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An Application of Combinatorial Methods

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AN APPLICATION OF COMBINATORIAL METHODS

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By

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Table of Contents

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Abstract

Probability theory is a branch of mathematics concerned with determining the long run frequency or chance that a given event will occur. This chance is determined by dividing the number of selected events by the number of total events possible, assuming these events are equally likely. Probability theory is simply enumerative combinatorial analysis when applied to finite sets. For a given finite sample space, probability questions are usually "just" a lot of counting.

The purpose of this thesis is to provide some in depth analysis of several combinatorial methods, including basic principles of counting, permutations and combinations, by specifically exploring one type of probability problem:

$\boldsymbol{C}$ ordered possible elements that are equally likely

$\boldsymbol{s}$ independent sampled subjects

$\boldsymbol{r}$ distinct elements, where $\boldsymbol{r} = 1, 2, 3, \ldots, \min(\boldsymbol{C}, \boldsymbol{s})$

we want to know P($\boldsymbol{s}$ subjects utilize exactly $\boldsymbol{r}$ distinct elements).

This thesis gives a detailed step by step analysis on techniques used to ultimately finding a general formula to solve the above problem.

CHAPTER 1

Introduction

1.1 Motivation

The term probability refers to the study of randomness and uncertainty. In any situation in which one of a number of possible outcomes may occur, the theory of probability provides methods for quantifying the chances, or likelihoods, associated with the various outcomes. The study of probability as a branch of mathematics goes back over 300 years, where it had its genesis in connection with questions involving games of chance (Devore, 1995). The theory thus developed for “heads or tails” or “red or black” soon found applications in situations where the outcomes were “boys and girls,” “life or death,” or “pass or fail,” and scholars began to apply probability theory to actuarial problems and some aspects of the social sciences (Miller and Miller, 1999). Later, probability and statistics were introduced into physics by L. Boltzmann, J. Gibbs, and J. Maxwell, and since last century they have found applications in all phases of human endeavor that in some way involve an element of uncertainty or risk (Miller and Miller, 1999).

In any probability problem, it is very important to identify all the different outcomes that could occur, i.e., total number of possible outcomes in an experiment. In many situations, it is reasonable to assume that all of the possible outcomes of the experiment are equally likely. In general, the total number of possible outcomes is calculated by various combinatorial methods. In other cases, calculating the total number

of possible outcomes is a matter of enumeration. How many sides does the coin have? Two, that's the number of total possible outcomes when a coin is flipped. A die has six faces, that's the total possible outcomes when a die is rolled. In a very simple experiment like betting on heads in coin flipping it is easy. There is one favorable outcome out of two. Betting on face three of a die is also easy: one favorable outcome out of six. However, to determine the number of favorable outcomes, outcomes that we are interested in, is more difficult in most cases. It usually requires simultaneous application of various combinatorial methods. We can see that the concept and application of combinatorial methods play an essential role in computing probability of an event of interest.

1.2 Problem Introduction

In analyzing various combinatorial methods, there is one type of probability problem that is particularly interesting, but possibly has not been fully tackled. Suppose there are 100 people attending a conference, we might want to know the probability that all of the 100 people have the same initial for their last name. Or we might want to know the probability that the 100 people have exactly 2 distinct initials for their last names. Or we might want to know the probability that the 100 people have exactly 26 distinct initials for their last names (since there are total 26 letters, the maximum distinct initials that the 100 people can have is 26). And so on. We can expect the total number of possible outcomes to be extremely large, when the independent sampled subjects, in this case the 100 people attending the conference, becomes large. We can also expect the outcomes of interest become very complicated as the number of both independent

sampled subjects and distinct outcomes become large. In case of this example with 2 distinct initials, we could have one person with one unique letter out of any of the 26 letters, and the rest of the 99 people having the same letter out of any of the remaining 25 letters (since one letter is already taken). Or we could have two persons with the same letter out of any of the 26 letters and the rest of the 98 people having the same letter out of any of the remaining 25 letters. Or we could have three persons with the same letter out of any of the 26 letters and the rest of the 97 people having the same letter out of any of the remaining 25 letters. And the process continues, until we have 50 people with the same letter and the other 50 people having the same letter that is different from the prior one.

