# MATHEMATICAL REPRESENTATIONS IN A SYNCHRONOUS ONLINE MATHEMATICS SPECIALIST PREPARATION PROGRAM

Theresa Wills
*George Mason University*
twills@gmu.edu

Deborah Crawford
*Frederick County Public Schools*
crawford@fcpsk12.net

Kate Roscioli
*Prince William County Public Schools*
rosciokm@pwcs.edu

Shruti Sanghavi
*Prince William County Public Schools*
shanghasd@pwcs.edu

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**ABSTRACT**

Universities are implementing more online courses (Yamagata-Lynch, 2014). However, instructors may feel a sense of trepidation in transitioning a mathematics class to a synchronous online platform because they do not want to compromise quality pedagogy (Herrington et al., 2001) for the convenience of an online environment (Wills, 2021). Some courses have successfully transitioned to a synchronous online environment while maintaining rich discussion and student collaboration (Baker & Hjalmarson, 2019); however, mathematics content courses include the additional challenge of incorporating problem solving with multiple representations. This paper focuses on how mathematical representations emerge in a synchronous online course for mathematics specialists.

**KEYWORDS**

synchronous instruction, multiple representations, discourse, distance learning, face-to-face instruction, rich tasks

The purpose of this paper is to show how students recorded their representations in both face-to-face (F2F) and synchronous online mathematics content courses in a mathematics specialist preparation program at George Mason University and to show the intentional instructional planning that encouraged students' use of multiple representations. We will guide readers through various mathematical representations (concrete, pictorial, and abstract) created in both F2F and online classrooms. Examples of the representations include pictures of student work and group posters presented in the F2F class and the student work visible on collaborative slides (e.g., Google Slides) in the online class. We will address the successes and challenges of implementing a mathematics education online course through the eyes of multiple stakeholders. Theresa Wills and Deborah Crawford are university instructors who have taught multiple mathematics courses in both F2F and online settings, and Deborah is also a district leader in Virginia. Shruti Sanghavi and Kate Roscioli are K–12 educators and alumni of George Mason University's Mathematics Educational Leadership (MEL) program. Shruti experienced a 100% online program, and Kate participated in a hybrid program with four mathematics courses taught in a F2F format and one taught online.

The National Council of Teachers of Mathematics (NCTM) (2014) states that "effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures as tools for problem solving" (p. 10). Representations come alive in F2F and online classrooms in many different formats, including drawings, physical manipulatives, formulas, tables, graphs, virtual manipulatives, and digital tools.

## Representations

Mathematical representations are essential components in mathematics classrooms. Representations such as drawings, concrete models, and abstract symbols are necessary components to help students build deep conceptual understanding (Berry & Thunder, 2017). Comparing representations through discussion helps make connections to the mathematical goals (Smith & Stein, 2011). Lesh et al. (1987) emphasized the importance of students moving flexibly between representations to understand the mathematical concepts fully.

## Discourse and Rich Tasks

Mathematical discourse involves the student to student discussion of models, representations, and strategies used in problem solving (Smith & Stein, 2011). Students must communicate and collaborate as they solve problems to develop a deep mathematical understanding (Steele, 1999; Walshaw & Anthony, 2008). Facilitating meaningful mathematical discourse is challenging because of the intricacies involved in the process (Stein, 2007). It requires student engagement with multiple, student-created representations and a teacher that possesses content knowledge, conceptual understanding, and a mindset to commit to changing their instruction (Smith & Stein, 2011; Firmender et al., 2014). It also requires the teacher to act as a facilitator to guide students' thinking and understanding in the classroom (Steele, 1998) as students discuss *how they arrive* at a solution, not just the solution (Stein, 2007). Discourse about rich tasks serves as a tool for equity as students can access the tasks through multiple entry points (Sealey, 2016) and the voices of all students are valued through their different representations of the problem situation.

Rich tasks serve as the vehicle through which students' mathematical thinking becomes visible. However, through mathematical discourse, students create a shared understanding of the big mathematical ideas in focus of the lesson (NCTM, 2014). Through discussion, students can compare and contrast multiple representations of different strategies used to solve a task and connect different representations to the underlying mathematical ideas and relationships (NCTM, 2014) which are intertwined with other mathematics teaching practices (Smith et al., 2017).

