ALIGNING MATHEMATICS CURRICULUM TO CREATE POTENTIAL FOR ACTIVE LEARNING IN PRE-K THROUGH EIGHTH GRADE TEACHER EDUCATION

G.E. PARKER
Dept. of Mathematics, James Madison University
Harrisonburg, VA 22807
parkerge@jmu.edu

Abstract

In this paper, we consider some issues surrounding the teaching of mathematics to pre-service teachers. In particular, we look at the possibilities for teaching elementary mathematics from an advanced standpoint and alignments of curriculum that have the capacity to enhance student involvement in the making of the mathematics. The particulars of the James Madison University curriculum are used to illustrate many of the points.

Introduction

Since I am about to ask you to consider my approach to teaching prospective teachers, let me explain the basis on which I offer the approach. For college mathematics instruction generally, I am committed philosophically to the principles that: 1) a student of mathematics should understand, beyond a superficial level, the mathematics he or she is trying to learn; and, 2) a college mathematics course should contain college level thinking—that is, the student should be challenged both to have ideas and to validate that which he/she creates or is asked to accept.

Thus, this must be extended to those courses for prospective teachers. To be consistent with this, prospective teachers cannot be allowed to deal with the mathematics as if they were children; the goal, rather, is to get them to understand it at a college level so that they can bring their learning theory and child psychology to bear on the planning. This helps them present the mathematics in a way that is both *age appropriate* for their students, and *consistent* with the form the mathematics will take in the long run. Indeed, I challenge my students to try to find ways to teach mathematics so that what their students eventually have to unlearn is minimized. In addition, when teaching within the mathematics major, I teach all my upper-level courses using the Moore method, a pedagogy that depends completely on student initiative and achievement. Even my "lecture-based" courses aim at getting students to make arguments for their problem solutions. My classroom experience has been largely that of finding ways to get students to discover, on their own, why things are correct. The "research" basis for the remarks that follow is philosophical and experiential. The experience base is real. However, if the philosophy does not

The Journal of Mathematics and Science: Collaborative Explorations Volume 7 (2004) 109 - 121

resonate, then the techniques that I have developed, while they may stand on their own, *would* have to be judged very robust to pass muster.

At James Madison University (JMU), prospective elementary and middle-school teachers routinely major in Interdisciplinary Liberal Studies (IDLS), a program developed specifically for the purpose of educating them and preparing them for teaching positions. The core of this program requires a three-course, nine-hour mathematics component. The first two courses in this sequence, *Fundamentals of Mathematics I* and *II* (*FM I*, *FM II*), have been a part of the university curriculum during my entire twenty-year stay at JMU, although their contents have recently been rethought during the development of IDLS. The third course, *Math 207*, is a recent addition. It was originally conceived as a "problem solving" course, although it is beginning to absorb material from *FM I* and *II* to free up time for deeper involvement in the topics of *FM I* and II. My experience is in teaching *FM I* and *II*; I have dealt with *Math 207* only at the committee level as the course was planned and revised, and in making tentative planning for the possibility of being assigned the course. The University catalogue commits the two courses to "sets, logic, numeration systems, number theory, probability and statistics, measurement, geometry and an introduction to computers." Probability and statistics are now deferred to Math 207 and an introduction to computers has, for at least a decade, been ignored as a priority since the computer as a tool is now addressed in the general education program. Historically, the topics have been split so that sets, logic, numeration systems, and the number theory that supports the algorithms of arithmetic are taught in *FM I* while geometry, measurement, and models for R are taught in *FM II*. Recent re-examination of the curriculum for the sequence resulted in increased commitment to geometry, and moving probability and statistics to the third semester to create the time for said commitment.