The thesis will give a comprehensive step by step analysis on the methods and techniques used to solve the above problem. A final formula will be given to compute probability of any event of this kind.

CHAPTER 2

Literature Review

Concepts of probability have been around for thousands of years, but probability theory did not arise as a branch of mathematics until the mid-seventeenth century. During the fifteenth century several probability works emerged. Calculations of probabilities became more noticeable during this time period even though mathematicians in Italy and France remained unfamiliar with these calculation methods (David, 1962). In 1494, Fra Luca Paccioli wrote the first printed work on probability, *Summa de arithmetica, geometria, proportioni e proportionalita* (David, 1962).

Probability theory was inspired by gambling problems. The earliest work was performed by Girolamo Cardano (1501-1576) an Italian mathematician, physician, and gambler. In his manual *Liber de Ludo Aleae*, Cardano discusses many of the basic concepts of probability complete with a systematic analysis of gambling problems. Unfortunately, Cardano's work had little effect on the development of probability because his manual, which did not appeared in print until 1663, received little attention (David, 1962).

In 1654, another gambler named Chevalier de Méré created a dice proposition which he believed would make money. He would bet even money that he could roll at least one twelve in 24 rolls of two dice. However, when the Chevalier began losing money, he asked his mathematician friend Blaise Pascal (1623-1662) to analyze the proposition. Pascal determined that this proposition will lose about 51% of the time.

Inspired by this proposition, Pascal began studying more of these types of problems. He discussed them with another famous mathematician, Pierre de Fermat (1601-1665) and together they laid the foundation of probability theory (David, 1962).

Laplace, one of the commanding figures of the mathematics and science during the 18^{th} century, made a natural division in the history of probability, since in his monumental *Théorie analytique des probabilities*, first published in 1812, he summed up his own extensive work and that of his predecessors (Parzen, 1960).

The literature of probability theory divides into three broad categories: (i) the nature (or foundations) of probability, (ii) mathematical probability theory, and (iii) applied probability theory (Parzen, 1960). In this thesis, we will mainly focus on mathematical probability theory.

The concept of the basic principles of counting, permutations and combinations is at the root of probability theory and many other specialties. The study of permutations began in ancient times. The number of ways of arranging an n number of items was known to be $n!$ for at least 2500 years. Before getting into detailed analysis of the combinatorial methods, we will start with several simple and fundamental definitions.

2.1 Experiment, Sample Space and Sample Point

It is customary in statistics to refer to any process of observation or measurement as an experiment. When we have defined all possible experimental outcomes, we have identified the sample space for the experiment. That is, the sample space (denoted by S) is defined as the set of all possible experimental outcomes. Any one particular

experimental outcome is referred to as a sample point and is an element of the sample space (Anderson, Sweeney and Williams, 1998).

Experiment	Sample Space
toss a coin	S = {Head, Tail}
select a part for inspection	S = {Defective, Nondefective}
roll a die	S = {1, 2, 3, 4, 5, 6}

2.2 Basic Principle of Counting

If an experiment can be described as a sequence of k steps in which there are n_1 possible outcomes on the first step, n_2 possible outcomes on the second step, and so on, then the total number of experimental outcomes is given by $\boldsymbol{(n_1)\ (n_2) \ldots\ (n_k)}$. That is the number of outcomes for the overall experiment is the product of the number of outcomes on each step (Anderson, Sweeney and Williams, 1998).

Example 2.1: How many possible outcomes are there when we roll a pair of dice, one red and one green?

Solution: With six outcomes possible on the roll of the red die ($n_1 = 6$) and six outcomes possible on the roll of the green dice ($n_2 = 6$), the counting rule states that there are $(n_1)\ (n_2) = 6 \times 6 = 36$ experimental outcomes.

Example 2.2: In how many different ways can one answer all the questions of a true-false test consisting of 20 questions?

