THE STABILITY OF BOATS: A SCIENCE, TECHNOLOGY, ENGINEERING, AND MATHEMATICS (STEM) EXERCISE

G. RUBLEIN
Dept. of Mathematics, The College of William and Mary
Williamsburg, VA 23187

Abstract

In what might be called genuine Science, Technology, Engineering, and Mathematics (STEM), an engineering construct subject to well-understood physical principles is analyzed mathematically to yield predicted behavior. In this article, we provide just such an example. The mathematics is at the high school level. Among other things, one actually sees an application of the quadratic formula. Experimental verification of the results may be realized with simple materials.

Introduction

In an extraordinary piece of geometric analysis, Archimedes studied the stability of a floating paraboloid of revolution. An account of this work may be found in Stein [1]. Heath's book contains still more material on the work of Archimedes [2]. In the context of the work in the present article, readers of either of these books will take note of the important role that the density of the boat material plays in questions of stability.

While studying Archimedes' work in a freshman seminar class, the idea of considering the stability of boats of a simpler structure arose, potentially making it easier to compare the mathematics of the problem with experimental results. The most obvious of these is a rectangular parallelepiped with a square cross section. Each of our boats will have a length substantially longer than its square edge. The use of a long boat allows us to deal only with the cross section of a partially immersed square. In the parlance of aircraft stability, our boat may roll, but is not permitted to yaw or pitch. A study of the stability of a boat in the shape of an arbitrary rectangular parallelepiped requires a deeper analysis.

In the photograph in Figure 1, we show three examples. The left-hand boat is a square cross-section plastic box with a tight fitting lid, while the central boat is solid pine. The right-
hand boat is a square cross-section boat of pine with a 3/8" steel spine fitted along its long axis.

Figure 1. Three square boats: How will they float?

The elementary geometric and algebraic methods that will be employed are appropriate to good high school students interested in physical or engineering applications. Any desired experimental materials can be cheaply constructed.

As a general matter, stability investigations deal with small perturbations from an "equilibrium" position of some physical system. One asks whether or not that system, a boat in this case, will tend to return to that equilibrium once slightly disturbed. To study our problem, we need specific help from Archimedes. These principles may be proved using modern methods:

A1. The weight of a rigid body acts at its center of mass. [Archimedes would have said "center of gravity."]
A2. A body floating in water displaces a quantity of water whose weight matches that of the boat.

A3. The (upward) supporting weight of the displaced water acts at its center of mass, a location commonly called the “center of buoyancy.”

A4. Two opposing forces acting on a rigid body will cause it to rotate if and only if the forces act along different lines (the lever principle).

Now, we need a disclaimer that will hold throughout the article. We will confine our investigation solely to those perturbations that leave the volume of displaced water fixed. Something like a down draft on a real boat would violate this constraint. Eventually, however, that boat would return to some calm water configuration in which, by Archimedes A2, the boat would displace precisely its weight in water. Our analysis could then be deemed to begin at this stage of the perturbation.

Alternatively, one may decide to examine only those configurations in which our square boat is initially floated with sufficient care that the weight of the displaced water volume does, indeed, exactly match the weight of the boat. Then, our analysis will deal with the stability of the chosen immersed configuration.

**The First Stability Question**

We begin with a long boat of square cross section whose center of mass falls midway along its long central axis. A simple way to accomplish this is to use a solid boat of uniform density. According to our agreement on the elongated shape of the boat, we need to work only with the plane cross section of the boat and of the water it displaces while floating.

We want to launch our boat by simply placing it in water. One possible configuration for this launch would put one of the long flat rectangular surfaces of the boat parallel to the water surface. (Passengers would then have a convenient place to walk.) Is this configuration stable if the boat encounters a very small wave?

In its strictly upright position, the boat will maintain its horizontal orientation because,
as in Figure 2, the supporting weight of the water and the weight of the boat act along the same vertical line. There is no rotation.

Figure 2. A boat in strict horizontal float.

If, however, under the influence of a mote of dust or a small ripple in the water, the boat lists slightly, say, to the right, we would see a configuration shown in Figure 3. For the sake of clarity, we have exaggerated the size of the list-angle, \( \theta \), in that figure.

Temporarily, we use a rectangular coordinate frame with origin at the lower left-hand corner of the boat, and with axes fixed to the edges of the boat. This is the "body-fixed" frame. In this frame, the coordinates of the center of mass of the boat are obviously the same, list or no list. They are \( B = (e/2, e/2) \), where \( e \) is the edge-length of the square cross section of the boat.

