A MATHEMATICIAN'S OVERVIEW OF THE VIRGINIA ELEMENTARY MATHEMATICS SPECIALIST PROGRAM

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Abstract

This article discusses the mathematics component of the Mathematics Specialist master's degree program in the "Virginia Mathematics Specialist Project" (VMSP). It includes my personal views on the significant mathematical knowledge and skills that Mathematics Specialists need, the mathematics that is taught in the Mathematics Specialist courses, and my thoughts on what appear to be the substantial mathematical abilities and aptitudes that are required by successful Mathematics Specialists in their work. The interpretations I present are highly personal and are undoubtedly dependent on my personal history, a short description of which is given (see Appendix A).

Background

I use "Virginia Mathematics Specialist Project" (VMSP) as a term covering the work done in Virginia over a seven-year period with a sequence of three Virginia MSP Specialist grants and two, five-year NSF projects: the TPC project, "Mathematics Specialists in K-5 Schools: Research and Policy Pilot Study"; and, the NSF Institute project, "Preparing Virginia's Mathematics Specialists." All this work was done under the umbrella of the Virginia Mathematics and Science Coalition (VMSC). The partnerships included six Virginia Institutes of Higher Education (IHE), the University of Maryland, and forty-five Virginia school divisions.

Mathematical Proficiency for All

The VMSP has led the effort to implement Mathematics Specialists in Virginia. The project has had three notable successes. The Commonwealth of Virginia has established a K-8 Mathematics Specialist endorsement. Eight universities have established Mathematics Specialist master's degree programs with 21-credit hours of common courses. Five years of research has now been completed and is discussed elsewhere in this issue. This article is focused on the mathematical core of the VMSP's master's degree programs, a sequence of five mathematics courses that each student takes. The courses are:

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- *Numbers and Operations* (*N&O*);
- *Rational Numbers and Proportional Reasoning* (*RN&PR*);
- *Algebra and Functions* (*A&F*);
- *Probability and Statistics* (*P&S*); and,
- *Geometry and Measurement* (*G&M*).

These courses align well with the content strands of elementary school mathematics as discussed in such documents as the National Council of Teachers of Mathematics (NCTM) *Principles and Standards* [1] and the Conference Board of the Mathematical Sciences (CBMS) report *The Mathematical Education of Teachers* [2]. The project started with only four courses. The *RN&PR* course was added when our experiences showed that our Mathematics Specialist students needed additional work with fractions and rational numbers that went significantly beyond that which was provided in the other four courses.

The Virginia endorsement is a K-8 Mathematics Specialist endorsement, but the program that I describe here is a K-5 program. There were many instances that arose in the program development where the breadth of the mathematics covered was limited in order to reach greater depth in the K-6 mathematics. The overall goal of the sequence of these courses is to provide future Mathematics Specialists with a *profound understanding* of the mathematics that is taught in our elementary schools. I use the term "profound understanding" to convey a significantly deeper understanding than the procedural competency often associated with mathematics courses. The term is borrowed from Ma's *Knowing and Teaching Elementary Mathematics*, but our usage here does not align precisely with it [3]. Our goal for these courses derives from our understanding of successful learning of school mathematics. This understanding aligns nicely with that presented by the *Mathematics Learning Study Committee* in their 2001 NRC report *Adding It Up* [4]. The term the committee used to designate successful mathematics learning was "mathematical proficiency." It consists of five interwoven strands:

- Conceptual understanding;
- Procedural fluency;
- Strategic competence;
- Adaptive reasoning; and,
- Productive disposition.

A valuable visual representation of the complex nature of the relations between these strands was presented in the form of a braid (see Figure 1).

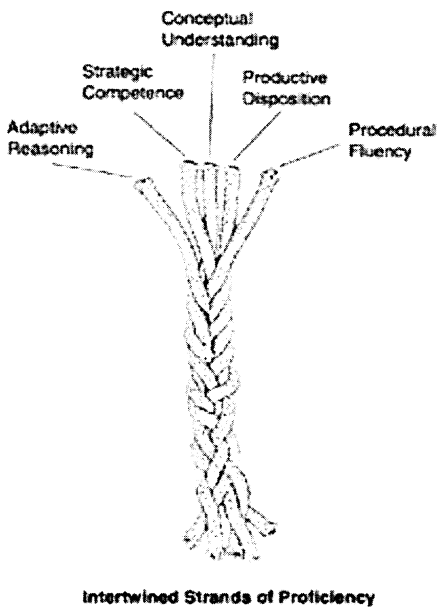


Figure 1. The five strands of mathematical proficiency represented by a braid.

