A Case of Misalignment of Reasoning, Affect, and Performance in the Transition-to-Proof

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Abstract

Learning how to prove is known to be difficult for undergraduate students. Understanding students' growth in the multiple arenas that make up proving is crucial for supporting them. Across four interviews over a semester, I examine one student who showed growth in his reasoning but whose proofs were still incorrect, yet he showed high levels of positive affect including confidence throughout. Investigating this single-subject case serves as an example of the interplay between development and performance. The question of whether we can say this student is a better prover than before—fundamentally, how to weigh reasoning versus affect versus performance—motivates the need for robust frameworks to characterize a student's progress in proving.

Keywords

Transition-to-Proof, Problem Solving, Affect, Undergraduate Students

Learning how to prove is well known to be difficult for undergraduate students (Moore, 1994), as there are multiple components that comprise the activity we call "proving" (Mejia-Ramos & Inglis, 2009). One such component is reasoning about logical statements in order to justify and write arguments, which is a shift from computation and exercises in students' mathematical experience (Smith et al., 2017). There is also a strong problem-solving component to proving (Stylianides et al., 2017; Savić, 2012), where the solution path is not apparent from the start and not all mathematical work and reasoning is included in the final written product.

Additionally, student affect (beliefs, attitudes, emotions, etc.) is also a component of learning how to prove successfully, as maintaining feelings of enjoyment (and a sense of self-efficacy) with mathematics during this process can be difficult (Smith et al., 2017). This is yet another transition for students, as they often come to transition to proof courses viewing themselves as "good at mathematics." However, as United States students have little prior experience with proving outside of high school geometry (Anderson, 1994), they often feel frustrated with this new mathematical work, as well as showing other forms of negative affect (Smith et al., 2017).

While much is known about students' errors (e.g., Selden & Selden, 1987), less is known about students' growth—how the learning process of proving unfolds over time. Understanding students' growth is crucial for helping undergraduate students through this difficult transition point in their upper-level mathematical career. Through more research on the various stages students step through while learning how to prove, instructors can better design transition-to-proof courses to support undergraduates along these expected pathways, as they grapple with these difficult mathematical ideas. There is also a need for frameworks to assess students' proving skills and processes (Savić, 2012; Selden & Selden, 2007): "We need a richer framework for keeping track of students' progress than the everyday one" (Selden & Selden, 2007, p. 1).

I present a short-term, longitudinal case of a student, Leonhard, whose reasoning, performance, and affect while learning how to prove are out of alignment in an unexpected way: the growth he shows in proving is not captured by his performance, yet he shows high positive affect throughout. I analyze his decision-making (reasoning), the correctness of his proofs (performance), and his emotions (affect) to illustrate how a student can have sophisticated decision-making and an overall high confidence yet not produce correct proofs. In doing so, the aim is not to fault Leonhard but to consider that written work, especially for proving, does not necessarily capture students' growth in crucial thinking processes—and that a robust framework to assess all the facets of students' proving skills and processes is needed.

Background & Conceptual Framework

There are multiple perspectives from which to approach research in proof (Stylianides, Stylianides, & Weber, 2017). One common perspective is to consider proving as a form of problem solving (e.g., Savić, 2012). However, the relationship between proving and problem solving is not purely that of one being a subset of the other.

Selden and Selden (2007) discussed two major sources of difficulty for students when writing proofs. The *formal-rhetorical* aspect of proving involves the logical structure of the proof, e.g., determining the first and last lines of a proof. Meanwhile, the *problem-centered* aspect of proving involves the decisions and key insights made to solve the embedded problem at the core of a proof (Raman, 2003), oftentimes with no set procedure. Both aspects are necessary for students to interpret mathematical statements and prove them, although students may favor one approach to proving over the other.

The formal-rhetorical versus problem-centered dichotomy parallels the notion of syntactic versus semantic proof production (Weber & Alcock, 2004). Under *syntactic proof production*, a person generates a proof by attending to the logical structure of a statement, oftentimes through manipulating symbols. In contrast, *semantic proof production* is where a person attends to the meaning of the mathematical objects and concepts in the statement to formulate the steps of a proof. While the specifics of a mathematical statement may lend themselves to one approach over another, it is important that students can work both syntactically and semantically in learning how to prove a variety of statements.

In terms of statements, students learn to determine the meaning of not only formal but also informal mathematical statements in the transition-to-proof. Formal statements use quantifiers, an *if-then* structure, logical operators such as *and*, *or*, and *not*, etc. Students must also learn how to unpack the meaning of *informal statements* (Selden, 2010; Selden & Selden, 1995), which are not written in their purely logical structure and may use words where mathematical meaning is inferred. For example, "All multiples of 6 are divisible by 3" is an informal statement in that to formally prove this, a person must infer the logical meaning of "all" and "are." There can be degrees of informality, in that one statement can be more informally worded than another. Students will see informal statements in their mathematical future: "Such statements are commonplace in everyday mathematical conversations, lectures, and books. They are not generally considered ambiguous or ill-formed, apparently because widely understood, but rarely articulated, conventions permit their precise interpretation by mathematicians and, less reliably, by students" (Selden & Selden, 1995, p. 127).

Students working with informal statements to identify their meaning and the analogue formal statement to then use in a proof is a crucial part of learning how to prove. Given a formal statement, there are a myriad of differently phrased equivalent informal statements. Selden (2010) reported students' struggles with informal statements: "When asked to unpack the logical structure of informally worded statements, but not to prove them, U.S. undergraduate mathematics students, many in their third or fourth year, did so correctly just 8.5% of the time" (p. 7). Yet, informal statements may help students build an intuitive understanding of the meaning of concepts and how they relate to each other (Selden & Selden, 1995).

Conceptual Framework: Reasoning, Performance, and Affect

I adopt the perspective of proving as a form of problem solving and draw from its literature base. Research on problem solving is vast and was a common theme of mathematics education research in the 1980s and early 1990s (Schoenfeld, 1992; Silver, 1985). Since Polya's (1945) work on problem solving, a number of theoretical frameworks for investigating problem solving have been created that build off that lineage (e.g., Carlson & Bloom, 2005; Garofalo & Lester, 1985). Schoenfeld (1992) identified five components of problem solving: a knowledge base, problem solving strategies and heuristics, monitoring and control, practices, and beliefs and affect.