The center of mass of the supporting displaced water is a different matter. Crucial to the resolution of the stability problem is the recognition that that center moves, relative to the fixed-body axes, because the shape of the displaced water changes when the boat lists. In particular, that shape shifts from rectangular to trapezoidal (see Figure 3). The diagram suggests the variability of the center of buoyancy of this trapezoid.
We need to locate the center of buoyancy, and there are a variety of ways of managing this technical matter. One way or another, one must deal with the center of mass of a continuous body—the displaced water. Archimedes had a method that relies on the principle of the lever and fundamentally does an integration. His “summation methods” are recounted in Archimedes: What Did He Do Besides Cry Eureka! [1]. We omit the details and simply give the answer that can be obtained by an ordinary integration.

We let $a$ and $b$ be the short and long legs of the displaced trapezoid. If we denote by $A$ the area of the trapezoid, then obviously $A = e(a + b)/2$. Using our body-fixed rectangular frame, the center of mass $(L, H)$ of the trapezoid will satisfy

\[
AL = \frac{e^2}{6}(a + 2b) \\
AH = \frac{e}{6}(b^3 - a^3)/(b-a)
\]

Students unfamiliar with either version of integration may, instead, employ yet another Archimedean “global” lever principle to find the center of gravity of the trapezoid. A rectangle of base $e$ and height $a$ has area $A_1 = ea$ and center of gravity at $(e/2, a/2)$. A triangle of base $e$ and height $b - a$ (mounted above the rectangle) has area $A_2 = e(b-a)/2$ and
center of gravity \((2e/3, a + (b - a)/3)\). Then according to the principle of the lever,

\[ AL = A_1(e/2) + A_2(2e/3) \]

and

\[ AH = A_1a/2 + A_2(a + (b - a)/3) \]

One may check that these give the same results as equation (1).

Let us write \(\hat{\rho}\) for the weight density of the boat material in some chosen units. Let \(\delta\) be the weight density of water in those same units and let \(\ell\) be the length of the boat. Then according to Archimedes A2,

\[ e^2 \ell \hat{\rho} = A\ell \delta, \]

so that \(\hat{\rho}/\delta = A/e^2\) is a quantity whose numerical value is independent of the units of measurement. Throughout the remainder of this article, we write \(\rho\) for the ratio \(\hat{\rho}/\delta\), the weight density of the boat material measured in units of the weight density of water. For short, we refer to \(\rho\) simply as the weight density of the boat.

As \(\rho\) is a (constant) physical characteristic of the boat material and \(e\) is a geometric constant characteristic of the boat shape, the quantity \(A = \rho e^2\) is also a constant, independent of the list-angle \(\theta\). Just to remind the reader, while the area of immersion of the boat is constant, the shape of that immersion is not. In much of the sequel, our stability results will be expressed in terms of the physical constant:

\[ \rho = A/e^2 \]

The dependence of the edge-lengths, \(a\) and \(b\), on the list-angle \(\theta\) is determined by the geometry of the trapezoid, a figure whose area, \(A\), is constant for our particular boat.

\[ b = a + e \tan(\theta) \]

\[ A = e \frac{a + b}{2} \]

so that,

\[ a = A/e - (e/2) \tan(\theta) \]

\[ b = A/e + (e/2) \tan(\theta) \]

It is, therefore, convenient to rewrite the formulas in (1) to describe the coordinates of the center of buoyancy as functions of the area and the list-angle:
The stability of boats...

\[
L = (e/2) + e^3 \tan \theta / (12A) \\
H = (e/6)[3A/e^2 + e^2 \tan^2 \theta / (4A)].
\] (2)

Then in a quite natural way, the density, \( \rho \), of the boat material, makes an appearance:

\[
L = (e/2)[1 + \tan(\theta) / 6\rho] \\
H = (e/2)[\rho + \tan^2(\theta)/12\rho].
\]

It remains now to locate the relative positions of the points \( B \) and \( W \) in a coordinate frame whose horizontal axis is parallel to the waterline. We need only rotate the boat-fixed frame counterclockwise through the list-angle. Then in this new frame, the respective horizontal and vertical coordinates of the boat center of mass, \( B \) are \((e/2)(\cos \theta + \sin \theta, -\sin \theta + \cos \theta)\). The coordinates of the center of buoyancy are \((L \cos \theta + H \sin \theta, -L \sin \theta + H \cos \theta)\).

