SEVEN PROPERTIES OF HIGHLY EFFECTIVE PROBLEMS

Thomas Ales
University of Lynchburg
ales_tm@lynchburg.edu

Kevin Peterson
University of Lynchburg
peterson@lynchburg.edu

Constantine Roussos
University of Lynchburg
roussos@lynchburg.edu

ABSTRACT
In an effort to provide more critical thinking opportunities in their courses, instructors are embracing the power of problem- and project-based learning (PBL). In this paper we address the importance of problem quality when utilizing PBL. We list seven important properties that a high-quality problem should have. We conclude with an example of a problem that possesses all seven properties.

KEYWORDS
problem-based learning, project-based learning, experiential learning, mathematics

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A goal of educators is to give their students an opportunity to think critically. According to Google's English dictionary (provided by Oxford Languages) the definition of critical thinking is “the objective analysis and evaluation of an issue in order to form a judgment.” The example sentence provided with the definition is surprising: "Professors often find it difficult to encourage critical thinking amongst their students." Thus, the motivation for this paper is understanding how educators can create an environment where students feel comfortable thinking critically.

One popular method is using Problem-based Learning (PBL). A search for this phrase in the digital library JSTOR returns over 700,000 articles and book chapters. These articles describe studies of many aspects of PBL but only one review paper by Yew and Goh (2016) mentioned anything about the effect of problem quality on student learning. We have found that an important part of “teaching” critical thinking is identifying problems that will hold a student’s attention and inspire them to persevere. It is an attractive idea to have a student work on a problem of their own devising so that they are “invested” in the experience. Unfortunately, student designed problems are often poorly defined and very difficult to solve. And personal experience tells us that a lot of effort spent on an unyielding problem does not encourage continued engagement and interest. When teaching critical thinking, we need to avoid situations where students feel there is no possibility of success. Hence, it is crucial to assign problems so that students have a “curated” critical thinking experience that strengthens both their mathematical knowledge and their intellectual confidence. Additionally, these curated experiences are less stress on the instructor because the educator knows what the result will be and is better able to provide the appropriate level of support required for each individual student.

The Setting

The majority of the problems and examples mentioned in this paper have been used in undergraduate courses taught at the University of Lynchburg. Over the past 23 years, we have developed, taught, and enhanced several project-based courses. Below we list a few of these courses with a brief description.

- Problem Solving in Mathematics (MATH 105) is a general education course that introduces students to the true problem-solving nature of mathematics. The focus is on using quantitative reasoning and intuitive logical thought techniques to solve problems rather than formal rigid processes.
- The Mathematics of Computer Science (MATH 231) introduces the theoretical and mathematical foundations of computer science through a combination of traditional lectures, problem sets, and six individual projects.
- Experimental Mathematics (MATH 350) is a project-based course that introduces students to the fine art of problem solving. The focus is on using computers, models, and examples to investigate problems to uncover possible solutions.
- Discrete Mathematics (MATH 330) covers topics such as counting, graph theory and cellular automata. It is a blend of traditional lectures, problem sets, and individual projects.
- Senior Research (MATH 451 and STAT 451) are the capstone courses for students majoring in mathematics or statistics. Students work on at least one major project throughout the semester. These projects allow students to dig deeper into key topics from earlier in the curriculum. Each student must write a research paper and present their work to the class.
• Statistical Methods (STAT 400) uses projects to explore topics such as estimation, inference, comparative analysis, analysis of variance, multiple comparisons, regression, analysis of covariance, and measures of fit regarding multiple regression models.

In addition, two of the authors, Ales and Peterson, have taught experiential-learning, project-based courses at Virginia’s Summer Residential Governor's School for Mathematics, Science, and Technology, which provide gifted high school seniors with immersive experiences in what it really means to be a mathematician.

All of these courses have a strong PBL component that involves weekly individual or group meetings with the students. All student anecdotal comments and feelings mentioned below come from conversations that took place during these meetings. Our extensive experience has motivated us to develop the following seven properties of problems that inspire critical thinking.

