MATHEMATICS PROFESSIONAL DEVELOPMENT WORKSHOP FOR MIDDLE SCHOOL TEACHERS: CONCEPT VERSUS MEMORIZATION

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Abstract

This article includes professional development topics for middle school mathematics and science teachers from two week-long Urban Teacher Institutes. These Institutes were held at J. Sargeant Reynolds Community College (JSRCC) and its partner institution, Virginia Commonwealth University (VCU), during the summers of 2007 and 2008, and were supported by a grant obtained by Dr. Harriet Morrison (JSRCC). Co-author Dr. Dewey Taylor directed the 2007 workshop, and both authors served as faculty leaders in both workshops. The workshops focused on teaching in an urban environment and "community mapping" (understanding the details of a certain locale to make the teacher more knowledgeable about the environments of both the students and the schools). The community mapping aspect of the workshops was led by Dr. Shirley Key of the University of Memphis. They featured content teaching and applications led by VCU faculty in mathematics, physics, forensics, engineering, mathematics education, and science education. This article focuses on the mathematics professional development strand in the workshop which featured conceptual learning with graphing calculator support as an alternative to the memorization of formulas.

Mathematics Concepts versus Memorization of Formulas

The discovery activities are outlined in the following sections. These activities were investigated either in the teacher workshop sessions or in the plenary presentations of the Urban Teacher Institute.

Find the Formula for the Area of an Ellipse *and Never Forget It!*

Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with x-intercepts $\pm a$ and y-intercepts $\pm b$

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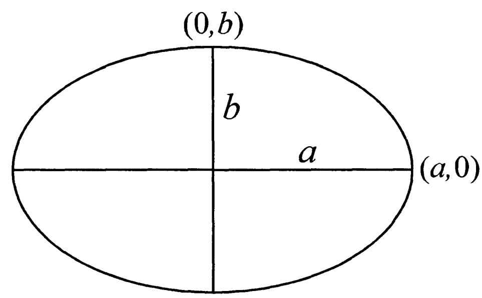
$$a^2y^2 + b^2x^2 = a^2b^2$$

$$a^2y^2 = a^2b^2 - b^2x^2$$

$$y^2 = \frac{b^2}{a^2}\left(a^2 - x^2\right)$$

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$



Using the following integral from calculus, we can compute the area,

$$A = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx\,.$$

Let's examine a special case with a = 3, b = 2

$$A = \frac{8}{3}\int_0^3 \sqrt{9 - x^2}\,dx$$

Since many teachers will either not have studied calculus or have forgotten it, we use the TI-83 calculator to demonstrate and run several programs which calculate the integral which, when multiplied by 4, gives the area. Setting the window to "zoom decimal," we first run the program "Riemann" to show the rectangular area which will approximate the actual areas. For this visual, we use n = 10 subdivisions to distinctly see the rectangles. Now, we run a program

"Integral" to get a very close area approximation with $n = 500$ divisions. This program will run in slightly more than a minute and will produce the area approximation of 18.84946 when the answer is multiplied by 4. We now ask the teachers to guess whether or not π is a factor of the answer for the area. Most will guess "yes." So, we now divide the answer by π and obtain 5.9999722 which we interpret approximately as an area of 6. The teachers now guess that 6 is the product of the length of two semi axes, namely $6 = 3 \times 2$. Now, the area in the given case can be guessed to be $A = \pi \times 3 \times 2$ and in the general case to be $A = \pi ab$. We now note that this generalization of the area is πa^2 of the circle which results when $a = b$ is the radius of the circle.

Arithmetic Sum Based on Concept of Average

We motivate this concept by asking: How much money would you make if you **averaged** \$50 per week for four weeks? Most teachers know that the result is $\$50 \times 4$ weeks which equals \$200. We recall that the average per week can represent each actual amount per week which could, for example, have been \$45, \$55, \$51, and \$49, respectively. Next, we pose the following problems and discuss the solutions.