Solution: Altogether there are $2 \times 2 \times 2 \times 2 \times \ldots \times 2 \times 2 = 2^{20} = 1{,}048{,}576$ different ways

in which one can answer all the questions; only one of these corresponds to the case where all the questions are correct and only one corresponds to the case where all the answers are wrong.

2.3 Permutations

Frequently, we are interested in situations where the outcomes are the different ways in which a group of objects can be ordered or arranged. For instance, we might want to know in how many different ways the 24 members of a club can elect a president, a treasurer, a secretary, or we might want to know in how many different ways six persons can be seated around a table. A permutation is an ordered selection made from a group of objects (Miller and Miller, 1999). For example, if you look at credit card codes you must look at the order. The order 5-3-7-5 is different from 3-7-5-5: the order in which you pick the numbers is significant. A permutation of (2, 3, 5, 4, 1) is an arrangement of the five objects 1, 2, 3, 4, and 5. It tells us that the second object is in the first position, the third object is in the second position, the fifth object is in the third position and so on.

Example 2.3: How many permutations are there of the letters a, b, and c?

Solution: The possible arrangements are *abc, acb, bac, bca, cab,* and *cba,* so the number of distinct permutations is six. We could have arrived at this answer without actually listing the different permutations. Since there are three choices to select a letter for the first position, then two for the second position, leaving only one letter for the third position, the total number of

permutation is $3 \times 2 \times 1 = 6$.

Generalizing the argument used in the above example, we find that n distinct objects can be arranged in $n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$ different ways. Therefore, the numbers of permutations of *n* distinct objects is ***n!***.

Example 2.4: In how many different ways can the five starting players of a basketball team to be introduced to the public?

Solution: There are $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways in which they can be introduced.

Example 2.5: If we have a total of 10 elements, the integers {1, 2, …, 10}, a permutation of three elements from this set is (5, 3, 4). In this case, $n = 10$ and $r = 3$. So how many ways can this completely be done?

Solution: 1. We can pretend to select the first member of all permutations out of *n* choices because there are n distinct elements from the generating set.

2. Next, since we have used one of the *n* elements already, the second member of the permutation has $(n-1)$ elements to choose from the remaining set.

3. The third member can be filled in $(n-2)$ ways since two have been used already.

4. The pattern continues until there are r members on the permutation. This means that the last member can be filled in $(n - r + 1)$ ways.

Summarizing, we find that a total of n*(n - 1)(n - 2)…(n – r + 1)* different permutations of *r* objects, taken from a pool of *n* objects, exist. If we denote this number by P_n^r and use the factorial notation, we can write

$$P_n^r = \frac{n!}{(n-r)!} \qquad \text{for r = 0, 1, 2, …, n.}$$

(* See Appendix A for proof)

In this example, we have *n* = 10 and *r* = 3, so to find out how many unique sets, we need to calculate $P_{10}^3 = \frac{10!}{(10-3)!}$ = 720.

Example 2.6: In how many different ways can three copies of one novel and one copy each of four other novels be arranged on a shelf?

Solution: If we denote the three copies of the first novel by a_1, a_2, and a_3 and the other four novels by *b, c, d,* and *e*, we find that with subscripts there are 7! different permutations of a_1, a_2, a_3, *b, c, d,* and *e*. However, since there are 3! permutations of a_1, a_2, and a_3 that lead to the same permutation of *a, a, a, b, c, d,* and *e*, we find that there are only $\frac{7!}{3!}$ = 7×6×5×4 = 840 ways in which the seven books can be arranged on a shelf.

2.4 Combinations

A combination, on the other hand, is an unordered selection made from a group of objects. For example, suppose you have fifty-two playing cards and select five cards

for a poker hand. It would not matter in which order the cards were drawn, because you could rearrange them in your hand. Since order does not matter, you are only interested in what is present, not in what order. Thus, {2, 4, 6} = {6, 4, 2}.

Let's denote the number of *r*-combinations of a set with *n* elements by $\binom{n}{r}$. We have

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad \text{for } r = 0, 1, 2, \ldots, n.$$

(* See Appendix B for derivation)

Example 2.6: In how many different ways can six tosses of a coin yield two heads and four tails?

