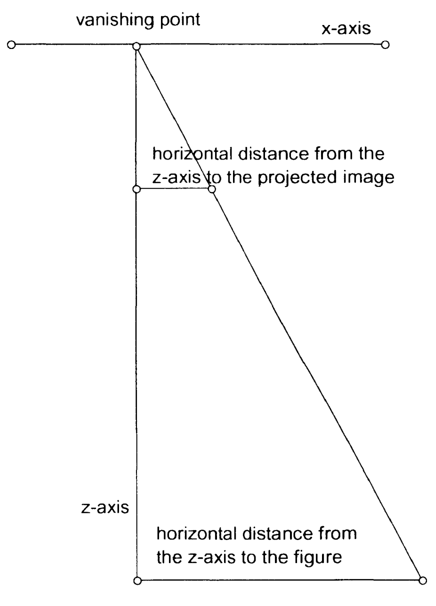
1. Draw the side view of the setup. The base of the triangle is the distance from the front of the figure to the vanishing point. The triangle then has a leg that extends the height of the front of the figure. The hypotenuse is a segment extending from the top of the face of the prism to the vanishing point. This view

will allow you to find the y- and z-coordinates. (As usual, $z$ is the distance from the vanishing point toward the object. The vertical distance is y and $x$ measures the horizontal distances parallel to the object.)




Side view

2. Draw the top view of the setup. The vanishing point is one of the vertices of the triangle and the other two vertices are found on either side of the front of the object. It is very likely that you will need to draw an altitude in this triangle.




3. Use similarity ratios to find lengths of each segment.

Similar triangles have corresponding angles that are congruent and corresponding sides that are proportional. The triangles we are looking at are similar because they have the same angle that has a vertex at the vanishing point. Parallel lines form the other two angles in each triangle. Since corresponding angles cut by parallel lines are congruent, we know that we have three congruent angles in our triangles. Because we have congruent angles, we know that the triangles must be similar, thus allowing us to set up proportions to solve for missing side lengths.

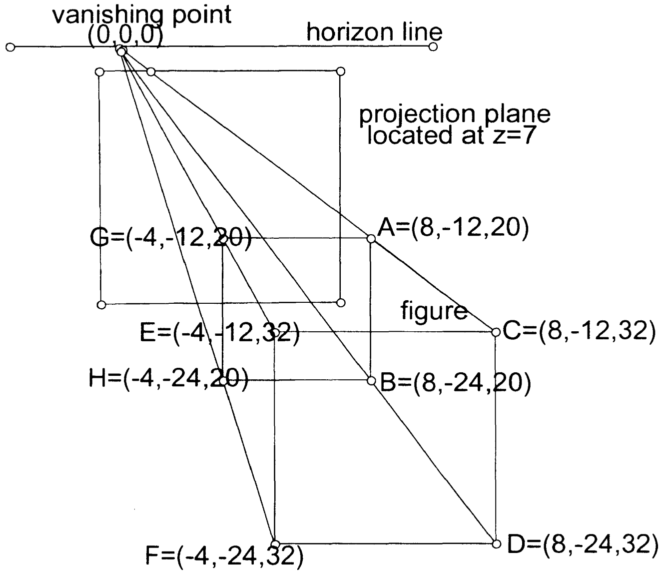
In our situation, we have drawn a perpendicular line through the vanishing point that goes through the projected plane and the real position of the figures. We can measure perpendicular distances from this line to any point in the projection. We also know how far the projection is from the vanishing point because we set these two positions. We are now in a position to set up our proportions.

$$\frac{\text{Distance from the perpendicular line to the projected y-value}}{\text{Distance from the perpendicular line to the figure's y-value}} =$$

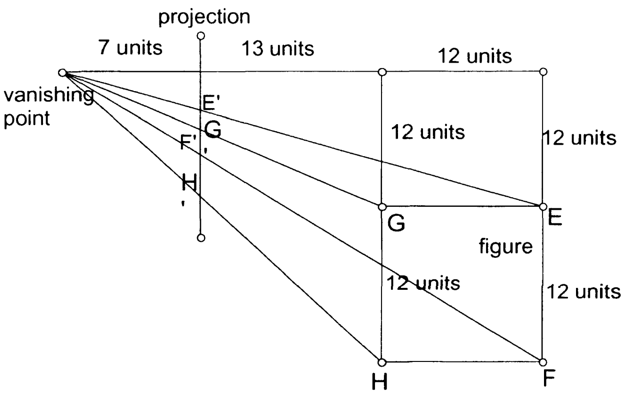
$$\frac{\text{Distance from the perpendicular line to the projected x-value}}{\text{Distance from the perpendicular line to the figure's x-value}} =$$

