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A STUDY ON INVESTMENT STRATEGIES FOR RETIREMENT PLANNING

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A STUDY ON INVESTMENT STRATEGIES FOR RETIREMENT PLANNING

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Abstract

The aim of the thesis is to construct an effective realistic retirement income plan for an individual investor. We propose realistic frameworks with specific inputs given by investor such as number of investment instruments, income, and length of the time period before retirement using Modern Portfolio theory. The aim is to develop a retirement framework using fundamentals of Modern Portfolio Theory as per investor's needs on asset allocation assuming investor's risk appetite reduces as he ages in life and worries for real retirement income planning by comparing different statistical models scenarios. In each of the Scenarios we have 3 changing probability profile scenarios to allow for flexibility to the investor to withdraw from the portfolio for personal needs with increasing probability, decreasing probability and uniform probability of withdrawal throughout the portfolio investment time horizon. The results clearly reveal that there is no one best model for different investors as each investor is different with different objective functions. The results also show that, Traditional method and Bootstrapping scenario results are not always the same implying investor should not expect historical returns from the securities to reflect the future.

4. Introduction

The goal of thesis is to develop a real retirement framework using fundamentals of Modern Portfolio Theory as per investor's needs on asset allocation, assuming investor's risk appetite reduces as he ages in life and worries for real retirement income planning, by comparing different statistical models scenarios. In each of the Scenario, we have 3 changing probability profile scenarios giving flexibility to investor to withdraw from the portfolio for personal needs with the increasing probability, the decreasing probability and the uniform probability of withdrawal throughout the portfolio investment time horizon. For instance, investor may withdraw for, big onetime expenses such as child's education or wedding, elderly parents moving in, big business loss. The first chapters of this paper titled "Introduction" give broad view of the theory to the reader including background, history, to give perspective on the Retirement Planning and Investment Management and significance of the thesis. The second chapter of this paper titled "Methodology" discusses Methodology and data in detail and exhibits how to establish the model using statistical software R for further analysis and interpretations. This part gives the reader a good understanding on the Accounting Framework, constraints, parameters and model building. The third chapter of this paper titled "Scenarios and results" is the analytical part of the research where we have discussed results and all scenarios with finer details and interpretations. The fourth chapter of this paper titled "Conclusion" concludes the research and discusses the benefits and audience for this research.

1.1 Basics of Retirement Planning

Retirement planning is one of the most imperative life events many of us will experience on the path to successful financial future. (Ortiz 2009) Uncertain global events, financial crisis, market panics are a few of the reasons compelling individuals to worry about retirement planning at one point in life. (Gruber

2002) Retirement planning is the process to comprehend how much cost it requires to live during retirement (when individuals leave the workforce and start living on social security income, after applying for social security benefits and how to keep growing the investments during one's retirement to meet any income shortfalls. A lot depends on how different individuals plan for their retirement and accumulate the wealth for retirement. (Ortiz 2009) There are essentially three stages for retirement planning. First stage is to accumulate the wealth for retirement by having the appropriate investment strategy meeting the retirement goals, second stage is one needs to have a plan on how much income to utilize year over year and to have enough cash flows for the buffer number of life years, and third stage is how to invest the remaining funds in order to keep the best flow of income during the retirement years. In the research work we are focusing on the first stage of retirement planning. Based on Lachman and Burack's (1993) general definition, retirement planning can be defined as the thoughts and behaviors undertaken to fulfill retirement goals. (Duggan 2007) The goal of retirees is to accumulate the amount of money to have comfortable retirement that will support spending needs for their expected life with some years of buffer for increased life meeting the present life goals. (Ortiz 2009) Retirees also would like to have some level of confidence in the investment plan that will hold in practical real life situations and not just hold to assumptions of undesirable investment results and theoretical models. With the ample choices available today internet by many retirement planning investment companies and financial planners, retirees would likely be overwhelmed as to which retirement plans will work best for them. Fidelity, J.D.Edwards, Money Tree, Vanguard, Financial Engines, and NETirement are examples of companies that offer net-based simulations to help retirees in planning and implementing the retirement model.

(Poterba 2007) Most financial planners use 3 legged stools to describe most common sources of retirement income for a retiree during retirement- Social Security Benefits, Defined Contribution, and Defined Benefits. We need to understand Social Security tax to understand the Social security benefits first leg of retirement income for a retiree. Social Security Tax is the deduction done from the payroll of working people and their employers, and also includes self-employed individuals. Social Security tax is

Old-Age, Survivors, and Disability Insurance (OASDI) federal program which is funded through the payroll taxes known as Federal Insurance Contribution Tax (FICA). (Gruber 2002) The Social Security tax is normally 6.2 percent of the income for the working people and the same for their employer. For Self-employed people Social security tax is 12.4 percent. For example, a single person with a salary of 50,000 USD will pay 6.2 percent of income into the Social security taxes and same contribution will be done by the employer. (Gruber 2002) Social security tax money collected by Internal Revenue Service (IRS) is formally entrusted to the Federal Old-Age and survivors Insurance Trust fund and Federal Disability Trust fund which is used to pay Social security benefits to the people who are retired, people who are disabled, dependents of beneficiaries, survivors of workers who have died.

(Nishiyama 2005) In most developed countries, the rapidly aging population, increasing life expectancies, reduced birth rates, with a rising proportion of retirees before full retirement age, has started placing considerable pressure on current social security programs. (Preston, Samuel 1975) With increased life expectancy with advancements in the medical facilities, longevity has certainly become a risk and challenge for retires on retirement planning. This demographic trend is likely to continue rendering benefits from social security uncertain for future retirees, unless we have higher percentage of younger working people paying towards the social security taxes. For instance (Stefhan 2005) study suggests the ratio of covered workers versus the number of beneficiaries under the U.S. Social Security program has been reduced significantly over the years. There were 35.3 million workers paying into the system in the year 1940, with only 222,000 beneficiaries, a ratio of 159 to 1. The numbers of workers increased to 154.3 million in 2003, with 46.8 million beneficiaries. – Ratio of 3.3: 1.

(Ortiz 2009) analysis suggest, that in the light of the current situation, the government is attempting to limit the social security commitments by moving towards the employee defined contribution schemes as supplemental saving vehicles for retirement income. Let us look at the other 2 legs of retirement income. Currently retirement plans belong to broadly 2 legs Defined Contribution and Defined Benefits. Second

leg of the Retirement income benefits is known as Defined Contribution plan. Defined Contribution (DC) plan is an investment account with tax benefits where an employee puts in annual contribution to the account in their name opened by the employer. Sometimes employer may also contribute in matching contribution to every dollar put in. In our retirement framework we will be using the DC for investment strategy for an investor. Third leg of the Retirement planning is Defined Benefit (DB) retirement plan. (Poterba 2007) Defined Benefit retirement plan is a pension plan in which employer pays retirement income in full.

1.2 Background - Basics of Investment Management

Essentially comprehending about retirement planning, raises important question, given the preferences of the investor, how should money be invested in different financial instruments using the fundamentals of Modern portfolio investment strategy and traditional investment strategy? This question revolves around risk and return relationship. Also it is our goal that with this thesis, the basic knowledge of investment management can be expanded, so it can be used later in the Methodology for building the accounting framework for retirement planning using the principles of investment management and Modern Portfolio theory. For understanding Investment Scenarios for Retirement framework, let us comprehend basic notion of Investment Management. This chapter aims to give a brief study of Investment Management and perspective of the Modern Portfolio theory. The background gives the historical view and view of different financial instruments that will discharge into the narrowed down problem question that are of significance to fulfill the purpose of this thesis. (Smith 1989) Financial well-being plays a pivotal role in overall well-being of an economy, organization and individual. This can be assessed qualitatively though but the perspective of quantative finance is required to make sense on the basis of numbers, statistics. Here comes the application of Mathematical ideas to financial markets. First let us introduce the idea of money and link it back to the value and then we will see different financial instruments and how to assess their value. (Mushkin 2006) Money is a means for trading/exchanging goods and does not generate value

by itself. The value of money is thus related to the physical goods itself. (Ordo 1995) 'Gold Standards' and 'Gold Certificates' is a famous example which was used in United States from 1882 to 1933. These certificates were freely convertible into gold coins. This means the money available to circulate is related to the physical value of the gold in the government vaults. This meant that the money is equal to the amount of gold available which is not related to goods available or value generated by the money. (Kydland 1995) This notion of 'Gold Standard' is not used in economics these days rather it is replaced by currencies free-floating in value and the monetary value of these currencies will be decided by the financial market. So what is financial market? (Cecchetti 2008), financial markets are places where financial securities such as bonds, shares and treasury bills are bought and sold. (Mushkin 1998) Financial markets are also defined as places where securities are traded may or may not have a physical location. To do the roles efficiently financial markets enable firms, organizations, individual investors to find financing for their respective businesses.

Financial instrument is a type of a financial property. The sum of all assets is the financial wealth. Examples: Currency, checkable deposits, bonds, stock shares, claims from loan contracts. (Soppe 1995) Investment Instruments or financial assets differ from the physical assets in terms of the following main points – Liquidity or Marketability – Physical Assets are not liquid when compared to financial assets. Liquidity reflects the feasibility of converting an asset quickly converted to cash when desired. Most financial assets are easy to buy /sell in the financial markets. Divisibility – Financial Assets are divisible while physical assets are not which means an investor can sell/buy a portion of asset. Termed Holding period – Holding period in financial assets vary from financial instrument to other. Holding period in physical assets is generally on long term horizon. Information – Information about financial assets is easy to obtain and more abundant. Information availability on Securities such as bonds and stocks is widely available which could influence the investment decisions for the investor. (MacKinlay1997) Stocks – It is the most common type of financial instrument most popular among investors with long term horizons for their investments. Stock represents ownership of interest in the corporations or equity in the company.

Bonds – It is a debt investment where investor loans money to the corporate that borrows the fund for the definite period of time with specified interest rate. Treasury Bills – These are treated as risk free securities ignoring inflation and default of the government. T- Bills will pay the fixed stated bond yield with certainty.

1.3 Different Investment Strategies

We understand that each investor is different and is at different phase in life for doing the retirement planning and hence there is a need to study and develop a customized investment planning tool for retirement meeting the real life challenging situations. Most of the retirement investment plan is based on one of the two strategies which is flawed and put the goals of individual retirees into danger. Let us first look at the two strategies most commonly used by the conventional advisers, retirement financial planners and major investment advising corporates and then let us look into the notion of what is different in the realistic study work on investment management scenarios for retirement planning. Two conventional retirement planning strategies are –

1. **Traditional Investing Strategies**

One strategy is to have portfolio with bond investment proportionate to age. The standard rule of thumb is to shift the portfolio from equities to bonds with proportion in bonds equivalent to age or subtract the age from 100 and let that be the non-bond allocation. This reduces the risk for investor when he starts aging in life and nearing to retirement as proportion of investment in risk free assets will increase. There is couple of problems with this income portfolio or traditional investing strategy. One problem is the inflation. (Duggan 2007) On the macroeconomic level and from public policy standpoint there are few other critical attributes affecting the retirement planning which are inflation rise impacting the income accumulation and planning, demise of defined benefits and rise of defined contribution and how it effects the planning

for retirees. Inflation is defined as a gradual increase in the cost of goods and services. Inflation contracts the purchasing power of the individuals over time because it decreases the value of currency. With lower interest rates on returns it is difficult for retirees to generate income which has been the case for last couple of years post housing recession in US in 2007. Every dollar invested in the market with rising inflation becomes less of value with time. (Jonathan 2004) That said inflation affects one's financial planning. Another problem in traditional investing is bond yield and income generating equities can sink to low levels of returns. (Bansal 2009) Treasury-bond market in US during 2008 recession sunk to historic low below inflation rate because of the panic in the market. Also risk appetite for different investors is different. People nearing retirement as per traditional investment strategy will be less invested in stocks and during recession can be severely impacted if they lose jobs. (Employee Benefit research 2007) nearly 45-50 percent of 55-65 year- old people held 70 percent of investments in equities. Investors in the traditional investing have potential problem of not having the risk defined for long term investment horizons.

2. Modern Portfolio Strategies

It is important to understand the characteristics of the second investment strategy that is Markowitz model and how non-linear programming can be used to optimize the investment model for retirement planning. Markowitz (1952, 1959), put forward a research work which is widely regarded as one of the foundational theories of financial economics. Stated in simple terms, the theory provides a method to analyze which Portfolio is better for investment that attempts to minimize the risk for a given level of the expected return (Markowitz, 1952, 1959, 1991), or maximize the returns for a given level of portfolio risk or maximum return/risk ratio, by choosing how one splits up investment pie and diversify the portfolio of assets. While expected return is based on the concept of the random variable shows the weighted average of the i th security in the observed time t. (Markowitz, 1952, 1959, 1991) In standard terms risk is a number which can be measured mathematically and refers to the fluctuations from the expected return. The notion of

MPT is a single-period theory on the choice of portfolio weights that provide the optimal trade-off between the mean and variance. Assets in the portfolio are periodically rebalanced to ensure the most optimal portfolio specific to the individual is aligned with the frontier.

(Markowitz 1952) It is important for us to understand the assumptions of modern MPT. (Markowitz 1959) Investors dislike risk which means investors are willing to take riskier investments in turn for higher expected return or accept lower expected return for less risky investments. There is no one exact risk-return formula; each investor has her own number for risk aversion characteristics depending on different stages in life and priorities. (Von Neumann & Morgenstern, 1947) The utility precisely refers to each investors objectives are different and invests catering to his needs and wants to maximize the utility function. (Markowitz 1959) Investors act rationally and utility function is based on only two dimensions expected return and variance of risk and nothing else. Investors consider each investment as the probability distribution of the expected returns over some holding period of the returns. This implies that historic returns are representative of future returns. Returns of assets are jointly normally distributed. Also there is no constraint on the minimum-maximum position of the asset and no distortion from costs, transaction fees, inflation or taxes is there. Portfolios are dynamic to less or more number of assets and assets can be traded in market at any point in time.

(Hull 2008) MPT theory has been a subject on increased criticism lately because of its underlying assumptions which have been challenged in the tough times of last big recession in US from 2007-2009. (Brinson 1986) Most of the risk and return trade-offs for the investors are defined based on the historical returns over last 10, 15 or 20 years. Historic returns and risks do not determine the future returns. Also Modern portfolio theory works on the principle of diversification. Risk depends on the correlation between returns from different securities in the portfolio, and the direction in which they go. On the periphery, adding the assets in the portfolio which are not completely correlated reduces the risk but not necessarily the return. (Simann 1997) With the less correlation when the part of the portfolio is down the

other part would go up and hence reduce riskiness associate with the portfolio. (Strivers 2009) But during the recession of 2008 most of the securities went down including the safe heaven investment gold too. The only benefit in the panic of diversification was some securities declined less than the others. Therefore past correlations are largely a product of coincidence, and probably won't hold in the future. In the scope of the paper we are not studying pitfalls of the modern portfolio theory but how we can use the fundamentals of different investment strategies helping investor meet his retirement goals.

3. Aims and objectives of study - Proposal on investment strategy for retirement planning

With the basic background on the investment strategies in the market it is therefore a need to create a retirement planning framework fitting individuals' needs and providing customized solutions. Since different investors have different risk appetite, different return objectives, are at different stages in life with different personal needs and situations we need to have specific retirement planning framework meeting the goals of investors. Generalized Industry financial investment models for everyone is not the solution of the retirees problems**.** It is the trend in the industry to sell what is generating good returns for bunch of people will produce same results for all the population. The well-known portfolio diversification strategies based on asset allocation are sold by financial planners, money managers to average investors who are naïve to understanding the finer details of the investments and generally make mistakes of choosing the financial plan which is most marketed in hope to generate great return on investments. With demise of Defined Benefits and industry trend towards the Defined contribution plans, it is imperative to study the different modeling frameworks for different investors to get to the most optimal solution required achieved. Application of real life challenges missing - This aspect is at the heart of the research, which is seen as the huge gap between the theoretical and application world. While retirement models may sell big on saving more now and not using the nest egg to have a comfortable retirement but real life challenges are different from theoretical frameworks.