**Property 1 – Easy to Understand**

There is a time and a place for complicated problem statements that force students to parse through the combination of language, variables, and notations to determine what the question is asking. Challenges such as these are excellent for the seasoned problem solver, but will rarely entice the average student to dive into a problem. This problem from the 1962 International Mathematical Olympiad (IMO Board, n.d.) is an excellent example of what to avoid when selecting a problem:

Consider the cube $ABCDA'B'C'D'$ where $ABCD$ and $A'B'C'D'$ are the upper and lower bases, respectively, and edges $AA', BB', CC', DD'$ are parallel. The point $X$ moves at constant speed along the perimeter of the square $ABCD$ in the direction $ABCDA$, and the point $Y$ moves at the same rate along the perimeter of the square $B'C'CB$ in the direction $B'C'CBB'$. Points $X$ and $Y$ begin their motion at the same instant from the starting positions $A$ and $B'$, respectively. Determine and draw the locus of the midpoints of the segments $XY$.

Of course, the simplicity of the problem statement does not imply that the problem is simple to solve but a well-worded, easy to understand problem is less likely to intimidate students before they attempt to solve it.

**Property 2 – The Result is not Given**

The student should have to make a conjecture. There is plenty of room in the curriculum for problems that say, “prove the following result”, but we have found that there is more interest if the result is not given. Instead, create a problem that asks students to explore a mathematical situation and make a conjecture on their own. For instance, a problem like:

Find a formula for $1 + 3 + 5 + \cdots + 2n - 1$ that depends on $n$ and prove that your conjecture is correct.

This problem encourages experimentation and has the potential to reveal a surprising solution, where a problem like:
Prove: \(2 + 4 + 6 + \cdots + 2n = n(n + 1)\)

has already revealed its surprise and only serves as an exercise on inductive techniques. Everyone loves a treasure hunt and much mathematics is learned by constructing examples and searching for patterns. This is the heart of mathematics!

**Property 3 – Easy to Do Examples**

The problem should be easy to model, by hand or by writing code, and the results of this modeling should be used to make or check the conjecture. An important part of problem solving is understanding the problem (Polya, 1957) and generating multiple examples is a great way to get a better feel for what is happening in a given problem.

When students are looking for a conjecture for the sum of the first \(n\) odds, for example, they almost immediately start writing down examples and searching for a pattern. Most students create something similar to the following table:

**Table 1**
*Sum of the First \(n\) Odd Numbers*

<table>
<thead>
<tr>
<th>(n)</th>
<th>Sum</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1 + 3</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>1 + 3 + 5</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>1 + 3 + 5 + 7</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>1 + 3 + 5 + 7 + 9</td>
<td>25</td>
</tr>
<tr>
<td>6</td>
<td>1 + 3 + 5 + 7 + 9 + 11</td>
<td>36</td>
</tr>
</tbody>
</table>

After these examples, students are ready to make the conjecture that 
\(1 + 3 + 5 + \cdots + 2n - 1 = n^2\). In fact, through this modeling, they also develop other insights into the sequence of squares. Generating examples gives students a chance to think algorithmically, gives them a way to check their result, and gives them an opportunity to move from specific examples to the general result.

**Property 4 – Know the Solution**

The problem should be tractable and the instructor should know the solution. If we do not know the solution, it is difficult to properly advise students or give good hints. Students can get frustrated without the appropriate support. An example of a problem that is easy to understand and students find interesting is the Collatz conjecture or the \(3x + 1\) problem which is stated as follows:

Define a sequence \(a_k\) in the following way: \(a_1\) is any positive integer and

\[
a_{n+1} = \begin{cases} 
\frac{a_n}{2} & \text{if } a_n \text{ is even} \\
3a_n + 1 & \text{if } a_n \text{ is odd}
\end{cases}
\]

Collatz states that this sequence eventually reaches 1 regardless of the starting integer (Lagarias, 2010).
When asked about the Collatz conjecture, Paul Erdös stated “[m]athematics is not yet ready for such problems” (Guy, 2004, p. 330). Jeffrey Lagarias, a University of Michigan professor who has written extensively about this problem, stated that the Collatz conjecture “is an extraordinarily difficult problem, completely out of reach of present-day mathematics” (Lagarias, 2010, p. 4). Clearly, this makes it quite difficult to properly aid students in pursuit of this problem. We think Randall Munroe’s (n.d.) website XKCD.com (see Figure 1) best describes what happens in these cases.