A thorough discussion of mathematical proficiency is given in *Adding It Up* [4]. Of course, a precise definition of mathematical proficiency and how to assess it is problematic and there is no universal agreement on this [5]. For our purposes, it suffices to observe that this vision of school mathematics with its five braided strands is far more complex than a view which sees procedural fluency as the primary goal of school mathematics. Preparing teachers who can nurture the development of mathematical proficiency in their students and preparing Specialists who can support teachers in these efforts is also vastly more complex than preparing teachers whose goal is procedural fluency.

I wish to contrast mathematical proficiency for all with what I believe was the *standard model* for mathematics education when I was a student in the 1950s. Then, it was widely

believed that students naturally rose to the level of their mathematical talent. Some students could succeed with fractions and others could not. Fewer students could succeed with Algebra I and very few with geometry. This ascending ladder continued through graduate school and beyond. It was believed that little, if anything, could be done to counter the students' natural upper limits. As a result of these beliefs, the instructors' responsibilities were sharply limited.

This model was also reflected in a recurring event that I observed in my early years as a mathematician. Senior colleagues frequently expressed the opinion that they doubted they had ever succeeded in teaching anyone other than the few students who were so talented that they practically did not require an instructor. These remarks can still be heard in mathematics departments, but not as frequently as they once were. With this model, the instructor's primary responsibilities were to challenge the students and maintain standards. With the very best students, this model was successful, but it failed with most students. This model was never made explicit or official. My interpretation of it is based on conversations and observations of mathematics teaching that I saw practiced. Others may wish to compare my observations with their own.

The goal of mathematical proficiency for all aspires to a student population composed of confident and capable problem solvers with substantial procedural and technical proficiency. This represents a dramatic change from the apparent educational goals of sixty years ago. When only the best students were expected to succeed in mathematics, it was possible to function with a relatively small pool of highly qualified mathematics teachers. This is no longer the case. It is impossible to overemphasize the impact of this change. It drives much in contemporary mathematics education and the development of the Mathematics Specialist concept in particular.

The type of knowledge and understanding that teachers require to nurture the development of mathematical proficiency in their students is not well understood, or at least it is not well documented in the literature with which I am familiar. However, significant progress has been made in recent years toward sketching the outlines of this knowledge. In this context, I mention the work of Liping Ma and her concept of "Profound Knowledge of Fundamental Mathematics" (PKFM) [3]. Even more significant is the ongoing, large scale project on "Mathematical Knowledge for Teaching" (MKT) of Ball, Hill, Bass, and their collaborators which aspires to being able to effectively assess this knowledge [2, 6-8]. We can assert with confidence that there is a tremendous gap between the knowledge and skills possessed by typical

elementary teachers today and the knowledge and skills they would require to foster the development of mathematical proficiency in all students.

Mathematics Specialists and Their Mathematical Needs

In the Virginia project, the Mathematics Specialist's job is seen as a way to strengthen teaching practice by providing school-based mathematics support to teachers and building-level administrators. The VMSP envisions Mathematics Specialists as a primary resource for addressing the knowledge gap that exists between the current reality in our schools and what we believe is required by our students' teachers. I will now sketch my understanding of the mathematical knowledge that Specialists will need to reach this goal.

The mathematics that is covered in the VMSP program and is taught in our schools is described in the course titles. Teachers and Specialists require an understanding and familiarity of this material that includes all aspects of mathematical proficiency and large amounts of mathematical flexibility, PKFM and MKT. A glimpse into what this means is given with a few examples. Specialists must be skilled in: interpreting students' mathematical work, both written and verbal; recognizing which solutions are valid and which are not; and, having informed opinions on what a given student knows and what the next steps are. They must have a deep knowledge of how children learn mathematics and know when specific pedagogical moves are developmentally appropriate. Teachers constantly choose from a variety of representations and explanations when teaching mathematics. In the best of circumstances, these choices are based on an understanding of the students' knowledge and learning styles and the teacher's knowledge of the strengths and weaknesses of the competing explanations and models. In fact, few teachers have this skill set.