Based on the background and the above research problem, objectives of this study shall be –

1. Determine the realistic investment scenarios for Retirement Planning for different investors using the fundamentals of Modern portfolio theory, tradition Investment strategy and real life situations and needs based on uncertain events in life and model their probabilities.

2. Compare the realistic investment scenarios with Bootstrap and No Bootstrap mechanisms to see if historical returns would be different from future returns.

3. Build a retirement accounting framework for investors suiting the real life needs and objectives set by investors.

4. Determine the retirement wealth accumulated for investor at the end of investment horizon customized based on the inputs provided by the investor and needs to give the realistic picture on portfolio wealth by running 10,000 simulations for each of the 48 scenarios.

1. Methodology

Portfolio theory has two dimensions Return and Risk. For large portfolio analysis for the study, let us understand the dimensions of portfolio theory Expected return and variances with the help of matrix algebra and linear models theory. We will illustrate the Simple interest returns, periodic returns, compounded returns and Annual effective returns for time value of money on equity returns with examples for each in the below section to understand the MPT dimensions of expected return and risk.

2.1 Simple Interest

Before we look at Mathematical notations, the mathematical expressions, distributions used in the research in the below chapters, let us take a simple example to understand the Modern Portfolio theory with investment in 3 stocks. Let us assume an investor has invested in two stocks Twitter, Blackberry. Let Twitter stock prices be $80 at start of the month and by the end of the month the stock price of Twitter be $85. Assuming Twitter stock investor does not get any dividends. The one month simple and gross returns will be: $85 - $80/ $80 = 0.0625. The gross returns would be 1+ 0.0625 = 1.0625. The one month investment in Twitter yielded 6.25 percent returns per month. Similarly we can calculate the monthly returns from the other stock Blackberry.

Let us assume V wealth invested is $100 in an equity which pays simple annual percentage of 10 percent. The future value FV for an investor after n years where n = 1, 5, 10 is as follows –

$$FV_1 = 100 \times (1.10)^1 = 110$$

$$FV_5 = 100 \times (1.10)^5 = 161.05$$

$$FV_{10} = 100 \times (1.10)^{10} = 259.37$$

Hence \$10 is paid in interest to the investor for investment in equity for over first year, \$61.05 is accrued to investor for over five years and \$159.37 is accrued to investor for over 10 years.

2.2 Compounding Returns

The rate of return on an investment is a profit over a period of time usually expressed as a proportion of the initial investment. Time period if expressed in years, then rate of return becomes Annual rate of return. We have explained Simple returns in chapter 2 above. Let wealth V invested in for n number of years. Let r be simple rate of return per annum from an asset i. Let us say compounding takes place every time at the end of the year, the Future Money after the end of n years is:

$$FV(n) = V(1+r_i)\times\ldots(1+r_i) \tag{2.1}$$

As time moves forward in increments, there is a return associated with the return at the end of each period. The greater frequency of compounding, the effective rates increases with each holding period but with small amounts. In the retirement accounting framework we have discretized monthly compounding returns computed for the length of number of retirement years. The reason for choosing the discretized compounding and not continuous compounding returns is that in a realistic retirement framework an average investor is not an inter-day trader and would not have time in real life to check into the retirement portfolio returns day in day out. He would rather prefer to check the returns generated once or may be twice in a year. For our accounting framework essentially we have developed to call for rebalancing once a year on investor's birthday. 'Rebalancing' is elaborated in detail later in the research. Monthly Discretized compounding is compounded 12 times in a year. For example, with monthly holding period returns, let say 1 percent is compounded 12 times then returns become $(1.01)^{12}-1$.In our framework let us say V is invested at the start of the portfolio. The discretized returns after a year would be compounded 12 times to get monthly returns compounded annually.

Equation (2.1) can be re-written as follows:

where n (number of years) = 1, m = number of periods = 12

$$FV\ (n) = V\ (1+r_i)^n \qquad (2.2)$$

Equation (2.2) can be re-written to have the periodic rate of return –

$$FV_n^{\ m} = V(1+r_i\ /\ m)^{m\times n} \qquad (2.3)$$

Using the Equation 2.3, in the below table

We have multiple compounding periods and respective returns on $100 investment.

Frequency Compounding	Periods	Future Value of $100 at 10 percent return
Annually	1	110
Quarterly	4	110.381
Monthly	12	110.471
Weekly	52	110.506
Daily	365	110.515

Table1: Frequency Compounding for periods and respective returns

As compounding frequency increases, (m increases), then the return from the investment due to compounding also increases.

2.3 Effective Annual Rate

Notation

r_A = Effective annual rate of return from as asset i

r_i = Simple annual rate of return from an asset i

The difference between two measures of return that is Simple annual return and Effective annual return calculations can be best understood with the help of an example. Suppose we start with previous example with $1000 investment and investor gets 10 percent rate of return, then the future value at the end of first quarter would be $1025. Then in the second quarter effect of compounding would be become apparent as the base would be $1025 and interest accrued in the second quarter would increase when compared with the first quarter interest accrued by $0.63. By the end of year the power of compounding would give $1103.81 in all. So the simple annual rate of return is 10 percent but the effective annual rate is more than the simple annual rate of return due to the power of compounding. The relationship between simple annual rate of return r_i and effective annual rate of return r_A with m times in n years is as follows -

Principal	1000
First Quarter	1025
Second Quarter	1050.625
Third Quarter	1076.891
Fourth Quarter	1103.813

Table 2: Quarterly compounding on $1000 invested

$$(1 + r_A) = (1 + r_i / m)^{m \times n} \tag{2.4}$$

Given the Effective annual rate r_A, we can solve for Simple annual return r_i as follows re-writing equation -

$$r_i = m[(1 + r_A)^{1/m} - 1)] \tag{2.5}$$

Given the Simple annual return r_i we can solve for Effective annual return r_A as follows re-writing equation -

$$r_A = (1 + r_i / m) - 1 \qquad (2.6)$$

We now consider a quick example to understand the relationship between periodic return, simple rate of return, compounding return. Let us assume an investor invests in equity for a year and interest on the investment is paid multiple times in the year periodically say. Therefore r_i / m becomes the periodic return on investment. Let us say investor invests $1000 in equity at 2.5 percent quarterly rate of return. How much will investor get at the end of the year?

Applying equations (1.4), (1.5), (1.6) above we will calculate the future value for the investor –
With 2.5 percent periodic return for 4 quarters, we get the simple annual return as 2.5 percent times 4 to get 10 percent.

$$FV = 1000\ (1 + 0.10/4)^4$$

$$= 1103.812$$

Putting the Simple annual rate of return r_i, future value, present value in the equation 1.8 we get Effective annual rate of return for the investor – r_A = 1103.812/1000 – 1 = 10.38 percent.

2.4 Portfolio Theory

We now have seen the Time value of money calculation on certain financial return measure such as Simple annual return, periodic return, and multiple compounding returns. With the basic knowledge on the return measures, portfolio investment theory, it is a good time to look deeper into calculations of

expected return and variance for an investor when he has invested money in large portfolio with the help of matrix algebra to make the computations easier to comprehend.. As stated earlier, investing is a trade-off between risk and returns expected. There are certain assumptions in the research. The first assumption is people have money to invest for retirement planning. Second assumption is people will invest in two asset classes – stocks and bonds for the research work. Third assumption is investor will withdraw money from the retirement portfolio and last assumption is investors risk appetite reduces as he ages in life. For the retirement framework we have assumed that in real life investor's risk appetite reduces as he ages in life. To find the optimum portfolio from these assets we will hold this portfolio for next period and set the objective function based on the preferences inputs by the investor. For building on the MPT theory alluded earlier let us consider that investor wants to invest in portfolio containing m risky assets (e.g., stocks, Bills, Securities, treasury bills, government bonds), $X_1, \ldots, X_m$. Next section gives insight on the Portfolio returns and risk dimensions.

2.4.1 Portfolio Expected Returns and Portfolio Risk

Before we discuss the Portfolio variance and Portfolio expected returns, let us see how simple returns are calculated for a security from one time period to a different time period building on the concept for calculating the portfolio returns thereafter.

Notation

$S_{i(\mathrm{t})}$ = price of the asset i at time t.

$S_{i(\mathrm{t}-1)}$ = price of asset i at the end of the month $t-1$

$r_i(t-1, t)$ = holding period return of asset i

We have collected the Discrete-time analog of Price of the securities to calculate the returns on assets for K periods. Consider purchasing an asset (e.g., stock, bond, Treasury bill, future, option, forward etc.) at time t for the price $S_{i(\mathrm{t})}$, and then selling the asset at time $\mathrm{t}-1$ for the price $S_{i(\mathrm{t}-1)}$. When we have no other cash flows (e.g., dividends) between t and $\mathrm{t}-1$, the rate of return over the period t to $\mathrm{t}-1$ is the percentage change in price of the security. The time frame from t and $\mathrm{t}-1$ is called the holding period and $r(\mathrm{t}-1, t)$ for the same time frame is called the holding period return. In principle, the holding period on the price differential can be any amount of time depending on the investor: one second; 20 minutes, 1 hour or 10 years anything. We will assume for our Accounting framework that the holding period is some increment of calendar time; e.g., one month or one year. We would generate the Monthly returns and use compounding on Monthly returns to calculate the Yearly returns on the securities. Let $S_{i(\mathrm{t})}$ denote the price at time period t of an asset that pays no dividends and let $S_{i(\mathrm{t}-1)}$ denote the price at the end of month $t-1$. Then the one-month simple net return on an investment in the asset between months $t-1$ and t is defined as ratio of the discrete-time analog price Lag time difference between the 2 time periods t and $t-1$ relative to the price at the previous month i.e. $t-1$.

$$r_i(t-1,t) = \frac{S_{i(\mathrm{t})} - S_{i(\mathrm{t}-1)}}{S_{i(\mathrm{t}-1)}} = \%\Delta S_{i(\mathrm{t})}. \tag{2.8}$$

$$r_i(t-1,t) = \frac{S_{i(\mathrm{t})}}{S_{i(\mathrm{t}-1)}} - 1$$

We can define the simple gross return as

$$1 + r_i(t-1,t) = \frac{S_{i(\mathrm{t})}}{S_{i(\mathrm{t}-1)}} \tag{2.9}$$

Since asset prices are non-negative and long-term investment in an asset is a limited liability, the smallest possible return on $1 invested is loss of $1 or we can say -100 percent. We will use Net return for the calculations below unless otherwise stated.

Now let us understand the Portfolio returns from two stocks for the investor. Portfolio return is explained in detail later in the chapter. A quick example on simple rate of returns is an investor put $10 in his savings account at time period ($t-1$) and at the time period t the amount becomes $12.So the rate of return for investor is 20 percent. We are assuming for the retirement framework we have ten stocks in which investor want to invest using the principles of modern portfolio theory. We will use Matrix algebra and linear models theory to explain the Expected return calculations below. The returns data for ten stocks is for certain period implying a sample from a population. Let us now consider a random sample $X_{i(1)}, X_{i(2)}, X_{i(3)}, \ldots., X_{i(n)}$ from a population for an asset i with mean $\overline{X_i}$ and variance S_i^2, then we know the relationship between population parameter and sample statistics is that the statistic $\overline{X_i}$ is an unbiased estimator of μ_i and S_i^2 is an unbiased estimator of σ_i^2. The alternative estimator for σ_i^2 is $\sigma_i^{\;2}$ which is the Maximum likelihood estimator with distribution assumption where

$$\sigma_i^{\;2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_i)^2 = \frac{n-1}{n}\mathrm{S}_i^2$$

Without the distribution assumption the maximum likelihood estimates of μ_i and σ_i^2 are $\overline{X_i}$ and S_i^2 respectively which are also the method of moments estimates. Using Markowitz portfolio theory Portfolio expected return from an asset i has following distribution -

$$r_i \overset{iid}{\sim} N(\mu_i, \sigma_i^{\;2}) \qquad (2.10)$$

Where r_i is the return from the i security and i = 1, 2, 3,…., 10 securities in the portfolio we have assumed for the retirement framework with Normal distribution. For the application purpose in the retirement framework we will us the unbiased estimator for mean that is $\overline{X_i}$ and for variance that is S_i^2.

Markowitz theory assumes known μ_i and σ_i^2. But since in practice μ_i and σ_i^2 are not known we estimate the parameters assuming the normal distribution and returns are i.i.d from the historical data. In the discrete time analog process as defined in the chapter 2 under the concept of Simple annual rate of return we discussed $r_i(t-1,t) = \dfrac{S_{i(\mathrm{t})} - S_{i(\mathrm{t}-1)}}{S_{i(\mathrm{t}-1)}} = \%\Delta\, S_{i(\mathrm{t})}$ and logarithm returns are

$\log(\dfrac{S_{i(t)}}{S_{i(t-1)}}) = \log(S_{i(t)}) - \log(S_{i(t-1)}) = \log(1 + r_i(t-1,t) \approx r_i(t-1,t)$ using the logarithm property of raw-logarithm equality when returns are very small which is common for short holding durations, the raw logarithm equality approximation gives results closer to raw returns which are independently and identically distributed as assumed. (Bayes and Shrinkage) asserts in his research work that if the μ_i and σ_i^2 are replaced by estimators μ_i and $\sigma_i^{\;2}$ then they perform poorly. Since we have assumed that returns in the assets are independent and identically distributed and are normally distributed, we will use the available data to estimate the properties of the statistical distribution but we will still take the expected return for a given asset to be a simple average of all the historical return values and the standard deviation to be the squared deviation from the average value. More on the expected return and standard deviation is discussed in the chapter below. For the retirement framework with ten stocks we define the 10×1 column vectors containing the asset returns and portfolio weights. Multiple returns can be put into single vector of Portfolio expected returns in the matrix notation. Let us define the Portfolio expected return but before that we need to define the expected return from an asset. The subscript w indicates that return portfolio is constructed using the weights $w_{(1)}, w_{(2)}, w_{(3)}, ..., w_{(10)}$. The expected value of a distribution can be thought

of as a measure of the center as we think of averages as the middle term. Hence by weighing the values of the random variable in the distribution we mean to have obtained a number than summarizes the typical expected value of the random variable. Therefore the expected return from an asset i is defined as below

$$E(\hat{r}_i) = \bar{X}_i \tag{2.12}$$

Where $E()$ denotes the expectation of the random variable. Though expected return or expected rate of return from an asset as defined above gives us some measure of the assets performance but it does not capture uncertainty in obtaining the comparable rate of return with the average. The indicator to quantify the deviation of rate of return from the expected value is the measure of the riskiness associated with the asset known as Variance from the asset i –

$$\hat{\sigma}_i^{\,2} = \mathrm{var}(\hat{r}_i) = \sum(|\, r_i - \bar{X}_i\,|^2) \tag{2.13}$$

Now for an investment in portfolio we have not a single asset but ten stocks. In the portfolio it is important to take into account not only the individual assets returns but also covariance of all assets and how they have impact on the rate of return. Also it is a good point to look at the distribution of the portfolio assets. The distribution of the each security in the modern portfolio theory is normally distributed hence the distribution for portfolio assets would be jointly normally distributed by the following principle. Multivariate distributions are used to characterize the joint distribution of a collection of m random variables. Consider a vector $X = X_1, \ldots, X_m$ which is said to have multivariate Normal distribution if the new random variable is the linear combination of the $Y = a_1X_1 + a_2X_2 + \ldots + a_mX_m$ which is normally distributed. For the sake of simplicity of our analysis, let us assume the assets returns in the portfolio are having the Gaussian distribution returns then the Portfolio distribution is multivariate normal.