The forces acting on the boat are its weight and the buoyant support of the water. These are equal and opposite, and both act vertically along lines perpendicular to the waterline. However, these vertical lines can be different from one another, giving rise to a rotation as in Archimedes' principle A4. Thus, by Archimedes' A1, the vertical line of action of the boat's weight passes through the coordinate of \( B \) along the direction of the waterline, namely \((e/2)(\cos \theta + \sin \theta)\). Meanwhile, by Archimedes' A4, the vertical buoyant force passes through the horizontal component of \( W, L \cos \theta + H \sin \theta \), along the direction of the waterline.

For stability of the clockwise listing boat, the center of buoyancy must fall to the right of the center of mass of the boat so as to exert a restoring rotation to the boat. Otherwise, the rotation will continue in the clockwise direction, tipping the boat even further.
Stability Condition: \[ L \cos \theta + H \sin \theta > \left( \frac{e}{2} \right) (\cos \theta + \sin \theta) \]  
Instability Condition: \[ L \cos \theta + H \sin \theta < \left( \frac{e}{2} \right) (\cos \theta + \sin \theta) \]  

Using the expressions for \( L \) and \( H \) as given by (2), the stability condition (3) is

\[
\left( \frac{e}{2} \right) [\cos \theta + \sin \theta/(6p) + \rho \sin \theta + \tan^2 \theta \sin \theta/(12p)] > \left( \frac{e}{2} \right) (\sin \theta + \cos \theta).
\]

The \( (e/2) \cos \theta \) terms cancel and we can factor out the common \( \sin \theta > 0 \) multiplier.

Then for stability,

\[
1/(6p) + \rho + \tan^2 \theta/(12p) > 1.
\]

That is,

\[
6 \rho^2 - 6 \rho + 1 + \tan^2 \theta/2 > 0.
\]

Now, the quadratic polynomial \( f(p) = 6 \rho^2 - 6 \rho + 1 \) has roots at \( 1/2 \pm \sqrt{3}/6 \). Moreover, \( f(p) \) is non-negative except for \( \rho \in \left( 1/2 - \sqrt{3}/6, 1/2 + \sqrt{3}/6 \right) \). This means the stability condition above is satisfied for all \( \rho \in \left( 0, 1/2 - \sqrt{3}/6 \right] \cup \left[ 1/2 - \sqrt{3}/6, 1 \right) \) and all non-zero values of \( \theta \). We would, of course, prohibit such large values of the perturbation angle \( \theta \) that would violate the trapezoidal nature of the perturbed geometry. This is a trivial matter here.

Our result: For small perturbation list-angles, a square cross-section boat with uniform density will float stably in the horizontal position for densities that satisfy:

\[ 0 < \rho \leq 1/2 - \sqrt{3}/6 \quad \text{and} \quad 1/2 + \sqrt{3}/6 \leq \rho < 1. \]

We call these the stable ranges for the horizontal configuration. For intermediate densities, that is, \( \rho \in \left( 1/2 - \sqrt{3}/6, 1/2 + \sqrt{3}/6 \right) \), the horizontal configuration is unstable.
The previous discussion has dealt with rightward perturbations. For stability at the horizontal equilibrium, we examined displacements in which the boat would enjoy a counterclockwise recovery from some disturbed rightward position. Formally, this meant that our list-angle, $\theta$, was deemed to be positive if taken in the clockwise direction.

From a physical point of view, there should be no distinction in the classification of stable and unstable configurations under left and right, or right perturbation. However, it is a simple matter to employ the analysis we have used to obtain clockwise recovery from a disturbed leftward position.

The analysis may be repeated verbatim while the sign of the list-angle, $\theta$, remains unspecified. To complete the argument so as to arrive at appropriate recovery/non-recovery inequalities, we remove a common factor, $\sin \theta$. For negative list-angles, that multiplier is negative. The directions of the resulting inequalities are, therefore, determined by the sign of the perturbing list-angle. The influence of the sign of the list-angle is now obvious. If the perturbing list-angle is positive (in the clockwise direction), recovery will be counterclockwise. If the perturbing list-angle is negative, recovery will be clockwise. We have, then, Theorem 1.