Solution: $\binom{6}{2} = \frac{6!}{2!\cdot 4!} = 15$

Example 2.7: How many different committees of two chemists and one physicist can be formed from the four chemists and three physicists on the faculty of a small college?

Solution: Since two of four chemists can be selected in $\binom{4}{2} = \frac{4!}{2!\cdot 2!} = 6$ ways and one of three physicists can be selected in $\binom{3}{1} = \frac{3!}{1!\cdot 2!} = 3$ ways, the counting rule shows that the number of committees is $6\times3=18$.

A combination of r objects selected from a set of n distinct objects may be considered a partition of the n objects into two subsets containing, respectively, the r objects that are selected and $n - r$ objects that are left. Often, we are concerned with the more general problem of partitioning a set of n distinct objects into k subsets, which requires that each of the n objects must belong to one and only one of the subsets. The order of the objects within a subset is of no importance.

The number of ways in which a set of n distinct objects can be partitioned into k subsets with n_1 objects in the first subset, n_2 objects in the second subset, …, and n_k objects in the kth subset is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}$$

(* See Appendix C for proof)

Example 2.8: In how many different ways can seven businessmen attending a convention be assigned to one triple and two double hotel rooms?

Solution:

$$\binom{7}{3, 2, 2} = \frac{7!}{3! \cdot 2! \cdot 2!} = 210$$

2.5 Permutations vs. Combinations

Let n be the number of objects from which you can choose and r be the number to be chosen.

- Permutation with replacement

When order matters and an object can be chosen more than once then the number of

permutations = n^r. For example, if you have the letters A, B, C, and D and you wish to discover the number of ways to arrange them in three letter patterns you find that there are 4^3 or 64 ways. This is because for the first slot you can choose any of the four values, for the second slot you can choose any of the four, and for the final slot you can choose any of the four letters. Multiplying them together gives the total.

- Permutation without replacement

When the order matters and each object can be chosen only once, then the number of permutations = $\frac{n!}{(n-r)!}$.

- Combination with replacement

When the order does not matter and an object can be chosen more than once, then the number of combinations = $\frac{(n+r-1)!}{r!(n-1)!}$. For example, if we want to pick 3 cards from a deck with replacement and we don't care about the order of the cards (we only care what 3 cards we get), we would have $\frac{(52+3-1)!}{3!(52-1)!}$ = 24804 ways to pick.

- Combination without replacement

When the order does not matter, but each object can be chosen only once, the

number of combinations = $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

In many probability related problems, the basic combinatorial methods illustrated above, including the counting rules, permutations and combinations are often used simultaneously. In this thesis, the concept and application of these methods will be further explained and analyzed.

CHAPTER 3

Problem Evaluation and Solution

3.1 Problem Outline and Notations

To address the type of probability problem introduced in Chapter 1, let's begin by outlining the problem and its corresponding notations. We have

$\boldsymbol{C}$ ordered possible elements that are equally likely

$\boldsymbol{s}$ independent sampled subjects

$\boldsymbol{r}$ distinct elements, where $\boldsymbol{r} = 1, 2, 3, \ldots, \min(\boldsymbol{C}, \boldsymbol{s})$

and we want to know P($\boldsymbol{s}$ subjects utilize exactly $\boldsymbol{r}$ distinct elements).

For instance, the problem introduced in Chapter 1, we would have $\boldsymbol{C} = 26$, i.e., a, b, c, …, z, since the alphabet has 26 ordered letters; $\boldsymbol{s} = 100$ people attending the conference; $\boldsymbol{r} = 1$, if all the 100 people have the same initial for their last name or $\boldsymbol{r} = 2$, if the 100 people have exactly 2 distinct initials for their last name or $\boldsymbol{r} = 3$, if the 100 people have exactly 3 distinct initials for their last name and so on. Notice that the upper limit for $\boldsymbol{r}$ equals $\min(\boldsymbol{C}, \boldsymbol{s})$. In case of $\boldsymbol{C} < \boldsymbol{s}$, 100 people can only have maximum 26 distinct initials for their last names. In case of $\boldsymbol{C} > \boldsymbol{s}$, 20 people can only have maximum 20 distinct initials for their last names.