## Representations, Discourse, and Synchronous Online Classrooms

A synchronous online classroom setting is a live experience that takes place via a video conference tool at a specified time. "In synchronous online courses in higher education, there is a tremendous pressure to ensure our students are engaged in their online learning environments." (Baker & Hjalmarson, 2019, p. 12). Rich tasks are a catalyst for engagement in mathematics education courses because they are designed to be accessible to all learners, are solved using various representations and strategies, and relate to students' lived experiences (Wolf, 2015). Regardless of the classroom format, students must have the ability to create and compare mathematical representations to fully explore and transmit conceptual understanding (Wills, 2019), which brings additional challenges for the planning and implementation of tasks in an online environment.

When teaching in a synchronous online format, instructors must anticipate student representations and strategies which may be shared using the available technological tools such as virtual manipulatives and collaborative slides (Wills, 2021). In order to ensure that these representations are accessible to everyone involved, instructors need to consider Technological Pedagogical Content Knowledge (TPACK). TPACK (Mishra & Koehler, 2006) describes the intersectionality of technological knowledge (creating the digital representation), pedagogical knowledge (knowing a variety of representations and when to use them), and content knowledge (mathematical knowledge and skills). Deficits in any of these three pieces of knowledge will result in incomplete or incorrect representations in the synchronous online classroom.

Additional challenges and opportunities arise in the types of representations used in the online classroom. Wills (2019) found various representations in synchronous online classes, including abstract, concrete, pictorial, and dynamic-pictorial (see Figure 1). Dynamic-pictorial representations are "pictorial models that use the advantages of technology to move representations on the screen in a way that could not be reasonably replicated using hand-held manipulatives" (p. 1). In other words, it moves during the discussion. Dynamic-pictorial representations (see Figure 2) are unique to synchronous online learning. They allow students to easily work with large quantities (e.g., candidates can copy and paste hundreds of squares efficiently) and easily make visual connections between models (e.g., candidates can duplicate a representation to show both a before and after manipulation efficiently).

## Situation

The MEL masters degree program, described in this paper, is offered in various formats. In one format all courses are offered 100% synchronously online, while another provides a hybrid experience for students including a mix of F2F and synchronous online courses.

**Figure 1**
*Multiple Representations in Multiple Formats*

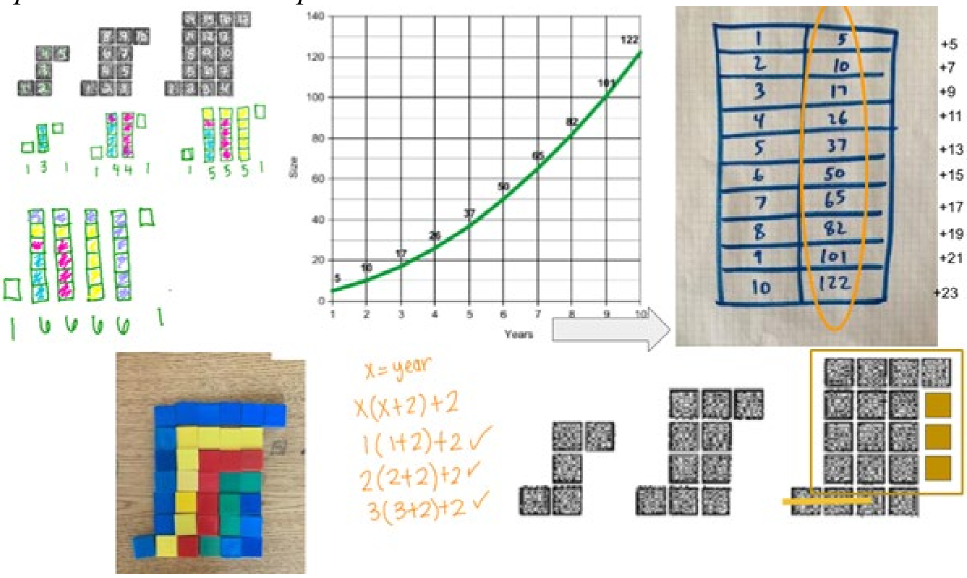
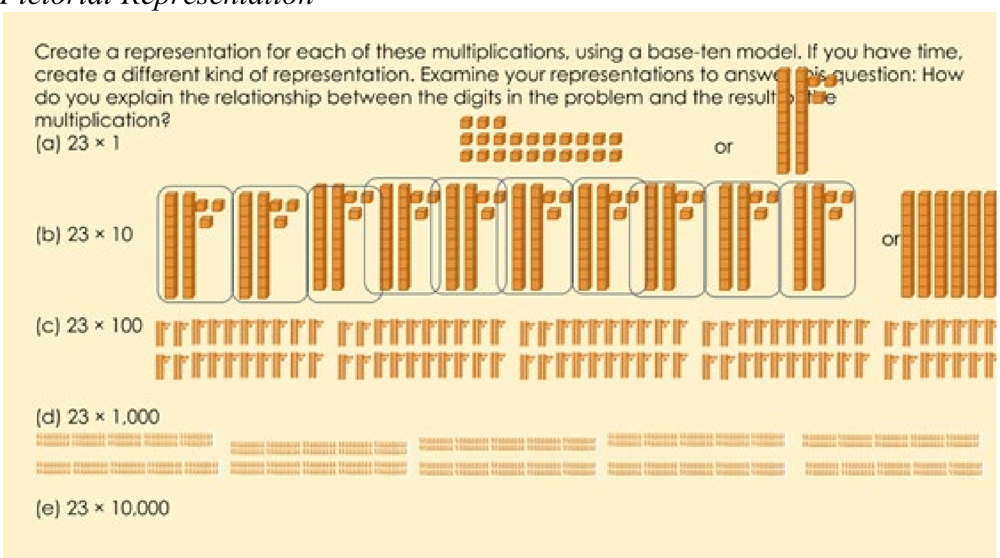