- *Find the average of 2, 4, 9.*
    - The average is $15/3 = 5$ which is the sum of the numbers divided by the number of terms, which defines "average."
- *Use the average to find the sum of the three numbers.*
    - Since the average 5 can represent each of the 3 terms, the solution is $5 \times 3 = 15$.
- *Find the average of 3, 5, 7, 9, 11, 13.*
    - The average is 8; namely, the sum of the terms which is 48 divided by the number of terms which is 6. We now observe that the sum is "arithmetic" which

means that each successive term is obtained by adding a constant difference (in this case 2) to the preceding term. We note that the average can be found in these arithmetic sums by averaging the first and last term, a fact which will be proved below.

- *Use the average to find the sum of the six numbers.*
    - The average is $(3+13)\div 2$. Since the number of terms is 6, the sum is $(3+13)\div 2\times 6 = 48$.
- *Prove that the sum S of* $1+2+3+\ldots+n$ *is given by* $S = \dfrac{n(n+1)}{2}$.

PROOF:

$$S = 1 + \quad 2 \quad + \quad 3 \quad + \ldots + n$$
$$S = n + (n-1) + n-2) + \ldots + 1$$

By adding the series written forward and backward, we obtain each successive term for *2S* is $n+1$ so that

$$2S = \underbrace{(n+1)+(n+1)+n+1)+\ldots+n+1}_{n\,terms} \text{ and } S = \frac{n(n+1)}{2}.$$

General arithmetic series may have the following form of:

$$S = S_1 + (S_1 + k) + (S_1 + 2k) + \ldots + S_n$$

$$S = S_n + (S_n - k) + (S_n - 2k) + \ldots + S_1.$$

Adding, we obtain:

$$2S = \underbrace{(S_1 + S_n) + (S_1 + S_n) + \ldots + (S_1 + S_n)}_{n \text{ terms}} = n(S_1 + S_n).$$

Thus, $S = \frac{n}{2}\left(S_1 + S_2\right)$.

Now, $S = \frac{n}{2}\left(S_1 + S_2\right) = n\frac{\left(S_1 + S_n\right)}{2}$. Many texts use the form $\frac{n}{2}\left(S_1 + S_2\right)$ instead of $n\frac{\left(S_1 + S_n\right)}{2}$ which tends to force memorization of the formula as opposed to conceptualizing the use of average.

Dividing by n, we obtain $\frac{S}{n} = \frac{S_1 + S_n}{2}$. Since $\frac{S}{n}$ is the definition of average, indeed the average of all the terms is the average of the first and last terms.

So, S = (number of terms) × (avg. term) where for arithmetic series:

AVG. TERM = AVG. OF FIRST AND LAST TERMS.

NOTE: The average is also the average of the second and next-to-last term, etc.

$$\text{AVERAGE} = \frac{\left(S_1 + k\right) + \left(S_n - k\right)}{2} = \frac{S_1 + S_n}{2}$$

However, this rule tends to be less useful since posed questions generally are stated so that the first and last terms are known.

We now pose one final problem: Find the sum of $S = 5 + 10 + 15 + 20 + \ldots + 100$. The average term is $105 \div 2$ which implies that $S = (105 \div 2) \times 20 = 1050$.

Concept of the Size of an Acre

Nearly every parcel of land bought or sold in this country is measured in acres. However, very few people, including college graduates, have a concept of the size of an acre. Rather than "looking up" the definition in terms of square feet or square yards, neither of which provides much enlightenment of the size concept, we propose to relate the size to that of a football field. We first have the teachers guess the relative size of an acre by comparison with standard sizes of a tennis court, a basketball court, a soccer field, or a football field. We settle on a comparison with the size of a football field. Nearly everyone will know that a football field is 100 yards long. Few (a coach or two) will know that the football field is $53\frac{1}{3}$ yards wide. We use the calculator to calculate the football field area to be $A_1 = 5330$ square yards. Using the definition that the area A_2 of an area is 4,840 square yards, we calculate the ratio $\frac{A_2}{A_1} = 1.10$ (approximately). Thus, a football field is approximately one and one tenth acres so that the size of a football field, not counting the end zones, is a reasonable approximation of an acre.