Figure 1
XKCD Comic about Collatz Conjecture

When instructors know the solution, they can more easily provide the support needed by each individual level of student. This support is the basis of the curated critical thinking experience.

Property 5 – From the World

The problem comes from a real-world situation or a game. It is motivating to have a tangible reason to study a problem, even if the implied connection is a bit farfetched. We have used the following three versions of the same problem in the Mathematics of Computer Science (MATH 231). Here is the original version without examples and extra explanations:

We begin with an \( n \times n \) grid with each cell containing either a 1 or a 0. We will call this the start configuration. Our simple rule for this automaton is that at each time step any cell containing a 0 that has at least two neighbors that contain a 1, becomes a 1. Here a neighbor is directly to the left, right, above or below a given cell, not diagonal. What is the least number of 1’s required in a start configuration to guarantee that the grid will eventually fill with 1’s?
Students found this problem somewhat interesting. During our first attempt at using this problem as part of a class activity, students needed to come up with three related problems. We found that this problem wasn’t very inspiring for them.

The next year, we changed the problem to an $n \times n$ grid of trees where some of the trees were on fire. The same rule as described above represented the “spreading” of the fire. There was certainly more interest in this version, and, as a result, the students proposed more interesting “new” problems.

Most recently, we have changed the problem from an $n \times n$ grid of trees to an $n \times n$ classroom of students. The goal in this scenario was to create a simple model of the spread of COVID-19. The problem situation starts with some infected students in the classroom and followed the same rules as outlined above. The students were very interested in this model. The problems proposed by students were more interesting and often included the obvious addition of probabilities to our rules for “infection”. That is, they naturally changed the problem from being automatically infected with two infected neighbors to a probabilistic rule that depended on the number of infected neighbors. There is true value to changing the statement of a problem to make it more applicable to how the students are experiencing the world around them while maintaining the problem’s mathematical integrity.

**Property 6 – Easy to Generalize**

Interesting problems naturally generate more problems. We are not interested in a “one and done” problem, but are instead looking for problems that have many (fairly) obvious, interesting modifications. One of our favorite problems is from a book called *Dueling Idiots and Other Probability Puzzlers* (Nahin, 2012). We modified the dueling scenario slightly as follows and call our version of the game Bagsy:

Consider a simple two-player game where the numbers $1-6$ are written on six ping-pong balls with one number on each ball. The numbered balls are placed in a bag. Player 1 reaches into the bag (without looking) and removes a ball. If that ball is numbered “1”, then Player 1 wins. If it is any other ball, Player 1 puts the ball back in the bag, shakes it and hands it to Player 2 who repeats the process. The first player to draw the ball numbered “1” wins.

The question is: What is the probability that Player 1 wins? Clearly, this problem has a plethora of interesting modifications that will change the probability that Player 1 wins. We mention the two most obvious: First, simply change the number of balls in the bag to $n$, and second, change the number of players to $k$. These generalizations give students the opportunity to think like a mathematician. The mathematics a problem generates can continue to engage the student long after the original problem is solved.

**Property 7 – The Problem Contains a Surprising Result**

Typically, the surprise when solving a mathematical problem in a class is that the proof requires some technique or result from a previous lesson or a previous course. We again use Bagsy (described above) as an example. In the original version of the game, it should be clear that it could take an arbitrarily long time to finish the game. In fact, the expression for the
probability that Player 1 wins is an infinite series. The nice surprise is that this series is a convergent infinite geometric series, and we can use well-known results from Calculus II to arrive at the answer. These results, often from previous courses, highlight the connectedness of mathematics.

Over years of teaching project-based courses and developing experiential learning modules we have found that problems with these seven properties do an excellent job of keeping a student’s attention while inspiring voluntary critical thought.