Based on Schoenfeld's problem solving work, I take a three-pronged approach to analyzing student growth in proving by looking at aspects of reasoning, performance, and affect. Within reasoning, I focus on students' decision-making for how they choose which proof technique to pursue when constructing a proof. Proof techniques include direct proof, cases, proof by contradiction, and proof by contrapositive. Students' rationales for their approaches do not of course encompass all of what it means to reason when attempting to prove a statement, but the act of decision-making is a clearly defined moment of reasoning. The choice to study this aspect of reasoning comes from more extensive findings about student proof development from Satyam (2018). Performance refers to whether the mathematical proof the student produced was correct or if there were invalid mathematical steps.

Lastly, within affect, I focus on emotions. Affect is generally thought of as the domain involving feeling (Middleton et al., 2017), including beliefs, attitudes, emotions, motivation, engagement, confidence, etc. Beliefs, attitudes, and emotions as a trio have commanded attention, but among these three, emotions remain a relatively understudied subfield (McLeod, 1992). Emotions may be described as "rapidly-changing states of feeling experienced consciously or occurring preconsciously or unconsciously" (DeBellis & Goldin, 2006, p. 135). Emotions can be seen as responses to events; they tend to be short in duration but can reach high intensity. This leads to methodological difficulties in collecting data on and studying them. However, understanding emotions is crucial for understanding other affective structures with strong ties to learning: repeated emotional responses of a kind (positive or negative) may influence deeper-seated affect, like attitudes and beliefs (Grootenboer & Marshman, 2016; McLeod, 1992). Emotion may therefore be a vehicle through which to enact affective change.

I examine aspects of one student's reasoning, performance, and affect over the course of a transition-to-proof class. The purpose of this case is to illustrate how growth in reasoning does not necessarily lead to correct work, even in a proof course, and is moreover not captured by written work, and to examine implications of this situation when coupled with high confidence.

Methods

This work is part of a larger study focusing on the cognitive and emotional aspects involved in the transition-to-proof (Satyam, 2018). The full set of participants were N = 11 undergraduate students taking a transition-to-proof course at a large, public Midwestern university. The transition-to-proof course was designed to ease the change from computation-based courses to upper-level mathematics courses that involve writing proofs. The content taught in the course included logic, quantifiers, proof techniques (direct proof, proof by cases, proof by contrapositive, proof by contradiction, mathematical induction), and it provided a sampling of topics from analysis, linear algebra, and number theory. The population was students majoring or minoring in mathematics.

A series of four, semi-structured, task-based interviews was conducted with each of the participants across one semester. Interviews were spaced two to three weeks apart. Students had seen all proof methods by the time of the first interview. Within each interview, participants were given two proof construction tasks, where they were given a statement and asked to write a proof for it.

Design of Proof Construction Tasks

All proof construction tasks were on basic number theory: properties of integers and real numbers, even and odd integers, divisibility, etc. Tasks were designed so that the content area would be the same and to minimize any special domain knowledge as much as possible; care should be taken, however, in making content-free claims about proving (Dawkins &

Karunakaran, 2016). One proof construction task in each interview used a definition to test students' skills at making sense of definitions; however, students had often been exposed to these definitions earlier through homework.

Tasks were also worded to incorporate some degree of informality, given the importance of informal statements in proving (Selden, 2010; Selden & Selden, 1995). An example of this can be seen in Task 3: *Suppose x, y, and z are positive integers. If x, y, and z are a Pythagorean triple, then one number is even or all three numbers are even*. The conclusion, *one number is even or three numbers are even*, is an informal statement, as it formally means exactly one of x, y, and z is an even integer (and exactly two of x, y, and z are odd integers) or x, y, z are all even integers. A more informal conclusion to the statement could be *one is even or all three are even*.

Data Collection

Participants were given fifteen minutes to construct a proof. Each proof construction task was administered as a think-aloud (Ericsson & Simon, 1980). Students were asked to verbalize their thinking, but the researcher did not ask questions while they were working in order to not interrupt their problem-solving process (Schoenfeld, 1985). Instead, a debrief was conducted with the participant immediately after each task, during which they were asked questions about their thought process, places where they perceived they were stuck, and other points of interest. Participants were not told whether their work was correct or not unless they asked after the interview was over.

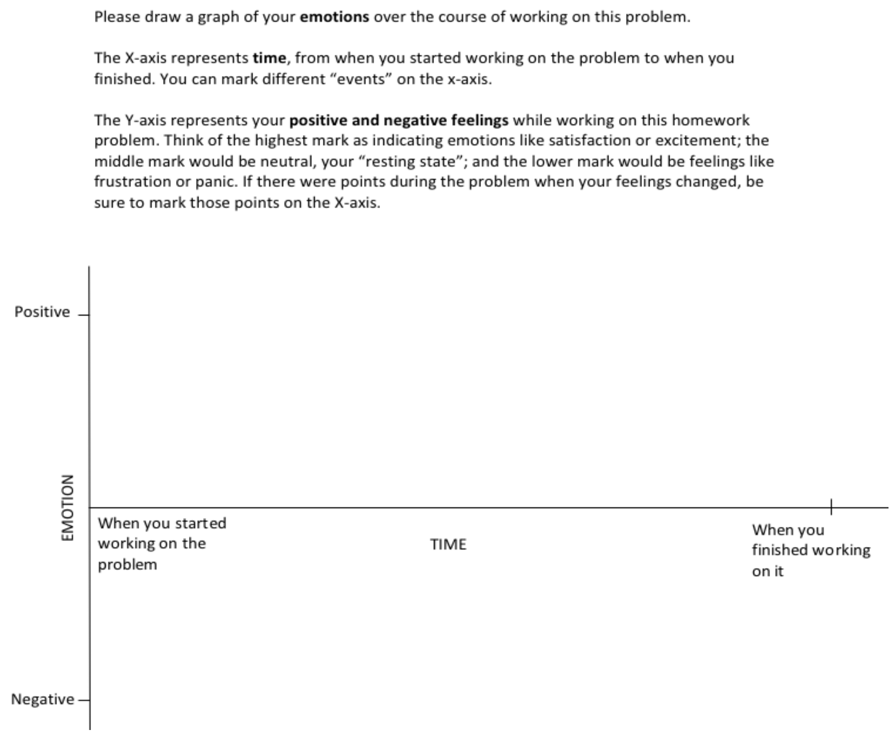
Participants were also asked after the task to talk about the emotions they experienced while constructing the proof, through an *emotion graph* task (adapted from McLeod et al., 1990 and Smith et al., 2017). Students drew by hand a graph of their emotions over the course of the task, where the x-axis represented time and the y-axis represented the intensity of emotion felt (see Figure 1). Students also textually annotated their graphs to describe what was happening at a certain point, the reason(s) for a shift in emotion, or specific emotions.