We now note that farmers approximate an area by "stepping off" seventy yards square. This measure of seventy yards square yields 4,900 square yards to approximate an acre. The ratio of 4,900 square yards to 4,840 square yards yields 1.01 acres, notably accurate to .01. Even if you forget the "farmers' measure," the football field measure will provide a good comparison.

Rational Numbers: Converting Repeating Decimals to Fractions

Demonstrating that converting terminating and repeating decimals to their rational number fraction representation will enable middle school mathematics teachers to recognize the equivalence of the definition of rational numbers in either form, namely $\frac{a}{b}$ where a and b are integers and $b \neq 0$, or a terminating or repeating decimal.

Of course, a fraction like $\frac{1}{4}$ can be divided out to yield .25, and conversely, .25 can be written as $\frac{25}{100}$ and reduced to $\frac{1}{4}$. Also, a number like $\frac{2}{7}$ can be divided out to yield $.295714\overline{295714}$ in a repeating decimal form.

However, converting a repeating decimal to a fraction is more difficult. While a repeating decimal like $.35\overline{35}$ can be represented as a geometric series $S = .35 + .0035 + .000035 + \ldots$, and summed by the formula $S = \dfrac{35/100}{1 - 1/100} = \dfrac{35/100}{99/100} = 35/100$, the student must memorize the formula $S = \dfrac{a}{1-r}$ where a is the first term $\frac{35}{100}$ and r is the common ratio $\frac{1}{100}$. We prefer that the student become familiar with the concept of subtracting the "infinite tails" of decimals as a general way of making these conversions as follows: Let $x = .35\overline{35}$, so that $100x = 35.\overline{35}$. Then, subtracting

$$\begin{array}{r} 100x = 35.\overline{35} \\ -\quad x = 00.\overline{35} \\ \hline 99x = 35 \end{array}$$

and $x = \dfrac{35}{99}$.

The student will quickly learn to adapt this process of creating and subtracting off "infinite tails" of other such repeating decimals in an example like the following:

$$x = .\overline{123}$$

so that $1000x = 123.\overline{123}$. Then, subtracting:

$$\begin{array}{r} 1000x = 123.\overline{123} \\ -\quad x = 000.\overline{123} \\ \hline 999x = 123 \end{array}$$

which implies that $x = \dfrac{123}{999} = \dfrac{41}{333}$.

One final example should provide ample concept reinforcement: Let $x = 6.321\overline{515}$. Then, $100x = 632.15\overline{15}$. Subtracting gives:

$$\begin{array}{r} 10000x = 63215.\overline{15} \\ -\quad 100x = 632.15\overline{15} \\ \hline 9900x = 62583 \end{array}$$

Thus, $x = \dfrac{62583}{9900}$.

Of course, this technique can be used to derive the aforementioned formula for the sum of a geometric series as follows:

$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + \ldots + ar^n + \ldots \\ rS &= \quad\; ar + ar^2 + ar^3 + \ldots + ar^n + \ldots \end{aligned}$$

Thus,

$$S - rS = (1-r)S = a \text{ so that } s = \frac{a}{1-r}.$$

Once the concept has been mastered, teachers can work in groups to create their own examples. The TI-83 graphing calculator (or other comparable calculator) can be used to check the answers. Setting the “mode key” to the maximum nine decimal places and using the “math key” to convert fractions to decimals and vice versa can provide an ample check.

Solving Systems of Linear Equations

Systems of two linear equations and two unknowns, that have exactly one solution, are taught in eighth grade mathematics and Algebra I. Systems of two linear equations and two unknowns that have either no solution or infinitely many solutions are not discussed. In this workshop, we wanted to explore all the different cardinalities of solution sets to systems of two linear equations and two unknowns, and then generalize these results to larger systems with more than two unknowns. In addition, methods to solve larger systems of linear equations by hand using techniques from linear algebra, as well as how to solve systems on the graphing calculator, were discussed.

To motivate this topic, we started with an activity where we asked the teachers the following questions: Does every system of two linear equations and two unknowns have a solution? Is it possible to find a system of two linear equations and two unknowns that has exactly two solutions? Most teachers guessed the correct answer to the first question, but not the second. The teachers were asked to come up with different graphs to try to illustrate these questions and make conjectures about how many solutions a system of linear equations can have. Even though the teachers were unable to draw two lines that intersected exactly twice, some of them were still hesitant to say that such a system of linear equations does not exist.