**Theorem 1.** Given a long boat of homogeneous material with square cross section whose density, $\rho$, falls in either of the two stable ranges above, there is a tolerance, $\epsilon = \epsilon(\rho) > 0$, such that the boat will remain stable in the horizontal configuration under any list perturbation with $|\theta| < \epsilon$.

**The Second Stability Problem**

Square boats of intermediate densities are unstable in the horizontal configuration. Easy experiments with such boats suggest that a “45 degree” orientation is stable. We examine this issue and Theorem 2 is the result.

**Theorem 2.** A long boat of square cross section will stably float in the 45 degree orientation against small perturbations if its density, $\rho$, falls in the range $9/32 \leq \rho \leq 23/32$.

Owing to different geometric shapes of the displaced water for light and heavy boats, we first consider densities less than $1/2$ and later densities greater than $1/2$. The proof of the
theorem for the singular case $\rho = 1/2$ is left to the reader.

That reader will notice a gap between this result and the range for stable horizontal float. There are two ranges of densities, $1/2 - \sqrt{3}/6 < \rho < 9/32$ and $23/32 < \rho < 1/2 + \sqrt{3}/6$ where neither the horizontal nor the 45 degree orientation is stable. (The author will provide a mathematical analysis in a subsequent paper.)

A. The case $\rho < 1/2$.

For a boat of density less than 1/2, the displaced water in the strict 45 degree orientation has the shape of an isosceles right triangle. If such a boat lists slightly to the right, the displaced water remains right triangular, but is no longer isosceles. Figure 4 shows such a displacement triangle with short leg $a$ and long leg $b$.

![Figure 4. A low-density boat listing from the 45 degree configuration.](image)
A natural “base line” against which to measure the list-angle is a diagonal of the square. We denote this list-angle by $\phi$. The geometry of the right triangle permits one to link $a$ and $b$:

$$a = b \tan \left( \frac{\pi}{4} - \phi \right) = \frac{b(1 - \tan \phi)}{1 + \tan \phi}. \quad (4)$$

As before, in order to locate the center of mass of the boat and center of buoyancy in the list orientation, we may use the rectangular coordinate frame fixed to the boat with origin at the deepest corner of the displaced water: “vertical” along the $a$ side, “horizontal” along the $b$ side. In this coordinate frame, the center of mass of the boat is, of course, $(e/2)(1, 1)$. The center of buoyancy is $(b/3, a/3)$. Before we pass to the waterline coordinates, it is useful to use an intermediate coordinate frame, also fixed to the boat. Thus, our new vertical will be the diagonal of the boat issuing from the deepest point of water displacement. The new horizontal will be a perpendicular to this diagonal, also passing through the deepest point (see Figure 5).

Figure 5. An interim coordinate frame for a low-density boat.
Using standard distance formulas, point to line, we can compute the coordinates of the center of mass of the boat, $B$, and the center of buoyancy, $W$, in this new frame:

$$B = e(0, \sqrt{2}/2)$$

$$W = 1/(3\sqrt{2})(b-a, b+a)$$

Rotating this intermediate frame counterclockwise through the list-angle, we come to respective horizontal coordinates along the waterline. For the center of buoyancy:

$$[(b-a)/(3\sqrt{2})]\cos(\phi) + [(b+a)/(3\sqrt{2})]\sin(\phi).$$

For the center of mass of the boat:

$$(e\sqrt{2}/2)\sin(\phi).$$

Which of these is larger? If the former, the boat is stable in the 45 degree position. If the latter, the boat is unstable.

From (3), we have $b-a = 2b \tan \phi/(1 + \tan \phi)$ and $b+a = 2b/(1 + \tan \phi)$. Hence, we can establish a criterion for stability:

$$(4/3)b/(1+\tan(\phi)) > e$$

Or,

$$(16/9)b^2/(1+\tan(\phi))^2 > e^2.$$ 

Or

$$(16/9)ab/(1-\tan^2(\phi)) > e^2.$$ 

Remembering that the area, $A$, of the displaced water is $A = ab/2$, we notice that the density, $\rho = A/e^2$, will appear. Thus, the stability criterion, for $\rho < 1/2$ and the boat in the 45 degree orientation, is

$$A > (9/32)(1-\tan^2 \phi) > e^2.$$
Then for any $\rho \geq 9/32$ and any perturbation angle, $\phi \neq 0$, this stability criterion is satisfied. Once again, we should observe a trivial constraint on the perturbation angle: It should not be so large as to spoil the triangular shape of the perturbed configuration. Then for boats of density less than 1/2, the density must fall in the interval $9/32 \leq \rho < 1/2$ in order that the 45 degree orientation be stable under small perturbation.