Essentially, there are three steps involved in computing the probability of interest. The first step is to obtain the total number of possible outcomes; the second step is to figure out the total number of combinations; the last and the most challenging step is to find the number of arrangements within each combination. Once we have the three elements, the probability of the event of interest would equal to

$$\frac{\text{(total number of outcomes)}\bullet\text{(\# of arrangements within each combination)}}{\text{total number of possible outcomes}}$$

The above formula comes from the definition of the classical probability concept, which states that if there are N equally likely possibilities, of which one must occur and n are regarded as favorable, or as a "success", then the probability of a "success" is given by the ratio $\frac{n}{\mathrm{N}}$ (Miller and Miller, 1999). Furthermore, we get the numerator by applying the basic principle of counting.

3.2 Total Number of Possible Outcomes

Since order matters for the outcomes (for the example in Chapter 1, the position of the letters matters: (a, a, b) ≠ (b, a, a)) and each of the C ordered possible elements can occur more than once (each letter can occur multiple times), implying with replacement, therefore, according to the rule of permutations, we have total C^S possible outcomes.

3.3 Total Number of Combinations

It is very straight forward to compute the total number of combinations to generate $\boldsymbol{r}$ distinct elements from $\boldsymbol{C}$ ordered possible outcomes. By simply applying the rule of combinations, we would have a total of $\binom{C}{r}$ possible combinations.

Let's denote $\boldsymbol{A}$ = number of arrangements within each combination, and plug in the formula given above, we have:

$$\text{P}(\boldsymbol{s} \text{ subjects utilize exactly } \boldsymbol{r} \text{ distinct elements}) = \frac{\binom{C}{r} \bullet A}{C^s}$$

3.4 Number of Arrangements Within Each Combination

In order to compute the number of arrangements within each combination, it would be a better idea to first of all gather some basic understanding of its pattern. To do so, we will begin by listing all the possible outcomes for given values of $\boldsymbol{C}$, $\boldsymbol{s}$ and $\boldsymbol{r}$. However, the pattern will become extremely complicated and the total number of possible outcomes will become extremely large when the values of $\boldsymbol{C}$, $\boldsymbol{s}$ and $\boldsymbol{r}$ are large. Therefore, we will proceed with small values of $\boldsymbol{C}$, $\boldsymbol{s}$ and $\boldsymbol{r}$, and continue the process by increasing $\boldsymbol{C}$, $\boldsymbol{s}$ and $\boldsymbol{r}$ systematically until we see a clear pattern for $\boldsymbol{A}$. To make it simple, we will let *1, 2, 3, …, n* be the $\boldsymbol{C}$ ordered possible outcomes which have equal probability.

3.4.1 $\boldsymbol{C} = 2$

- $\boldsymbol{s} = 2$

1,1	2,1
1,2	**2,2**

$$P(\boldsymbol{r} = 1) = \frac{\binom{2}{1}\bullet(1)}{2^2} = \frac{2}{4} \qquad P(\boldsymbol{r} = 2) = \frac{\binom{2}{2}\bullet(2)}{2^2} = \frac{2}{4}$$

- $\boldsymbol{s} = 3$

1,1,1	2,1,1
1,1,2	2,1,2
1,2,1	2,2,1
1,2,2	**2,2,2**

$$P(\boldsymbol{r} = 1) = \frac{\binom{2}{1}\bullet(1)}{2^3} = \frac{2}{8} \qquad P(\boldsymbol{r} = 2) = \frac{\binom{2}{2}\bullet(6)}{2^3} = \frac{6}{8}$$