Data collected and analyzed here include the audio- and video-recorded think-aloud and debrief portions of the interviews, student written work, and their emotion graphs. From the audio-recordings, the interviews were then transcribed. Students' verbal responses were analyzed for their reasoning, and emotion graphs were analyzed qualitatively for dips and rises. A coding rubric was developed for assessing performance (correct, partially correct, or incorrect) on the proof construction tasks but is not used here due to the single case structure of this study.

Case Study

In this work, I examine a single participant, Leonhard, as a case study. In keeping with case study methodology, this work does not generalize nor is it representative of the data set. Leonhard serves as a *unique case* (Yin, 2009) of a phenomenon and is why I discuss a singular case (rather than compare and contrast multiple cases). Across the set of participants, some participants showed strong reasoning, performance, and affect from the start, some struggled with these throughout, and some showed gradual growth across these three metrics. I have chosen Leonhard's case in particular due to his atypicality from the expected development: he grows in reasoning but not in performance, yet shows high affect throughout. His case serves as an example where reasoning and performance are in misalignment, showing how assessing a student's growth in proving can be difficult.

Figure 1
Blank Emotion Graph as an Instrument



Leonhard was a white male freshman majoring in mathematics. He wanted to be either a high school teacher or a mathematician in aerospace engineering. Leonhard had many thoughts about mathematics, which he effusively shared. He chose his own pseudonym, Leonhard, after Leonhard Euler, which shows the extent to which he enjoyed and identified with mathematics.

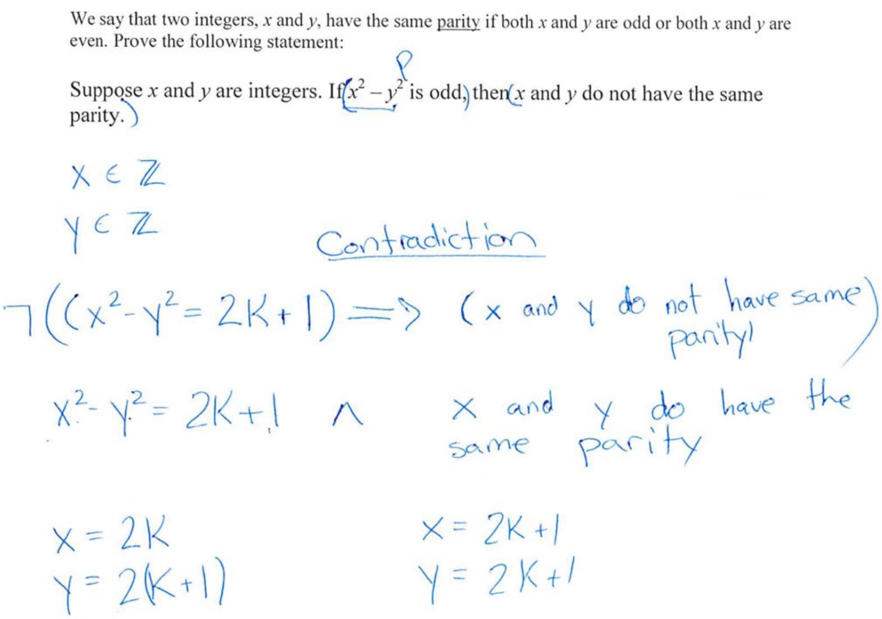
Results

I trace through a task from each of Leonhard's four interviews to illustrate his affect and the growth in his decision-making as reasoning in response to each task. As there were two tasks to choose from for each interview, I selected the tasks in the following way. The first three tasks concern proof by contradiction, so we may see how Leonhard's reasoning particular to that technique changed. The last task concerns proof by contrapositive, to show that his decision-making extended to other proof techniques as well. Leonhard had seen all proof techniques in class by the first interview.

Interview 1: Little Rationale for Choice of Proof Technique

In the beginning, Leonhard's baseline practice was to choose proof techniques based on what he knew and was familiar with. The first task of the first interview was to prove the statement: *Suppose x and y are integers. If* $x^2 - y^2$ *is odd, then x and y do not have the same parity.* The definition of two numbers having the same parity—both being even or odd—was given in the task. Leonhard was stuck on how to start the proof. Having seen all standard proof techniques in class at this point (direct proof, cases, etc.), he decided to use proof by contradiction, despite not being sure how to negate the conclusion. He carefully wrote down the parts of the statement to find its negation (see Figure 2). His rationale for his choice of proof technique was, “A lot of time in class whenever we’re proving an implication, we use contradiction, I guess, so that’s why it’s my first thought.” He used contradiction because he noticed the instructor often used it in class, and he was used to it.

Figure 2
Student Work in Interview 1


We say that two integers, x and y, have the same parity if both x and y are odd or both x and y are even. Prove the following statement:

Suppose x and y are integers. If $x^2 - y^2$ is odd, then x and y do not have the same parity.