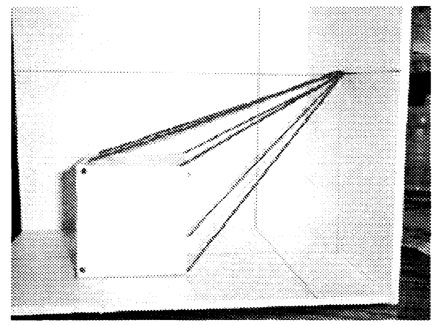
$$\frac{\text{Distance from the vanishing point to the projected z-value}}{\text{Distance from the vanishing point to the figure's z-value}} =$$

**Example 1**

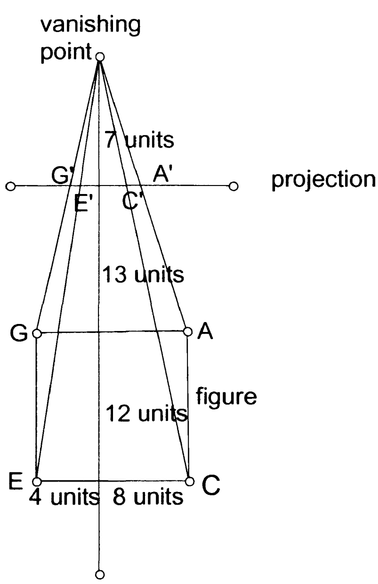


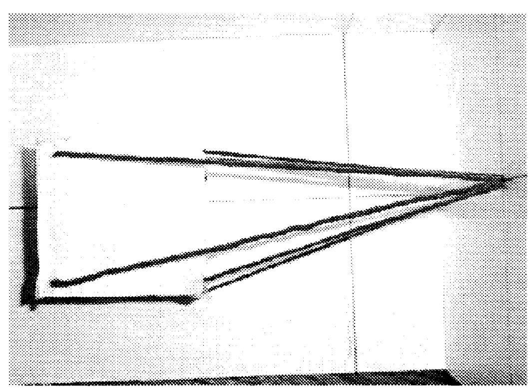

Step 1: Draw the side view of the entire projection.



Step 2a: Draw the top view of the projection.



Step 2b: Draw the top view of the projection.

Step 3: Use proportions of similar triangles to find lengths and positions of points.

Let's first find the coordinates of G'. Looking at the side view, we can determine the y- and z-values in our projection. The z-value from the vanishing point to the projection plane is 7 units long. The z-value from the vanishing point to the side of the box is 20 units. We also know that from the y-axis to the top of the box is 12 units. We do not know how far it is from the y-axis to the position of the top of the box in our projected plane. We can set up a proportion to find that distance.

$$\frac{7}{20} = \frac{G'}{12} \qquad 63 = 20G' \qquad 3.15 = G'$$

Since we are finding the distance of this segment from the y-axis, the value that we got is the actual coordinate. We only have to check to see if it is above or below the axis and in this case, it is below so the y-coordinate for this point is -3.15. Using this process, we can find all of the y-coordinates in this projection.

| Projection points | y-coordinates | z-coordinates |
|---|---|---|
| A' | -3.15 | 7 |
| B' | -8.4 | 7 |
| C' | -1.97 | 7 |
| D' | -5.25 | 7 |
| E' | -1.97 | 7 |
| F' | -5.25 | 7 |
| G' | -3.15 | 7 |
| H' | -8.4 | 7 |

Our diagram includes the points E',F',G' and H', but since we are working with a cube that is positioned parallel to the projection plane, we can reason that there are two points that share the same y-values. Using this deduction, A', B', C', and D' can be found.

Notice that all of the z-values are the same. This is indicating that the projection is contained in one plane that runs perpendicular to the z-axis.

Once we have the y- and z-values, we just repeat the above process using the top view to find the x-values.

To find A', we first find the perpendicular distance from the vanishing point to the projected plane which is 7 units. Then, we find the perpendicular distance from the vanishing point to the point on the figure which is 20 units. The next distance that is needed is the perpendicular distance from the z-axis to the point on the figure which is 8 units. This distance is unknown in the projection.