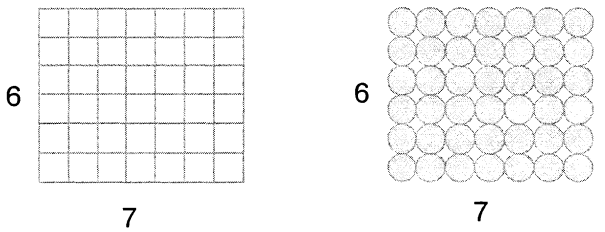
Mathematics Specialists become their schools' mathematics authorities and it is essential that they have the kind of knowledge referred to above at a much deeper level than is within reach of our teachers. A very long list of examples could be given of places where a profound knowledge of mathematics is essential for the Specialist. For example, in elementary school mathematical fallacies are quite frequently taught as fact. The errors range from the obvious to the subtle. Specialists must be able to recognize what is mathematically correct and what is false. They must be able to discuss and explain these issues with teachers, administrators, and parents. They must be able to advise and lead on issues of mathematics assessment, mathematics

curriculum, mathematics special education, mathematics for ESL students, and mathematics for the gifted and talented.

The braided strands in the illustration of mathematical proficiency are much too simple to capture the situation for Specialists. All of the Specialist's strands involve mathematics, although many are not primarily mathematics. The mathematics proper is woven into what I call the "mathematical landscape." This construct is similar to the "landscape of learning" that Fosnot and Dolk discuss in their *Young Mathematicians at Work* series [7]. In the mathematical landscape, I place primary emphasis on the connectivity. I picture the collection of mathematical concepts, ideas, results, and procedures being represented as hills and mountains. Most of the landscape is hidden from us using any one viewpoint; but, the landscape is connected by a complex web or network of pathways. There are often many paths connecting different mountaintops, and the journeys along different pathways provide the students with different understandings and knowledge. It is knowledge of this network of connections that provides individuals with their mathematical flexibility and power. The network reveals the mathematical relationships between topics. The mathematical representations and models that we use provide different viewpoints. Each representation offers a distinct view of a part of the mathematics. Mathematical relationships are formed by combining different viewpoints and mathematical representations.

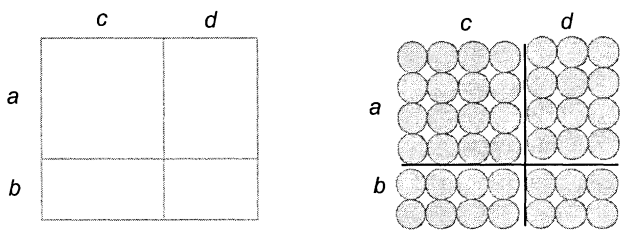
I will illustrate my understanding of this landscape and the types of knowledge that a Specialist needs by peeking at this landscape through one multifaceted example, that of multiplication and area/array models. In this example, the Specialist's knowledge of the landscape should be highly connected and include knowledge of the following topics and links between them: multiplication; areas of rectangles including area and array models, and decomposition and recomposition of numbers; the distributive property and other laws of arithmetic; and, our base 10 number system.

1. A basic understanding of multiplication as repeated addition can be developed *using either* the area of rectangles or arrays of discrete objects, as illustrated below.



2. However, children must develop the spatial structure of rows and columns implicit in this model before the model can become the basis for significant generalization and abstraction. Case 19 in "Measuring Space in One, Two, and Three Dimensions" shows some of the difficulties students (third graders) may have drawing small arrays [9].

3. It is possible to use the area model in the development of the distributive property.



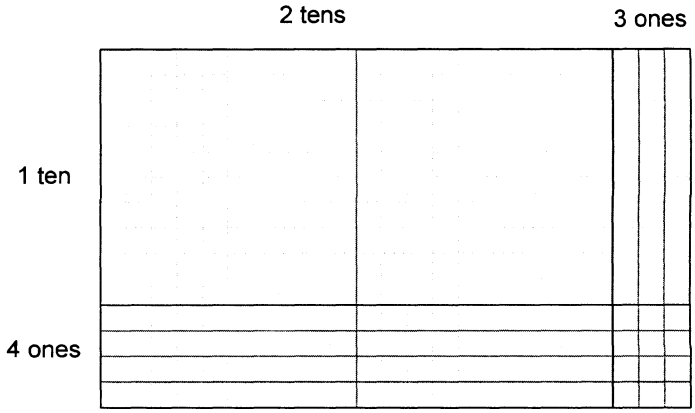
$$(a + b)\bullet(c + d) = ac + ad + bc + bd.$$

However, to do this, the students need to understand:

- multiplication, arrays;
- the area model; and,
- the fact that when a rectangle is decomposed into rectangular pieces, the area of the large rectangle equals the sums of the areas of the pieces.