**Figure 2**
*Dynamic Pictorial Representation*




## Stakeholders and Diverse Perspectives

The authors of this paper have various experiences in online and F2F mathematics content courses and describe these unique perspectives throughout the paper to explain the complexities of planning, implementing, and participating in online mathematics courses.

### *University Instructors*

Deborah and Theresa, both university instructors, taught both F2F and synchronous online sections of the same course to prepare mathematics specialists. They co-planned their classes to ensure that the online sections incorporated the same tasks and activities as the F2F section. This planning ensured that the content and pedagogy remained consistent and that the rich tasks, representations, and mathematical discussions were not compromised in the online course.

***District Supervisor***

Deborah is also a school division mathematics supervisor who hires mathematics specialists as classroom mathematics teachers, coaches, Title 1 mathematics teachers, STEM specialists, and other locally defined roles. Since teachers tend to teach or coach in the same way they were taught (Wiliam, 2011), she wants to ensure that the candidates learn mathematics content that models the Mathematical Teaching Practices (NCTM, 2014) and mathematics teacher leadership attributes such as coaching the Process Standards (NCTM, 2000).

***Students***

Kate and Shruti were both candidates in MEL masters degree program. Shruti was part of a 100% synchronous online cohort, while Kate experienced a hybrid instructional model with only one content course taught in the synchronous online setting and the other four content courses were F2F. They noticed that the structure of facilitating a task did not differ significantly in either format. Both had the experience of incorporating multiple representations when working with rich tasks in all of the mathematics content courses. Another critical part of facilitating a rich task is the discourse, which could be challenging in a synchronous online environment. However, through breakout rooms and collaborative slides, the experience was not very different from a F2F setting in which candidates sit around a classroom table. Shruti explained that although she had anticipated feeling disconnected from the other cohort members in an online environment, she found that, due to the synchronous format, the experience was collaborative with a strong focus on discussions. As a result, she never felt that her peers or the professors were not supporting her.

## Themes

Through discussions, interviews, and journaling, these four stakeholders discovered three essential themes that were paramount for encouraging mathematical representations in the F2F and synchronous online classes: community, expectations, and mathematical discourse. These themes will be discussed below, first according to the similarities in both F2F and synchronous online settings and then by the characteristics exhibited only in the synchronous online environment. Each stakeholder provides unique perspectives and insights into each theme.

### Community

Building a classroom community is critical in all mathematics classrooms, including synchronous online environments (Fisher et al., 2020; Garrison, 2015). Students require interaction and collaboration when exploring various strategies, perspectives, and representations. Theresa and Deborah intentionally planned activities that valued mistakes, persistence, and celebrated risks in solving problems using representations outside of the candidates' comfort zones. From the first day of class, instructors used differentiated "getting to know you" activities for synchronous online students using interactive slides and small breakout groups to ask questions about the technology. In this way, instructors were able to pre-assess the technology, mathematics, and other skills that the candidates would need throughout the class. Candidates had varied levels of expertise; some were technology experts; some were primary grade experts, and some were formulas and abstract notation experts. When instructors created heterogeneous groups based on expertise, they noticed characteristics such as patience, productive struggle, and willingness to make mistakes. These same traits were evident in different groups' abilities to create multiple representations for rich tasks.