P

$x \in \mathbb{Z}$
$y \in \mathbb{Z}$

Contradiction

$\neg((x^2 - y^2 = 2K+1) \Longrightarrow$ (x and y do not have same parity))

$x^2 - y^2 = 2K+1 \;\wedge$ x and y do have the same parity

$x = 2K$ $\quad$ $x = 2K+1$
$y = 2(K+1)$ $\quad$ $y = 2K+1$

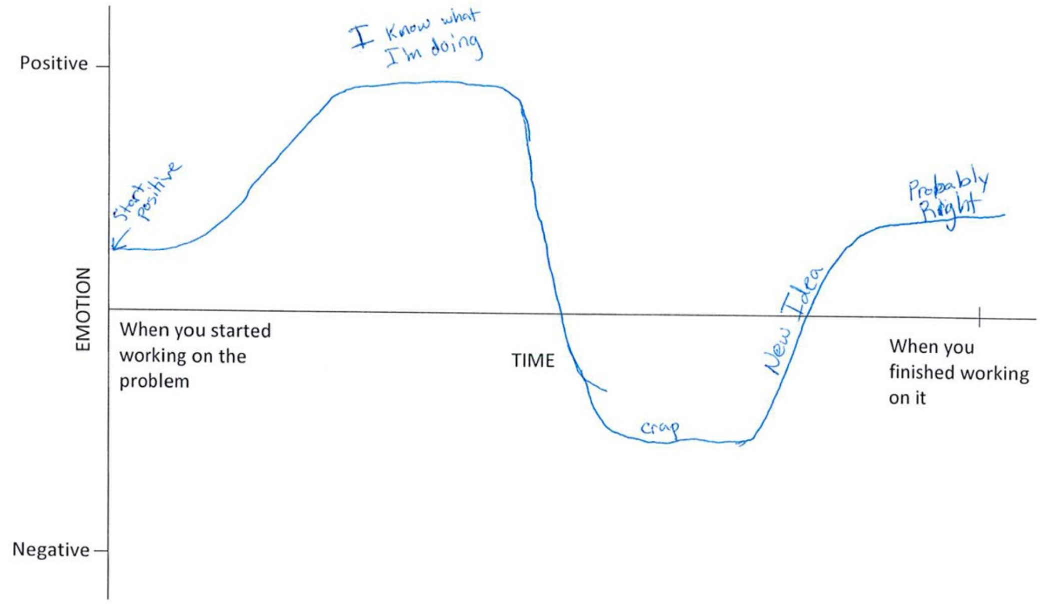
Note. The start of Leonhard’s work on this task is shown (not his complete work), as he tried to find the negation of the statement and mistakenly used the same variable for both x and y.

Leonhard set up the proof well, but ultimately, it was not a fully correct proof: while x and y are both even (or both odd) in his approach, he made an error in using the same variable for both x and y, which implies they are the same number. He changed the variables in his proof later down (not shown) but then went back and changed $y = 2k$ to $2(k + 1)$ so “he’d have something left over” to reach a contradiction that an even integer would equal an odd integer. For these reasons, his errors led to his proof being incorrect.

Leonhard’s emotion graph for this first task revealed big shifts in emotions throughout this attempt (see Figure 3). His emotions grew to a peak early on, remaining positive for a period of a time (“I know what I’m doing”). He then realized something in his work was wrong as indicated by the dip below the x-axis, but then the graph ended slightly positive (“probably right”).

In summary, Leonhard used proof by contradiction because it was what was done in class, even when he found it difficult to take the negation. His proof contained errors, so it was not correct. His emotions dropped negatively when he was stuck, but he felt positively about his work in the end.

Figure 3
Emotion Graph in Interview 1



Interview 2: Choosing a Proof Technique Based on Fluency

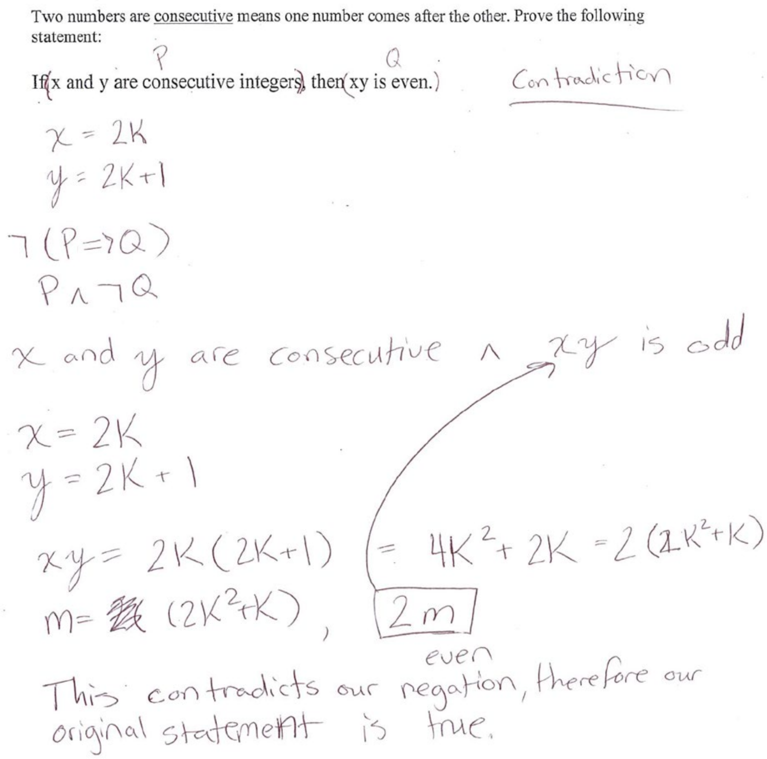
In the first task of the second interview, the statement to prove was: *If x and y are consecutive numbers, then xy is even.* As students had already been exposed to the definition of consecutive integers in class, a more informal definition for "consecutive number" using everyday language was intentionally given. Moreover, the definition of consecutive integers x and y as $y = x + 1$ leads to $xy = x(x + 1) = x^2 + x$, which does not contain enough information without further work to be shown as an even integer. The task was intentionally chosen for this disconnect between the definition of consecutive integer and the solution path. As seen in Figure 4, Leonhard wanted to use direct proof but became stuck, as he was unsure if direct proof would work.

Leonhard immediately knew to not use the direct definition of consecutive integers but instead set $x = 2k$ and $y = 2k + 1$, albeit leaving off that k must be an integer as well. When asked why he used $2k$ and $2k + 1$, he explained that his thought process was that an odd integer comes after an even integer and an even integer comes after an odd integer. Leonhard also took liberties in assuming that x was the even integer; for a fully correct proof for students at this level, he should have done another case where x was an odd integer and y the subsequent even integer or potentially use a "without loss of generality" argument.