(Markowitz, 1959), stated that portfolio risk is not the variance of the individual assets but the covariance of the whole portfolio. The smaller the covariance between the securities, as covariance is the direction of the assets, the smaller the volatility of the portfolio overall. The more the assets move in the same direction, the more chances of the economic turmoil bringing them down together and hence increasing the risk associated with them. The diversification is most imperative to the modern MPT theory. On the periphery, adding the assets in the portfolio which are not completely correlated reduces the risk but not necessarily the returns. Market-risk or Systematic risk is not tied to the diversification but unsystematic risk can be controlled with diversification. That is why Diversification in the econometric financial industry is freely coined as 'Free Lunches'. When wide variety of investment instruments is mixed in the portfolio, it reduces the impact of one security in the overall portfolio performance. It ranges from negative 1 to positive one. Each asset in the portfolio is selected based not on its own goodness but how its position in the portfolio changes relative to the every other asset in the portfolio. We have already defined the 10×1 matrix for expected returns from each security and weights allocated to each security in the portfolio. Let us derive the covariance matrix now for calculating the portfolio expected return and portfolio variance.

When we have $S_{i,j}$ = $S_{j,i}$ and that when $j = i$, we have $S_{j,i} = \mathrm{S}_i^2$ where $S_{j,i}$ is the covariance between two assets i and j is defined as below -

$$S_{i,j} = \Sigma(r_i - \bar{X}_i)(r_j - \bar{X}_j)$$

A covariance matrix Σ is as follows –

$$\Sigma = \begin{pmatrix} S_{1,1} & \cdots & S_{1,n} \\ \vdots & \ddots & \vdots \\ S_{n,1} & \cdots & S_{n,n} \end{pmatrix} \tag{2.14}$$

Re-writing equation (2.14) for 10 stocks we have covariance matrix as follows –

$$\Sigma = \begin{pmatrix} S_1^2 & \cdots & S_{1,n} \\ \vdots & \ddots & \vdots \\ S_{n,1} & \cdots & S_n^2 \end{pmatrix} \tag{2.15}$$

The portfolio is invested in ten assets with the total wealth of V. Let W_i denote the wealth invested in each asset. We have assumed positive values for W_i our framework which implies we are having no short selling for our retirement framework.

So total wealth invested in the portfolio be as follows-

$$\sum_{i=1}^{n} W_i = V \tag{2.16}$$

Let w_i denote the proportion of the wealth invested in asset relative to the total wealth invested in the portfolio where (i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10) stocks in our retirement framework. So proportion of wealth invested in i th asset is –

$$w_i = \frac{W_i}{V} \tag{2.17}$$

Therefore proportion of wealth invested in all 10 stocks sum to 1 so that

$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 + w_8 + w_9 + w_{10} = 1$.

This implies

$$\sum_{i=1}^{n} w_i = 1 \tag{2.18}$$

The portfolio value T_p at any time period t can be expressed as using the equations (2.9) an (2.17) from the i security as below -

$$T_p(t) = \sum_{i=1}^{n} w_i \times V \frac{S_{i(t)}}{S_{i(t-1)}} \tag{2.19}$$

Re-writing using equation (2.17)

$$T_p(t) = \sum_{i=1}^{n} W_i \frac{S_{\mathrm{i}(t)}}{S_{\mathrm{i}(t-1)}} \tag{2.20}$$

Where we know the portfolio value $T_p(0)$ at time period 0 is W from equation (2.16). Therefore the portfolio rate of return r_p at any time t is given as below –

$$r_p(t) = \frac{T_p(\mathrm{t}) - T_p(0)}{T_p(0)} \tag{2.21}$$

$$r_p(t) = \frac{\sum_{i=1}^{n} W_i \frac{S_{i(t)}}{S_{i(t-1)}} - W}{W}$$

$$r_p(t) = \sum_{i=1}^{n} \frac{W_i}{W} \frac{S_{i(t)}}{S_{i(t-1)}} - \sum_{i=1}^{n} \frac{W_i}{W}$$

$$r_p(t) = \sum_{i=1}^{n} w_i \frac{S_{i(t)} - S_{i(t-1)}}{S_{i(t-1)}}$$

$$r_p(t) = \sum_{i=1}^{n} w_i r_i \tag{2.22}$$

Therefore as derived in equation (2.22), the rate of return of portfolio is the weighted average of the rates of the return from the assets where weights are determined by proportion of investment in each asset. The expected return of the portfolio r_p is derived as below using the equation (2.7) -

$$E(\hat{r}_p) = E(\sum_{i=1}^{n} w_i r_i)$$

$$E(\hat{r}_p) = \sum_{i=1}^{n} w_i \bar{X}_i \tag{2.23}$$

Re-writing the equation (2.23) for ten stocks –

$$E(\hat{r}_p) = w_1\bar{X}_1 + w_2\bar{X}_2 + w_3\bar{X}_3 + w_4\bar{X}_4 + w_5\bar{X}_5 + w_6\bar{X}_6 + w_7\bar{X}_7 + w_8\bar{X}_8 + w_9\bar{X}_9 + w_{10}\bar{X}_{10} \tag{2.24}$$

The portfolio variance $\mathrm{S}_p{}^2$ for the rate of return is given by –

$$\hat{\mathrm{S}}_p{}^2 = \mathrm{E}(|\sum_{i=1}^{n} w_i (r_i - \bar{X}_i)|^2)$$

$$\hat{\mathrm{S}}_p{}^2 = \mathrm{E}((\sum_{i=1}^{n} w_i (r_i - \bar{X}_i)(\sum_{j=1}^{n} w_j (r_j - \bar{X}_j))$$

$$\hat{\mathrm{S}}_p{}^2 = \mathrm{E}((\sum_{i=1}^{n} w_i (r_i - \bar{X}_i)(\sum_{j=1}^{n} w_j (r_j - \bar{X}_j))$$

$$\hat{\mathrm{S}}_p{}^2 = \sum_{i,j=1}^{n} w_i w_j E((r_i - \bar{X}_i)(r_j - \bar{X}_j))$$

Using equation (2.15) -

$$\hat{\mathrm{S}}_p{}^2 = \sum_{i,j=1}^{n} w_i w_j S_{i,j} \tag{2.25}$$

Re-writing equation (2.25) –

$$\hat{S}_p^{\ 2} = w^t \Sigma w \tag{2.26}$$

Where w^t is $[w_{1,} w_{2,\ldots\ldots,} w_n]$ and Σ is defined in equation (2.14)

The variance of the portfolio is defined by the covariance matrix and weights of the asset classes, where Σ is the positive semi-definite variance-covariance matrix of the asset returns for m different assets. Sharpe (2000) states that portfolio return is the weighted average of the expected return of the individual assets, depending on the weights of the individual assets, an asset will have smaller or larger impact on the portfolio return overall. Brinson (1986) states that different assets differ in terms of their expected return and expected return of an asset is just based on future predicted performance of the asset but what influences the expected return of the asset is how volatile is the asset. It is impossible to state the expected return of the asset accurately. Hence the objective is to predict the expected return of each asset to predict the expected return of the portfolio as stated and explained with the help of examples in the chapter above. The standard deviation is nothing but the volatility or uncertainty associated with each asset. It measures how spread is the data of the returns for an asset. The standard deviation of the portfolio expected return is the relation between the correlation between different assets, standard deviation of each asset and the proportion of the weights invested in the assets. If we were having three assets in the portfolio the variance of the portfolio will depend on the three variance terms and six covariance terms. So what is the relationship between return and risk - Risk taking ability of an investor drives the return and so risk is something to be managed and not avoided. Hagstrom (2001) stated that one cannot expect high returns without exposing oneself to some kind of risk. Risk and return go hand in hand and depending on investor's risk appetite or loss taking ability, returns can be maximized.

2.4.2 Rebalancing

Since every point on the curve represents an optimal portfolio, for every level of return there is a portfolio that offers a lowest level of risk and similarly for every level of risk there is a single portfolio that offers maximum return. (Sharpe 2009). Thus it is certain investor for a given level of risk strives for a portfolio along the frontier curve to generate maximum returns on his investments. The combination of assets in the portfolio have distribution of returns which is not fixed over a period of time, therefore weights need to be reallocated to the assets in response to the macroeconomic factors to mitigate the systematic risk.

2.4.3 Asset Allocation

Asset allocation is an investment strategy which aims at distributing the investment instruments in the portfolio to balance the risk reward trade-off according to an investor's risk appetite, investment horizons and individual's goals. Asset allocation is at the heart of diversification and independence of the assets in the overall portfolio. A well-diversified portfolio in the investor's portfolio of invested securities provides the variation in the distribution of the returns and hence helps in reducing the risk. Brinson, Hood, and Beebower (1986), asserts that asset allocation is the dominant driver of a portfolio's investment returns over long horizons. Investors follow different rules of thumb to maximize the return at a pre-specified risk level by following different strategies on investments of assets. Some investors attain optimization by allocating assets in different securities from different industries and some believe in identifying global stocks to mitigate the risk. Siegel (2003) recommends investors to allocate heavily to equities over long horizons to maximize the returns. Kim and Wong (1997) also find 100 percent equity strategy dominant over all other strategies for long horizon investors. Other studies like Booth and Yakoubov (2002) and Blake, Cairns, and Dowd (2001) do not support such strong conclusions on the investments in equity. They suggest that DC plan participants should pursue a well-diversified strategy till retirement. Shoven (1968) considers that optimal location of asset allocation between risk-free and risky assets in the pension funds in order to give maximum tax benefits. Hickman (2001) used US market data from period (1926-1997) examined relative performance of bills, bonds, and stocks by employing sampling with replacement

and estimating period-by-period return differentials to conclude that for investors with holding periods of 20 years or more, investing in any asset class other than equity results in substantially less expected terminal wealth, while imparting little risk reduction benefits in compensation. The old school of thought on asset allocation is to subtract age from number 100 and invest the result number in stocks and remaining percentage in bonds. For example, 30 year old investor should invest 30 percent in bonds and (100-30) = 70 percent in stocks. ‘Bonds to age’ asset allocation strategy is what we will apply in the retirement accounting framework for retirement income strategy. The primary notion behind the asset allocation strategy is simply negatively correlated to the age and risk equation. Our hypothesis is in the realistic life framework as age advances investor’s risk appetite starts reducing. The risk taking ability of the investor reduces with the more number of years added to the age and he starts to worry about the real problems in life. Historically returns from the stocks are higher with higher variations in the distributions of returns when compared to bonds returns. During the early years of the retirement planning the variation association with stocks does not scares the investor from taking the risk.

2.4.4 Bootstrapping Resampling Procedure

The bootstrapping is a statistical technique in which inference about the population from the sample data can be modeled using the resampling procedure that is to resample from the sample data and to perform the inference on the same. As for example assume we want to measure the height of all the people in the global population. It is not possible to measure the height of each individual, so rather we can sample a small part of the population with let say N data points and estimate the mean. But with this single estimate we won’t get the sense of variability in the mean. Hence we would need more samples for appropriate statistical inference or can implement the bootstrapping procedure and have resamples computed from the sample with replacement to have the bootstrap distribution. The advantages of the Bootstrap Technique are: No prior assumption on the distribution of the data, does not rely on the large sample size in contrast to central limit theorem. However there are certain disadvantages entitled to the

Bootstrapping technique of certain assumptions which are hidden under the simplicity of this approach of assumptions being made on the independence of the samples. The simple reason we have chosen Bootstrapping technique is to counter the prior normal distribution assumption of the portfolio returns. Without assuming the distribution from the returns we would take the empirical distribution in the non-parametric bootstrapping approach and push the returns downward in the accounting framework. This would account for the fact that no assumption on the distribution is made to see if future returns would be same as historical returns and to illustrate if it is evident based on the results to say that investor has no reason to worry if bootstrapping results are similar to the multivariate normal distribution results, and last but least no assumption of distribution on the samples which would have created the bias with the bootstrapping distribution used.

Bootstrapping Technique used:

1. $r_{i(1)}, r_{i(2)}, r_{i(3)} ..., r_{i(144)}$ Historical returns from the last 13 years of the data with 144 observations for an asset i.
2. Sample Procedure – Using sampling procedure with replacement generate the returns for the each asset for n retirement years for investment. The generated returns are from the historical data randomly chosen with replacement. $r_{i(1)}, r_{i(2)}, r_{i(3)} ..., r_{i(\mathrm{n})}$
3. Use generated returns in the Retirement accounting framework.
4. Repeat the above 1, 2, 3 steps for n years

2.4.5 Monte Carlo Simulation Procedure

A Monte Carlo Simulation method is based on the simple idea that in more involved engineering studies and in realistic models the parameters of the probability densities are not known; however the estimates of the parameters are known. For instance, the mean of the variable of interest μ , can be estimated by $\bar{X}$ =

X /n, however the μ is not equal to X , although Central Limit Theorem states it is close. (Boyle 1977) 'The close enough for the engineering study' would lead to gross under-estimation of the risk as probability of 1/1000 would be close enough for one kind of study but certainly not for the other. Monte Carlo simulations offer full probability distributions for the variables and hence full information with simplicity. Also results from it are more reliable with reduced modeling risk as we do not assume any probability distribution. Hence we would use the Monte Carlo Simulation method in the Retirement accounting framework to compute the Portfolio Value at the end of the retirement horizon of n years. Runs in the Simulation are the number of times the entire algorithm is run. A disadvantage with the Monte Carlo simulation is that entire Simulation process could be very time consuming. The total time to calculate the variable of interest with the simulations could further increase with complexity in the computing modeling algorithm. Also one needs large number of Scenarios say 1000 or 10,000 depending on the problem situation to keep the uncertainty within acceptable levels. For the study we have chosen 10,000 runs, as standard error for the mean of the distribution is:

$$SE= \frac{S}{\sqrt{N}} \tag{2.27}$$

Where

(Chen 2000) The standard error of running Monte Carlo simulation is the estimate of the standard deviation of the many values returned from running the Monte Carlo simulations. The standard deviation is nothing but the volatility or uncertainty associated with each asset. It measures how spread is the data of the returns for an asset. The standard deviation of the portfolio expected return is the relation between the correlation between different assets, standard deviation of each asset and the proportion of the weights invested in the assets. From the above formula, one can reduce the error term by either increasing the numerator or by decreasing the denominator. Technically speaking, the case increasing the numerator can be done by improvising on the distribution's volatility estimates. This can be achieved by improving the distribution of the returns, variance reduction or other numerical techniques. In the other case of the

denominator increasing we mean to increase the number of runs for the simulations. In this formulation to reduce the error no efforts are made to harness the distribution. This method is generally implemented when no clarity on the distribution is there. Simulation studies of this nature are said to be ‘Brute force’ computations. The Accounting framework we are using the second approach as we do not know the distribution of the Portfolio value. We will run 10,000 runs for each Scenario to reduce the risk error estimates and to get better accuracy of the distribution of interest.

2.5 Data

Data to investigate consists of randomly chosen ten stocks and five bonds representing different sectors such as Technology, Aviation, Beverages, Restaurants, Apparel, Oilfield Services, Internet and online retailing, Medical equipment, Telecommunications and equipment, Food, Energy, Hotel and Entertainment. However the stocks have been picked randomly from Yahoo Finance with no intent on Diversification. Data represents ten stocks such as Apple, Qualcomm, Amazon, Pepsi, Marriott International, Johnson and Johnson, British Airways, Coach, Schlumberger, Mc Donald and is analyzed on monthly returns scale on time period between 2001 and 2013 with 144 observations. Dataframe created to store the monthly returns data from ten stocks in the required processing format. R libraries and packages installed for retrieving returns and for getting returns from Yahoo finance and for storing into Dataframe for further analysis. Treasury Bonds data solicited from the US Department of the Treasury on time periods same as stocks of Treasuries such as 1 Year, 3 Year, 5 Year, 7 Year, 10 Year. Throughout this paper data is referred to as Stocks data and Bonds data. Code is flexible for addition of stocks, bonds and other investment instruments as per the specific needs of the investor. The annualized monthly yield is converted to monthly yield for the accounting framework using the formula used above in equation (2.4),(2.5). The annualized effective yield return given for bonds treasury, we can calculate the monthly discretized compounded returns. Bond returns from the Annualized treasury bond yield curves will give

the positive Bond return values. A bond's yield is always positive, but its actualized return can be negative due to capital losses. Bond has 4 yields – one of them is coupon yield. Bond yield is nothing but return on the investment. A tricky concept with the Bond yields is that bond yields and prices of the bonds are inversely proportional. The simplest formula to calculate the bond yield is coupon amount/ price. So when price of the bond increases and coupon return decreases. Bonds prices changes on daily basis and sometimes understanding fluctuations in the bond market prices on daily basis is a confusing concept for the investors to understand. The fluctuations in the bond prices could be very small. Let us consider an example to illustrate this – If an investor buys a bond with a 10 percent coupon at its par value of $1000, then the bond yield is 100/1000 = 10 percent. But if the price changes to $1200 then the bond yield shrinks to 8.33 percent and if the price of the bond changes to 800 dollars then the bond yield changes to 12.5 percent. How it benefits the investor is buyer of the bond wants to pay less price to nurture more bond yield, in the above example buyer would want to pay $800 to get 12.5 percent bond yield. On the other hand, seller wants to sell the bond at a higher price, since he has already locked the interest and he hopes price of the bond goes up as he can cash out by selling the bond in future. Taking the logarithm of the monthly compounded returns of bonds help us apply the raw-logarithm equality approximation in case of return being very small which is common for bond trades for short duration will ensure value closer to the return $\log\ (1+r_i) \sim r_i$. Take Logarithm of bonds, since returns bond yields are always positive. The only way investor could have negative bond yields on the bonds investment is when he were receiving negative interest payments or if somehow bond had market value of less than $0, both of the situations are unlikely. Since one gets the positive values then one would work on taking the antilog so to get back to the original numbers. This way we achieve the approximate raw logarithm equality, when bond yields are very small. We will generate the returns using the multivariate normal distribution using the $\log(z)$ returns of the bonds. Then we get the anti-logarithms for the Actual bond returns.