Fundamentals of Mathematics I

The students that populate *FM I* and *II* are preponderantly freshmen and sophomores who intend to major in IDLS; indeed, for the past three years, JMU has had an effective filter on registration for the courses, and the course has been largely limited to IDLS majors. Generally, *FM I* is the first mathematics course a student in it takes. Hence, there can be no realistic expectation of the students in the course bringing with them any mathematical maturity, and the students must thus be taught within the course to meet whatever standards of rigor are demanded. To my mind, this makes the teaching of logic and language a mandatory first topic. To try to give "relevance" to the task, I teach the logic unit as analysis of language. I proselytize the need for them as teachers to make connections between language arts and mathematics rather than to build barriers. I give them the syntactical vocabulary, "and"; "or"; "it is not the case that"; "if...,then..."; and, "there is...so that..." I give them the sentence constructs of statements and open sentences. I give them the standards for truth for statements formed using "and"; "or"; "it is not the case that"; "if...,then..."; and, "there is...so that..." Then I put them to work on real sentences with the goal of recognizing the logical construction of the sentences and recognizing which words give meaning within the sentences. There are two expected outcomes. One is that the students realize that grammar can be used poorly or used well. Those sentences in which grammar is used poorly (often simply because of idiom!) usually have more than one plausible interpretation in logical form. This has the capacity to teach the importance of using grammar carefully (a language art), and also gives a context in which to study logical equivalence as we address the question, "Might the structural ambiguity have any bad consequences?" Second is the fact that the conscious separation of syntax from semantics affords a discussion of why knowing what the words mean (what the semantic content is) is crucial to making a judgment about the truth or falsity of the statement. This provides a pulpit from which to preach the importance of definition in mathematics. Having made the initial application of logic on sentences from ordinary language, I finish the unit by studying sets. This can be an effective bridge because we identify the primitive words "set, element, ordered pair, first co-ordinate, second co-ordinate," and describe (not define!) how the words are used. This is a first attempt at axiomizing and can serve as a basis for comparison when number algebra is axiomized later in the course. When we describe the definition of sets through language in our third protocol, the function of open sentences in mathematics is reinforced. When definitions are made for "subset, intersection, union," and "complement," the student gets to experience how semantic content is introduced into mathematics.

At this stage of the course, we have in place: 1) standards for sentence construction in ordinary language; 2) truth standards associated with sentences we might make; 3) a common language through which to form logical structure; and, 4) a common language for the use of sets. The connection with ordinary language is the only one of these topics that is not revisited later in the course.

Despite the omnipresence of computing devices, I still regard the algorithms for arithmetic as the primary source from which elementary/middle school students learn number sense and develop the capacity to see the importance of structure within mathematics. I devote over half of *FM I* to their study. In the first half of this portion, students are asked to develop explanations for the "correctness" of the algorithms from counting, thus making the trip in the same direction as the elementary school mathematics curriculum. In the second half of the algorithm portion, the students are given the field axioms, model place value numbers as polynomials in the base of the system with digit coefficients, model fractions as products of natural numbers with reciprocals of natural numbers. Then, they are asked to justify the algorithms as a deductive consequence of the algebra of the field axioms when applied to the models. Thus, the students first get to experience the development of the algorithms in a sequentially analogous manner to that of the students they will teach. Then, they get to start with what is commonly the long-term goal of teaching arithmetic structurally, number algebra, and demonstrate that the ideas of number algebra are sufficient to drive the algorithms of arithmetic, regardless of where the axioms might have originated. Students often confuse the purpose of *FM I*, seeing it as learning the "mathematics" that they "learned" in grade school. I see it as vital that any such notion be disabused, that my expectation is that they already know the processes we are studying. What I want for them is to be able to make sense of, as mathematics, what they learned in grade school. I try to make it clear that we are doing the mathematics in an adult mindset so that they will understand it in a way that will give them the flexibility to pitch the mathematics to the individuals in their care as they become aware of individual differences and various stages of maturation. This is my way of walking the tightrope of balancing grade-school content with college credit.