- $\boldsymbol{s} = 4$

1,1,1,1	2,1,1,1
1,1,1,2	2,1,1,2
1,1,2,1	2,1,2,1
1,1,2,2	2,1,2,2
1,2,1,1	2,2,1,1
1,2,1,2	2,2,1,2
1,2,2,1	2,2,2,1
1,2,2,2	**2,2,2,2**

$$P(\boldsymbol{r} = 1) = \frac{\binom{2}{1} \bullet (1)}{2^4} = \frac{2}{16} \qquad P(\boldsymbol{r} = 2) = \frac{\binom{2}{2} \bullet (14)}{2^4} = \frac{14}{16}$$

- $\boldsymbol{s} = 5$

1,1,1,1,1	2,1,1,1,1
1,1,1,1,2	2,1,1,1,2
1,1,1,2,1	2,1,1,2,1
1,1,1,2,2	2,1,1,2,2
1,1,2,1,1	2,1,2,1,1
1,1,2,1,2	2,1,2,1,2
1,1,2,2,1	2,1,2,2,1
1,1,2,2,2	2,1,2,2,2
1,2,1,1,1	2,2,1,1,1
1,2,1,1,2	2,2,1,1,2
1,2,1,2,1	2,2,1,2,1
1,2,1,2,2	2,2,1,2,2
1,2,2,1,1	2,2,2,1,1
1,2,2,1,2	2,2,2,1,2
1,2,2,2,1	2,2,2,2,1
1,2,2,2,2	**2,2,2,2,2**

$$P(\boldsymbol{r} = 1) = \frac{\binom{2}{1} \bullet (1)}{2^5} = \frac{2}{32} \qquad P(\boldsymbol{r} = 2) = \frac{\binom{2}{2} \bullet (30)}{2^5} = \frac{30}{32}$$

3.4.2 $\boldsymbol{C} = 3$

- $\boldsymbol{s} = 2$

1,1	2,1	3,1
1,2	**2,2**	3,2
1,3	2,3	**3,3**

$$P(\boldsymbol{r} = 1) = \frac{\binom{3}{1} \bullet (1)}{3^2} = \frac{3}{9} \qquad P(\boldsymbol{r} = 2) = \frac{\binom{3}{2} \bullet (2)}{3^2} = \frac{6}{9}$$

- $\boldsymbol{s} = 3$

1,1,1	2,1,1	3,1,1
1,1,2	2,1,2	3,1,2*
1,1,3	2,1,3*	3,1,3
1,2,1	2,2,1	3,2,1*
1,2,2	**2,2,2**	3,2,2
1,2,3*	2,2,3	3,2,3
1,3,1	2,3,1*	3,3,1
1,3,2*	2,3,2	3,3,2
1,3,3	2,3,3	**3,3,3**

$$P(\boldsymbol{r} = 1) = \frac{\binom{3}{1} \bullet (1)}{3^3} = \frac{3}{27} \qquad P(\boldsymbol{r} = 2) = \frac{\binom{3}{2} \bullet (6)}{3^3} = \frac{18}{27}$$

$$P(\boldsymbol{r^*} = 3) = \frac{\binom{3}{3} \bullet (6)}{3^3} = \frac{6}{27}$$

- $\boldsymbol{s} = 4$

1,1,1,1	2,1,1,1	3,1,1,1
1,1,1,2	2,1,1,2	3,1,1,2*
1,1,1,3	2,1,1,3*	3,1,1,3
1,1,2,1	2,1,2,1	3,1,2,1*
1,1,2,2	2,1,2,2	3,1,2,2*
1,1,2,3*	2,1,2,3*	3,1,2,3*
1,1,3,1	2,1,3,1*	3,1,3,1
1,1,3,2*	2,1,3,2*	3,1,3,2*
1,1,3,3	2,1,3,3*	3,1,3,3
1,2,1,1	2,2,1,1	3,2,1,1*
1,2,1,2	2,2,1,2	3,2,1,2*
1,2,1,3*	2,2,1,3*	3,2,1,3*
1,2,2,1	2,2,2,1	3,2,2,1*
1,2,2,2	**2,2,2,2**	3,2,2,2
1,2,2,3*	2,2,2,3	3,2,2,3
1,2,3,1*	2,2,3,1*	3,2,3,1*
1,2,3,2*	2,2,3,2	3,2,3,2
1,2,3,3*	2,2,3,3	3,2,3,3
1,3,1,1	2,3,1,1*	3,3,1,1
1,3,1,2*	2,3,1,2*	3,3,1,2*
1,3,1,3	2,3,1,3*	3,3,1,3
1,3,2,1*	2,3,2,1*	3,3,2,1*
1,3,2,2*	2,3,2,2	3,3,2,2
1,3,2,3*	2,3,2,3	3,3,2,3
1,3,3,1	2,3,3,1*	3,3,3,1
1,3,3,2*	2,3,3,2	3,3,3,2
1,3,3,3	2,3,3,3	**3,3,3,3**