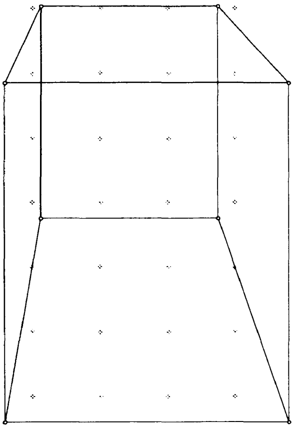
$$\frac{7}{20}=\frac{G'}{4} \qquad \frac{7}{20}=\frac{A'}{8}$$

$$28=20G' \qquad 56=20A'$$

$$1.4=G' \qquad 2.8=A'$$

Again, we need to observe whether this distance is to the right or left of the x-axis and in this case, it is to the right making the coordinate positive. We can repeat this process to find the rest of the coordinates in the projection.

| **Projection points** | **x-coordinates** | **y-coordinates** | **z-coordinates** |
|---|---|---|---|
| A' | 2.8 | -3.15 | 7 |
| B' | 2.8 | -8.4 | 7 |
| C' | 1.75 | -1.97 | 7 |
| D' | 1.75 | -5.25 | 7 |
| E' | -.875 | -1.97 | 7 |
| F' | -.875 | -5.25 | 7 |
| G' | -1.4 | -3.15 | 7 |
| H' | -1.4 | -8.4 | 7 |



This is what the projection would look like in our example.

In our case, these distances are very easy to find since we have set the vanishing point to the origin. We can use the given coordinates rather than having to subtract two coordinates to find the distance. If the vanishing point were not the origin, then you would have to subtract the line's position from the position of the point in question. It is also important to remember that you are working with one specific triangle at a time since the similarity ratio will vary for different triangles. The distances that you are finding can then be added or subtracted from the axes to find the coordinates of each point.

**Finding the Coordinates of the Figure**

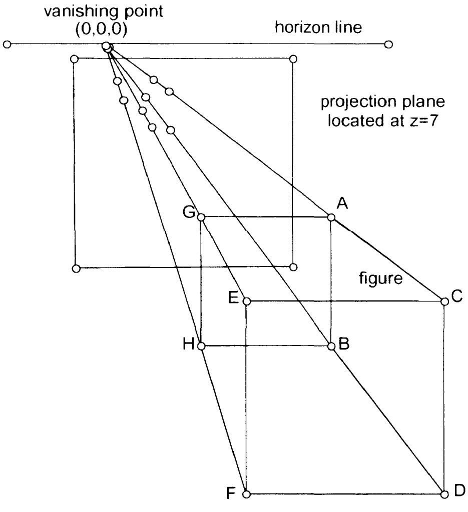
When you are given a projection, you can find the coordinates of the figure, as they would be positioned in real life. First, you must set the distance from the vanishing point to the projection. It is logical when working with a painting as a projection to use an arm's length for this distance since the vanishing point in this instance would be the artist's eye, the projection his or her canvas, and the objects being painted would be behind the canvas. Once this point is set, you must also know the real length of one of the objects in the projection.

To find the vanishing point of the picture, find the intersection of the major lines of the painting. Once you find this point, you can use it as the origin and draw the x- and y-axis on the projection plane. You can then measure distances perpendicular from these axes and use the set z-value that corresponds to the placement of the projected plane to find the coordinates of each point. Using these points, the origin and the information that you know about the figures, you can find real-life placements from the painting.

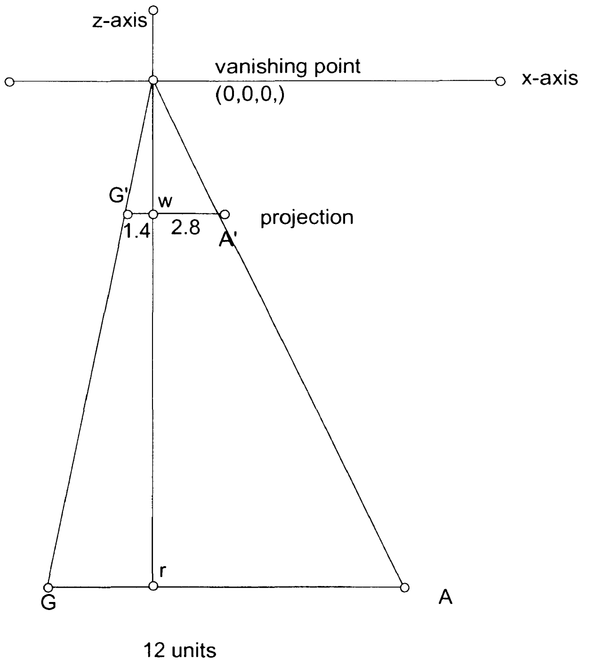
**Example 2**

Using the same prism as above, we will find the position of each of the coordinates in the real figure given only the projected coordinates and the knowledge of the distance between two of the points in real space.