Affectively, I work really hard at having the "Algorithms from Counting" unit contrast strongly to the "Algorithms as a Consequence of Number Algebra" unit. In the "Algorithms from Counting" unit, everything is developed intuitively and, as ideas take shape, we make some attempts at formalizing the ideas with the language of the theory of sets. In the "Algorithms as a Consequence of Number Algebra" unit, the formal mathematics is dictated at the beginning, and everything that is accepted after the models are made is accepted because of arguments based on the axioms.

In the first unit, the progression is:

1) devise a scheme of comparison of sets to "define" counting
2) devise a numeration scheme that replaces the comparison with symbols from which the count can be recovered
3) devise a numeration scheme based on place value
4) define + and – in terms of counts through the language of the theory of sets
5) define * and ÷ in terms of + and –
6) execute the algorithms for +, –, *, and ÷ within the place value numeration systems they make
7) verify that the algorithms give correct answers
8) justify why they give correct answers

In the second unit, the progression is:

1) state the field axioms using the primitive words number, +, and *
2) make the connection to the numeration system by declaring the placeholders to be numbers, give a rule for + and * on the placeholders (in the process showing why the placeholder 0 must be the identify for + and the placeholder 1 must be the identity for *), declare the place value and powers of the place value to be numbers
3) define the number for which a numeral stands
4) interpret each step of each algorithm within the algebra and justify it

As stated earlier, the macrostructure of the two units described above is designed to have the students look at the algorithms from the "same" basis as the students, to look back at the algorithms from the basis that is the long-term goal of the curriculum, and to create a context in which to reinforce the use of the theory of sets to formalize ideas and give an experience in the making of proofs. The microstructure also affords opportunities. I develop place value numeration as a tool to symbolize the ideas developed for counting, and counting is initially developed, completely intuitively, on the notion of "matching." I never articulate "matching" formally (despite having the means to do so within the set theory they have been given via the primitive words "ordered pair, first co-ordinate, second co-ordinate," thus making the definition of function available), using instead demonstration to give the word meaning. The word "count" is *established* to describe the result of "matching" when a special outcome occurs. Thus, a matching between / / / / and # # # # can be demonstrated by physically pairing the objects from the first listing with those of the second listing in an "acceptable" way and all objects of both listings are parts of pairs. Interpreting the lists within the theory of sets and using that language to formalize what the lists are is as far as I go with actual mathematics.

Grouping to make comparisons more convenient is a short step away, and we are on our way to creating place value numeration by deciding on a count (which I call the "organizer") to be the standard by which grouping is done, creating numerals to represent the counts less than the organizer and a placeholder, and a place value scheme for making the numerals to represent counts. The creation of such a numeration system is then repeated each class day until the end of the unit as the class operates under the following rule: the organizer cannot be //////////, and the organizer we choose today cannot have the same count as the organizer we chose the previous class. The students are thus put in a position to do their arithmetic as a consequence of the structure established. The long-term lesson is that the algorithms work because of the structure of place value numeration, not just because of the choice of ////////// as the organizer. When the ideas for +, –, *, and ÷ are decided upon, I have the students formalize them using the language of the theory of sets together with the word "count." The symbols + and – are thus shown to be more basic, from the counting point of view, than * and ÷. Once the definitions are made, students are asked to add, subtract, multiply, or divide on the basis of the definition they have made. Then, they are reminded of the appropriate algorithm from grade school and the following teaching dynamic recurs:

1) the students execute the algorithm to get an answer;
2) the students create a count to show that the answer the algorithm gave **is** correct;
3) the students create a count that shows **why** the algorithm gave the correct answer.

The first step reinforces mastery of the procedure that defines the algorithm; the second indicates that it gives correct answers; and, the third argues it gives correct answers. Once an operation is defined, but before the algorithm is brought to their attention, students are asked to find sums or differences or products or quotients. And interesting sidelight is that they seldom produce counts that suggest the algorithm. A context exists to suggest that perhaps the algorithms being taught might not be so "natural" after all, and that a young student may have ideas that make sense even when they don't "fit."