Shruti remembers that her age and inexperience with computer programs and websites did not adversely impact her synchronous online learning experience because of the supportive community she was participating in. "At the beginning of each class, the professors would ask them to provide an update about their lives with pictures and a short narrative. It was so wonderful to know when people were getting engaged, receiving promotions, or having babies" (S. Sanghavi, personal communication).

Kate also benefitted from participating in synchronous online communities. She enjoyed the random breakout room feature in the synchronous online class because candidates were able to work with different people and hear multiple in-depth perspectives. In F2F classes, she sat with the same group and did not get to know everyone else in the class. Both formats engendered camaraderie among the students, thus generating another support layer for the cohorts' students.

Building a community is a purposeful act prompted by instructors through activities, observations, and student groupings. As the communities grew, students felt safe taking risks and using digital means of connecting to collaborate and create mathematical representations.

## Expectations and Norms

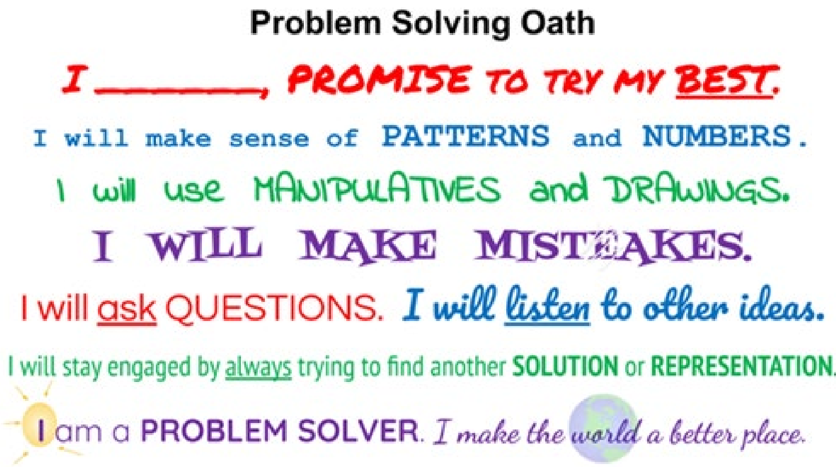
Instructors were explicit in setting expectations and norms to encourage students to create multiple representations. They modeled and practiced these expectations regularly in both F2F and synchronous online classroom settings.

### *Problem Solving Oath*

The problem solving oath (see Figure 3) was an intentional structure implemented in both F2F and synchronous online classes. Students read the problem solving oath aloud in F2F classes and interacted with the oath in synchronous online classes by finding a line in the oath they would focus on during the work time on that particular day. Part of this oath reminded students to consider many different representations and misconceptions. Kate remembers that when everyone said the oath, they committed to using multiple representations.

Deborah was explicitly looking for various types of representations to present during the whole group discussion. When she looked across the representations used by a diverse class of learners, she found many concrete, pictorial, abstract, and even dynamic-pictorial representations. Theresa describes the purposeful planning for encouraging different representations. They found that it was important that they provided a shared space for displaying the representations and reinforced the norm that multiple representations were required. Theresa also anticipated both student voice during class discussions and the multiple modalities necessary for interacting with the representations. For example, a candidate could take a photo of their paper-and-pencil work, share a video of their procedure, create shapes using the tools on the interactive slides, or provide a screen capture of a virtual manipulative. Once the candidates' representations were visible, they could implement the rest of the problem solving oath by asking questions and finding another solution or representation. This structure resulted in all students engaging in the task for the entire work time and provided a plethora of representations for the whole group discussion.

**Figure 3**
*Problem Solving Oath. Reproduced with permission from theresawills.com*




***Collaboration***

Another expectation was that candidates collaborate in small groups as they developed mathematical representations. Instructors in the F2F class observed the collaboration by listening to table discussions, watching candidates point to drawings in a notebook, and seeing pairs of students build a model with manipulatives. Similarly, it was observed in the synchronous online class through listening to small group discussions (each participant used a headset with a microphone within a breakout room), watching candidates use a virtual arrow to point at drawings on a shared slide, and seeing candidates share images of homemade manipulatives. Through collaboration, candidates made connections to different representations even as they were still emerging during an activity, as they developed a deeper conceptual understanding of mathematics.