We followed up this activity with a worksheet containing three systems of two linear equations with two unknowns, one with exactly one solution, one with no solution, and one with infinitely many solutions. The teachers were asked to solve all three systems using algebra only. Even though the teachers were able to work down to some "end result," they were not able to interpret their answers correctly. For example, ending up with an equality of $2 = 2$ or $0 = 2$ did not make sense.

In an effort to get the teachers to interpret their results on their own, we asked the teachers to graph each system of linear equations. Suddenly, everyone started to make sense of the answers that they had gotten algebraically. They were able to conjecture that every time a system of linear equations has no solution, one will always end up with an equality at the end that is mathematically absurd; i.e., $0 = 2$. Similarly, the teachers were able to notice that if two equations differed by a nonzero scalar only, then they were in fact the same equation, giving the system infinitely many solutions.

It is easy to remember that every system of linear equations has either exactly one solution, infinitely many solutions, or no solution by simply thinking about a system of two equations and two unknowns. Every pair of lines must either intersect in a point, be scalar multiples of each other (the same line), or be parallel, thus giving exactly one, infinitely many, or no solutions, respectively.

After the teachers were comfortable with this, we moved to systems of three equations and three unknowns. We discussed the different cardinalities of possible solution sets thinking of the three planes modeled by configurations within the classroom. Referencing the planes containing two adjacent walls as plane A and plane B, respectively, and the plane containing the floor as plane C, we could see that planes A and C intersect in the baseline and planes B and C meet in another baseline, while planes A and C meet in the corner line. Now, we can see that these three lines meet in a point where the corner intersects the floor. So, a unique solution is possible. To illustrate the case where three planes intersect in a line, we swing the corner door ajar and let plane D be the plane containing the door. Then planes A, B, and D intersect in the corner line. We can also indicate a no solution possibility which occurs when two planes such as the ones containing the ceiling and the floor are parallel. Of course, all three equations might represent the same plane within which infinite solutions exist. Hence, we need no additional props other than the visualization within the room configuration to model the possible solutions for three linear equations in three unknowns. The teachers were again able to be convinced that every system of three equations and three unknowns was going to have either exactly one solution, infinitely many solutions, or no solution. The teachers were asked to solve the following system:

Starting with $z=3$ and using back substitution, we get $2y-21=-17$ which implies that $y=2$. Finally, substituting both of these values into the first equation yields $x+2+6=9$, hence $x=1$.

When asked to solve the system

$$\begin{aligned} x+y+2z&=9 \\ 2y+4y-3z&=1 \\ 3x+6y-5z&=0 \end{aligned}$$

by hand, it was clear that some alternative methods would be necessary.

Since we wanted to be able to solve systems that involved any number of equations and unknowns, we introduced the teachers to matrices, Gaussian elimination, and row echelon (and reduced row echelon) form. After learning these ideas, the teachers were led to an understanding of how to use the three elementary row operations:

1) Multiply a row through by a nonzero constant;
2) Interchange two rows; and,
3) Add a multiple of one row to another row.

The teachers were asked to first solve a system of two equations and two unknowns using these three rules, and then solve the above system of three equations and three unknowns. This was enough to notice that the work involved is tedious and that the use of a graphing calculator would be handy.

We used the TI-83 and TI-84 calculators for this lesson, but any comparable calculator could be used. After learning how to input matrices into the calculator, we used the REF and RREF commands under the MATRIX menu to provide the output matrix in row echelon and reduced row echelon forms, respectively. When using RREF, one can read the solution for the system directly from the screen of the calculator for a system that has a unique solution. Similarly, systems that have no solution are easy to recognize as well. We need only look for a row in the output matrix that has all zeros except for the last entry. For example, to solve the system, we calculate