An analogous "mind expansion" occurs in the second unit as the savvy students realize that, if they already knew algebra as a deductive device, the algorithms would be superfluous, and the rules for the arithmetic of common fractions would be afterthoughts. Good flashpoints occur: when the students are asked to make algebraic sense out of "the placeholder that is not a count" and 0*□ is not covered in the axioms (the role of theorems in mathematics arises); and, when +

and * are axiomized, but * and ÷ are seemingly omitted. Regarding the latter, the role of definition in mathematics arises and a tension between counting as a foundation — + and – are more basic than * and ÷ —and algebra as a foundation— + and * are more basic than – and ÷. The distributive law emerges as the engine that makes things go. Also, when fractions come up as a consequence of the field axioms, I remark that they were not apparent from counting and that the fundamental intuition for them comes from geometry. Redeeming this observation in *FM II* by emphasizing the role that congruence plays in counting parts to make fractions thus becomes a priority for the second course.

The Fundamentals of Mathematics II

The curriculum for *FM II* contains geometry and the decimal and fraction models for real numbers with, as noted in the introductory comments, a recent mandate for not shortchanging geometry. I commit two units to geometry, with the organization created to first emphasize synthetic geometry and then to study the geometry thus created with the added assumption that number lines may be created so that distance between points is consistent with congruence of the line segments with those points as endpoints. For the unit on the number systems, I build on the question, "Can we communicate, through language, what the objects in the numeration system are?" The aim is to plant it in the minds of the students the possibility that the system, if you're not already used to it (and the students they will teach will not be!) is really quite complicated.

Classroom dynamics for *FM II* are completely different than those for *FM I*. In *FM II*, you know that each student has passed *FM I*, so there is reasonable expectation of some maturity. On the other hand, any class is likely to be very heterogeneous; one can expect students from four to eight different teachers in *FM I*, most who do not share my prejudices about the level of rigor that should be demanded. I think it is important to try to turn this into a positive by giving them a stake in voicing their version of what they bring forward from *FM I*. I express my desire to make sure that we have a basis for communication within mathematics and suggest that a clear use of language can provide such a basis. My expectation (which I do not share with the students) in *FM II* is that I will have the opportunity to re-teach logic and language in the context of geometry, and I consider time well spent.

By making synthetic geometry the first unit in *FM II*, I get at least two teaching opportunities. First, in making mathematics from the pictures, or more specifically, from the tools that make the pictures, the students are, as in the counting unit of *FM I*, starting from the place at which their students will be. Second, since they are given the task of articulating the ideas suggested by the pictures using the language from logic and the theory of sets, I have a chance to involve them in the making of the axioms. They get to experience the distinction between axiom and theorem, and primitive word and definition, as *participants*. I consider this an opportunity to give the students an experience in the possibilities and difficulties in making constructivist learning happen.

The problem I set for the class is to connect the primitive words "point" and "line segment" to the tools (end of half of a compass and edge of a meter stick). They must make mathematics that articulate the notions associated with the tools using the primitive words "point, line, plane, line segment," and "congruent":

1) making a picture by letting the line segment go "on and on";
2) looking at the surface on which the point maker leaves a track when adhering to the straightedge;
3) giving meaning to "same size."

Goals I have for the class are that they recognize the power that the language of sets gives them and that when first principles are clearly articulated, other ideas become consequent to those principles. For instance, that angle makes sense, because the relationships among points, lines, line segments, and planes are clear, is supposed to be important. Once the structures are in place, the students return to the concrete by making constructions in which they interpret deductively the mathematics they have created. The outcome that I hope will influence their teaching is that number, particularly fraction, can be given meaning through geometry. The progression is:

1) identify the tools;
2) make principles using the words point, line, plane, line segment, and congruent, as well as the words from logic and the theory of sets to give meaning to the primitive words that are deemed consistent with the pictures, the tools make;
3) enhance the vocabulary for the subject by defining (at least) ray, circle, angle, and parallel;
4) make arguments from the principles that justify the outcome of constructions using the tools;

5) interpret appropriate constructions as creating number ideas.