4. In the area model, the factors in a product are lengths while the units for the product are square units of area. The units change! This naturally leads to the questions: Can the language of the area model effectively be adapted to discuss multiplication generally? What are the mathematical issues involved here? *Note that when points on the real line are used to model the real numbers, the product of numbers must be represented as a point on the line.*

5. The algorithm for multiplication of two-digit (and multi-digit) numbers rests upon the distributive property and is often explained using area or array models. To explain the computation 14 x 23, a drawing similar to this (or base 10 manipulatives) is often used.

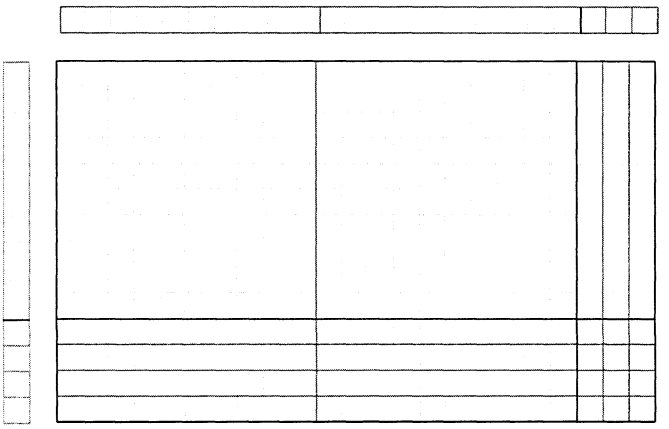


The pictorial representation illustrates the distributive property and shows that

$$14 \times 23 = (10 + 4) \times (20 + 3) = 10 \times 20 + 10 \times 3 + 4 \times 20 + 4 \times 3.$$

The naturalness of the area model seems to largely disappear when multiplying numbers with three or more digits because we cannot effectively draw accurate representations. One would hope that Specialists have encountered and worked through this issue. This next step of multiplying three-digit numbers seems to lead to significant abstraction. Substantial mathematical knowledge for teaching seems to be required here.

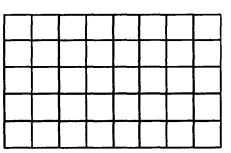
6. The problem of units, referred to in #4 above, reappears when base 10 materials are used to multiply decimals. For example, in the problem 1.4 x 2.3, base 10 materials are sometimes used in the following manner:



In the frame on the outside, the rods are often incorrectly described as designating units while the small squares are said to designate tenths. In the array, the large squares are units, the rods are tenths, and the small squares are hundredths. The mathematical issue here is that, in the frame, 2-dimensional pieces are used to measure lengths and these same pieces appear in the rectangle as units of area. Serious confusion can result at this point and serious misunderstandings will likely result in the minds of our students if these mathematical errors are not addressed.

7. When arithmetic is extended to negative numbers, the array model is not an area model and, if used to include discussion of (a + b) x (c + d) where the terms may be either positive or negative, the model must be extended to become a signed area model. In my experience, this fact is almost never addressed.

8. Arrays also appear in work on fractions, decimals, and percents. The following problem where multiple solutions are sought is typical. What fraction, decimal, and percent of the large rectangle is shaded?



The Program's Courses

This multiplication example displays a significant amount of mathematical knowledge that one would hope Specialists possess. The topics which were mentioned are not discussed in most ordinary mathematics classes. Moreover, it seems to me that the understanding needed with each of these numbered items is typically something that will not be directly transmitted in a lecture, but requires thoughtful reflection and discussion by the learner. The Mathematics Specialist courses provide the participants with constant opportunities to reflect on and discuss such matters. Students in these courses must constantly explain their reasoning concerning problems, concepts, and solutions. They must react to solutions from other students. They read many case studies of student work from real classrooms (typically from *Developing Mathematical Ideas*) and they are expected to react to the student work, try to discern what understanding the students exhibit, and suggest appropriate pedagogical next steps [9].

The program was designed specifically to prepare teachers to serve as Mathematics Specialists in elementary schools. The development was done by teams of mathematicians and mathematics educators from the higher education partners and school mathematics faculty and supervisors. As would be expected, differences of opinion as to what mathematics is needed by Specialists occurred frequently, but everyone's voice was heard and consensus compromises were reached. The curriculum that emerged represents a broad consensus within the development teams on what Mathematics Specialists need to perform their jobs. The original disagreements did not, however, disappear and they continue to resurface seven years into the project. For example, active discussions persist on whether it is best to go deeper or to cover more material in *Numbers and Operations*.