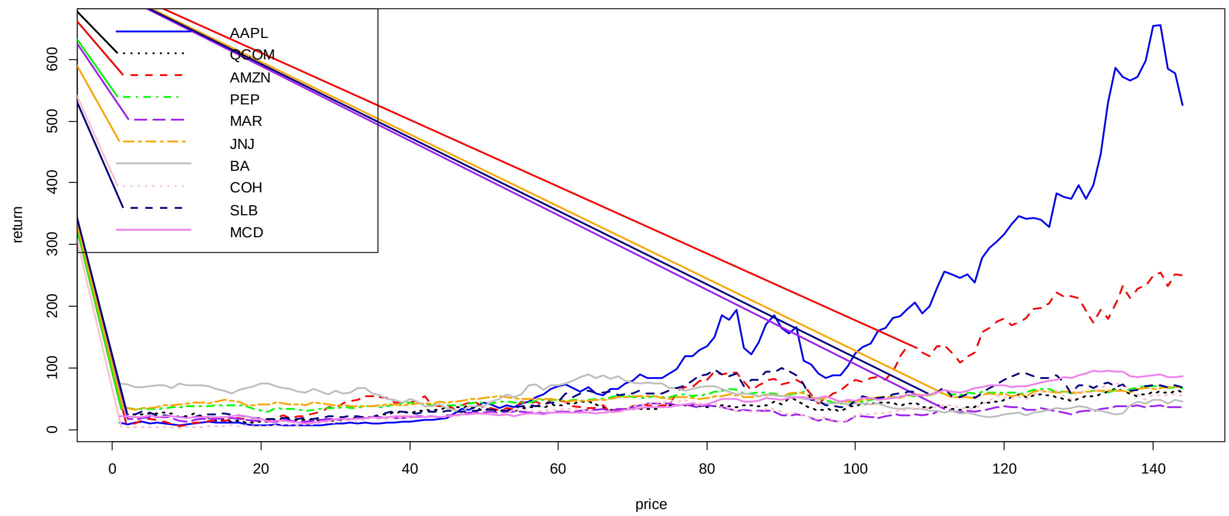

FIG. 1 For ten Assets chosen randomly above is the Return versus Price graph with each line representing individual asset returns.

2.6 Distribution for factors in accounting framework

To be able to model a phenomenon, in terms of the random variable X , with a cumulative distribution function we are concerned with a behavior of the random variable and what properties fit the random variable to fit distribution and use the variable further in the model. The selection of a distribution for a variable depends on the absence or presence of the data set with respect to it means value. When we have a situation with data symmetrically distributed around its mean while the frequency of the data farther away from the mean gradually diminishes then we fit a normal, logistic distributions. When we have a situation where we have larger values farther away from the mean than the smaller values then we use distributions for left skewed or right skewed relevant distributions fit. There are techniques for distribution fitting example parametric methods such as Methods of moments, Maximum likelihood with which we estimate the parameters of the distribution. Beta distribution is critical to understand here as in the later chapters we will use Beta Distribution to simulate the distributions for variables. We will explain this later when we use these expressions in the Accounting Framework.

2.6.1 Simulating Income, Bonus and 401k Monthly Distribution

We will simulate the income data for K periods using the Beta distribution package in R to generate the data. For estimating the parameters for the Beta Distribution for Income we need to estimate the Mean and Variance. Beta is a suitable model for modeling random behavior of percentages. It is non-negative continuous distribution. We use Beta distribution to model random percentages changes for random variables in our model i.e. income, Bonus, and Inflation factor. We use randomly generated with Mean 1 percent and variance 0.01 percent in current dollars or randomly generated with Mean 4 percent and variance 0.01 percent in nominal dollars. We have solicited Per capita income data from Bureau of labor statistics from 1980 to 2013. Bonus is performance based measure given to employees as a reward once in a year. We have estimated Bonus using beta distribution with mean 4 percent and variance 0.01 percent basing it on income. We have used fixed percent 4 percent of monthly contribution on annualized basis for simulating the Monthly contribution basing it on income.

2.6.2 Withdrawal Logic

Getting most out of the savings plan is the dream of every investor planning for retirement.
Though many financial planners would argue on try never to take money out of the retirement portfolio and also contribute as much as one can. Having maximum percentage of base salary contributed to the 401k account, expecting on to accumulate the stash cash almost exceeding million dollars with retirement income lasting gracefully for entire life expectancy, having a beautiful house, exciting vacations, retiring abroad and educating one's kids is a dream of every investor. But real life story is not that simple. For investor who has predicaments or real life situations when he feels cash starved looks to the retirement portfolio to make ends meet. In real life situations one may need cash to fund to different life emergencies such as health care of elderly family members, economic crisis causing no job situation, pregnancy of

wife, kid's education to name few uncertain events situations in life. It's a daunting task in real life situations to make ends meet. The immense task for the investor is to seek for help from someone to make a good optimal retirement plan which is individual specific and serves him best. There are generalists in the group to offer financial planning services. They are supposed to help investor with the investing, budgeting, planning, retirement planning, insurance needs and so on. People in the industry like financial planners, money managers, mortgage analysts, personal finance analysts offer services to help investors with generalized optimal portfolios. However investor needs to pick up from the list of the candidates selectively who offer best pro-service suiting the individualistic needs customized specifically for the investor. It's time now we understand in detail the methodology to handle the uncertain events in life and to model the probability of these events. As discussed earlier each investor is different and hence we cannot enforce the generalized uncertain modeling events logic on all. To handle this unique situation we implemented the below uncertain events withdrawal logic with changing probability profiles scenarios:

1. Initialize the withdrawal variable.
2. Scenario1 – Changing probability profile Uniform discrete uncertain events probabilities.

There is a likelihood of uncertain event happening at every time period in the investor's investment horizon. Since Event can happen any time period as there is a very small probability of events happening at all-time points, we cannot time the uncertain events; we need to have a discretized distribution to assign equal weights at all time periods. The discrete Uniform Distribution puts equal mass on each of the outcomes where N any specified number is. In our Accounting framework N is $12\times n$ where 'n' is the number of retirement years and 12 is the number of months in a year.

$$P(X = x \mid \mathrm{N}) = 1/\mathrm{N}, \mathrm{x} = 1, 2, 3, ..., \mathrm{N}$$

3. Scenario 2 – Changing probability profile Increasing risk uncertain events probabilities

We know there is a likelihood of uncertain event to happen at any time period during the investor's horizon. In a Changing probability profile Scenario 2, there is a second kind of investor who expects the uncertain events probability in correlation to his age. He expects the probability of uncertain events to go up as he ages in life. For instance he thinks he would need money in later years for kid's education, elderly members health care life needs and so on. We would use STEPFUNC to increase/Decrease the probability by constant to the Uniform Probability.

4. Scenario 3 – Changing probability profile Decreasing risk uncertain events probabilities

In a Changing probability profile Scenario 3, there is a third kind of investor who expects the uncertain events probability to decrease as he ages in life. For instance he would need more cash in initial years when he is settling down for events like wife's pregnancy, house renovation.

Changing Probability profile Scenario depicted graphically on next page

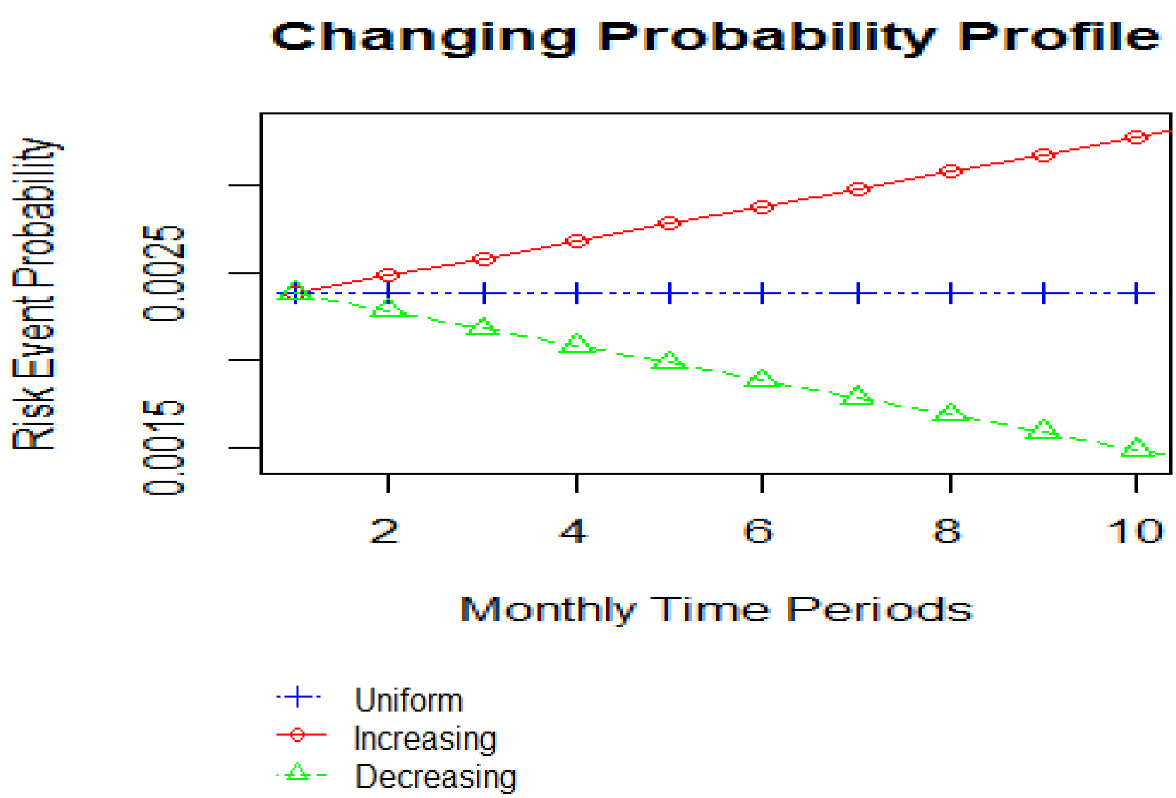

Fig 2: On the Y axis we have Risk event probability for in investor's life. On the X axis we have Monthly time periods implying there is a risk event probability at every time period. The blue color line displays the uniform probability of risk event happening in investor's life time , red color line displays the increasing risk event probability and green line displays the decreasing risk event probability in investor's life.

5. When Uncertain events happen generate the probability of True

We would use discrete Binomial Distribution as it is based on the idea of Bernoulli trial. A random variable is said to have a Bernoulli trial if:

$$f(k;p) = (\, p \text{ if } k=1 \text{ or } 1-p \text{ if } k=0\,)$$

Where p is the probability of success and 1-p is the probability of failure. The value of 'p' where X = 1 is termed as a success of event and value of X = 0 is often termed as a failure of event. In 'n' identical Bernoulli trials are performed, as in the case of coin tossing then the distribution of all the independent events is defined as total number of successes in n trials which is Binomial distribution. Generate the true event using the binomial distribution with a trial at each time point for each scenario. In our Accounting framework we wish to determine the number of successes in the n independent trails to generate the probability of TRUE event. TRUE event would generate the uncertain event at the time period in 420 time periods. At this time period Investor would need money and withdraw some cash from the portfolio.

6. When True event generated investor withdraws money on below logic -

Investor would take out money from the portfolio when we have TRUE event generated. We have implemented the withdrawal logic using the Location and Scale family distribution by setting the lower bound and upper bounds on the withdrawal of money. A location-scale family is a family of distributions formed by translation and rescaling of a standard family member.

Suppose $f(x) = (1/\sigma)\, f((x - \mu)/\sigma)$

Where μ and σ are constants.

- $f(x \mid \mu, \sigma)$ is termed as location scale family

- If σ *= 1, we have location family;* $f(x \mid \mu) = \mathrm{f}(\mathrm{x} - \mu)$
- *If* μ *= 0 we have Scale family;* $f(x \mid \sigma) = \mathrm{f}(\mathrm{x}/\sigma)/\sigma$

For our accounting framework investor can take let say 10 percent of income as lower bound of withdrawal distribution and 90 percent of portfolio value as the upper bound. (Using Theorem 3.5.6, in the book George Casella) Let μ be any real number and σ be any positive real number and Let f(.) be any pdf. Then X is a random variable with pdf (1/σ) f((x - μ)/σ) if and if only there exists a random variable Z with pdf f(z) and X = σ Z + μ. If Investor needs money, Generate a distribution on Portfolio Value set to lower bounds and upper bounds with Income × 0.10 as LB and portfolio value × 0.90 as Upper bound. Z is a withdrawal variable with beta distribution to have random percentage values between 0 and 1. We have assumed Mean 15 percent and variance 0.1 percent for the withdrawal variable. As stated in equation above for any random variable when we want to impose lower bounds and upper bounds on a distribution we use the Location and Scale family distribution. When for random variable Z value is 0 then lower bound is the μ and when the Z value is 1 the upper bound of the distribution of a new variable becomes σ + μ. The lower bound for the accounting framework is no less than 0.10 × income and upper bound is no more than 0.90 × portfolio value depending on the value of the Z withdrawal variable distribution.

2.6.3 Retirement Accounting Framework

Initialize the Individual's annual wages and beginning value to invest in the Portfolio. Let say W = total Annual wages of the individual at a certain age A_1 when the investment planning commenced. Let say investor wants to invest in m = number of assets. For the sake of simplicity, let the Asset be in the vector form as - $X_1, \ldots, X_n$ = assets in which investor want to invest in. Let us also assume the C_0 = capital that can be invested in dollars and C_{end} = capital at the end of the period in dollars. Money invested in the stocks would be distributed using the Portfolio Optimizer nonlinear solver program logic. Let w_i =

weight of security in portfolio as output from the solver. First we would calculate the expected Returns and standard deviation for the portfolio from the historical data using the Method described in chapter 2 above. Then we would use the estimated the Mean and variance from Modern Portfolio Theory. We would then generate the Returns using the Multivariate normal distribution assumption using estimated Mean and Variance calculated above. Let B = bonus or supplemental wages earned once in a year and I_m = monthly contribution from 401k in the stocks which is a certain percentage of income. At time point $t-1$ = starting time point, after declaring all the input variables required such as Inflation, bonus, monthly contribution, allocation for bonds and stocks, we would calculate the returns as-

$$r_{t+1} = V(1+r\ ) + w(\mathrm{I}) \tag{2.28}$$

Where r = Return at t time point after 1 month

$V_{t=1}$ = Portfolio Value at t time point after 1 month

$I_{mt=1}$ = Monthly contribution at t time point after 1 month

Similarly we can calculate the Portfolio Value for 11 remaining periods. We can store the Portfolio Value return, Returns generated and Monthly contribution in a variable. Repeat the loop 12 times to get the Portfolio Value at the end of the year. Withdrawal logic of life events changing probability profile would be applied every month to see if there is a need to take out cash by investor and hence portfolio value stored in a variable would be calculated accordingly. Once we come out of the inner loop we add the Yearly B = bonus or supplemental wages earned once in a year and C_m = monthly contribution from 401k in the stocks which have been generated using the Beta Distribution and logic of simulation. We sum up the final Portfolio value at the end of time period t_{12} .We will rebalance the portfolio with the Portfolio Value $V_{t=12}$ and run the outer loop for K periods. K = total number of periods in years.

