RIGHT TRIANGLES OF GIAN FRANCESCO MALFATTI

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Introduction.

Every triangle circumscribes a unique triple of circles, each of which is tangent to the other two. Figure 1 shows a right triangle which circumscribes three circles as described.

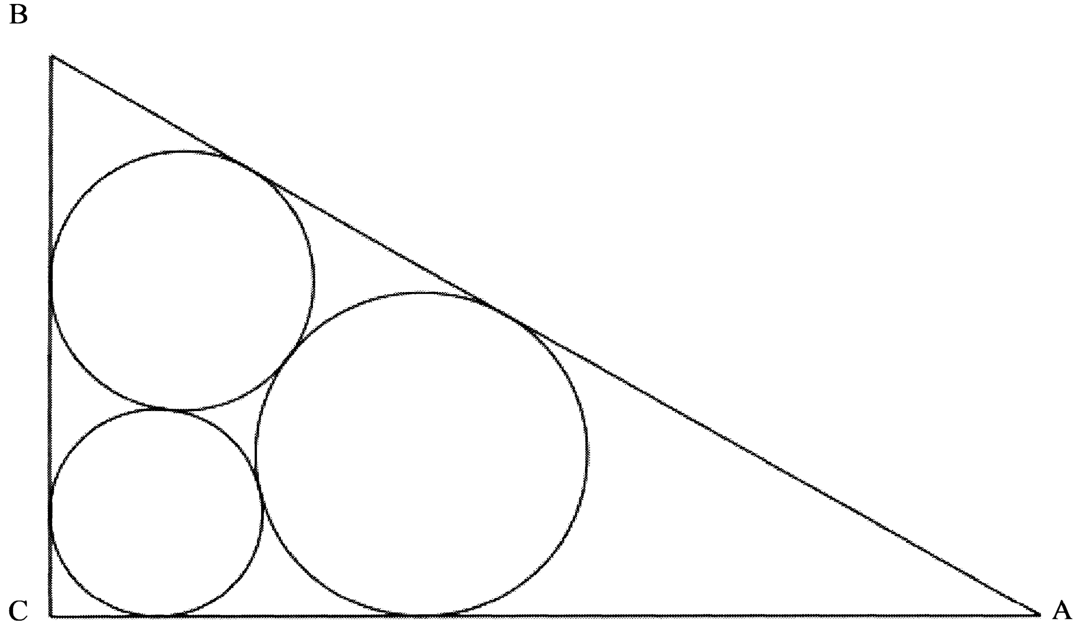

Figure 1. Right triangle with its circles.

Such circles are named Malfatti circles to honor the Italian mathematician Gian Francesco Malfatti who, in 1803, wrongly conjectured that the greatest area that can be bounded by three circles drawn within any triangular region is the area contained by the three Malfatti circles of the triangle. Using the search engine *Google*™ to search for "3 circles in a triangle" produced an enormous amount of information about the geometry of triangles and their Malfatti circles. Thus, it should be clear that no startling contributions to the subject are to follow.

Motivation

To this mathematics teacher, the most interesting problems are those that arise naturally from the material that he is teaching, that are easy to pose, and that quickly lead from the familiar to mathematical places new to him. So it was that the teacher (i. e., the author *of this article*) wondered how to write a program with *Mathematica* to create a figure like the one above. It was obvious that he needed to locate his triangle in the *xy*-plane and then to find the coordinates of the centers and the lengths of the radii of the three circles. Figure 2 shows the triangle above with the addition of the centers of the circles and the radii to the points of tangency between the circles and the sides of the triangle.