The course, *Numbers and Operations* (*N&O*), is a prerequisite for all other courses in the program. It closely follows the *Developing Mathematical Ideas* (*DMI*) numbers and operations materials, but this is supplemented by adding problems and additional work on topics, such as arithmetic with different bases [9]. This course sets the tone for all the mathematics courses where students are pushed to question, explain, and understand. Throughout the program, the standard the faculty and students are held to is that everyone must understand both how to solve problems and how to justify their solutions. A successful feature of the program is that students do not pretend they understand things which they do not.

The *N&O* course primarily treats whole number and fraction arithmetic, but the project staff recognized early on that additional work was needed on fractions, rational numbers, and proportional reasoning. There are no *DMI* materials appropriate for this course and several texts (Lamon, Fosnot and Dolk, and Smith, Silver, and Stein) are used and supplemented with additional activities [5, 7, 11]. This course develops a deep understanding in this strand—a strand dominating a large fraction of the middle school curriculum.

The course, *Rational Numbers and Proportional Reasoning* (*RN&PR*), is followed by the course *Algebra and Functions* (*A&F*) that is primarily based on the *DMI* algebra materials. This course stresses early algebraic thinking, generalization, the development of the laws of arithmetic for the integers (and to a far lesser degree the rational numbers), functions, and symbolic algebraic arguments. Because their initial understanding is often limited, many of the students do not progress far in developing algebraic reasoning. An illustrative example showing this limitation is that, after completing this course, not all students who took a geometry course were able to find the length of an edge of a square of known area without assistance. Quadratic functions had been introduced (not treated extensively), but these students' ownership of the function and inverse function concepts was still very limited.

The course entitled *Probability and Statistics* uses the *DMI* text on data and a variety of materials on probability, including especially the NCTM *Navigations* text [9, 12]. The primary emphasis here is the development and use of the elementary tools of descriptive statistics, together with an introduction to probabilistic reasoning that develops such concepts as events, sample spaces, repeated trials, and independence. I judge the part of the course focusing on data to be quite successful. I have found the probability piece, especially that part dealing with conditional probability, to be highly challenging for many of the students. We have no research to document this, but it is my opinion that the largest obstacle to learning this material well is the students' limited fluency in and ownership of proportional reasoning.

Finally, *Geometry and Measurement* (*G&M*) covers the K-8 geometry and measurement topics with a strong emphasis on measurement in dimensions one, two, and three. The *DMI* geometry and measurement materials are used for approximately half of the course [9]. Activities from other sources, especially the Virginia Department of Education professional development materials and the unpublished text of Pitt, Timmerman and Wall, extend the course well beyond the limits of the *DMI* course [13, 14]. The standard K-8 area and volume formulas are all

discussed and derived (or given intuitive justifications). The Pythagorean Theorem, similarity, congruence and transformational geometry are all explored. The van Hiele model for how children learn geometry is discussed and a strong emphasis is given through developmentally appropriate student-centered activities. This is an area of real weakness for most of our Specialist students.

My concluding remarks for this section are:

- The program is intended to prepare K-5 Specialists. The operative interpretation here is that K-6 mathematics is covered in depth. My judgment is that, for the majority of the students, the program meets this goal well. The mathematics of grades 7 and 8 is discussed in these classes, but is not treated with the same depth as the K-6 curriculum.

- The program includes much discussion of MKT and it attempts to develop a solid familiarity with the K-6 mathematical landscape. Typically, graduates leave being well prepared to serve as K-5 Specialists. However, only those graduates who entered the program with a strong mathematical preparation for teaching middle school mathematics are well prepared to serve as 6-8 Specialists.

Mathematical Aptitudes and Abilities

In this final section, I offer a few observations and personal thoughts concerning the mathematical abilities and aptitudes that are needed by highly successful Mathematics Specialists. They are based on my extensive experiences in the program. I have been involved in designing and teaching all of the mathematics courses, and I have directed many of the final practicum projects of University of Virginia (UVA) graduates. In the practicum projects, students are asked to research, design, and implement a project in which they practice the work of a Mathematics Specialist. The projects provide the faculty with excellent opportunities to assess the students' potential as Mathematics Specialists. I have also been engaged in the admission process of more than 150 applicants to UVA's program. This has provided me with the opportunity to develop informed opinions about what this population looks like on paper. The combination of all of these experiences has allowed me to form opinions on the mathematical aptitudes, abilities, and skills that I would like Specialists to possess, and on the impact our program has had on individual teachers.