4. Scenarios and results

Let us apply the above explained algorithm to the different Scenarios demonstrated below.

3.1 Optimizing Scenarios

We have three retirement framework Optimizing scenarios in our framework. Each investor being different has unique investment goals tailoring to the needs suiting him. (Sharpe 1952) There are different kind of optimal portfolios investor can choose from for example Maximize return, minimize the risk, maximum reward-to-risk ratio or Maximum Sharpe ratio portfolio explained later in the section. There are several ways of determining the efficient frontier for the investor depending on the Objective functions specified. Our modeling framework gives flexibility to the investor to choose between the optimal efficient frontiers choices and returns received from each in terms of the portfolio value at the end of the investment horizon and make a pragmatic call. Let us look in detail at each of the optimizing scenarios. We have used general non-linear solver package available in R to solve the linear/non-linear objective function optimization problems.

3.1.1. Return maximized portfolios

In the return maximized portfolios, investor is return lover and is looking for investments to make big returns for the risk investor is willing to take in life. As stated in equation (2.12) above the portfolio return is given by the below function. In the accounting framework the objective function for the non-linear solver is set to maximizing the return for stocks and returns from bonds with inequality constraint of risk between 0 to 5 percent. No short selling is allowed. All the weights of the assets are between 0 and 1 with no maximum allocation concentration implemented.

$$E(\hat{r}_p) = E(\sum_{i=1}^{n} w_i r_i)$$
$$E(\hat{r}_p) = \sum_{i=1}^{n} w_i \bar{X}_i \quad (2.23)$$

$$\max imize = E(\hat{r}_p) = \sum_{i=1}^{n} w_i \bar{X}_i$$

Subject to

$$0 \le \sqrt{(w^t \Sigma w)} \le 0.05$$
$$\sum_{i=1}^{n} w_i = 1$$
$$where$$
$$w_i \ge 0, i = 1, 2, 3, ..., n$$

3.1.2. Risk minimized portfolios

In the risk minimized portfolios, investor is interested risk averse and is interested in portfolio which gives the lowest level of portfolio risk. We will use variance of the returns as the measure of portfolio risk. Using equation as stated above:

$$\mathrm{S}_p{}^2 = w^t \Sigma w \quad (2.26)$$

$$\text{minimize} = \hat{\mathrm{S}}_p = (\sum_{i,j=1}^{n} w_i w_j S_{i,j})^{1/2} = \sqrt{(w^t \Sigma w)}$$

Subject to

$$0.02 \le \sum_{i=1}^{n} w_i \bar{X}_i \le 0.04$$

$$\sum_{i=1}^{n} w_i = 1$$
$$where$$
$$w_i \ge 0, i = 1, 2, 3, ..., n$$

Solver objective function for minimizing the risk, would be set to minimize the risk associated with the variance of returns from the stocks only as bonds are considered to be risk free assets in the portfolio. No short selling is allowed. All the weights of the assets are between 0 and 1 with no maximum allocation concentration implemented. The inequality constraint on returns is set to returns between 2-4 percent.

3.1.3. Maximum Sharpe ratio portfolios

In the scenario when investor is looking for maximum return-to-risk ratio or in other words is looking for portfolio that represents the best expected return per unit of risk is thus best efficient portfolio. Because of the graphical representation it is also called as the Tangency portfolios, as the portfolio with the maximum Sharpe ratio is the point where the line through the origin is tangent to the efficient frontier as point has the unique property of best mean-standard deviation ratio on the frontier. We have used non-linear optimization solver in R as we have different objective functions with different scenarios analysis. No short selling is allowed. All the weights of the assets are between 0 and 1 with no maximum allocation concentration implemented. (Edwin 2003) Efficient frontier is simply the line of the risk-reward curve. It essentially answers the question on how to figure out best level of diversification. There is a relationship between return and risk and to identify the degree of risk that would generate different levels of returns is the essence of efficient frontier. In the retirement accounting framework, every individual has different risk taking appetite and is hence submitted to different return levels. For investors who are risk takers can generally expect higher returns as return and risk are positively correlated. The most efficient frontier is the one which gives highest returns possible for the specified level of the risk. Each point on the line represents an optimal portfolio as a trade-off between risk and return. Suppose investor wants to maximize the returns at a pre-specified risk level, at the same risk level there is no other portfolio better than the frontier. Optimal portfolio could be in any of the 3 directions with high return/high risk or low return/low risk or medium return/medium risk. (Marnix Angels, 2005) In the traditional MPT, theory risk represents the dimension on X-axis and return represents the dimension on the Y-axis. Portfolios below the curve are in efficient portfolios because for the same risk one could achieve a better return. Each point

on the curve represents a different efficient portfolio depending on the risk and return combination investor prefers. As we go from lower bottom to upper right hand side of the efficient frontier there is high return risk combination.

Sharpe ratio = mean / standard deviation

Using equations (2.23, 2.24, 2.26)

$$\max imize = \frac{E(\hat{r}_p)}{\hat{S}_p} = \frac{\sum_{i=1}^{n} w_i \bar{X}_i}{\sqrt{(w^t \Sigma w)}}$$

Subject to

$$\sum_{i=1}^{n} w_i = 1$$
$$\textit{where}$$
$$w_i \geq 0, i = 1, 2, 3, \ldots, n$$

3.2 All Scenarios with rebalancing and optimization for investor

In the accounting framework we have used following 8 scenarios with Optimizations and rebalancing logics explained above. There is one Scenario with No Optimization and No rebalancing which is control scenario case in the research. In this scenario we have equal weights allocation to all the asset classes throughout the investment horizon. For each of the 8 Scenarios we would have 3 Scenarios each making a total of 24 Scenarios for ‘Changing probability’ life events profiles of Uniform risk, Increasing risk and decreasing risk of life events during the investors retirement planning cycle.

Table Representation below of 24 Scenario

Scenarios	Objective Function	Rebalancing	Probability profiles
Scenario 1			
	Control No optimization	No rebalancing	Uniform risk
	Control No optimization	No rebalancing	Increasing risk
	Control No optimization	No rebalancing	Decreasing risk
Scenario 2			
	No optimization	Rebalances percentage in bonds to age.	Uniform risk
	No optimization	Rebalances percentage in bonds to age.	Increasing risk
	No optimization	Rebalances percentage in bonds to age.	Decreasing risk
Scenario 3			
	Max Return	Rebalances with no care on what percentage in bonds.	Uniform risk
	Max Return	Rebalances with no care on what percentage in bonds.	Increasing risk
	Max Return	Rebalances with no care on what percentage in bonds.	Decreasing risk
Scenario 4			
	Max Return	Rebalances percentage in bonds to age	Uniform risk
	Max Return	Rebalances percentage in bonds to age	Increasing risk
	Max Return	Rebalances percentage in bonds to age	Decreasing risk
Scenario 5			
	Min Risk	Rebalances with no care on what percentage in bonds.	Uniform risk
	Min Risk	Rebalances with no care on what percentage in bonds.	Increasing risk
	Min Risk	Rebalances with no care on what percentage in bonds.	Decreasing risk
Scenario 6			
	Min Risk	Rebalances percentage in bonds to age.	Uniform risk
	Min Risk	Rebalances percentage in bonds to age.	Increasing risk
	Min Risk	Rebalances percentage in bonds to age.	Decreasing risk
Scenario 7			
	Sharpe Ratio	Rebalances with no care on what percentage in bonds.	Uniform risk
	Sharpe Ratio	Rebalances with no care on what percentage in bonds.	Increasing risk
	Sharpe Ratio	Rebalances with no care on what percentage in bonds.	Decreasing risk
Scenario 8			
	Sharpe Ratio	Rebalances percentage in bonds to age.	Uniform risk
	Sharpe Ratio	Rebalances percentage in bonds to age.	Increasing risk
	Sharpe Ratio	Rebalances percentage in bonds to age.	Decreasing risk

Table3: Design - All the 8 Scenarios have 3 each sub scenarios with 3 changing Probability profiles of Uniform risk, increasing risk, decreasing risk

3.3 No Bootstrap and with Bootstrap Scenarios

One of the assumptions of Modern Portfolio theory is Asset returns are jointly normally distributed random variables. (Sharpe 1952). This implies the 2 dimensions standard deviation and expected returns are sufficient to describe the return distribution of each asset class. Moreover, this assumption implies that if we combine all the asset classes into portfolio, then portfolio returns are normally distributed. However empirically we know that returns in equity and other markets are not normally distributed. There are large deviations (3 to 6) observed in the returns from the mean. While model can be summed with any jointly elliptical distribution which is symmetrical in nature however we know that asset returns are not symmetrical in nature. We generate returns in the accounting framework by using the Mean and variance from the historical returns from all the asset classes as the estimates to, create the predictive distribution and generate the returns using the multivariate normal distributions.

Though the objective of the research is not to criticize the Modern Portfolio theory by any means, jointly normal distribution of the returns in the statistical framework cannot be let there. Return distribution of stocks in different markets have been investigated by many researchers. Many empirical studies show the interval ling effect on the statistical moments of the stock returns and are often not symmetrical. As an example, following studies of asymmetry of returns can be mentioned are : Folger, Groves and Richardson found that increasing the investment horizon leads to lower Skewness of stock returns. Simkowitz and Beedles (1978) studied the impact of increasing the portfolio size on the Skewness of the stock returns. Beedles (1986) investigated on the data of Australian equity returns significant positive Skewness in the returns. Grubel (1968) analyzed the internationally diversified portfolios and proved positive Skewness in the stocks returns. Lau and Wingender (1989) analyzed in theoretical methodology how the Skewness and the kurtosis of the stock returns are affected by the length of the holding interval which has very different impacts on the returns and the logarithm returns of the stocks. Kane (1982) studied the Mean and Variance models are adequate with the compact distributions and when portfolio

decisions are made frequently so that risk parameter becomes sufficiently small. Lee, Leuthold and Cordier (1985) investigated the impact of investment horizon on four statistical moments Mean, variance, Skewness and kurtosis. They found that return and risk increase with investment horizon, whereas Skewness and kurtosis are generally negatively correlated to horizon. Based on the results of the empirical studies time series of stock returns is characterized by excess kurtosis and Skewness.

Hence modeling the volatile behavior of returns based on independent/ Gaussian distribution would be inadequate. As the investor is much interested in the portfolio value after the investment horizon much closer to reality and approximations, we would challenge the jointly normal distribution assumption of the MPT theory. We have already introduced Bootstrapping and resampling procedure we would use for the research earlier sections. Efron (1998) studied and introduced bootstrapping method when actual distribution of the variable is not known and it relies on the actual distribution of the data, rather than artificially generated from the probability distributions. We would run 24 Scenarios design for each with Bootstrap and without Bootstrap logic to get the expected wealth or portfolio value distribution. Results are compared from all the 48 scenarios in the below sections

3.4 Scenario testing and algorithm application

We then applied above Algorithm –

3.4.1 Scenario Testing 1

Scenario 1. No optimization. No rebalancing (this is the control).

Let us say we want to apply above algorithm to calculate the Investment plan for retirement planning for American lower middle class individual. According to US Census bureau statistics (2011) class models the lower middle class is located roughly between the 52nd and 84th percentile of society and mean gross

annual personal incomes from about $32,500 to $60,000. Let the initial income to start with be W = $32000 and C_0 = 10000.We collected the data from Bureau of Economic Analysis and Bureau of labor statistics All classes income data from 1967-2012 in Nominal and Current dollars. Mean percent change in the income = 0.299 percent and Variance = 0.053 percent. We then Simulated the income of this individual applying Beta distribution simulation logic explained above to estimate the shape Parameters α = shape parameter 1 of Beta distribution. β = shape parameter 2 of Beta distribution.

$$\alpha = 0.96$$

$$\beta = 2.24$$

For simulating the Bonus and Monthly contribution from the Income we use the Beta distribution again -

B = bonus or supplemental wages earned once in a year

I_m = monthly contribution from 401k in the stocks.

Considering the demographics for this individual and statistics on the same from Bureau of Labour statistics on Employee compensation we have based the B as 4 percent of total wages and I_m as 4 percent of total wages. We simulated the data for the Yearly Bonus and Monthly contribution data. We would simulate the Inflation index as well. Inflation data is simulated based on the CPI index. Average percent change for last 10 years history is accounted for calculating the mean and variance to estimate the parameters of the Beta distribution.

A generalized investment plan will not support individual investor's needs as all the investors are different in terms of the financial situations and life stages each is at. It is imperative to create the Investment strategy for Retirement tailoring to each individual's personal needs. There is no one master formula for Risk-Return trade-off. Each individual has to create the Investment strategy suiting their needs. Also it is not correct to have same level of risk tolerance and return needs throughout the investor's life cycle of investing. The risk appetite of investor is not a constant number and keeps changing through

the length of the investment and as the personal financial situations changes. For our model we want to use the traditional MPT optimal model to maximize the return at a pre-specified risk level. Also we assume that Investor's risk appetite reduces as investor ages and would want to keep the portfolio diversified into the Asset classes – Stocks and Bonds basis the risk appetite. Let say investor wants to maximize the return for the pre-specified risk level. We do not want to cap the returns and let the objective function be maximizing the returns at the pre-specified risk level varying for each investor (as investor could be at different stage in life and has a risk appetite which suits her). Risk appetite of the individual investors decreases as they start to age in life. For the Scenario testing example we have No optimization and No rebalancing. This is the control experiment. Equal weights would be assigned to all the securities and model would be executed for the investment horizon to see the portfolio value in the end. We ran the model for 10,000 runs with Bootstrap and No Bootstrap methods to compare the results on the best optimal solution for the investor. Results and interpretations are discussed in detail in the below section.

3.4.2 Results and interpretations

The tables below exhibits the results from the 4 different Scenarios for Investor based on the changing probability profiles for Life events for all the objective scenarios Maximizing return, Minimizing risk, Maximizing the Sharpe ratio, Control design all with rebalancing and no rebalancing. In each table below we have all the objective function results for Bootstrap and No bootstrap categorized by changing probability profiles. MVN is multivariate normality.

Increasing Probability results

We know there is a likelihood of uncertain event to happen at any time period during the investor's horizon. In a Changing probability profile Scenario 1, there is a kind of investor who expects the uncertain events probability in correlation to his age. He expects the probability of uncertain events to go

up as he ages in life. For instance he thinks he would need money in later years for kid's education, elderly members health care life needs and so on. Below are the results for the Increasing probability results.