In the second geometry unit, the mathematics of numbers is merged with the mathematics made during the first geometry unit by articulating a ruler axiom and a protractor axiom, and stating the axiom for similar triangles. The ruler and protractor axioms allow the definition of measures of line segments, angles, and rectangles and the geometry developed in the first unit carries these ideas as a basis for fractions and to the measure of polygons. The axiom for similar triangles gives a comprehensive look at a geometric basis for fractions; the study of area provides a geometric alternative to thinking of multiplication as repeated addition as well as making plausible that multiplication could make sense for any pair of numbers that represent length. Investigation of trapezoids is a good place to use algebra to clarify consequences of geometry. Establishing meanings for the measure of a circle and the measure of its interior develops clearly stated approximation schemes. The Pythagorean theorem is a high point of the unit. A plausible argument is available from the structures created in the first unit using the structures developed in the second unit. Application of the theorem gives an application of number algebra and establishes existence of lengths which are not fractions of integer units (in itself a magnificent result). This gives background for the study of the structure of the numbers that will take place in the third unit.

A progression is:

1) articulate a ruler axiom, a protractor axiom, and the axiom for similar triangles;
2) define distance and defend direct construction of fractions from the axiom for similar triangles;
3) define measure for triangles, articulate the ideas for Cavalieri's principle, and measure polygons;
4) develop the Eudoxus principle to measure the lengths of circles and the areas of their interiors;
5) prove the Pythagorean theorem and explore its consequences.

For the third unit, I have the students study the place value and fraction models for the numbers. The first problem I set for the students is: "using only the language of the theory of sets and the existence of the natural numbers, express the qualities that will allow you to explain through language what the symbols in each system stand for."

For the place value model, I don't rest until they have identified the importance of position relative to the decimal point, position relative to the place values thereby identified, the placeholders, and the restrictions on "to the left of the decimal point" that are not in force "to the right of the decimal point." For the fraction model, it usually doesn't take long to identify "what is the numerator and what is the denominator" as the key issues. Once they have constructed objects using the language of the theory of sets to formalize the two structures, I have them go back and forth between the formalisms and the notation with which they will teach their students. Then we set out to articulate, in both models, comparison principles from which $<$ can be defined, and algorithms for + and *.

In the place value model, there are deep thoughts to be had at every turn. In studying comparison, the students find that $\overline{0}.0\overline{9}$ is less than $\overline{0}.1\overline{0}$, but that there is no element of the model between them; where to go from there is always fun. When they appropriate the ideas from the algorithm for + for natural numbers, they find that those ideas do not apply unless the part of the numeral "to the right of the decimal point" "terminates", and a resolution of the question of defining + on the rest of the model is challenging. I seldom have time to do * on the place value model, but when I do, the same extension problem arises. This affords either a second chance to try to understand its resolution or else reinforcement of the likelihood that its resolution is very important (we'd like to be able to add or multiply any two numbers or we can't even meet the first expectation for an algebra!). These extension problems have no direct application to the elementary mathematics curriculum; what they do is teach a respect for the complexity of the place value model regarding arithmetic and, by analogy, plant the need for patience with young students struggling with the model at their own levels.

In the fraction model, studying comparison leads to the problems associated with dual representations and gives context to the reasonableness of reducing fractions to lowest terms. That both comparison and addition of fractions depend directly on multiplication of natural numbers has the capacity to be an engaging idea. The question, "How can we recognize when elements of the different models might represent the same number?" can be used to raise additional questions. In addition, it provides connections between what they brought with them and what you have them thinking about.