$$\mathrm{P}(\boldsymbol{r} = 1) = \frac{\binom{3}{1}\bullet(1)}{3^4} = \frac{3}{81} \qquad \mathrm{P}(\boldsymbol{r} = 2) = \frac{\binom{3}{2}\bullet(14)}{3^4} = \frac{42}{81}$$

$$\mathrm{P}(\boldsymbol{r^*} = 3) = \frac{\binom{3}{3}\bullet(36)}{3^4} = \frac{36}{81}$$

- $\boldsymbol{s} = 5$

1,1,1,1,1	1,2,1,1,1*	1,3,1,1,1*	2,1,1,1,1*	2,2,1,1,1*	2,3,1,1,1	3,1,1,1,1*	3,2,1,1,1	3,3,1,1,1*
1,1,1,1,2*	1,2,1,1,2*	1,3,1,1,2	2,1,1,1,2*	2,2,1,1,2*	2,3,1,1,2	3,1,1,1,2	3,2,1,1,2	3,3,1,1,2
1,1,1,1,3*	1,2,1,1,3	1,3,1,1,3*	2,1,1,1,3	2,2,1,1,3	2,3,1,1,3	3,1,1,1,3*	3,2,1,1,3	3,3,1,1,3*
1,1,1,2,1*	1,2,1,2,1*	1,3,1,2,1	2,1,1,2,1*	2,2,1,2,1*	2,3,1,2,1	3,1,1,2,1	3,2,1,2,1	3,3,1,2,1
1,1,1,2,2*	1,2,1,2,2*	1,3,1,2,2	2,1,1,2,2*	2,2,1,2,2*	2,3,1,2,2	3,1,1,2,2	3,2,1,2,2	3,3,1,2,2
1,1,1,2,3	1,2,1,2,3	1,3,1,2,3	2,1,1,2,3	2,2,1,2,3	2,3,1,2,3	3,1,1,2,3	3,2,1,2,3	3,3,1,2,3
1,1,1,3,1*	1,2,1,3,1	1,3,1,3,1*	2,1,1,3,1	2,2,1,3,1	2,3,1,3,1	3,1,1,3,1*	3,2,1,3,1	3,3,1,3,1*
1,1,1,3,2	1,2,1,3,2	1,3,1,3,2	2,1,1,3,2	2,2,1,3,2	2,3,1,3,2	3,1,1,3,2	3,2,1,3,2	3,3,1,3,2
1,1,1,3,3*	1,2,1,3,3	1,3,1,3,3*	2,1,1,3,3	2,2,1,3,3	2,3,1,3,3	3,1,1,3,3*	3,2,1,3,3	3,3,1,3,3*
1,1,2,1,1*	1,2,2,1,1*	1,3,2,1,1	2,1,2,1,1*	2,2,2,1,1*	2,3,2,1,1	3,1,2,1,1	3,2,2,1,1	3,3,2,1,1
1,1,2,1,2*	1,2,2,1,2*	1,3,2,1,2	2,1,2,1,2*	2,2,2,1,2*	2,3,2,1,2	3,1,2,1,2	3,2,2,1,2	3,3,2,1,2
1,1,2,1,3	1,2,2,1,3	1,3,2,1,3	2,1,2,1,3	2,2,2,1,3	2,3,2,1,3	3,1,2,1,3	3,2,2,1,3	3,3,2,1,3
1,1,2,2,1*	1,2,2,2,1*	1,3,2,2,1	2,1,2,2,1*	2,2,2,2,1*	2,3,2,2,1	3,1,2,2,1	3,2,2,2,1	3,3,2,2,1
1,1,2,2,2*	1,2,2,2,2*	1,3,2,2,2	2,1,2,2,2*	**2,2,2,2,2**	2,3,2,2,2*	3,1,2,2,2	3,2,2,2,2*	3,3,2,2,2*