Increasing Probability						
	Mean	Median	Min	Max	Skewness	Variance
Scenario 1 MVN	$121,769	$111,789	$33,669	$727,749	1.81	193353.678
Scenario 1 Boot	$118,004	$108,893	$22,447	$510,608	1.86	189851.918
Scenario 2 MVN	$123,270	$113,148	$20,825	$736,112	2.28	216690.250
Scenario 2 Boot	$120,646	$110,605	$14,403	$612,760	2.09	199782.181
Scenario 3 MVN	$122,512	$112,424	$27,003	$703,407	1.97	200999.789
Scenario 3 Boot	$87,860	$81,923	$28,307	$349,933	1.81	84622.810
Scenario 4 MVN	$119,787	$112,766	$15,406	$474,884	1.55	171122.869
Scenario 4 Boot	$119,716	$110,705	$25,313	$779,930	2.08	188885.852
Scenario 5 MVN	$119,529	$110,646	$31,011	$591,817	1.92	185158.090
Scenario 5 Boot	$122,249	$111,399	$16,030	$806,767	2.31	235496.678
Scenario 6 MVN	$113,199	$100,500	$64,361	$719,633	3.16	196080.696
Scenario 6 Boot	$168,570	$112,306	$3,900	$1,070,800	2.13	2786796.197
Scenario 7 MVN	$120,868	$109,043	$4,412	$987,655	3.32	248811.416
Scenario 7 Boot	$118,857	$109,998	$38,749	$658,895	1.88	174097.563
Scenario 8 MVN	$120,329	$111,436	$31,797	$604,109	1.91	184058.160
Scenario 8 Boot	$121,466	$110,712	$15,730	$595,247	2.21	217090.765

4: Results for increasing probability uncertain events in life. The results include all the objective functions scenarios. For all the Scenarios above we have used same color coding vertically to choose the best model based on different statistics.

Decreasing Probability results

In a Changing probability profile Scenario 2, there is a second kind of investor who expects the uncertain events probability to decrease as he ages in life. For instance he would need more cash in initial years when he is settling down for events like wife's pregnancy, house renovation.

Decreasing Probability						
	Mean	Median	Min	Max	Skewness	Variance
Scenario 1 MVN	$87,227	$80,575	$38,552	$322,638	2.27	210837.089
Scenario 1 Boot	$124,342	$115,331	$16,906	$618,876	1.90	208867.280
Scenario 2 MVN	$117,851	$108,994	$26,522	$474,229	1.79	182346.080
Scenario 2 Boot	$122,430	$111,154	$20,395	$637,252	2.24	227519.460
Scenario 3 MVN	$121,941	$112,196	$8,472	$564,464	2.00	218341.253
Scenario 3 Boot	$89,766	$82,990	$21,967	$450,184	2.09	98903.960
Scenario 4 MVN	$121,902	$111,470	$18,452	$666,981	2.10	227433.610
Scenario 4 Boot	$120,437	$112,267	$21,077	$494,483	1.71	177738.128
Scenario 5 MVN	$121,875	$112,301	$9,176	$701,732	2.19	199683.860
Scenario 5 Boot	$120,777	$110,580	$8,648	$963,654	2.68	211241.352
Scenario 6 MVN	$123,649	$112,130	$3,789	$667,871	2.38	252868.180
Scenario 6 Boot	$182,172	$122,783	$2,858	$1,082,064	2.07	3232408.452
Scenario 7 MVN	$121,979	$110,164	$4,600	$628,342	2.46	222255.674
Scenario 7 Boot	$119,882	$111,148	$16,115	$541,655	1.65	183706.532
Scenario 8 MVN	$121,135	$110,816	$17,311	$552,794	2.04	220984.608
Scenario 8 Boot	$75,473	$69,858	$40,169	$465,433	5.77	45569.441

Table 5: Results for deccreasing probability uncertain events in life. The results include all the objective functions scenarios. For all the Scenarios above we have used same color coding vertically to choose the best model based on different statistics.

Uniform Probability Results

There is a likelihood of uncertain event happening at every time period in the investor's investment horizon. Since Event can happen any time period as there is a very small probability of events happening at all-time points, we cannot time the uncertain events; we need to have a discretized distribution to assign equal weights at all time periods. The discrete Uniform Distribution puts equal mass on each of the outcomes where N any specified number is. In our Accounting framework N is $12\times n$ where n is the number of retirement years and 12 is the number of months in a year.

Uniform Probability						
	Mean	Median	Min	Max	Skewness	Variance
Scenario 1 MVN	$120,539	$111,030	$28,801	$545,469	2.03	200122.023
Scenario 1 Boot	$87,227	$80,575	$38,552	$322,638	1.60	88845.725
Scenario 2 MVN	$119,945	$111,510	$28,454	$602,460	1.87	178447.105
Scenario 2 Boot	$120,070	$110,435	$21,540	$593,438	1.70	194269.378
Scenario 3 MVN	$121,452	$110,373	$7,962	$843,654	2.47	229527.228
Scenario 3 Boot	$88,991	$82,325	$31,853	$346,805	1.94	90078.017
Scenario 4 MVN	$119,713	$110,637	$18,146	$539,397	1.81	189573.160
Scenario 4 Boot	$121,706	$112,345	$8,165	$633,885	1.89	199094.440
Scenario 5 MVN	$120,610	$112,792	$13,573	$469,475	1.46	174891.240
Scenario 5 Boot	$121,023	$112,919	$7,722	$549,636	1.64	172067.336
Scenario 6 MVN	$115,270	$108,358	$34,287	$406,553	1.13	160977.488
Scenario 6 Boot	$129,842	$109,646	$69,867	$1,064,257	4.16	453117.460
Scenario 7 MVN	$119,626	$110,329	$13,652	$487,310	2.01	183783.690
Scenario 7 Boot	$120,580	$110,246	$11,706	$506,710	1.85	203816.132
Scenario 8 MVN	$121,281	$111,574	$25,075	$680,245	1.96	210350.650
Scenario 8 Boot	$121,398	$111,941	$38,456	$481,613	1.76	194101.925

Table 6: Results for Uniform probability uncertain events in life. The results include all the objective functions scenarios. For all the Scenarios above we have used same color coding vertically to choose the best model based on different statistics.

3.4.3 Case Study

Let us say we have 3 investors named Mark, Phil, Michelle all 30 years old, wanting to retire at 65 and are all equally worried about the source of retirement income when they retire. They all are aware about the uncertainties related to social security income which is one leg in the retirement income. They all dream to retire in a comfortable manner with all the real life priorities not compromised. They want to start a retirement portfolio planning by investing $10,000 to start with in the portfolio. They all are working individuals and make same annual income. For the sake of unbiased interpretations of the retirement framework results we would use examples of investors with similar financial situations. They all being hard working individuals do not have much time to check into their retirement portfolio more than once or twice a year.

Let say we have selected the option for calling for rebalancing or no rebalancing for them on their birthdays every year. Now all these investors have given certain key inputs on the objective functions and money withdrawal strategies in life based on individual life situations. Mark thinks he would want to maximize the return and is willing to take the risk for the same. He thinks he would need to withdraw cash from the portfolio in the later years to meet medical expenses of grandparents and parents in the family. Phil thinks he would want to minimize the risk as given his personal problems he does not want to increase risk for return in the retirement planning. Also he thinks given his family situation he would want to withdraw money during the early years of portfolio planning more than the later years. Michelle thinks that there is trade-off between return and risk and for maximizing the return she has to take every unit more risk. She thinks her investment objective should be maximizing the Sharpe ratio and on withdrawal strategy she thinks uncertain events can happen any time period in life. So she wants to go with uniform probability on withdrawal of money for uncertain events in life.

Let us look at the results from the Table1, Table2 and Table3 for our 3 investors.
For Mark with his objective strategy and withdrawal anticipation, we see the 5 best models are Scenario1 MVN, Scenario2 MVN, Scenario 3 MVN, Scenario 4 MVN and Scenario 6 Boot. Now when if compare the 6 parameters Scenario 4 MVN and Scenario 3 MVN would work good for him as Scenario 3 and Scenario 4 both have objective function of maximizing the return, have the lowest variance and Skewness in the 5 best models. If we look at the Mean and Median both the parameters results are in top 5 best models for Mark.

For Phil with his objective strategy and withdrawal anticipation, we see the 5 best models are Scenario 1 Boot, Scenario 3 MVN, Scenario 4 Boot, Scenario 6 MVN, Scenario 6 Boot. Let us compare the 6 parameters now for best 5 models. If we go by Mean, Median and Skewness, then Scenario 1 boot,

Scenario 3 Boot are the best models. But if we go by Median, Skewness and variance then Scenario 4 and Scenario 6 Boot are the best models also inclined with Phil's objective strategy on minimizing the risk.

For Michelle with his objective strategy and withdrawal anticipation, we see the 5 best models are Scenario 3 MVN, Scenario 4 Boot, Scenario 5 Boot, Scenario 6 MVN, Scenario 8 Boot. Let us compare the 6 parameters now for best 5 models. If we go by Mean, Median and Skewness, then Scenario 8 Uniform MVN, Scenario 8 Boot are the best models. But if we go by Median, Skewness and variance then Scenario 5 MVN and Scenario 6 Boot are the best models also inclined with Michelle's objective strategy on maximizing the sharpe ratio.

5. Conclusion

The aim of the thesis was to construct an effective realistic retirement income plan for individual investor by proposing realistic frameworks, specific to inputs given by investor such as number of investment instruments, income, and length of the time period before retirement using Modern Portfolio theory. The real life situations are different for each investor and hence the notion of building this retirement framework was to come up with effective real strategies suiting the needs of each investor giving close to realistic picture on the wealth accumulated during the investment horizon.

This effective and realistic retirement framework certainly has the scope for added complexity. There are certain additional aspects which could have been added in the retirement methodology such as Dividend's logic, Net present value of the portfolio value in current dollars, more data points on historical data for stocks and bonds for effective bootstrapping results for added complexity in the study. For statistical complexity we would want to use different methods to estimate parameters of modern portfolio theory and check on the performance. These are few examples of the incremental work we think could be done in the higher stages of the research. The scope of the study is within the boundaries of first stage retirement income planning and best investment strategy suiting investors needs and could be expanded to other two stages. Second stage would be one need to have a plan on how much income to utilize year over year and to have enough cash flows for the buffer number of life years and third stage would be how the remaining funds should be invested in order to keep the best flow of income during the retirement years. The study has some limitations and constraints, which to some extent affect the scope and validity but still the objective of the thesis can be achieved.

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7. Appendix: A brief overview of the Code and its use in the model with the help of an example.

The Scenario Example we have taken for illustration is -

MAXIMUM RETURN. REBALANCES WITH NO CARE ON WHAT PERCENTAGES IN BONDS

for both the cases with BOOTSTRAP and without BOOTSTRAP.

6.1 DATA Section

There are 2 data sections in the Code. First part consists of Variables and second part consists of constraints enforced on the variables. The variables with data examples are shown below:

1. **INCOME:** STARTING INCOME OF INVESTOR AT THE TIME OF RETIREMENT PLANNING
2. **PORTFOLIO1.VALUE:** PORZTFOLIO WEALTH INITIAL INVESTMENT TO START WITH FOR RETIREMENT PORTFOLIO PLANNING.
3. **VALUE1:** PORTFOLIO VALUE ALLOCATION FOR STOCKS.
4. **VALUE2:** PORTFOLIO VALUE ALLOCATION FOR BONDS.
5. **RETURN.STOCKS:** RETURN STOCKS
6. **RETURN.BONDS:** RETURN BONDS
7. **MONTHLY.CONTRIBUTION:** MONTHLY CONTRIBUTION
8. **BONUS:** BONUS CONTRIBUTION
9. **INFLATION:** INFLATION FACTOR. RANDOMLY GENERATED WITH MEAN 2 PERCENT AND VARIANCE 0.01 PERCENT.
10. **WITHDRAW**: INITIALIZE THE WITHDRAW VARIABLE. ASSUME MEAN AND VARIANCE FOR WITHDRAWAL DISTRIBUTION WITH 15 PERCENT MEAN AND VARIANCE 0.1 PERCENT. RBETA(1,18.975,107.525)

11. STEPFUNC: STEPFUNC WOULD INCREASE/DECREASE THE TIME PERIOD PROBABILTIIES BY CONSTANT TO THE UNIFORM PROBABILTIY

EXAMPLE - SCENARIO 3 with Bootstrap

1.INCOME: 35000

2.PORTFOLIO1.VALUE: 10000

3.VALUE1: PORTFOLIO.VALUE times WEIGHTS for each stocks

4.VALUE2: PORTFOLIO.VALUE times WEIGHTS for each bonds

5.RETURN.STOCKS: Initialize the returns from each stock to be c(0,0,0,0,0,0,0,0,0,0)

6.RETURN.BONDS: INITIALIZE THE RETURNS FROM EACH BOND TO BE c(0,0,0,0,0)

7.MONTHLY.CONTRIBUTION : LET SAY MONTHLY CONTRIBUTION IS 4 PERCENT OF ANNUAL INCOME. THEREFORE IT IS $0.04/12 \times$ INCOME

8.BONUS.CONTRIBUTION : INITIALIZE THE BONUS CONTRIBUTION WITH CERTAIN PERCENTAGE OF INCOME.ONE DOES NOT GET THE BONUS YEAR ONE STARTS WORKING. LET SAY BONUS CONTRIBUTION IS RANDOMLY GENERATED WITH MEAN 4 PERCENT AND VARIANCE 0.01 PERCENT USING rbeta (1, 0.01365, 0.44135)), 5.

9. **INFLATION** : RANDOMLY GENERATED WITH MEAN 2 PERCENT AND VARIANCE 0.01 PERCENT.

10. **WITHDRAWAL LOGIC:** INITIALIZE THE WITHDRAW VARIABLE as 0. IMPOSE TIGHTER BOUNDS WITH LOWER BOUND AS 10 PERCENT INCOME AND UPPER BOUND OF PORTFOLIO VALUE AT THAT TIME POINT.