To complete the argument for the case $\rho < 1/2$, we need only mention that removal of the factor $\sin \phi$ from our inequalities will enforce the symmetry of the stability criterion for negative angular perturbations, $\phi$.

B. The case $\rho > 1/2$.

Now, the perturbed cross section is a five-sided figure (see Figure 6). The (perturbed) waterline cuts off a short leg of length $a$ and a long leg of length $b$. As before, the edge-length of the boat is denoted $e$.

Figure 6. A high-density boat listing from the 45 degree configuration.

We may continue with a brute-force analysis employing some unpleasant algebra.
and trigonometry to verify the stability of the 45 degree orientation for densities $1/2 < \rho < 23/32$. It is, however, more efficient to rely on the work already done on the low density case. We need a “dual” to that argument.

Let us review the low-density configuration, the perturbed version of which is shown in Figure 4. Use $x$’s generically for coordinates along the waterline. In the perturbed Figure 4, $A$ is the area below the waterline, $x_1$ the coordinate of its CG. The boat of area $e^2$ has coordinate $x_0$ for its CG and the pentagonal region with area $e^2 - A$ has coordinate $x_2$ for its CG. By the lever principle (or an ordinary integration), $A x_1 + (e^2 - A) x_2 = e^2 x_0$.

Or better,

$$A(x_1 - x_0) + (e^2 - A)(x_2 - x_0) = 0.$$ 

Now, to study stability in the low density 45 degree configuration, compute the total moment about the CG of the boat after clockwise angular perturbation $\phi$. The boat itself contributes no moment, and the water contributes a counterclockwise moment $A(x_1 - x_0)$. Let us assume that the density, $\rho$, satisfies $9/32 \leq \rho < 1/2$ so that the 45 degree orientation, as we have shown above, is stable. Then $A(x_1 - x_0) > 0$, so that $(e^2 - A)(x_2 - x_0) < 0$. The latter inequality tells us that if the figure were reflected in the waterline, the immersed area, $(e^2 - A)$ with coordinate $x_2$ for its CG would produce a clockwise moment about the CG of the boat. That is, a boat with density $1 - \rho$ (and immersed area $e^2 - A$) when perturbed to the left, would tend to be restored by the resulting moment. Hence, any density, $1/2 < \rho \leq 23/32$ would produce a stable 45 degree configuration.

**Experimental Epilogue**

One could look for physical verification of the stability analysis above. It is obvious that the boats we are studying are far from a practical design. The discipline of naval architecture, however, relies on the use of complex models whose stability analysis employs numerical methods, together with much more attention to the dynamics of response to perturbations. Among other things, disturbances in heavy weather entail a stability investigation of a very much more elaborate nature than the one we have presented. Nevertheless, consideration of center of buoyancy and center of mass, as in the work above, will always play a role in the design of a boat intended for a particular application.
Experimental validation of the fine details of the calculations above presents interesting challenges. For instance, for a very low density boat, the merest lateral shift of the center of gravity of the boat from its geometric center is bound to cause a list from the horizontal configuration, making it difficult for a square boat of real material (wood for example) to exhibit true horizontal stability. The same idea holds for the 45 degree orientation of square boats of intermediate density. In the end, then, one should look for "nearly horizontal" or "nearly 45 degree" stability of actual square boats.

The photograph in Figure 7 shows a flotilla of our three boats. The low-density plastic box boat is nearly horizontal as is the pine boat with the steel spine. The intermediate pine boat is clearly in the (near) 45 degree configuration. The three boats have the following respective densities: .21 for the plastic box boat, .82 for the steel spine boat, and .35 for the pine boat.

![Figure 7. A flotilla of three boats of varying densities.](image)

One may achieve a closer physical approximation to the mathematical prediction by making "adjustments." The photograph in Figure 8 shows the plastic box boat with some small correcting patches which, presumably, move the center of gravity of the boat more in coincidence with the geometric center of the boat.
Further Work

As indicated above, the stability “gaps” in the horizontal configuration can be resolved. Triangular boats should be easier to study. The role of density should play a prominent role in all of these studies. Students may construct the boats with inexpensive materials and check the mathematics against experimental results.

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References