$$\begin{aligned} x+y+2z&=9 \\ 2y+4y-3z&=1 \\ 3x+3y+bz&=0 \end{aligned}$$

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 3 & 6 & 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 5.5 & 0 \\ 0 & 1 & -3.5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The last line indicates that 0 = 1 which shows that this system has no solution, just the same as what happened in the smaller systems that the teachers are familiar with that have two equations and two unknowns. Finally, in the situation where the system has infinitely many solutions, the output matrix will have fewer nonzero rows (meaning nonzero in the reduced row echelon form of the coefficient matrix) than there are variables in the system, as illustrated in the system shown below:

$$x+y+2z=9$$
$$2y+4y-3z=1$$

$$\begin{pmatrix} 1 & 0 & 5.5 & 17.5 \\ 0 & 1 & -3.5 & -8.5 \end{pmatrix}$$

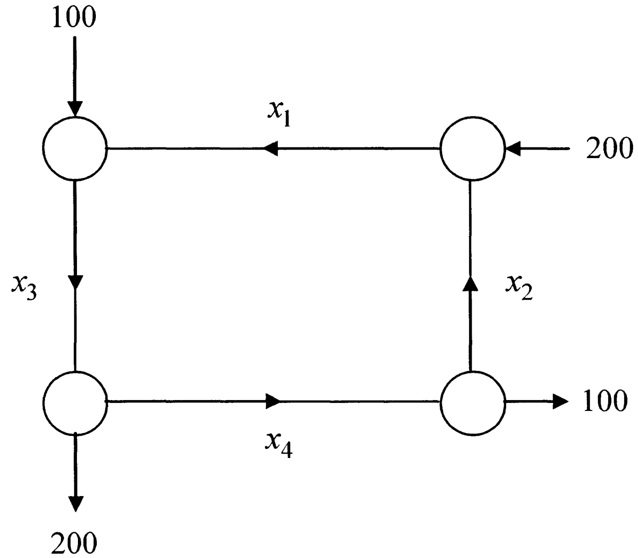
This system can be written now as

$$x+5.5z=17.5$$
$$y-3.5z=-8.5.$$

Let $z=t$, where t is any real number. Then $y=-8.5+3.5t$ and $x=17.5-5.5t$.

The use of the Gaussian elimination has the advantage of allowing one to solve any system of linear equations. Indeed, using the "inverse matrix method" for findings, as is taught for use on the Virginia *SOL*, is limited to systems that have a unique solution.

As an application that could be used for a post-*SOL* activity, we studied traffic flow through an intersection. Given an intersection, we can set up a system of *linear equations* following the idea that the flow of cars into the intersection has to equal the flow of cars out of the intersection. The teachers can easily set up the system of linear equations, using the calculator to find the reduced row echelon form. Consider the following network of streets.



Starting at the upper left and moving counterclockwise, the system of linear equations that corresponds to this diagram is:

$$\begin{cases} x_1 + 100 = x_3 \\ x_3 = x_4 + 200 \\ x_4 = x_2 + 100 \\ x_2 + 200 = x_1 \end{cases} \Rightarrow \begin{cases} x_1 - x_3 = -100 \\ x_3 - x_4 = 200 \\ -x_2 + x_4 = 100 \\ x_1 - x_2 = 200 \end{cases}$$

The calculator quickly produces the reduced row echelon form of the matrix for this system as:

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & 0 & 1 & -1 & 200 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution of the system can now be written as

$$\begin{aligned} x_4 &= t, \text{ where } t \text{ is any real number} \\ x_3 &= 200 + t \\ x_2 &= -100 + t \\ x_1 &= 100 + t. \end{aligned}$$

From this, we can see that the flow of traffic on the street labeled x_4 controls the flow of traffic in the entire network. For example, if there are 100 vehicles per hour moving along the street labeled x_4, then the flow of traffic on the remaining streets would be $x_1 = 200$, $x_2 = 0$, and $x_3 = 300$ vehicles per hour. This example also offers the opportunity for the workshop teachers to observe that in this situation, t must be a "whole number" since it represents a number of cars.

Conclusion

Although all of the topics in this paper are not new to teachers, our purpose was to get the teachers to think about math that they may have already known in a slightly different way. Learning how to generalize and think about special cases is an important tool in mathematics and is often difficult for teachers to do.

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