6.2 CONSTRAINTS AND LOGIC SECTION

We will use the Withdrawal logic as explained above in the results and interpretation of scenario section. In the Retirement Accounting Framework, we have Returns generated from Bonds and Returns generated from stocks. We will then use the changing Probability profile to generate the TRUE uncertain event probability in life for all the scenarios of Uniform, Increasing and decreasing probability scenarios. In the accounting framework, we will have variables initialized in the data section above and finally Portfolio value is calculated at the end of the retirement period. Following code is for Scenario 3 exhibiting the Maximum Return with No care on what percentage in Bonds for with Bootstrap scenario with the data figures mentioned in Appendix data section.

```
# SCENARIO 3.  With Bootstrap. MAX RETURN. REBALANCES WITH NO CARE OF
WHAT PERCENTAGE IN BONDS.

# INSTALL PACKAGES

#install.packages("stockPortfolio") # BASE PACKAGE FOR RETREIVING
RETURNS
#install.packages("quadprog")       # NEEDED FOR SOLVER QP
#install.packages("quantmod")       # FOR GETTING RETURNS FROM YAHOO
FINANCE
#install.packages("mvtnorm")        # SIMULATING THE NORMAL
DISTRIBUTION DATA
#install.packages("mnormt")         # SIMULATING THE NORMAL
DISTRIBUTION DATA
#install.packages("Rsolnp")         # NON LINEAR SOLVER
```

```
# CALL LIBRARIES

library(stockPortfolio)              # BASE PACKAGE FOR RETREIVING
RETURNS
library(quadprog)                    # NEEDED FOR SOLVER QP
library(quantmod)                    # FOR GETTING RETURNS FROM YAHOO
FINANCE
library(mvtnorm)                     # SIMULATING THE NORMAL
DISTRIBUTION DATA
library(mnormt)                      # SIMULATING THE NORMAL
DISTRIBUTION DATA
library(Rsolnp)                      # NON LINEAR SOLVER


# GET THE STOCKS DATA FROM YAHOO FINANCE
# MORE STOCKS CAN BE ADDED FOR DIVERSIFICATION
# PRICE QUOTES, MONTHLY RETURNS ARE HAVING HISTORICAL DATA STARTING
FROM YEAR 2001 TILL CURRENT STOCK QUOTE
# FREQUENCY CAN BE CHANGED TO HAVE YEARLY OR DAILY RETURNS AND STOCK
QUOTES
```

```
stocks <-
c("AAPL","QCOM","AMZN","PEP","MAR","JNJ","BA","COH","SLB","MCD")
# STOCKS VECTOR
returns.stocks <- getReturns(stocks, freq ="month", start="2001-01-
01",end="2013-01-01")         # FETCHING THE DATA IN DATAFRAME FROM
YAHOO FINANCE
asset.stocks.ret <- data.frame(returns.stocks$R)
# TO GET STOCK RETURNS
asset.stocks.interim <- data.frame(returns.stocks$full)
# TO GET STOCK MONTHLY PRICES
asset.stocks.df <-
asset.stocks.interim[,c(7,14,21,28,35,42,49,56,63,70)]



# GET THE BONDS DATA FROM YAHOO FINANCE
# MORE BONDS CAN BE ADDED FOR DIVERSIFICATION
# PRICE QUOTES, MONTHLY RETURNS ARE HAVING HISTORICAL DATA STARTING
FROM YEAR 2008 TILL CURRENT BOND QUOTE
# FREQUENCY CAN BE CHANGED TO HAVE YEARLY OR DAILY RETURNS AND BOND
QUOTES


bonds <- c("TR1YR","TR3YR","TR5YR","TR7YR","TR10YR")
# BONDS VECTOR
```

```
asset.bonds.ret <- read.csv(file =
"C:/Users/megha/Documents/Bonds.csv",header=TRUE)            # FETCHING
THE DATA IN DATAFRAME FROM TREASURY YIELDS

# COVARIANCE AND EXPECTED RETURNS OF STOCKS

V.stocks = cov(asset.stocks.ret)
mu.stocks <- apply(asset.stocks.ret, 2, mean)
mu.stocks <- as.matrix(mu.stocks)



# COVARIANCE AND EXPECTED RETURNS OF BONDS

V.bonds = cov(asset.bonds.ret)
mu.bonds <- apply(asset.bonds.ret, 2, mean)
mu.bonds <- as.matrix(mu.bonds)



##################################
# ACCOUNTING FRAMEWORK
##################################

# ESTIMATE BETA PARAMETERS
```

```
estBetaParams <- function(mu, V) {
  alpha <- ((1 - mu) / V - 1 / mu) * mu ^ 2
  beta <- alpha * (1 / mu - 1)
  return(params = list(alpha = alpha, beta = beta))
}

# EQUALLY WEIGHTED PORTOFOLIO TO START WITH

x <- rep(1/15,15)

# SCENARIO BASED SIMULATIONS FOR AN INVESTOR FOR PORTFOLIO VALUE

end.value1 <- rep(0, 1)

for(sims1 in 1:1){

  # INDIVIDUAL INVESTOR INPUTS FOR THE ACCOUNTING FRAMEWORK
```

```
  ageInit <- 30                              # INITIAL AGE AT WHICH
INVESTOR STARTS THE PORTFOLIO PLANNING
  ageFinal <- 65                             # FINAL AGE AT WHICH INVESTOR
WISHES TO RETIRE
  bondInitial <- ageInit*0.01                # BONDS ALLOCATION FOR THE
PORTFOLIO TO START WITH
  bondFinal <- ageFinal*0.01                 # BONDS ALLOCATION FINAL THE
PORTFOLIO
  stockInitial <- 1- bondInitial             # STOCKS ALLOCATION FOR THE
PORTFOLIO TO START WITH
  stockFinal <- 1 - bondFinal                # STOCKS ALLOCATION FINAL FOR
THE PORTFOLIO
  n <-  ageFinal - ageInit                   # TOTAL NUMBER OF YEARS FOR
WHICH RETIREMENT PLANNING NEEDS TO BE DONE


  # REBALANCING ON BDAY LOGIC
  # ADDING A YEAR TO BDAY DATE AND CALL FOR REBALANCING


  portfolio.start.date <- as.POSIXlt(as.Date('2013-01-01'))      #
PORTFOLIO START DATE
```

```
  bday.date <- as.POSIXlt(as.Date('2013-01-01'))                #
INVESTOR's BDAY DAY WHEN REBALANCING IS DONE ONCE EVERY YEAR



  # STARTING INCOME OF INVESTOR AT THE TIME OF RETIREMENT PLANNING
  # RANDOMLY GENERATED WITH MEAN 1% AND VARIANCE 0.01% IN CURRENT
DOLLARS,rbeta(1,0.089,8.811).
  # RANDOMLY GENERATED WITH MEAN 4% AND VARIANCE 0.01% IN CURRENT
DOLLARS,rbeta(1,0.113,2.726).


  income = 35000
  income.hold <- income



  # PORZTFOLIO WEALTH INITIAL INVESTMENT TO START WITH FOR RETIREMENT
PORTFOLIO PLANNING.

  portfolio1.value = 10000
  portfolio1.value.hold <- portfolio1.value
  portfolio1.value.hold
```

```
# PORTFOLIO VALUE ALLOCATION FOR STOCKS.

value1 <- portfolio1.value*x[1:10]
value1.hold <- value1
value1.hold


# PORTFOLIO VALUE ALLOCATION FOR BONDS.

value2 <- portfolio1.value*x[11:15]
value2.hold <- value1
value2.hold


# RETURN STOCKS
# ASSUMED THE MULTIVARIATE NORMAL DISTRIBUTION FOR RETURNS FROM
STOCKS.

return.stocks <- c(0,0,0,0,0,0,0,0,0,0)
return.stocks.hold <- return.stocks
return.stocks.hold
```

```
  # RETURN BONDS
  # ASSUMED THE MULTIVARIATE NORMAL DISTRIBUTION FOR RETURNS FROM
BONDS.

  return.bonds <-  c(0,0,0,0,0)
  return.bonds.hold <- return.bonds
  return.bonds.hold



  # MONTHLY CONTRIBUTION
  # INITIALIZE THE MONTHLY CONTRIBUTION WITH CERTAIN PERCENTAGE OF
INCOME.
  # LET SAY MONTHLY CONTRIBUTION IS 4% OF ANNUAL INCOME.

  monthly.contribution = income * 0.04/12
  monthly.contribution.hold <- monthly.contribution
  monthly.contribution.hold



  # BONUS CONTRIBUTION
  # INITIALIZE THE BONUS CONTRIBUTION WITH CERTAIN PERCENTAGE OF
INCOME.ONE DOES NOT GET THE BONUS YEAR ONE STARTS WORKING.
  # LET SAY BONUS CONTRIBUTION IS RANDOMLY GENERATED WITH MEAN 4% AND
VARIANCE 0.01% USING rbeta(1, 0.01365, 0.44135)),5.
```

```
bonus = 0
bonus.hold <- bonus
bonus.hold



# INFLATION FACTOR
# RANDOMLY GENERATED WITH MEAN 2% AND VARIANCE 0.01%.

inflation = 0.02
inflation.hold <- inflation
inflation.hold



# NET PRESENT VALUE
# NPV OF PORTFOLIO VALUE IN CURRENT DOLLARS.

npv.portfolio.value = 0
npv.portfolio.value.hold <- npv.portfolio.value
npv.portfolio.value.hold
```

```
####### UNCERTAIN EVENTS PROBABILITY SCENARIOS AND WITHDRAWAL LOGIC


# INITIALIZE THE WITHDRAW VARIABLE.
# ASSUME MEAN AND VARIANCE FOR WITHDRAWAL DISTRIBUTION WITH 15% MEAN
AND VARIANCE 0.1%.RBETA(1,18.975,107.525)
# IMPOSE TIGHTER BOUNDS WITH LOWER BOUND AS 10% INCOME AND UPPER
BOUND OF PORTFOLIO VALUE AT THAT TIME POINT.


withdraw <- 0
withdraw.hold <- withdraw
withdraw.hold


# SCENARIO 1 - UNIFORM DISCRETE UNCERTAIN EVENTS PROBABILITIES.
# ASSIGN PROBABILITIES OF 12*n TIME POINTS USING THE DISRETE UNIFORM
DISTRIBUTION.
# 12 MONTHS TIMES THE NUMBER OF RETIREMENT YEARS

prob1 <- rep(1/(12*n),12*n)
plot(prob1)
```

```
# STEPFUNC WOULD INCREASE/DECREASE THE TIME PERIOD PROBABILTIIES BY
CONSTANT TO THE UNIFORM PROBABILTIY

stepfunc <-  abs(1/(12*n)*0.50)/(12*n)

# SCENARIO 2 - INCREASING FUNCTION UNCERTAIN EVENTS PROBABILITIES.

prob2 <- 1/(12*n)
prob2.hold <- prob2

for(i in 1:12*n){
  prob2 <- (1/(12*n) + stepfunc*i)
  prob2.hold <- rbind(prob2.hold,prob2)
}
plot(prob2.hold)

# SCENARIO 3 - DECREASING FUNCTION UNCERTAIN EVENTS PROBABILITIES.

prob3 <- 1/(12*n)
```

```
prob3.hold <- prob3

for(i in 1:12*n){
  prob3 <- (1/(12*n) - stepfunc*i)
  prob3.hold <- rbind(prob3.hold,prob3)
}
plot(prob3.hold)




# SOLVER FUNCTION FOR MAXIMIZING THE RETURN AT A PRE-SPECIFIED RISK.

# OBJECTIVE FUNCTION

fnobj1 <- function(x){
  return = -(t(x[1:10]) %*% mu.stocks) - (t(x[11:15]) %*%mu.bonds)
  obj = -return
}




# INEQUALITY CONSTRAINT FUNCTION
inequality.const <- function(x){
  sqrt( t(x[1:10]) %*% V.stocks %*% x[1:10])
}
```

```
  # LINEAR STEP FUNCTION.

  #stepInit <- abs(bondInitial-bondFinal)/(ageFinal-ageInit)




  ##################################
  # RETIREMENT INCOME FRAMEWORK
  ##################################



  for(j in 1:n){
    for(i in 1:12 ){



      return.stocks <-
asset.stocks.ret[sample(nrow(asset.stocks.ret),size=1,replace=TRUE),]
      return.stocks.hold <- rbind(return.stocks.hold,return.stocks)


      # RETURNS FROM BONDS BOOTSTRAP


      return.bonds <-
asset.bonds.ret[sample(nrow(asset.bonds.ret),size=1,replace=TRUE),]
```

```
return.bonds.hold <- rbind(return.bonds.hold,return.bonds)

# GENERATE THE TRUE EVENT USING BINOMIAL DISTRIBUTION WITH A
TRIAL AT EACH TIME POINT FOR SCENARIO1.

binom1 <- rbinom(1,1,1/(12*n))

# GENERATE THE TRUE EVENT USING BINOMIAL DISTRIBUTION WITH A
TRIAL AT EACH TIME POINT FOR SCENARIO2.

#binom2 <- rbinom(1,1,(1/(12*n) + stepfunc*i))

# GENERATE THE TRUE EVENT USING BINOMIAL DISTRIBUTION WITH A
TRIAL AT EACH TIME POINT FOR SCENARIO3.

#binom3 <- rbinom(1,1,(1/(12*n) - stepfunc*i))
```

```
      # IMPOSE TIGHTER BOUNDS WITH LOWER BOUND AS 10% INCOME AND UPPER
BOUND OF PORTFOLIO VALUE AT THAT TIME POINT.


      if (binom1 == 1){
        withdraw <-
(0.90*portfolio1.value*rbeta(1,18.975,107.525)+0.10*income)
        withdraw.hold <- rbind(withdraw.hold,withdraw)


        # GEOMETRIC AVERAGE RATE OF RETURN TO CALCULATE THE ANNUALIZED
RATE OF RETURN


        value1 <- value1*(1+(return.stocks/12)) + x[1:10] *
monthly.contribution - x[1:10] * withdraw
        value1.hold <- rbind(value1.hold,value1)


        # GEOMETRIC AVERAGE RATE OF RETURN TO CALCULATE THE ANNUALIZED
RATE OF RETURN


        value2 <- value2*(1+(return.bonds/12)) + x[11:15] *
monthly.contribution - x[11:15] * withdraw
        value2.hold <- rbind(value2.hold,value1)
```

```
      }



      else if (binom1 == 0){
        withdraw <- 0
        withdraw.hold <- rbind(withdraw.hold,withdraw)

        # GEOMETRIC AVERAGE RATE OF RETURN TO CALCULATE THE ANNUALIZED
RATE OF RETURN

        value1 <- value1*(1+(return.stocks/12)) + x[1:10] *
monthly.contribution
        value1.hold <- rbind(value1.hold,value1)

        # GEOMETRIC AVERAGE RATE OF RETURN TO CALCULATE THE ANNUALIZED
RATE OF RETURN

        value2 <- value2*(1+(return.bonds/12)) + x[11:15] *
monthly.contribution
        value2.hold <- rbind(value2.hold,value1)
      }
```

```
    }

    # ACCOUNTED FOR INFLATION ON YEARLY BASIS.

    inflation <- rbeta(1,4.929,192.2)
    inflation.hold <- rbind(inflation.hold,inflation)


    # YEARLY INCOME RISE TO KEEP PACE WITH INFLATION AND ALSO BASED ON
THE INDIVIDUAL's PERFORMANCE

    income <- income + income * (rbeta(1,0.113,2.726))
    income.hold <- rbind(income.hold,income)


    # YEARLY STATIC MONTHLY CONTRIBUTION OF 401K BY THE EMPLOYER IN
THE INVESTOR ACCOUNT

    monthly.contribution <- income * 0.04/12
    monthly.contribution.hold <-
rbind(monthly.contribution.hold,monthly.contribution)
```

```
    # YEARLY BONUS GIVEN ON INCOME SIMULATED USING A DISTRIBUTION AS
BONUS IS A FUNCTION OF PERFORMANCE TOO.HENCE NOT STATIC

    bonus <-  income * 0.04 * (rbeta(1, 1.496, 35.904))
    bonus.hold <- rbind(bonus.hold,bonus)



    # PORTFOLIO VALUE(GROSS ANNUAL RETURNS) AT THE END OF THE YEAR IS
RETURN FROM BONDS,STOCKS AND MONTHLY COMPOUNDED ANNUAL RETURN ON
WEALTH INVESTED.

    portfolio1.value <-  sum(value1) + sum(value2) + bonus
    portfolio1.value.hold <-
rbind(portfolio1.value.hold,portfolio1.value)



    # CONSERVATIVE PORTFOLIO STRATEGY - WITH LOW RISK TOLERANCE AND
DECREASING PERCENTAGE INVESTED IN STOCKS.


    r <- solnp(pars = x,fun=
fnobj1,ineqfun=inequality.const,ineqUB=0.05,ineqLB=0.00,LB=rep(0,15),U
B=rep(1,15))
  }
```

```
  # NET PRESENT VALUE OF PORFOLIO VALUE AND INCOME IN CURRENT DOLLARS

  cum.inflation.out <- 1 + cumprod(inflation.hold)
  npv.portfolio.value <- portfolio1.value.hold/cum.inflation.out



  # SIMULATION RESULTS OF PORTFOLIO VALUE

  end.value1[sims1] <- portfolio1.value.hold[n+1]
}

write.table(end.value1, file
="C:/Users/megha/Documents/output3aboot.csv", row.names = FALSE,
append = FALSE, col.names = TRUE, sep = ",")


# YEARLY OUTPUTS FROM THE DIFFERENT FACTORS
inflation.hold
income.hold
portfolio1.value.hold
bonus.hold
monthly.contribution.hold
npv.portfolio.value
```

```
hist(end.value1)
```