| Projection points | x | y | z |
|---|---|---|---|
| A' | 2.8 | -3.15 | 7 |
| B' | 2.8 | -8.4 | 7 |
| C' | 1.75 | -1.97 | 7 |
| D' | 1.75 | -5.25 | 7 |
| E' | -.875 | -1.97 | 7 |
| F' | -.875 | -5.25 | 7 |
| G' | -1.4 | -3.15 | 7 |
| H' | -1.4 | -8.4 | 7 |




Step 1: Graph a top view of the points in the projection plane.
Step 2: Sketch a top view of the position of points in real space and fill in any information given.




Step 3: Set up proportions to find missing distances.

Use the distance formula to find the distance from the *VP* to *G'*,

$$d = \sqrt{(-1.4)^2 + (-3.15)^2 + 7^2}$$
$$d = \sqrt{60.8825}$$
$$d = 7.803$$

7.803 units is the distance from *VP* to *G'*.

Using that distance, find the distance from the *VP* to *G*.

$$\frac{\text{Distance from } A' \text{ to } G'}{\text{Distance from } A \text{ to } G} = \frac{\text{Distance from } VP \text{ to } G'}{\text{Distance from } VP \text{ to } G}$$

$$\frac{4.2}{12} = \frac{7.803}{x}$$
$$4.2x = 93.636$$
$$x = 22.29$$

22.29 is the distance from *VP* to *G*.

Use the distance found to set up another proportion.

$$\frac{\text{Distance from } VP \text{ to } G'}{\text{Distance from } VP \text{ to } G} = \frac{\text{Distance from the } z\text{-axis to } G'}{\text{Distance from the } z\text{-axis to } G}$$

$$\frac{1.4}{x} = \frac{7.803}{22.29}$$

$$7.803x = 31.206$$
$$x = 3.999$$

3.999 is the distance from the z-axis to *G*.

If the distance from the z-axis to *G* is 3.999, then the distance for the z-axis to *A* is 12-3.999=8.001.

Use the distance found to set up another proportion.

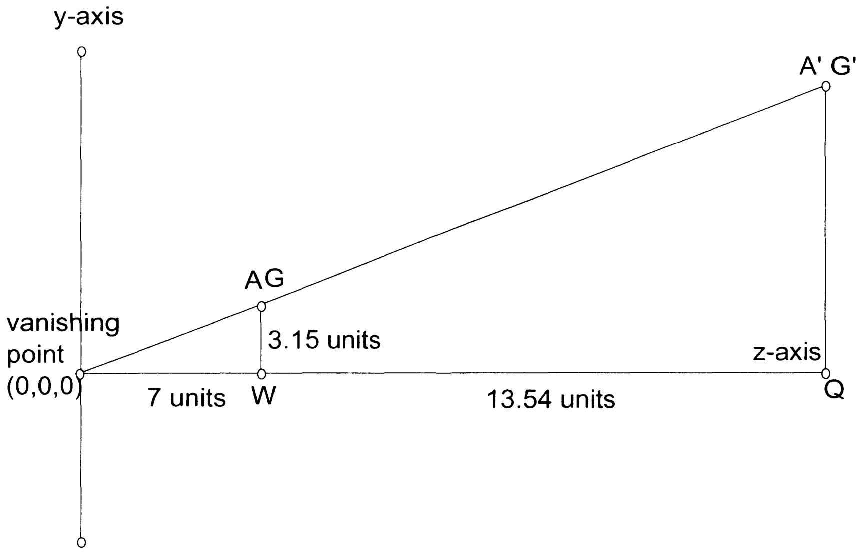
$$\frac{\text{Distance from } VP \text{ to } w}{\text{Distance from } VP \text{ to } r} = \frac{\text{Distance from } VP \text{ to } G'}{\text{Distance from } VP \text{ to } G}$$

$$\frac{7}{x} = \frac{7.803}{22.9}$$
$$160.3 = 7.803x$$
$$20.54 = x$$

The distance from the vanishing point to *r* (adjacent point on the z-axis) is 20.54.

Therefore, the known coordinates for *G* are $x = -3.999$, $z = 20.54$, and the known coordinates for *A* are $x = 8.001$ and $y = 20.54$.

We now need to set up the side view of the projection. This is easier than the first calculations because we already know the distance from the vanishing point to the figure.