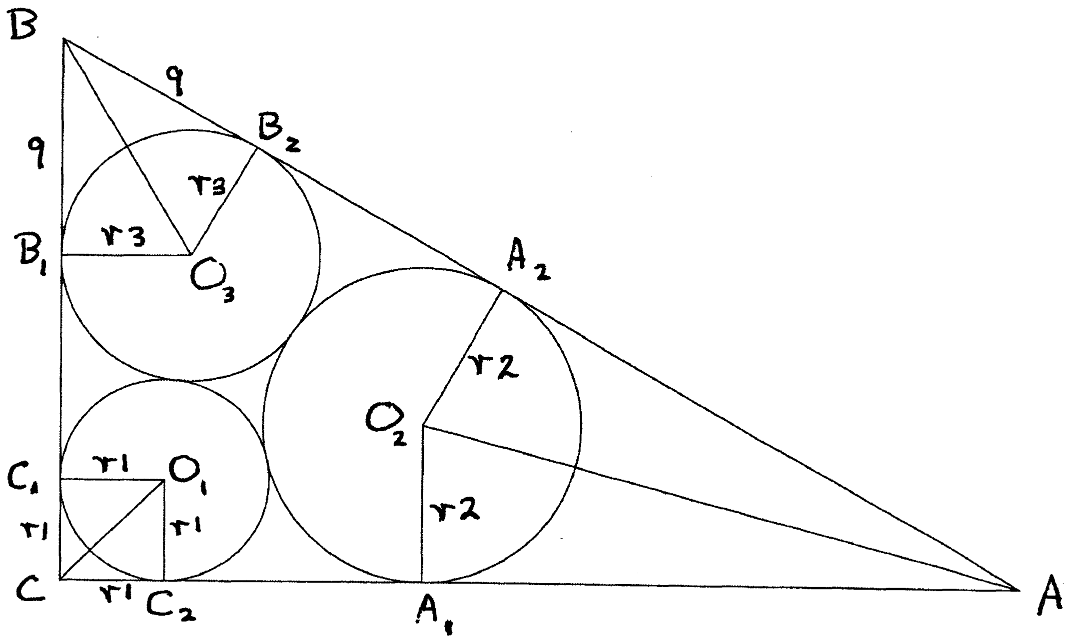

Figure 2. Right triangle with its Malfatti circles.

Facts from Grade 9 Geometry

The notation used in the statements that follow is derived from Figure 2. Although summer will remove many of these statements from the rising grade 10 memory, the facts and ideas with which the statements are concerned were once current and familiar in grade 9 geometry class. The centers of the three circles are O_1, O_2, and O_3 and the corresponding radii are r1, r2, and r3.

1) Tangent segments AA_1 and AA_2 have the same length. In this case, $AA_1 = AA_2 = p$. Also $BB_1= BB_2 = q$ and $CC_1 = CC_2 = r1$. Note that CC_1 and CC_2 will also have the same length, but that length is the same as radius r1 only because angle ACB is a right angle.

2) Rays AO_2, BO_3, and CO_1 bisect their angles BAC, ABC, and BCA, respectively.

3) B_1 C_1, A_1 C_2, and A_2 B_2 are common external tangents for their circles and have lengths $2\sqrt{r1 * r3}$, $2\sqrt{r1 * r2}$, and $2\sqrt{r2 * r3}$, respectively.

If one starts with a correctly given triple of parts that determines the congruence of triangles, one (in theory) ought to be able to compute the radii and the coordinates of the centers of the Malfatti circles of a triangle. If a triangle is a right triangle as shown in Figure 6, one can write five equations which, when solved, will supply the information needed to write the program to create the figure. Under the assumption that triangle ABC of Figure 2 is a right triangle with all sides and angles known, these five equations hold true:

$$p + q + 2\sqrt{r2 * r3} = AB,$$
$$r1 + p + 2\sqrt{r1 * r2} = AC,$$
$$r1 + q + 2\sqrt{r1 * r3} = BC,$$
$$\tan(\angle A/2) = r2/p, \text{ and } \tan(\angle B/2) = r3/q.$$

If the triangle is taken to represent the general case, six equations are required and an enormous amount of algebra is necessary to achieve a solution. Goldilocks might have said, "The general triangle is too hard and the equilateral triangle is too easy. The right triangle is just right."

Malfatti Circles in Right Triangles

Grade 9 geometers at the top of their game should understand the thinking that went into the five equations above. Even though the difficulty of the Malfatti circles in a right triangle is "just right", the algebra involved in solving the five equations is still quite challenging. However, *Mathematica* can do the algebra as well as draw the figures.

Here are two examples to argue the richness of the blend of analytic geometry and technology in problems on Malfatti circles.

Example 1. Find the radii and centers of the Malfatti circles in a 30°- 60°- 90° right triangle with sides of lengths 5, $5\sqrt{3}$, and 10 units. Then, draw the triangle with its circles.

Solution. Since a figure is needed in explaining the solution, it makes sense to place the cart before the horse in this instance. So here is ΔABC with right $\angle C$ in Figure 3; the program for creating the figure will follow.