He was then stuck again over what technique to use: direct proof versus proof by contradiction. He chose to use proof by contradiction, saying, "I decided to do contradiction because I know how to do it." Leonhard decided what method to use based on what he felt he could do at that point in time, i.e., his sense of fluency with proof techniques. Interestingly, the direct proof is embedded in here; his finding that xy is even is the conclusion to the direct proof. Given that direct proof was the more efficient proof, Leonhard's work suggests he felt more comfortable with proof by contradiction.

Leonhard's proof was overall correct, albeit missing details we want to see in students at this level, and his affect matches this. His emotion graph (see Figure 5) shows that this was a positive experience overall, with little variation in emotion. He was slightly confused at the beginning in deciding between direct proof or proof by contradiction, but he felt that he knew what he was doing after that. His annotation of "Yeah! (I got this)" reveals his sense of pride as he completed his proof.

Figure 4
Student Work in Interview 2


Two numbers are consecutive means one number comes after the other. Prove the following statement:

P Q

If (x and y are consecutive integers), then (xy is even.) Contradiction

$x = 2K$
$y = 2K+1$

$\neg(P \Rightarrow \neg Q)$
$P \wedge \neg Q$

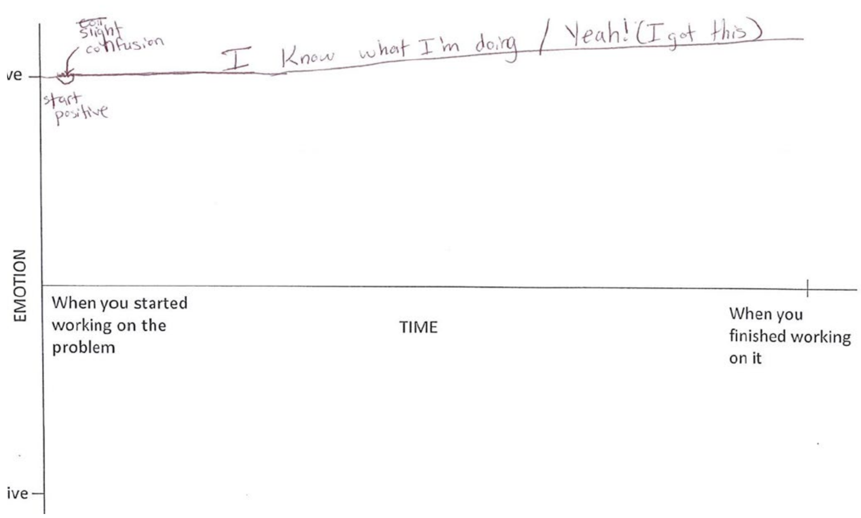
x and y are consecutive $\wedge$ xy is odd

$x = 2K$
$y = 2K+1$

$xy = 2K(2K+1) = 4K^2 + 2K = 2(2K^2+K)$
$m = (2K^2+K)$, $\boxed{2m}$ even

This contradicts our negation, therefore our original statement is true.

Figure 5
Emotion Graph in Interview 2



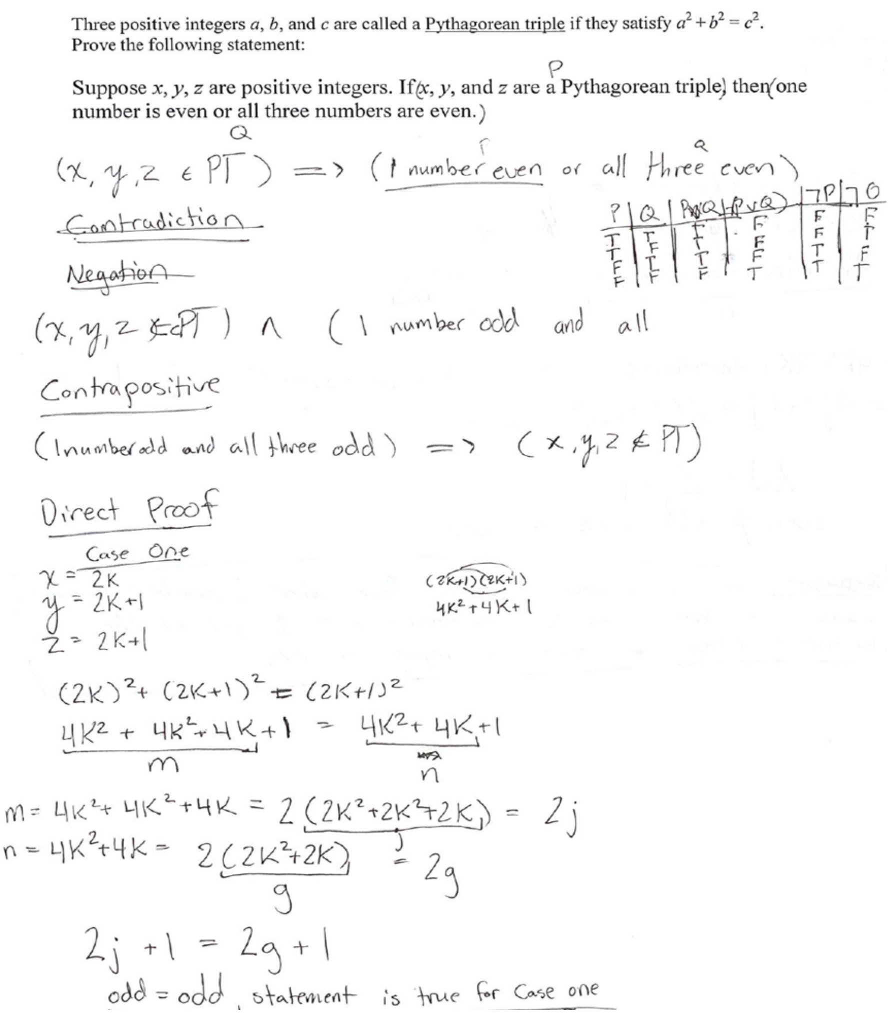
In summary, Leonhard used a technique that he felt he knew how to do well (proof by contradiction), even though it was not the simplest one and the direct proof was embedded in his work. His work was generally correct, and his affect was positive with no dips, except for slight confusion at the start, which abated when he decided on a technique and went with it.