Side view

Notice that from the side view, A and G are at the same height. One portion will find the missing length.

$$\frac{\text{Distance from } VP \text{ to } W}{\text{Distance from } VP \text{ to } Q} = \frac{\text{Distance from } W \text{ to } A'}{\text{Distance from } Q \text{ to } A}$$

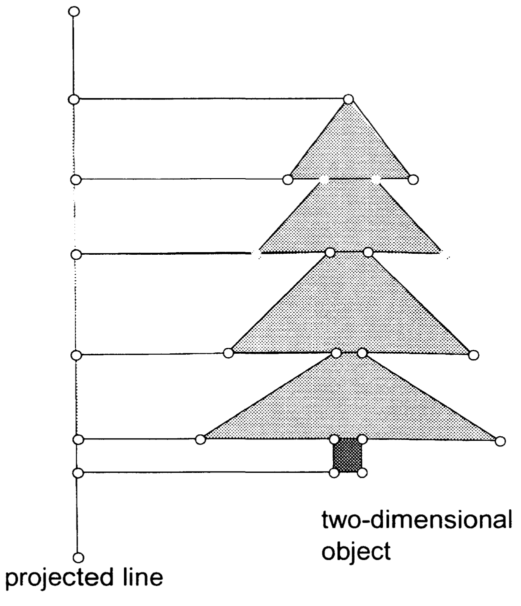
$$\frac{7}{20.54} = \frac{3.15}{x}$$
$$7x = 64.701$$
$$x = 9.234$$

The distance from $Q$ to $A$ is 9.234 units.

Therefore, the ordered triples for the two points are $A = (8.001, 9.243, 20.54)$ and $G = (-3.999, 9.243, 20.54)$.

We can repeat this process to find all of the coordinates in this case because we are working with a cube. The cube creates a special case. The above process will vary slightly for each new situation based on what is known, but the process and technique remains essentially the same.

When objects are projected from two-space to one-space, certain features are lost. The same is true when a projection is made from three-space to two-space. With the transition from two-space to three-space, many placements can be made as they originally were in the figure. This reverse projection is a tool that can be used to recreate positions, but not to reproduce every detail of the original object.




When the tree outline to the left is projected to one-space, the projection becomes a line. It is impossible to recreate the original tree based only on the projected line.

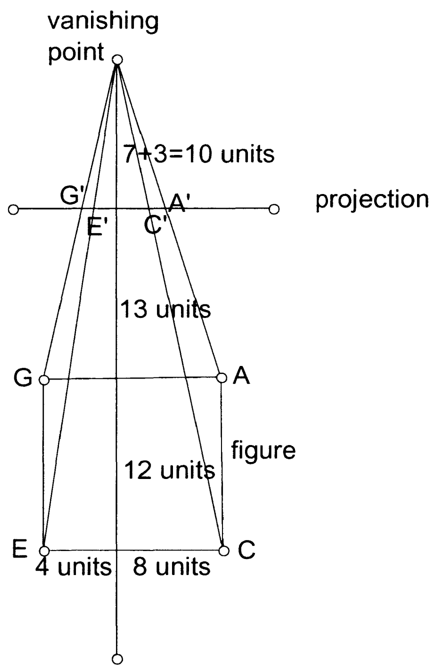
**From a Different Point of View**

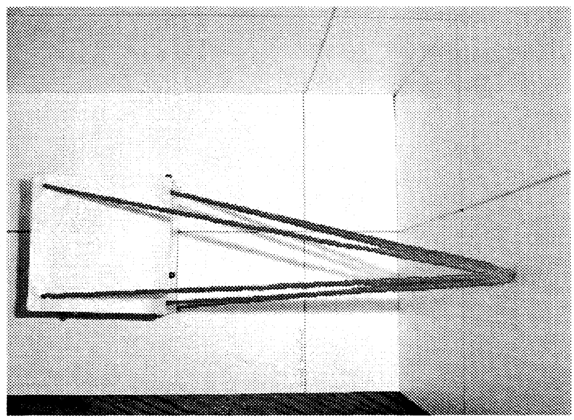
Once we know the real position of the figures in the projection, we can change the vanishing point and create a new projection plane. This will alter what the picture looks like in the projection, but will still hold true for the properties of perspective. The process will work exactly as it did in Example 1 except the vanishing point is not at the origin so additional calculations will be needed to get coordinates.

**Example 3**

We will use the same figure and projection plane as in the first example, but change only the vanishing point to (5,10, -3).

Step 1: Draw the top view of the projection.



Step 2: Use proportions to find missing distances.