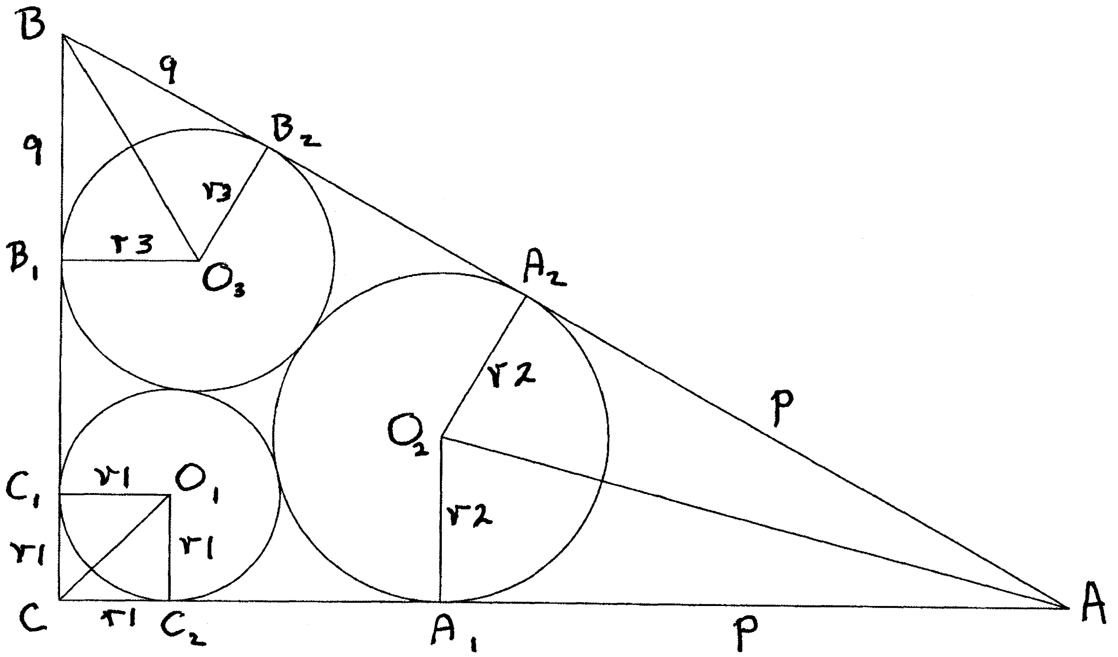

Figure 3. A 30° - 60° - 90° right triangle.

In ΔABC, $\angle A = 30°$, $\angle B = 60°$, and $\angle C = 90°$ with $AB = 10$, $AC = 5\sqrt{3}$, and $BC = 5$. The five equations written in the symbols developed above are:

$$
\begin{aligned}
&p + q + 2\sqrt{r2 * r3} = 10,\\
&r1 + p + 2\sqrt{r1 * r2} = 5\sqrt{3}\\
&r1 + q + 2\sqrt{r1 * r3} = 5,\\
&\tan(\angle A/2) = \tan(30°/2) = r2/p = 2 - \sqrt{3}, \text{ and}\\
&\tan(\angle B/2) = \tan(60°/2) = r3/q = 1/\sqrt{3}.
\end{aligned}
$$

The last two equations may be rewritten as $p = (2 + \sqrt{3})\, r2$ and $q = \sqrt{3}\, r3$. Substitution of these expressions for p and q in other equations leaves only the following three equation to be solved for the radii:

$$
\begin{aligned}
&(2 + \sqrt{3})r2 + \sqrt{3}\, r3 + 2\sqrt{r2 * r3} = 10,\\
&r1 + (2 + \sqrt{3})r2 + 2\sqrt{r1 * r2} = 5\sqrt{3}, \text{ and}\\
&r1 + \sqrt{3}\, r3 + 2\sqrt{r1 * r2} = 5.
\end{aligned}
$$

Mathematica gives the speedy numerical solution r1 = 0.928434, r2 = 1.44996, r3 = 1.15499. Figure 4 lists the instructions which lead to the solution. It follows that p = $(2 + \sqrt{3})$r2 = 5.41131 and q = $\sqrt{3}$ r3 = 2.0005.

```
NSolve[{(2 + √3)r2 + √3 r3 + 2√(r2 * r3) == 10,
  r1 + (2 + √3) r2 + 2√(r1 * r2) == 5√3 , r1 + √3 r3 + 2√(r1 * r3) == 5},
{r1, r2, r3}]
{{r1 → 0.928434,      r2 → 1.44996,      r3 → 1.15499}}
```

Figure 4. Instructions for finding the three radii with *Mathematica*.