Interview 3: Proof by Contradiction as a Default Choice

As time progressed, there was clear growth in Leonhard's reasoning related to the proof techniques he pursued in a problem. This task from the third interview provides an example of where Leonhard cycled through a few options for proof techniques, as seen in his written work (see Figure 6). The statement to prove was: *Suppose x, y, and z are positive integers. If x, y, and z are a Pythagorean triple, then one number is even or all three numbers are even.*[1] He used proof

Figure 6

Student Work in Interview 3


Three positive integers a, b, and c are called a Pythagorean triple if they satisfy $a^2+b^2=c^2$.
Prove the following statement:

Suppose x, y, z are positive integers. If (x, y, and z are a Pythagorean triple) then (one number is even or all three numbers are even.)

(x, y, z ∈ PT) => (1 number even or all three even)

Contradiction

Negation

(x, y, z ∉ PT) ∧ (1 number odd and all

Contrapositive

(1 number odd and all three odd) => (x, y, z ∉ PT)

P	Q	P⇒Q	¬(P∨Q)	¬P	¬Q
T	T	F	F	F	F
T	F	T	F	F	T
F	T	T	F	T	F
F	F	F	T	T	T

Direct Proof

Case One

$x = 2K$
$y = 2K+1$
$z = 2K+1$

$(2K+1)(2K+1)$
$4K^2+4K+1$

$(2K)^2 + (2K+1)^2 = (2K+1)^2$

$\underbrace{4K^2 + 4K^2 + 4K} _{m} + 1 = \underbrace{4K^2 + 4K}_{n} + 1$

$m = 4K^2 + 4K^2 + 4K = 2\underbrace{(2K^2+2K^2+2K)}_{j} = 2j$

$n = 4K^2+4K = 2\underbrace{(2K^2+2K)}_{g} = 2g$

$2j + 1 = 2g + 1$

odd = odd, statement is true for Case one

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[1] See Design of Proof Construction Tasks in the Methods section above for an explanation for the phrasing of this task.

by contradiction but then became stuck when writing the negation, because his negation of the conclusion did not make sense to him: "One number is odd and all three numbers are odd" did not seem possible, and he stopped writing the negation midway through his work. He had negated the "or" when it was in fact not a logical operator; the task was intentionally structured to check if students thought about the meaning or took the negation mechanically. The correct formal negation was "none or exactly two of *x, y,* and *z* are even integers." He switched to proof by contrapositive but realized he had the same issue with how to negate the conclusion as before. He then switched to direct proof. While he again used the same variable k in setting *x,y,* and *z* equal to even or odd integers, he realized his mistake near the end but did not change his answer as it would not change his overall result.

His rationale for using proof by contradiction in the beginning was: "I'm biased towards contradiction so I usually like to do that…my mind goes straight there. I like it the most because…at some point you usually run into something that just comes out sounding weird." Leonhard admitted that proof by contradiction was his go-to technique; it was his favorite and so he tended to use it. He liked proof by contradiction for its unique nature in producing something nonsensical. He later remarked on his proof by contrapositive attempt, "I don't know what possessed me to write this [contrapositive]," because he ran into the same issue, needing to negate the conclusion. Leonhard knew he liked certain techniques over others and had some rationale grounded in the techniques themselves, namely that a proof by contradiction results in a nonsensical claim and that proof by contrapositive has no advantage over proof by contradiction here. His rationale was still relatively general, however, in that proof by contradiction was a technique he liked and his fondness for it drove his usage of it.

His use of direct proof as his third attempt suggests he came to it through a process of elimination. He posited that his underlying idea may have been to check which proof techniques did not work well here and see what was left over: "I guess this was a good way of crossing out the things that you can't do so you can find the things that you can do."

Unfortunately, Leonard's proof was not correct. He started with one of the cases in the conclusion, reached a point where an odd integer was equal to an odd integer, and thought this meant he had shown the statement. Leonhard had used backwards reasoning on one case and shown there was logical consistency, but this was not a proof.

Leonhard's emotion graph shows this was a positive experience for him (see Figure 7). While there was a dip in emotion when he was confused ("eh, what"), his emotions grew steadily more positive as he continued on. His experience was so positive that he labeled a period of time as "The Zone," annotating his self-talk on the graph, "I'm doing it! I'm doing it! Almost there." His annotations also show his confidence, with humor ("Dope, I'm smart.") The "oops" near the end referred to his realization that he had used the same variable k in all three of *x, y,* and *z,* but he felt it did not fundamentally affect the correctness of his work. His emotion graph suggests that Leonhard was confident about his work and that he thought it was correct.

Interview 4: A Rationale Based on the Statement

By the fourth interview, Leonhard's rationales for his choice of proof technique displayed more precision. In the second task of the last interview, the statement was: *If x, y are positive real numbers and* $x \neq y$, *then* $\frac{x}{y} + \frac{y}{x} > 2$. He was stuck over how to start; he then identified the assumption and conclusion, tested a couple examples for x and y, and then tried proof by contrapositive (see Figure 8).