 # SCENARIO 3. With Bootstrap. MAX RETURN. REBALANCES WITH NO CARE OF WHAT PERCENTAGE IN BONDS.

```
# INSTALL PACKAGES

#install.packages("stockPortfolio") # BASE PACKAGE FOR RETREIVING
RETURNS
#install.packages("quadprog")       # NEEDED FOR SOLVER QP
#install.packages("quantmod")       # FOR GETTING RETURNS FROM YAHOO
FINANCE
#install.packages("mvtnorm")        # SIMULATING THE NORMAL
DISTRIBUTION DATA
#install.packages("mnormt")         # SIMULATING THE NORMAL
DISTRIBUTION DATA
#install.packages("Rsolnp")         # NON LINEAR SOLVER


# CALL LIBRARIES

library(stockPortfolio)             # BASE PACKAGE FOR RETREIVING
RETURNS
library(quadprog)                   # NEEDED FOR SOLVER QP
```

```
library(quantmod)                    # FOR GETTING RETURNS FROM YAHOO
FINANCE
library(mvtnorm)                     # SIMULATING THE NORMAL
DISTRIBUTION DATA
library(mnormt)                      # SIMULATING THE NORMAL
DISTRIBUTION DATA
library(Rsolnp)                      # NON LINEAR SOLVER



# GET THE STOCKS DATA FROM YAHOO FINANCE
# MORE STOCKS CAN BE ADDED FOR DIVERSIFICATION
# PRICE QUOTES, MONTHLY RETURNS ARE HAVING HISTORICAL DATA STARTING
FROM YEAR 2001 TILL CURRENT STOCK QUOTE
# FREQUENCY CAN BE CHANGED TO HAVE YEARLY OR DAILY RETURNS AND STOCK
QUOTES



stocks <-
c("AAPL","QCOM","AMZN","PEP","MAR","JNJ","BA","COH","SLB","MCD")
# STOCKS VECTOR
returns.stocks <- getReturns(stocks, freq ="month", start="2001-01-
01",end="2013-01-01")          # FETCHING THE DATA IN DATAFRAME FROM
YAHOO FINANCE
```

```
asset.stocks.ret <- data.frame(returns.stocks$R)
# TO GET STOCK RETURNS
asset.stocks.interim <- data.frame(returns.stocks$full)
# TO GET STOCK MONTHLY PRICES
asset.stocks.df <-
asset.stocks.interim[,c(7,14,21,28,35,42,49,56,63,70)]



# GET THE BONDS DATA FROM YAHOO FINANCE
# MORE BONDS CAN BE ADDED FOR DIVERSIFICATION
# PRICE QUOTES, MONTHLY RETURNS ARE HAVING HISTORICAL DATA STARTING
FROM YEAR 2008 TILL CURRENT BOND QUOTE
# FREQUENCY CAN BE CHANGED TO HAVE YEARLY OR DAILY RETURNS AND BOND
QUOTES


bonds <- c("TR1YR","TR3YR","TR5YR","TR7YR","TR10YR")
# BONDS VECTOR
asset.bonds.ret <- read.csv(file =
"C:/Users/megha/Documents/Bonds.csv",header=TRUE)           # FETCHING
THE DATA IN DATAFRAME FROM TREASURY YIELDS
asset.bonds.ret <- log(asset.bonds.ret)
```

```
# COVARIANCE AND EXPECTED RETURNS OF STOCKS

V.stocks = cov(asset.stocks.ret)
mu.stocks <- apply(asset.stocks.ret, 2, mean)
mu.stocks <- as.matrix(mu.stocks)


# COVARIANCE AND EXPECTED RETURNS OF BONDS

V.bonds = cov(asset.bonds.ret)
mu.bonds <- apply(asset.bonds.ret, 2, mean)
mu.bonds <- as.matrix(mu.bonds)


##################################
# ACCOUNTING FRAMEWORK
##################################

# ESTIMATE BETA PARAMETERS

estBetaParams <- function(mu, V) {
  alpha <- ((1 - mu) / V - 1 / mu) * mu ^ 2
  beta <- alpha * (1 / mu - 1)
```

```
  return(params = list(alpha = alpha, beta = beta))
}

# EQUALLY WEIGHTED PORTOFOLIO TO START WITH
x <- rep(1/15,15)

# SCENARIO BASED SIMULATIONS FOR AN INVESTOR FOR PORTFOLIO VALUE
end.value1 <- rep(0, 1)
for(sims1 in 1:1){

  # INDIVIDUAL INVESTOR INPUTS FOR THE ACCOUNTING FRAMEWORK

  ageInit <- 30                         # INITIAL AGE AT WHICH
INVESTOR STARTS THE PORTFOLIO PLANNING
```

```
  ageFinal <- 65                          # FINAL AGE AT WHICH INVESTOR
WISHES TO RETIRE
  bondInitial <- ageInit*0.01             # BONDS ALLOCATION FOR THE
PORTFOLIO TO START WITH
  bondFinal <- ageFinal*0.01              # BONDS ALLOCATION FINAL THE
PORTFOLIO
  stockInitial <- 1- bondInitial          # STOCKS ALLOCATION FOR THE
PORTFOLIO TO START WITH
  stockFinal <- 1 - bondFinal             # STOCKS ALLOCATION FINAL FOR
THE PORTFOLIO
  n <-  ageFinal - ageInit                # TOTAL NUMBER OF YEARS FOR
WHICH RETIREMENT PLANNING NEEDS TO BE DONE



  # REBALANCING ON BDAY LOGIC
  # ADDING A YEAR TO BDAY DATE AND CALL FOR REBALANCING


  portfolio.start.date <- as.POSIXlt(as.Date('2013-01-01'))      #
PORTFOLIO START DATE
  bday.date <- as.POSIXlt(as.Date('2013-01-01'))                 #
INVESTOR's BDAY DAY WHEN REBALANCING IS DONE ONCE EVERY YEAR
```

```
# STARTING INCOME OF INVESTOR AT THE TIME OF RETIREMENT PLANNING
# RANDOMLY GENERATED WITH MEAN 1%
AND VARIANCE 0.01% IN CURRENT DOLLARS,rbeta(1,0.089,8.811).
# RANDOMLY GENERATED WITH MEAN 4% AND VARIANCE 0.01% IN CURRENT
DOLLARS,rbeta(1,0.113,2.726).


income = 35000
income.hold <- income



# PORZTFOLIO WEALTH INITIAL INVESTMENT TO START WITH FOR RETIREMENT
PORTFOLIO PLANNING.

portfolio1.value = 10000
portfolio1.value.hold <- portfolio1.value
portfolio1.value.hold



# PORTFOLIO VALUE ALLOCATION FOR STOCKS.

value1 <- portfolio1.value*x[1:10]
value1.hold <- value1
value1.hold
```

```
# PORTFOLIO VALUE ALLOCATION FOR BONDS.

value2 <- portfolio1.value*x[11:15]
value2.hold <- value1
value2.hold



# RETURN STOCKS
# ASSUMED THE MULTIVARIATE NORMAL DISTRIBUTION FOR RETURNS FROM
STOCKS.

return.stocks <- c(0,0,0,0,0,0,0,0,0,0)
return.stocks.hold <- return.stocks
return.stocks.hold



# RETURN BONDS
# ASSUMED THE MULTIVARIATE NORMAL DISTRIBUTION FOR RETURNS FROM
BONDS.

return.bonds <-  c(0,0,0,0,0)
```

```
return.bonds.hold <- return.bonds
return.bonds.hold



# MONTHLY CONTRIBUTION
# INITIALIZE THE MONTHLY CONTRIBUTION WITH CERTAIN PERCENTAGE OF
INCOME.
# LET SAY MONTHLY CONTRIBUTION IS 4% OF ANNUAL INCOME.

monthly.contribution = income * 0.04/12
monthly.contribution.hold <- monthly.contribution
monthly.contribution.hold



# BONUS CONTRIBUTION
# INITIALIZE THE BONUS CONTRIBUTION WITH CERTAIN PERCENTAGE OF
INCOME.ONE DOES NOT GET THE BONUS YEAR ONE STARTS WORKING.
# LET SAY BONUS CONTRIBUTION IS RANDOMLY GENERATED WITH MEAN 4% AND
VARIANCE 0.01% USING rbeta(1, 0.01365, 0.44135)),5.

bonus = 0
bonus.hold <- bonus
bonus.hold
```

```
# INFLATION FACTOR
# RANDOMLY GENERATED WITH MEAN 2% AND VARIANCE 0.01%.

inflation = 0.02
inflation.hold <- inflation
inflation.hold



# NET PRESENT VALUE
# NPV OF PORTFOLIO VALUE IN CURRENT DOLLARS.

npv.portfolio.value = 0
npv.portfolio.value.hold <- npv.portfolio.value
npv.portfolio.value.hold



####### UNCERTAIN EVENTS PROBABILITY SCENARIOS AND WITHDRAWAL LOGIC


# INITIALIZE THE WITHDRAW VARIABLE.
```

```
# ASSUME MEAN AND VARIANCE FOR WITHDRAWAL DISTRIBUTION WITH 15% MEAN
AND VARIANCE 0.1%.RBETA(1,18.975,107.525)
# IMPOSE TIGHTER BOUNDS WITH LOWER BOUND AS 10% INCOME AND UPPER
BOUND OF PORTFOLIO VALUE AT THAT TIME POINT.


withdraw <- 0
withdraw.hold <- withdraw
withdraw.hold



# SCENARIO 1 - UNIFORM DISCRETE UNCERTAIN EVENTS PROBABILITIES.
# ASSIGN PROBABILITIES OF 12*n TIME POINTS USING THE DISRETE UNIFORM
DISTRIBUTION.
# 12 MONTHS TIMES THE NUMBER OF RETIREMENT YEARS

prob1 <- rep(1/(12*n),12*n)
plot(prob1)


# STEPFUNC WOULD INCREASE/DECREASE THE TIME PERIOD PROBABILTIIES BY
CONSTANT TO THE UNIFORM PROBABILTIY


stepfunc <-  abs(1/(12*n)*0.50)/(12*n)
```

```
# SCENARIO 2 - INCREASING FUNCTION UNCERTAIN EVENTS PROBABILITIES.

prob2 <- 1/(12*n)
prob2.hold <- prob2

for(i in 1:12*n){
  prob2 <- (1/(12*n) + stepfunc*i)
  prob2.hold <- rbind(prob2.hold,prob2)
}
plot(prob2.hold)


# SCENARIO 3 - DECREASING FUNCTION UNCERTAIN EVENTS PROBABILITIES.

prob3 <- 1/(12*n)
prob3.hold <- prob3

for(i in 1:12*n){
  prob3 <- (1/(12*n) - stepfunc*i)
  prob3.hold <- rbind(prob3.hold,prob3)
```

```
}
plot(prob3.hold)




# SOLVER FUNCTION FOR MAXIMIZING THE RETURN AT A PRE-SPECIFIED RISK.

# OBJECTIVE FUNCTION

fnobj1 <- function(x){
  return = -(t(x[1:10]) %*% mu.stocks) - (t(x[11:15]) %*%mu.bonds)
  obj = -return
}



# INEQUALITY CONSTRAINT FUNCTION
inequality.const <- function(x){
  sqrt( t(x[1:10]) %*% V.stocks %*% x[1:10])
}



# LINEAR STEP FUNCTION.
```

```
#stepInit <- abs(bondInitial-bondFinal)/(ageFinal-ageInit)



##################################
# RETIREMENT INCOME FRAMEWORK
##################################


for(j in 1:n){
  for(i in 1:12 ){


    # RETURNS FROM STOCKS MULTIVARIATE NORMAL DISTRIBUTION

    return.stocks <- rmvnorm(1, mean=mu.stocks, sigma=V.stocks)
    return.stocks.hold <- rbind(return.stocks.hold,return.stocks)

    # RETURNS FROM BONDS MULTIVARIATE NORMAL DISTRIBUTION

    return.bonds <- rmvnorm(1, mean=mu.bonds, sigma=V.bonds)
    return.bonds <- exp(return.bonds)
    return.bonds.hold <- rbind(return.bonds.hold,return.bonds)
```

```
# GENERATE THE TRUE EVENT USING BINOMIAL DISTRIBUTION WITH A
TRIAL AT EACH TIME POINT FOR SCENARIO1.

binom1 <- rbinom(1,1,1/(12*n))

# GENERATE THE TRUE EVENT USING BINOMIAL DISTRIBUTION WITH A
TRIAL AT EACH TIME POINT FOR SCENARIO2.

#binom2 <- rbinom(1,1,(1/(12*n) + stepfunc*i))

# GENERATE THE TRUE EVENT USING BINOMIAL DISTRIBUTION WITH A
TRIAL AT EACH TIME POINT FOR SCENARIO3.

#binom3 <- rbinom(1,1,(1/(12*n) - stepfunc*i))

# IMPOSE TIGHTER BOUNDS WITH LOWER BOUND AS 10% INCOME AND UPPER
BOUND OF PORTFOLIO VALUE AT THAT TIME POINT.

if (binom1 == 1){
```

```
        withdraw <-
(0.90*portfolio1.value*rbeta(1,18.975,107.525)+0.10*income)
        withdraw.hold <- rbind(withdraw.hold,withdraw)


        # GEOMETRIC AVERAGE RATE OF RETURN TO CALCULATE THE ANNUALIZED
RATE OF RETURN


        value1 <- value1*(1+(return.stocks/12)) + x[1:10] *
monthly.contribution - x[1:10] * withdraw
        value1.hold <- rbind(value1.hold,value1)


        # GEOMETRIC AVERAGE RATE OF RETURN TO CALCULATE THE ANNUALIZED
RATE OF RETURN


        value2 <- value2*(1+(return.bonds/12)) + x[11:15] *
monthly.contribution - x[11:15] * withdraw
        value2.hold <- rbind(value2.hold,value1)
      }



      else if (binom1 == 0){
```

```
        withdraw <- 0
        withdraw.hold <- rbind(withdraw.hold,withdraw)


        # GEOMETRIC AVERAGE RATE OF RETURN TO CALCULATE THE ANNUALIZED
RATE OF RETURN



        value1 <- value1*(1+(return.stocks/12)) + x[1:10] *
monthly.contribution
        value1.hold <- rbind(value1.hold,value1)


        # GEOMETRIC AVERAGE RATE OF RETURN TO CALCULATE THE ANNUALIZED
RATE OF RETURN


        value2 <- value2*(1+(return.bonds/12)) + x[11:15] *
monthly.contribution
        value2.hold <- rbind(value2.hold,value1)
      }

    }
```

```
    # ACCOUNTED FOR INFLATION ON YEARLY BASIS.

    inflation <- rbeta(1,4.929,192.2)
    inflation.hold <- rbind(inflation.hold,inflation)



    # YEARLY INCOME RISE TO KEEP PACE WITH INFLATION AND ALSO BASED ON
THE INDIVIDUAL's PERFORMANCE

    income <- income + income * (rbeta(1,0.113,2.726))
    income.hold <- rbind(income.hold,income)


    # YEARLY STATIC MONTHLY CONTRIBUTION OF 401K BY THE EMPLOYER IN
THE INVESTOR ACCOUNT

    monthly.contribution <- income * 0.04/12
    monthly.contribution.hold <-
rbind(monthly.contribution.hold,monthly.contribution)


    # YEARLY BONUS GIVEN ON INCOME SIMULATED USING A DISTRIBUTION AS
BONUS IS A FUNCTION OF PERFORMANCE TOO.HENCE NOT STATIC
```

```
    bonus <-  income * 0.04 * (rbeta(1, 1.496, 35.904))

    bonus.hold <- rbind(bonus.hold,bonus)



    # PORTFOLIO VALUE(GROSS ANNUAL RETURNS) AT THE END OF THE YEAR IS
RETURN FROM BONDS,STOCKS AND MONTHLY COMPOUNDED ANNUAL RETURN ON
WEALTH INVESTED.



    portfolio1.value <-  sum(value1) + sum(value2) + bonus

    portfolio1.value.hold <-
rbind(portfolio1.value.hold,portfolio1.value)



    # CONSERVATIVE PORTFOLIO STRATEGY - WITH LOW RISK TOLERANCE AND
DECREASING PERCENTAGE INVESTED IN STOCKS.



    r <- solnp(pars = x,fun=
fnobj1,ineqfun=inequality.const,ineqUB=0.05,ineqLB=0.00,LB=rep(0,15),U
B=rep(1,15))
```

```
  }

  # NET PRESENT VALUE OF PORFOLIO VALUE AND INCOME IN CURRENT DOLLARS

  cum.inflation.out <- 1 + cumprod(inflation.hold)
  npv.portfolio.value <- portfolio1.value.hold/cum.inflation.out

  # SIMULATION RESULTS OF PORTFOLIO VALUE

  end.value1[sims1] <- portfolio1.value.hold[n+1]
}

write.table(end.value1, file ="C:/Users/megha/Documents/output3a.csv",
row.names = FALSE, append = FALSE, col.names = TRUE, sep = ",")

# YEARLY OUTPUTS FROM THE DIFFERENT FACTORS

inflation.hold
```

```
income.hold

portfolio1.value.hold

bonus.hold

monthly.contribution.hold

npv.portfolio.value

hist(end.value1)
```

Thank you.