If the coordinates of the vertices of the triangle are taken to be $(5\sqrt{3}, 0)$ for A, (0, 5) for B, and (0, 0) for C, reference to Figure 3 will reveal the coordinates of the centers of the circles. The coordinates of O_1 are (r1,r1) = (0.928434, 0.928434), the coordinates of O_2 are $(5\sqrt{3}$ - p, r2) = (3.24894, 1.44996), and the coordinates of O_3 are (r3, 5 - q) = (1.15499, 2.9995). Now that the coordinates of the centers and radii of the three circles have been computed, all input information needed for *Mathematica* to draw the 30°-60°-90° right triangle with its Malfatti circles is available. Here is the program that produced Figure 3:

```
r1 = 0.928434; r2 = 1.44996; r3 = 1.15499;

q = √3 1.1549879938480403
2.0005

5 - q

2.9995

p = (2 + √3) * 1.44995658969149
5.41131

5 √3 - p
3.24894

list1 = {{0, 0}, {5 √3, 0}, {0, 5}, {0, 0}};

plot1 = ListPlot[list1, PlotJoined → True,
    AspectRatio → Automatic, PlotStyle → GrayLevel[0], Axes → False];

list2 = {{r1, 0}, {r1, r1}, {0, r1}, {0, 0}, {r1, r1}};

plot3 = ListPlot[list2, PlotJoined → True,
    AspectRatio → Automatic, PlotStyle → GrayLevel[0], Axes → False];


list3 = {{0, 5 - q}, {r3, 5 - q}, {r3 (1 + 1 / 2), 5 - q + r3 * √3 / 2}};

plot4 = ListPlot[list3, PlotJoined → True,
    AspectRatio → Automatic, PlotStyle → GrayLevel[0], Axes → False];

plot5 = ListPlot[{{0, 5}, {r3, 5 - q}}, PlotJoined → True,
    AspectRatio → Automatic, PlotStyle → GrayLevel[0], Axes → False];

plot6 = ListPlot[{{5 √3 - p, r2}, {5 √3, 0}}, PlotJoined → True,
    AspectRatio → Automatic, PlotStyle → GrayLevel[0], Axes → False];

list4 = {{5 √3 - p, 0}, {5 √3 - p, r2}, {(5 √3 - p) + r2 / 2, r2 (1 + √3 / 2)}};

plot7 = ListPlot[list4, PlotJoined → True,
    AspectRatio → Automatic, PlotStyle → GrayLevel[0], Axes → False];

plot2 = ParametricPlot[{{.928434 (1 + Cos[t]), .928434 (1 + Sin[t])}, {3.24894 + 1.44996 Cos[t],
      1.449969 (1 + Sin[t])}, {1.15499 (1 + Cos[t]), 2.9995 + 1.15499 Sin[t]}},
    {t, 0, 2 π}, PlotStyle → GrayLevel[0], Axes → False];

Show[plot1, plot2, plot3, plot4, plot5, plot6, plot7]
```