Figure 7
Emotion Graph in Interview 3

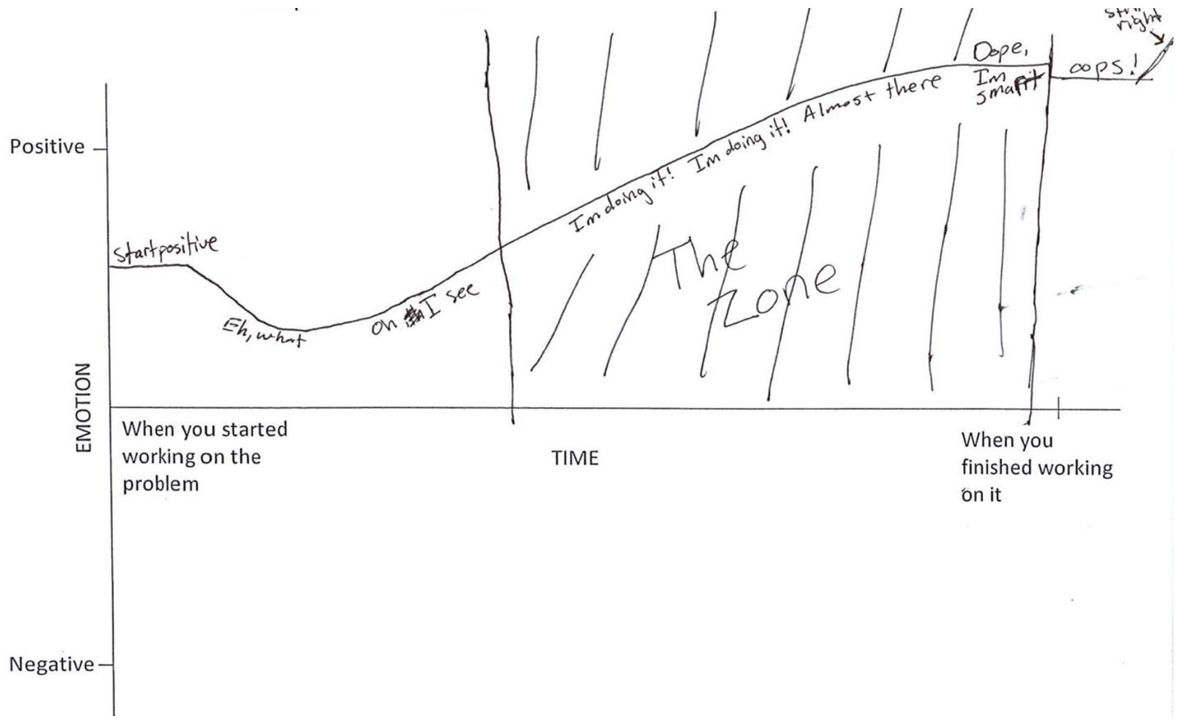
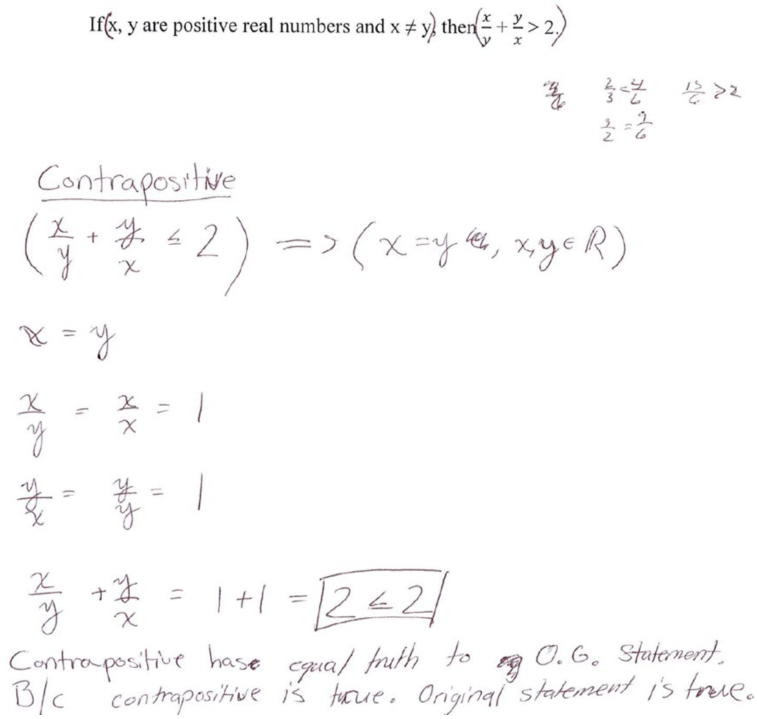


Figure 8
Student Work in Interview 4

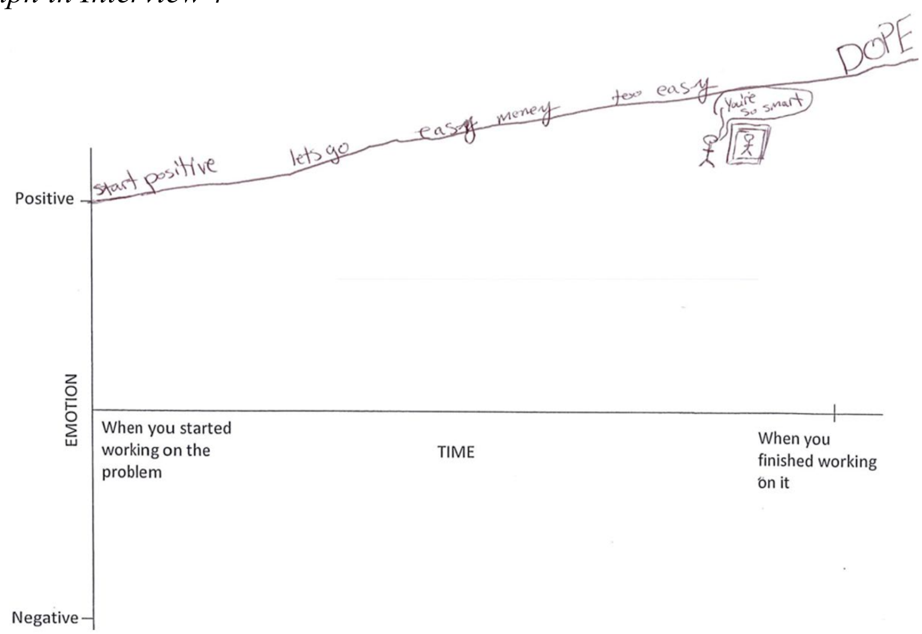


His rationale for proof by contrapositive was, “You can’t really do much with x not equal to y. But you can do a whole lot with x is equal to y.” He also explained why proof by contradiction would not be helpful: “The contradiction wouldn’t give me anything to work with.” He wanted to start with $x = y$ because he saw how an equality was more useful than having objects not equal to each other when proving. Neither direct proof nor proof by contradiction would provide an equality here. We see that Leonhard decided which proof technique to use based on specifics of the statement to be proven. His rationale also specifically explained why another proof technique (proof by contradiction) would be less useful here. In summary, he had a rationale for why his chosen proof technique was a helpful approach and why other techniques would be less helpful.

Although his rationale for why to use contrapositive was coherent and his affect overwhelmingly positive, his proof was incorrect. He used backwards reasoning to work off the conclusion (of his contrapositive) and then reached a true statement ($2 \leq 2$); he had still not realized that this was not the same as showing the original statement is true.