A progression is:

1) identify decimal point, left of the decimal point, right of the decimal point, place value, and placeholder as critical ideas for giving meaning to decimal notation;
2) create structures within the theory of sets to articulate these ideas and achieve facility going back and forth between notation and meaning for the notation;
3) articulate a comparison principle to define <, construct "numbers" between "numbers," and address the problem of the existence of different numbers that have no number between them;
4) articulate an algorithm for + and deal with its deficiencies;
5) articulate an algorithm for * and deal with its deficiencies;
6) identify numerator, denominator, and fraction line as critical ideas for giving meaning to fraction notation;
7) create structures within the theory of sets to articulate these ideas and get facility going back and forth between notation and meaning for the notation;
8) articulate a comparison principle to define < and address the problem that there are different "numbers" which cannot be compared;
9) articulate algorithms for + and * on the model; and,
10) relate the models.

If you are still reading, it has perhaps occurred to you that the course plan through which you have been led does not mesh particularly well with most of the books designed to support courses like *FM I* and *FM II*. Most such books have a chapter or two on logic and sets, but the emphasis is typically on truth tables and Boolean diagrams, with little emphasis on how grammar is used to create logical structure and the distinction between structure and semantics. Many of these books work with different bases for place value numeration, but usually emphasize changing from one to another rather than the system itself having qualities that are not base dependent. In addition, they often deal with the axioms for the numbers as an algebraic structure, but use them to consolidate examples rather than as a basis for deduction. These books treat the principles of synthetic geometry informally and use the synthesis of a metric with a geometry, but usually make no big deal about geometry dictating number ideas or number ideas being useful for articulating geometric properties. The books available use both the decimal and fractions models, but usually act as if it is obvious what the symbols stand for and why objects within the separate models must stand for the same thing. The main point of teaching the way I do is to put students in a position where they have no option other than to get involved in thinking about why things are the way they are. As a result, it might seem as if having no text is a reasonable choice. On the other hand, IDLS students have a common career goal—to teach—and they, once in the work force, will be expected to deal with curricular materials adopted by the jurisdictions in which they teach. Thus, I feel an obligation to make a textbook a part of their course, both to nurture the use of materials and to teach skepticism about materials, at least until one has thought about them.

I use the text in one of two ways. Some semesters, I will introduce a unit by leading a discussion in which the class clarifies an idea from which we hope to make mathematics. I will then suggest that perhaps the book has treated those ideas and suggest that they find the parts of the book pertinent to them. If students succeed in such a task, we will go to the selected parts, make them the focal point of our discussions, and then either adopt them as a part of our mathematics, reject them for just cause, or do whatever is necessary to adapt the ideas to our standards for language. An alternative method is to have the class develop the mathematics independent of the book, then finish off the unit by sending them to the book with the purpose of their finding how the book treated the mathematics we had developed. When I use this method, I purposefully neglect something the book has covered that fits with our unit. When the students find one or more of "the neglected ideas," we discuss it (them), adopt or reject each one, and the students experience using multiple sources for expanding on what they have in hand. If the students don't find what I know is there, I make sure at least one idea they haven't uncovered makes its way onto the unit test, and I get to teach something affective when I put out the ensuing fire.

Conclusion

This paper is already rather long, but one aspect sorely lacking is a collection of demonstrative instances of how one might handle a particular idea. Toward that end, if the philosophy offered above were to interest you, feel free to contact me. The references I have used ask questions that have triggered, or comment upon, some of the ideas that I have attempted to make a part of my teaching of prospective teachers [1-5]. ■

References

[1] P.R. Halmos, "The Problem of Learning to Teach: The Teaching of Problem Solving," *The American Mathematical Monthly*, **82**(5) (1975) 466-470.

[2] P.R. Halmos, "The Heart of Mathematics," *The American Mathematical Monthly*, **87**(7) (1980) 519-524.

[3] *Everybody Counts—A Report to the Nation on the Future of Mathematics Education*, National Research Council, Washington, DC, 1989.

[4] G.E. Parker, "Getting More from Moore," *PRIMUS*, **II**(3) (1992) 235-246.

[5] L. Ma, *Teachers' Understanding of Fundamental Mathematics in China and the United States*, Lawrence Erlbaum Associates, Mahwah, NJ, 1999.