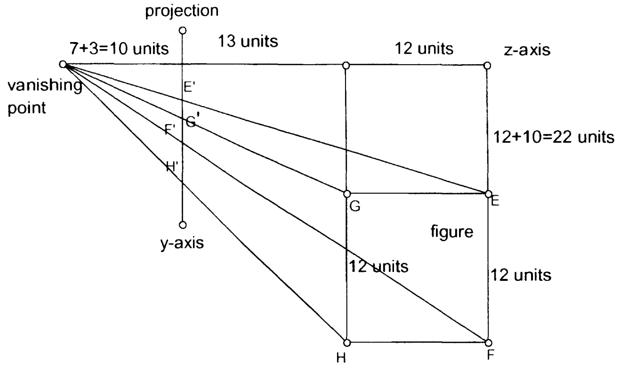
$$\frac{10}{23}=\frac{A'}{8} \qquad \frac{10}{23}=\frac{G'}{4} \qquad \frac{10}{35}=\frac{C'}{8} \qquad \frac{10}{35}=\frac{E'}{4}$$

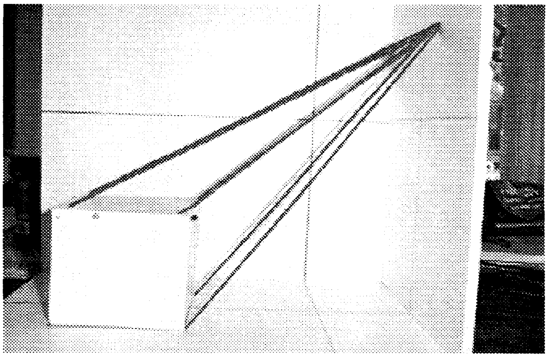
$$80=23A' \qquad 40=23G' \qquad 80=35C' \qquad 40=35E'$$
$$3.45=A' \qquad 1.74=G' \qquad 2.29=C' \qquad 1.14=E'$$

This is not the distance from this obj
to the x-axis, but rather to the line tl
runs through x=5. To get the
coordinates, you must add and subtra
values from 5.

The x-coordinate for A' and B' is 5+3.45=8.45
The x-coordinate for G' and H' is 5-1.74=3.26
The x-coordinate for C' and D' is 5+2.29=7.29
The x-coordinate for E' and F' is 5-1.14=3.86

Step 3: Draw the side view of the projection.



Note that these distances represent not the y-coordinates, but rather the distance from the perpendicular line running through y=10. In order to get the coordinates, these distances need to be added or subtracted from 10.

*Notice that each y-coordinate corresponds to two points in our figure.

Step 4: Use proportions to find the missing distances.

$$\frac{10}{23}=\frac{A'}{22} \qquad \frac{10}{23}=\frac{B'}{34} \qquad \frac{10}{35}=\frac{C'}{22} \qquad \frac{10}{35}=\frac{D'}{34}$$

$$220=23A' \qquad 340=23B' \qquad 220=35C' \qquad 340=35D'$$
$$9.57=A' \qquad 14.78=B' \qquad 6.29=C' \qquad 9.71=D'$$

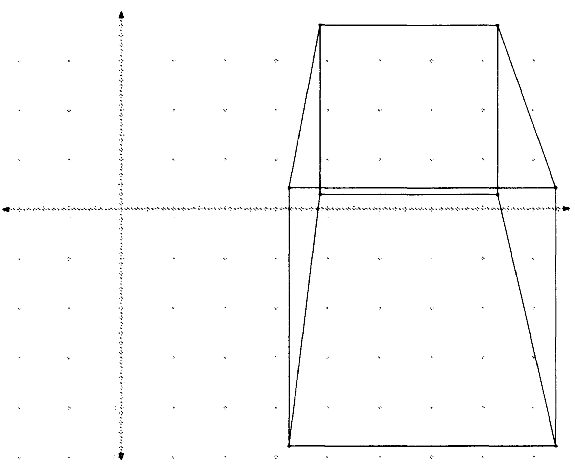
The y-coordinate of A' and G'=10-9.57=.43
The y-coordinate of E' and C'=10-6.29=3.71
The y-coordinate of H' and B'=10-14.78=-4.78
The y-coordinate of F' and D'=10-9.71=.29

Since the z-values stayed the same (same projection plane), we now have all of our coordinates for the new projection.