Leonhard’s emotion graph in Figure 9, however, depicts a student who feels comfortable and confident with proving. His graph was entirely positive; he started the graph at the positive tick-mark, and the graph rose even more. Although he was not sure how to start, it did not appear to impact his emotions based on the graph drawn afterwards. His annotations, “easy money” and “too easy,” suggests not only that he wrote this proof with ease, but that he enjoyed it.

Figure 9
Emotion Graph in Interview 4



Looking Across Leonhard’s Reasoning, Performance & Affect

Over the course of these four interviews, the rationales Leonhard gave for why he chose the proof techniques that he did became more nuanced. He moved from choosing certain techniques because it was done in class (no rationale), to what he was comfortable with, to deciding based on the particulars of the statement itself. By the end of the series of interviews, Leonhard also articulated why other proof techniques would not be helpful (so as to not go down that path). Leonhard showed clear growth in his reasoning for how he decided which proof technique to pursue.

However, if we look at his performance, Leonhard's work was oftentimes incorrect. Across the four tasks shown here, he only proved one statement correctly (Interview 2); he was partially correct in Interview 1, and his work for both Interview 3 and Interview 4 was incorrect. In fact, across the entire set of eight tasks (two per interview), the one task from Interview 2 was the only statement he proved entirely correctly. Moreover, his work on the last two interviews (all four tasks) was all incorrect due to substantial errors or missing crucial pieces of the proof. Leonhard would repeatedly work from the conclusion until he found a statement that was logically consistent, e.g., an even integer is equal to an even integer, and took that to mean he had proved the statement. Given that the interviews were weeks apart and Leonhard continued to use this logic, this is evidence his misconception had not been dislodged.

Interestingly, Leonhard's perception was that his work was correct. Looking across the set of emotion graphs, Leonhard's affect was overwhelmingly positive. They paint a portrait of a person who is confident with and feels at ease proving. He recovered from dips in emotion, felt good about writing proofs ("I'm doing this"), referenced being "in the zone," and believed in his abilities. Leonhard genuinely enjoyed doing this work; he displayed the positive affect we hope to see in students regarding proving. That his work was oftentimes incorrect and he did not realize it is troublesome.

Discussion

Through this case of Leonhard, we explored one transition-to-proof student's reasoning, performance, and affect over a series of four tasks and interviews. Over time, Leonhard's rationales in deciding which proof techniques to pursue became more sophisticated while his performance declined, yet his affect was quite positive. He went from using one proof technique (proof by contradiction) for everything, at first because it was done in class to later because he felt the most comfortable with it, to analyzing the structure of the statement itself for what technique would make sense. He also articulated why other techniques would not work well. Leonhard showed relatively favorable affect through many of the tasks, in that he had a positive orientation to his work: he felt at ease, enjoyed proving, and displayed confidence about his proofs and his competencies. However, Leonhard's work was often incorrect, with major logical flaws regarding backwards reasoning and about what it meant to prove a statement. While he had a positive orientation towards his work, he did not notice major logical flaws in his work.

Leonhard is an example of a student who has strong positive affect towards proving and their reasoning—specifically their rationale for their decisions, is strong—but these do not necessarily lead to correct work. There is a difference between reasoning and execution: can we say Leonhard knows how to prove or that he is better at proving than when he started? How do we weight reasoning versus performance versus affect here?

This work—the misalignment of reasoning, performance, and affect—highlights multiple implications for the transition-to-proof. First, thinking that reaching a true statement (often of the form $1 = 1$ or $2k = 2j$) is equivalent to proving a statement is true is a stubborn and pervasive error. In noticing that two sides match, students have verified that the mathematical situation is valid, that there are no inconsistencies—but writing a formal proof to in fact prove the statement is different. Further research is needed on this particular error, on how to help students notice when they make this error in their work, see why it is incorrect, and how to fix their proof. One recommendation is for transition-to-proof classes to more regularly task students to read and critique sample proofs with errors such as this one and discuss them. Misconceptions like these may in fact be developmental stepping-stones in learning how to prove, and rather than attempt to dislodge and replace such errors, we can help students refine and reorganize their knowledge (Smith et al., 1994). Continued work with students could include differentiating between the mathematical process of proving and the final written product (Karunakaran, 2018) and could reinforce the importance of keeping track of the conclusion one wishes to show.

Second, what do we do with students who are in fact overly confident about their work, not realizing they are making errors? On one hand, overconfidence with one's work can lead to not noticing errors, as happened here. More caution would have helped to catch errors. On the other hand, students who are overconfident tend to at least put down a written solution; because their thinking is now visible, their errors can be addressed. Meanwhile, students who are underconfident may doubt their thinking and not write down much or any of their thoughts. It is difficult for instructors to know that this is the case and determine how to help without talking to the students. This also brings up questions about the role of confidence in mathematics, whether overconfidence is beneficial for learning how to prove in that the positive affect helps students move forward through what may otherwise feel paralyzing. This has implications for students who come from backgrounds that have been historically marginalized in mathematics in the United States (African Americans, Native Americans, underrepresented Asians, Latinos, women, etc.), on whom mathematical confidence has not culturally been bestowed by society. Lundeberg et al. (1994) found that undergraduate men were more overconfident over incorrect answers than women. One recommendation is for instructors to address what makes for a healthy sense of confidence in proving—and provide strategies for all students in dealing with under- and over-confidence, but with special attention to gender and racial dynamics.

Third, the misalignment in reasoning, performance, and affect indicates the continued need for a framework for assessing students' proving (Savić, 2012; Selden & Selden, 2007) that encompasses these multiple components. While not typically thought of as part of the work of proving, affect can be a supplementary or even central component, much like how beliefs and affect are components of Schoenfeld's (1992) problem solving framework. Skills assessed should include common ones such as applying definitions and taking negations but also skills seen in this case, such as interpreting informal statements, negating informal statements, and differentiating valid statements from one's conclusion. Processes assessed should include how students choose a proof technique; a framework for students' development in this domain is provided in Satyam (2020). Such a framework would support the characterization of and assessment of students' proving as a process over short and potentially longitudinal timescales.

Lastly, this case serves as a reminder that progress in learning how to prove does not always manifest itself in performance as measured by objective correctness. Through interviews, Leonhard's more nuanced decision-making and positive affect shone through. Assessing a student solely through their written work does not capture the thinking and reasoning behind their choices that may have been valid, which, when taken alone, is valuable growth in proving.

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