Shruti remembers that her cohort could interact in real-time, which included seeing each other, communicating, and answering questions together every day of the program. They were continuously able to share their thoughts about a task, question, or assignment. She could see how others answered the problem, asked questions, made side comments, or offered a different solution or strategy. She instantly had a couple of people to bounce ideas off of and knew that they would support her no matter what.

During the whole group discussion, instructors could also see evidence of collaboration. Instructors were deliberate about how they facilitated a discussion by asking questions that required students to make connections with other peers' work. Deborah noticed the candidates collaborated to collect, organize, display, and interpret their data to make decisions about a rich task, scenario, or game. Teams created slides in the class deck to share out their mathematical thinking in a virtual gallery walk. Groups visited each team's slides, giving feedback through comments, symbols such as emojis, emboldening or highlighting, and via the virtual classroom chat box (see Figure 4).

**Figure 4**
*Solution Slide with Peer Feedback Via Emojis*


***Ownership***

A unique characteristic of synchronous online learning was a greater sense of ownership by candidates. In the F2F classes, the slides were static, but since candidates had full editing rights in the synchronous online class, they could add slides, change them, paste screenshots and create unique virtual representations. Deborah recalls that candidates used household items for physical manipulatives and various technology tools to simulate objects' physical movement. Many times, they color-coded their virtual manipulatives to represent their thinking. Others began with drawings and sketches that they uploaded to the slide. If another student wanted to draw on a sketch, they could quickly duplicate it and share a different representation. While many virtual tools will allow students to upload pictures of their mathematical strategies, interactive slides allow for more flexibility as students can upload, modify, duplicate, and collaborate within the same document. The affordance of the interactive slides was critical in obtaining many mathematical representations for a rich task.

## Representations within Mathematical Discourse

In both F2F and synchronous online environments, candidates engaged in mathematical discourse around representations developed from rich tasks. The most significant difference between the synchronous online and F2F experiences was the type of representations used for problem solving. Synchronous online students used homemade or virtual manipulatives in place of traditional, hand-held manipulatives. Shruti explored multiple representations through the mathematical tasks in every course. She solved a task using her strategy and posted it to a shared slide as she watched other strategies emerge alongside of hers, and then she tried to connect her work with the work of others through discussion. By communicating with other candidates, she was able to identify the similarities and differences as she developed her conceptual understanding of a mathematical procedure or concept.

Similarly, Kate observed that no matter the location, whether it was at your table in a physical classroom or in a virtual breakout room, representations were used as a springboard for discussions. Both F2F and synchronous online classes began with small group discussions about incomplete representations. However, an advantage to the synchronous online class was duplicating an incomplete representation and modifying it without altering the original work.

Because of this, she experienced shared ownership in synchronous online classes as she modified and shared a different representation.

Facilitating a productive mathematics discussion requires intense multi-tasking by the instructor. Deborah and Theresa watched the representations emerge on the group slides in real-time as groups collaborated in breakout rooms. From the representations viewable on the shared slides, they could see the access point and first representation based on the comfort level and initial problem solving strategies used across different groups. They chose groups to listen to as they discussed their emerging solutions. They observed shifts in thinking as students shared their ideas as well as by how they responded during a small group discussion. They also used feedback to differentiate their responses to advance the thinking of individuals or groups. One group might receive scaffolding to bridge candidates to the next level, while another group might be challenged to think about a related question to extend their thinking beyond the task. The instructors also took copious notes while selecting and sequencing the pieces of student work to present during the whole group discussion. To alleviate instructor overload during the busy class session, Theresa found it critical to anticipate the mathematical strategies (Smith & Stein, 2011), the technical requirements, the applications being used, and also possible candidate misconceptions (Wills, 2021).

## Conclusion

Similar to F2F courses, synchronous online mathematics courses must elicit multiple student-created representations of mathematical understanding. Three themes, community, expectations, and mathematics discourse should be explicitly planned before implementation to ensure that students have the required physical, social, and virtual resources to create and share their representations. Students who have a strong sense of community are more likely to participate and share their misconceptions as they explore problem solving. Clear expectations provide the structure for small group time and ensure that students explore multiple representations. Finally, mathematics discourse is the glue that brings the various representations together to form a clear image of the mathematics goal being explored. All of these themes can and should be implemented in synchronous online mathematics courses.

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