Beneficial Impact—The program has had a beneficial impact on every teacher who has completed it. I believe they are all better teachers than they were when they began the program.

Some of them are truly outstanding. The changes are the result of their new knowledge of mathematics and mathematical knowledge for teaching, as well as changes in their beliefs on teaching practice. In most cases, this has been dramatic and it began with the first course, *Numbers and Operations*.

Mathematics Specialist Position—The position of the Mathematics Specialist is very complex and demanding. In addition to the mathematics qualifications that I have written about, it requires the personal skills to work productively with students, teachers, administrators, and parents. A knowledgeable Specialist without these skills and personality traits may be an excellent teacher and a bad Specialist. They can easily damage the quality of instruction in a school, and administrators must pay close attention to these matters when selecting candidates to be Specialists. The importance of these issues can scarcely be overemphasized.

Profound Understanding of Fundamental Mathematics—The heart of a Mathematics Specialist's job centers on improving mathematics teaching and learning in the schools. Very often, this work will rest on their "profound understanding of fundamental mathematics," mathematical issues that are not understood by other teachers in the school. This role regularly requires a technical knowledge of mathematics, an intimate familiarity with the landscape of school mathematics, and a significant amount of mathematical flexibility. My experience has convinced me that not all elementary teachers can rise to the required level and that it is imperative that we do not endorse teachers below this level.

Because graduation from the state approved degree programs leads to an endorsement as a Mathematics Specialist, it is critical that these programs exercise standards that limit the number of unqualified graduates. I urge my colleagues to continue to investigate and discuss this issue. In my work with UVA students, I have gained valuable insights relevant to this situation.

- Successful completion of some courses with the title *Mathematics for Elementary Teachers* does not guarantee that the student has the abilities that I am advocating.
- Knowledge of college algebra and precalculus is not a prerequisite for Mathematics Specialists, but successful completion of such courses typically indicates possession of the sought after abilities.
- Standardized examinations, such as the GRE quantitative examination, are not precise tools for evaluating mathematical ability, but students with GREQ scores of

400 and below have typically struggled in our program for mathematical reasons. Students with GREQ scores above 500 have not struggled for mathematical reasons. When students take the GRE multiple times, their scores may vary significantly and when students prepare for this examination, their scores can rise dramatically. I believe that an appropriate cut score on the GREQ test lies somewhere between 400 and 500. Due to the variability of the scores, applicants in this range, but below the cut score, should be encouraged to take the test again.

Appendix A
Author's Background and Involvement in the VMSP

I grew up in the 1940s and 1950s in rural northern Idaho. This setting provided me a nearly ideal constructivist, hands-on environment to learn the mathematics and physical science of elementary and middle school; an environment where all problems came with a context that was compelling to me. The resulting childhood experiences started me on my way to becoming a mathematician (now retired) and lay the groundwork for my philosophy of mathematics education; a philosophy emphasizing enquiry, activities, and problem solving.

My path to becoming a (pure) mathematics professor included the standard bachelor's, master's, and doctorate degrees (including an M.S. in Biometry from Catholic University in 1964). I joined the mathematics faculty at the University of Virginia in 1970 and by the mid-1980s, I had become involved in working with mathematics teachers and schools. In 2000, I and a few school partners began work on conceptualizing a master's program for Mathematics Specialists, work which eventually became the Virginia Mathematics Specialist Project. When this partnership expanded and funding was received to develop a formal program, I led the project's development of its mathematics curriculum. I served on the development teams for each of our five basic mathematics courses and then taught each of these courses. In the years 2007-2009, I was the advisor for seventy-five students who graduated from the UVA Mathematics Specialist M.Ed. program.

Eight of the courses in the program are shared by UVA and our partner IHE. They were developed by teams consisting of school mathematics teachers and mathematics administrators, mathematicians, and mathematics educators. The curriculum decisions that were made reflected the committee's understanding of the Mathematics Specialist position and the knowledge and skills, including mathematical, pedagogical, and leadership, that the Specialists need to perform their jobs. My interactions with the development teams and with the students who have completed the program have shaped most of the opinions expressed here.

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