**Problem Example**

We now give an example of an interesting problem (with a solution) that has all the properties mentioned above. This problem developed from a daily task of one of the authors. When walking their dog, they typically take a handful of pistachio nuts to snack on during the outing. As anyone who has eaten pistachios knows, there are two kinds of pistachios: Type 1 are those that have a split in their shell that makes them easy to open and Type 2 are those that have no split and require reconstructive dental work after attempting to open. During the walk, the author would reach in their pocket and pull out a random pistachio. If it is a Type 1 pistachio, they open it, eat it and toss the compostable shells. If it is a Type 2 pistachio, they return it to the same pocket, mix up the nuts and choose a new one. The following question arises: On average, how many times is a Type 2 pistachio chosen before all the Type 1 pistachios are consumed?

To make the problem simpler to talk about, we turn it into a game of chance. Imagine the following game:

Fill a bag with ten black marbles and one red marble, shake the bag and choose one marble without looking. If that marble is black, remove it from the bag and choose again. If the chosen marble is red, collect one dollar, return the red marble to the bag and choose again. The goal is to accrue as much cash as possible before all the black marbles have been chosen.

The question is: On average, how much money would you expect to win per game?

Let $E(n)$ represent the expected winnings when the bag contains $n - 1$ black marbles and one red marble. The formula for this quantity is quite simple (the term in the sum for selecting zero red marbles is omitted):

$$E(n) = 1 \times P \text{(picking red exactly once)} + 2 \times P \text{(picking red exactly twice)} + \cdots$$

Clearly, each game can be arbitrarily long. A very lucky (and unlikely) player may continue to choose only the red marble indefinitely, meaning the expression for $E(n)$ is an infinite series.

We wrote code (using MATLAB) to model this game. We were curious if there were any obvious patterns in the experimental results for different values of $n$, and wanted to have data to check conjectures. The code generated results for $n = 2$ to 16 marbles (one of which is red). The model “plays” the game 10,000 times for each $n$ and calculates the average winnings. The table of results is provided below.
Table 2

<table>
<thead>
<tr>
<th>Experimental Expected Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n)</td>
</tr>
<tr>
<td>(E(n))</td>
</tr>
</tbody>
</table>

Initially, the only noticeable pattern is that the expected value grows very slowly.

Next, we attempted to calculate cases of \(E(n)\) directly, but quickly recognized the folly in such an approach. Nevertheless, there is value in calculating \(E(2)\) directly. Notice that when starting with two marbles (one black and one red) the game is over as soon as you choose the black marble. Furthermore, the probability of choosing the red marble exactly \(k\) times before choosing the black marble is \(\left(\frac{1}{2}\right)^k \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^{k+1}\). This means \(E(2)\) is

\[
E(2) = 1 \cdot \left(\frac{1}{2}\right)^2 + 2 \cdot \left(\frac{1}{2}\right)^3 + 3 \cdot \left(\frac{1}{2}\right)^4 + \cdots = \frac{1}{2} \sum_{i=1}^{\infty} \frac{i}{2^i} = 1
\]  

(2)

This “scaled” geometric series is fairly well-known. Notice that the result matches our model quite nicely. It is left to the reader to calculate \(E(3) = 1.5\). Attempts were made to compute \(E(n)\) for higher values of \(n\), but such calculations proved prohibitively difficult. A new approach was needed, but a conjecture was in sight.

Instead of calculating the probabilities directly, we tried to develop a simple recursive relationship by removing one marble. We split the problem into the two obvious parts; whether a red or black marble was chosen first. Clearly, with probability \(\frac{1}{n}\) the red marble could be chosen and replaced or with probability \(\frac{n-1}{n}\) a black marble could be chosen (and subsequently removed) leaving us with \(n-1\) marbles in the bag. This thinking yields the following expression as a conjecture for the expected value:

\[
E(n) = \frac{1}{n} (E(n) + 1) + \frac{n-1}{n} E(n-1)
\]  

(3)

Solving for \(E(n)\) in (3) we arrive at:

\[
E(n) = E(n-1) + \frac{1}{n-1}
\]  

(4)

Using our result for \(E(2)\) and the above recurrence relation, we arrive at the following surprising result:

\[
E(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}
\]  

(5)

which is the partial sums of the harmonic series.