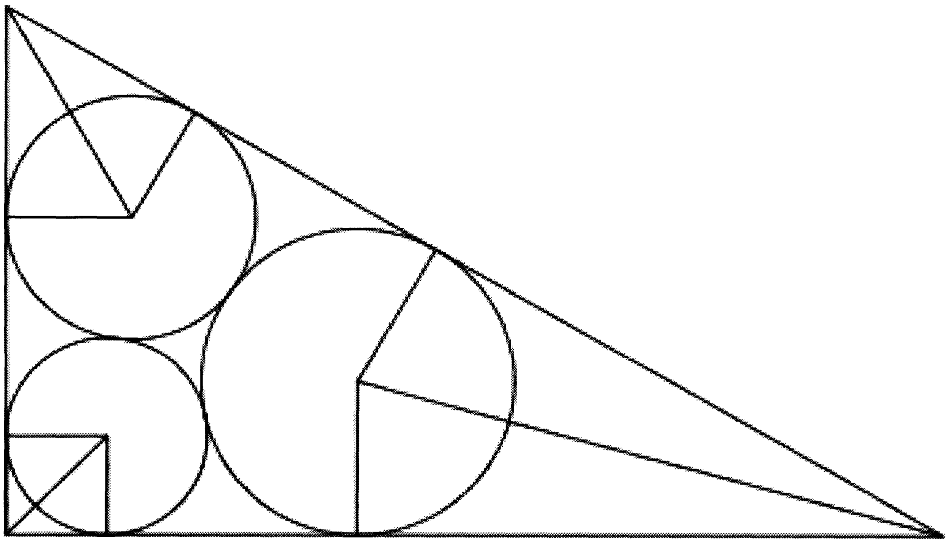

Figure 5. Program for the 30°-60°-90° right triangle.

Here is the second example. Since it involves an isosceles right triangle, it is quite a bit simpler than the first example.

Example 2. Find the radii and centers of the Malfatti circles in a 45°- 45°- 90° right triangle with sides of lengths 10, 10, and $10\sqrt{2}$. Then, draw the triangle with its circles.

Solution. Here is ΔABC with right angle at C as shown below in Figure 6. The notation is the same as that in Figure 3, but the symmetry of the isosceles triangle offers significant simplifications. Thus, AC = BC = 10, AB = $10\sqrt{2}$, $\angle A = \angle B = 45°$, and $\angle C = 90°$. Also, $AA_1 = AA_2 = BB_1 = BB_2 = p$ and $\tan(\angle A/2) = \tan(\angle B/2) = \tan 22.5° = \sqrt{2} - 1$.

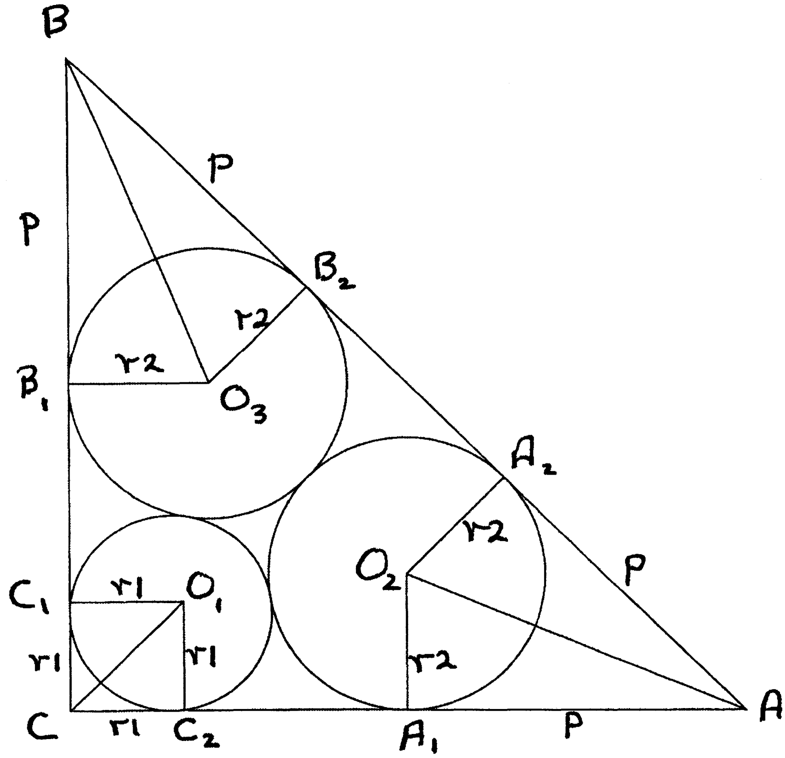

Figure 6. Isosceles right triangle.

Symmetry requires that r2 = r3 and reduces the five equations of the first example to only three in this case. The three equations to be solved are:

$$\begin{aligned} &p = (\sqrt{2} + 1)r2, \\ &2p + 2r2 = 10\sqrt{2}, \text{ and} \\ &r1 + p + 2\sqrt{r1 * r2} = 10. \end{aligned}$$

Then, *Mathematica* wastes little time in solving for r1, r2, and p.

```
Clear[r1, r2, p]

NSolve[{p == (√2 + 1) r2, p + r2 == 5 √2, r1 + p + 2 √(r1 * r2) == 10}, {r1, r2, p}]
{{r1 → 1.48847, r2 → 2.07107, p → 5.}}
```

Figure 7. Solution of the three equations with *Mathematica*.