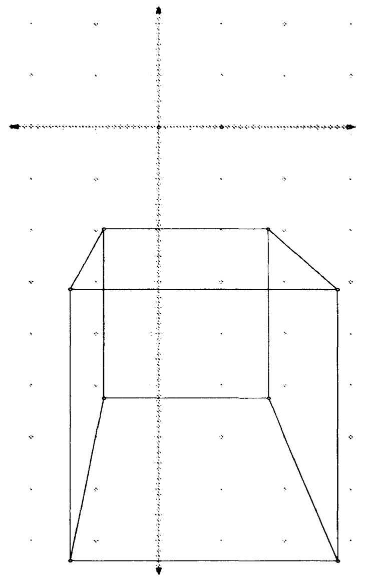
| name | x | y | z |
|---|---|---|---|
| A' | 8.45 | .43 | 7 |
| B' | 8.45 | -4.78 | 7 |
| C' | 7.29 | 3.71 | 7 |
| D' | 7.29 | .29 | 7 |

| | | | |
|---|---|---|---|
| E' | 3.86 | 3.71 | 7 |
| F' | 3.86 | .29 | 7 |
| G' | 3.26 | .43 | 7 |
| H' | 3.26 | -4.78 | 7 |

Below, you can compare what the projection looks like with the original vanishing point compared to the new vanishing point.



vanishing point (5,10,-3)



vanishing point (0,0,0)

## Botticelli's *Adoration of the Magi*

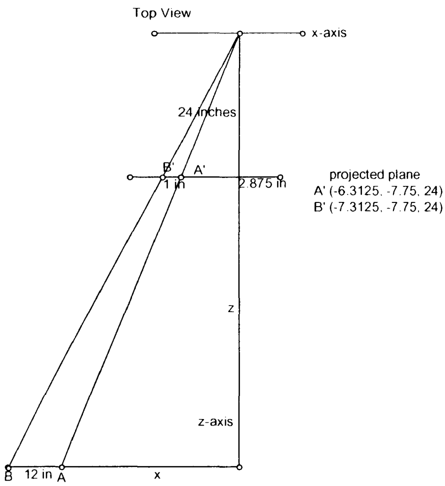
I chose to work with a painting by Sandro Botticelli, a Renaissance artist. He was born in Florence in 1445 and showed a talent for painting from an early age. Although he did get many famous commissions during his lifetime, his greatest recognition as a Renaissance master came centuries after his death. During his lifetime, the arts were flourishing and many wealthy people were commissioning portraits and other artwork for their estates. A wealthy merchant commissioned a religious work by Botticelli, but also wanted it to contain portraits of the de Medici family and the group of artists of which Botticelli was a member. The piece was called *Adoration of the Magi*, and featured portraits of Lorenzo, Piero, Giovanni, and Giuliano de Medici, as well as artists Poliziano and Pico della Mirandola. Botticelli is pictured in the lower right corner wearing a yellow cloak, and looking outward. In addition to this painting, he produced many other famous works, including *The Birth of Venus*, *Virgin and Child*, *The Annunciation*, and many more [6].

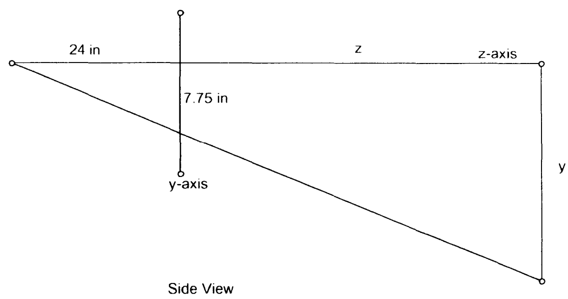
[7]

**Finding Real-Life Distances**

When you are given a painting as a projection, no calculations are needed to find the coordinates of each point. You should first find the vanishing point by following the major lines of the painting to their intersection point. This point can be called (0,0,0). Since we are working on a plane, we can draw perpendicular lines through the origin to create the s- and y-axes. The coordinates of each point can be found by measuring to the left and right of these axes.

To find the actual position of the objects in real space, the next step is to set the distance from the vanishing point to the projection. A logical distance would be an arm's length from the projection [8]. The artist most likely stood this distance away from his or her canvas as the projection was created. I have chosen to set this distance at 24 inches. Next, an assumption must be made about the painting, such as the length of a foot, the width of a hand, etc. I have chosen to assume that a foot in the painting is twelve inches long. As always, it is good practice to draw the scenario.