Before we prove that (3) is true, we will check to see how well our model matches the proposed solution found in (5). We display two graphs below that show the model and the proposed solution on the same graph for \(2-100\) marbles. The first graph (Figure 2) was created using 1,000 trials in the model and the second (Figure 3) using 10,000 trials.
Now that we are convinced that this result is likely to be correct, we will prove that the recurrence relation found in (3) is true.

**Main Result**

The expected winnings, $E(n)$, when playing the game with $n - 1$ black marbles and 1 red marble satisfies the following recurrence relation $E(n) = \frac{1}{n} (E(n) + 1) + \frac{n-1}{n} E(n - 1)$. 
Proof:
We start with the definition for the expected winnings as given in (1).

\[ E(n) = \sum_{i=1}^{\infty} i \cdot P(i,n) \]

Here \( P(i,n) \) is the probability of drawing the red marble \( i \) times when there are \( n \) total marbles at the start. The individual probabilities \( P(i,n) \) can be broken down in the following way because drawing a red marble first and drawing a black marble first are mutually exclusive events:

\[
P(i,n) = P(i,n|\text{draw black first}) + P(i,n|\text{draw red first})
\]
\[
= \frac{n-1}{n}P(i,n-1) + \frac{1}{n}P(i-1,n)
\]

(6)

We now rewrite \( E(n) \) using (6), properties of summations, (1), \( i = i-1 + 1 \), and the fact that for each \( n \) we have \( \sum_{i=0}^{\infty} P(i,n) = 1 \) to obtain the following:

\[
E(n) = \sum_{i=1}^{\infty} i \cdot \left( \frac{n-1}{n}P(i,n-1) + \frac{1}{n}P(i-1,n) \right)
\]
\[
= \frac{n-1}{n} \sum_{i=1}^{\infty} i \cdot P(i,n-1) + \frac{1}{n} \sum_{i=1}^{\infty} i \cdot P(i-1,n)
\]
\[
= \frac{n-1}{n} E(n-1) + \frac{1}{n} \left( \sum_{i=1}^{\infty} (i-1) \cdot P(i-1,n) + \sum_{i=1}^{\infty} P(i-1,n) \right)
\]
\[
= \frac{n-1}{n} E(n-1) + \frac{1}{n} \left( \sum_{j=0}^{\infty} j \cdot P(j,n) + \sum_{j=0}^{\infty} P(j,n) \right)
\]
\[
= \frac{n-1}{n} E(n-1) + \frac{1}{n} (E(n) + 1)
\]

which is the desired result.

Conclusion

This result is both surprising and satisfying. It is rare that a problem from the real world generates such an interesting solution. We see that this problem possesses all seven of the important properties required to entice critical thinking in students. It is certainly easy to understand, there is no conjecture provided, the problem is easily modeled either by writing code or by actually playing the game, the problem is accessible to both undergraduates and strong high school students with some understanding of probability and expected value, it is motivated from a real-world situation, has a surprising result (i.e., the solution is the partial sums of the harmonic series), and it is easily generalized.

Below you will find a few problems that are motivated by the original pistachio problem but stated in terms of colored marbles.

1. The most obvious generalization is to add more red marbles. This turns out to have a very nice solution as well. The solution is left to the reader.
2. We can add players and have them choose a marble in turns. The player that has the most money wins. Find the probability that Player 1 wins or find the expected value for each player.

3. Create a new game. Start with \( n \) black marbles and \( k \) red marbles and one player. The player chooses one marble at a time and removes it from the bag (regardless of color). The player wins if they are able to remove all the black marbles before removing all the red marbles. Find the probability that the player wins. This is very similar to the old Price is Right game “Strike Out”.

4. Play the same game as above but every time you choose a red marble, you remove it and add a black marble. The player wins if they are able to remove all the black marbles before removing all the red marbles. Find the probability that the player wins.

Hopefully, the reader can see that we could fill the page with interesting problems that are related to the original.

As you employ PBL in your classes, we hope you now have an appreciation for the importance of problem selection and how the right problems can help you help students acquire critical thinking skills. With this in mind, we end this article with a short list of interesting books that contain great examples of high-quality problems. We purposefully did not include most of Martin Gardner’s books, as most people are already familiar with those gems.

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