It follows that the radii of circles O_1, O_2, and O_3 are r1 = 1.48847, r2 = 2.07107, and r3 = 2.07107, respectively. Reference to Figure 6 reveals that the coordinates of O_1 are (r1, r1) = (1.48847, 1.48847), the coordinates of O_2 are (10 - p, r2) = (5, 2.07107), and the coordinates of O_3 are (r3, 10 - p) = (r2, 10 - p) = (2.07107, 5). The program which produced Figure 6 is listed below.

```
Clear[r1, r2, p]

p = 5; r1 = 1.48847; r2 = 2.07207;

list3 = {{0, 0}, {10, 0}, {0, 10}, {0, 0}};

list4 = {{0, r1}, {r1, r1}, {r1, 0}};

plot5 = ListPlot[list4, PlotJoined → True,
   AspectRatio → Automatic, PlotStyle → GrayLevel[0], Axes → False];

plot3 = ListPlot[list3, PlotJoined → True,
   AspectRatio → Automatic, PlotStyle → GrayLevel[0], Axes → False];

plot6 = ListPlot[{{0, 0}, {r1, r1}}, PlotJoined → True,
   AspectRatio → Automatic, PlotStyle → GrayLevel[0], Axes → False];

plot7 = ListPlot[{{0, 10}, {r2, 5}}, PlotJoined → True,
   AspectRatio → Automatic, PlotStyle → GrayLevel[0], Axes → False];

list5 = {{0, 5}, {r2, 5}, {(1 + 1/√2) r2, 5 + (1/√2) r2}};

plot8 = ListPlot[list5, PlotJoined → True,
   AspectRatio → Automatic, PlotStyle → GrayLevel[0], Axes → False];

plot9 = ListPlot[{{10, 0}, {5, r2}}, PlotJoined → True,
   AspectRatio → Automatic, PlotStyle → GrayLevel[0], Axes → False];

list6 = {{5, 0}, {5, r2}, {5 + r2/√2, r2 (1 + 1/√2)}};

plot10 = ListPlot[list6, PlotJoined → True,
   AspectRatio → Automatic, PlotStyle → GrayLevel[0], Axes → False];

plot4 = ParametricPlot[{{1.48846 (1 + Cos[t]), 1.48846 (1 + Sin[t])},
    {5 + 2.07107 Cos[t], 2.07107 (1 + Sin[t])}, {2.07107 (1 + Cos[t]), 5 + 2.07107 Sin[t]}},
   {t, 0, 2 π}, PlotStyle → GrayLevel[0], Axes → False];

Show[plot3, plot4, plot5, plot6, plot7, plot8, plot9, plot10]
```

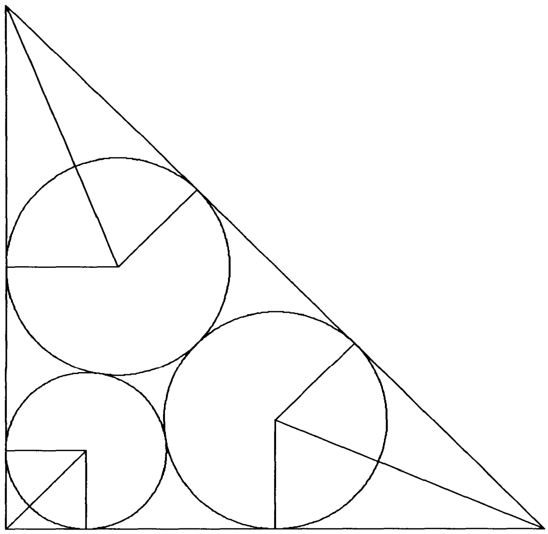

Figure 8. Program for the 30°-60°-90° right triangle.

Suggestions for Other Problems.

One good thing about teaching geometry is that fun and work often overlap. Such was the case with the circles and triangles of Gian Francesco Malfatti. The summer's assigned work was to use the Internet to seek enrichment material for the geometry class of 2011-12. The fun was learning more geometry (new to this teacher, old to many others), and then using the computing power of *Mathematica* to achieve the results described above. Should there be readers who found these ideas to be of interest, more fun awaits them in the 3 - 4 - 5 right triangle and in isosceles triangles with nice integer sides.