Use the distance formula to find the distance from the vanishing point to B'.

$$d = \sqrt{(-7.3125)^2 + (-7.75)^2 + (24)^2}$$
$$d = 26.26in$$

Set up proportions of similar triangles from the top view to find the other missing distances.

$$\frac{1}{12} = \frac{26.26}{w} \qquad \frac{26.26}{315.12} = \frac{24}{24+z} \qquad \frac{6.3125}{x} = \frac{24}{288.02}$$

$$1w = 315.12 \qquad 315.12 = 26.26(24+z) \qquad 1818.13 = 24x$$
$$w = 315.12 \qquad z = 264.02 \qquad x = 75.76$$

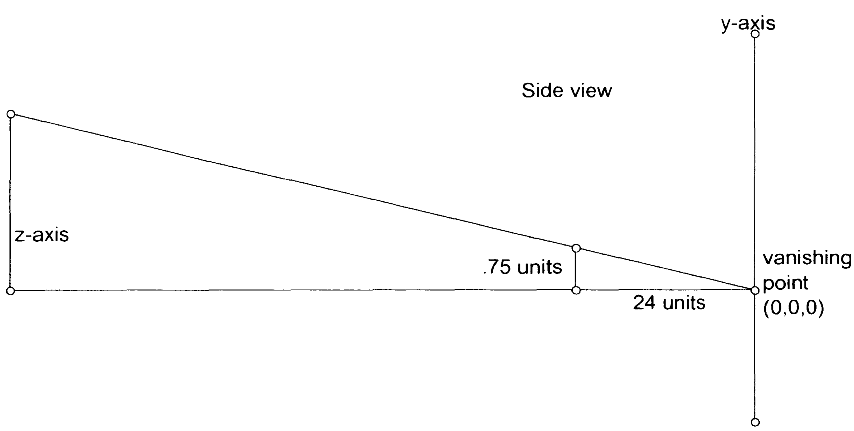
Set up proportions of similar triangles using the side view and what was found from the top view to find the missing distances.

$$\frac{24}{288.02} = \frac{7.75}{y}$$
$$24y = 2232.155$$
$$y = 93.006$$

Therefore, the coordinates to the toe and heel in real space would be:

| Heel | Toe |
|---|---|
| (-87.76, -93.006, 288.02) | (-75.76, -93.006, 288.02) |

Use the same steps as described to find the coordinates of the foot in the painting. Using this coordinate, you can also find any object's position that lies in the same x, y, or z plane. For example, it is possible to find the height of the object, any object that is placed on the same horizontal surface, or any object that is in the same perpendicular plane. In this case, I have also decided to find the height of the man.




$$\frac{24}{288.02} = \frac{.75}{s} \qquad 24s = 216.015 \qquad s = 9.0006$$

Based on the assumption that his foot is twelve inches long, his height is 93.006+9.0006=102.007 inches or about 8.5 feet.

His actual height kneeling, based on the assumption that his foot is twelve inches long, is 102.007 inches. The assumption made is not reasonable since a fifteenth century man is probably not over eight feet tall. I chose to recalculate his height using the assumption that his foot is eight inches long. This new assumption made his height 68.007 inches, which is about five and a half feet tall. This second calculation is much more reasonable. This example speaks to the challenges that arise when trying to recreate the position of three-dimensional objects based on a two-dimensional projection.

## Summary

As you can see using a series of relatively simple steps, it is possible to recreate the actual location of objects as they appear in a painting. You can also place three-dimensional objects in a two-dimensional plane without the use of a camera. This mathematical process, in conjunction with technology, has given birth to advancements that are changing the way we live. Filmmaking, computer graphics, modern creative media, and virtual reality are a few examples of what is possible due to our understanding of perspective [8].

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## References

[1] R. Smith, *An Introduction to Perspective*, DK Publishing, Inc., New York, 1998.

[2] K. Heng Ser Guan and H. Aslaksen, "Perspective in Mathematics and Art," Internet: http://www.math.nus.edu.sg/aslaksen/projects/perspective/intro.html

[3] "Egyptian Life Photograph," Internet: http://nefertiti.iwebland.com/portraiture

[4] G. Crack, "Photograph of the Battle of Hastings," Internet: http://battle1066.com/bayeux1.shtml

[5] G. Williams, *Linear Algebra with Applications*, Jones and Bartlett Publishers, Sudbury, MA, 2001.

[6] *Sandro Botticelli*, Uffizi Gallery, Internet: http://www.televisual.it/uffizi/s_bottic.html

[7] M. Dalrymple, "Photograph of Adoration of the Magi," Internet: http://www.artchive.com/artchive/B/botticelli/adoration.jpg.html

[8] M. Frantz, *Mathematics and Art*, Indiana University-Purdue University Indianapolis, 1997, Internet: http://www.math.iupui.edu/m290/