1,1,2,2,3	1,2,2,2,3	1,3,2,2,3	2,1,2,2,3	2,2,2,2,3*	2,3,2,2,3*	3,1,2,2,3	3,2,2,2,3*	3,3,2,2,3*
1,1,2,3,1	1,2,2,3,1	1,3,2,3,1	2,1,2,3,1	2,2,2,3,1	2,3,2,3,1	3,1,2,3,1	3,2,2,3,1	3,3,2,3,1
1,1,2,3,2	1,2,2,3,2	1,3,2,3,2	2,1,2,3,2	2,2,2,3,2*	2,3,2,3,2*	3,1,2,3,2	3,2,2,3,2*	3,3,2,3,2*
1,1,2,3,3	1,2,2,3,3	1,3,2,3,3	2,1,2,3,3	2,2,2,3,3*	2,3,2,3,3*	3,1,2,3,3	3,2,2,3,3*	3,3,2,3,3*
1,1,3,1,1*	1,2,3,1,1	1,3,3,1,1*	2,1,3,1,1	2,2,3,1,1	2,3,3,1,1	3,1,3,1,1*	3,2,3,1,1	3,3,3,1,1*
1,1,3,1,2	1,2,3,1,2	1,3,3,1,2	2,1,3,1,2	2,2,3,1,2	2,3,3,1,2	3,1,3,1,2	3,2,3,1,2	3,3,3,1,2
1,1,3,1,3*	1,2,3,1,3	1,3,3,1,3*	2,1,3,1,3	2,2,3,1,3	2,3,3,1,3	3,1,3,1,3*	3,2,3,1,3	3,3,3,1,3*
1,1,3,2,1	1,2,3,2,1	1,3,3,2,1	2,1,3,2,1	2,2,3,2,1	2,3,3,2,1	3,1,3,2,1	3,2,3,2,1	3,3,3,2,1
1,1,3,2,2	1,2,3,2,2	1,3,3,2,2	2,1,3,2,2	2,2,3,2,2*	2,3,3,2,2*	3,1,3,2,2	3,2,3,2,2*	3,3,3,2,2*
1,1,3,2,3	1,2,3,2,3	1,3,3,2,3	2,1,3,2,3	2,2,3,2,3*	2,3,3,2,3*	3,1,3,2,3	3,2,3,2,3*	3,3,3,2,3*
1,1,3,3,1*	1,2,3,3,1	1,3,3,3,1*	2,1,3,3,1	2,2,3,3,1	2,3,3,3,1	3,1,3,3,1*	3,2,3,3,1	3,3,3,3,1*
1,1,3,3,2	1,2,3,3,2	1,3,3,3,2	2,1,3,3,2	2,2,3,3,2*	2,3,3,3,2*	3,1,3,3,2	3,2,3,3,2*	3,3,3,3,2*
1,1,3,3,3*	1,2,3,3,3	1,3,3,3,3*	2,1,3,3,3	2,2,3,3,3*	2,3,3,3,3*	3,1,3,3,3*	3,2,3,3,3*	**3,3,3,3,3**

$$P(\boldsymbol{r} = 1) = \frac{\binom{3}{1}\bullet(1)}{3^5} = \frac{3}{243} \qquad P(\boldsymbol{r^*} = 2) = \frac{\binom{3}{2}\bullet(30)}{3^5} = \frac{90}{243}$$

$$P(\boldsymbol{r} = 3) = \frac{\binom{3}{3}\bullet(150)}{3^5} = \frac{150}{243}$$

3.4.3 $\boldsymbol{C} = 4$

- $\boldsymbol{s} = 2$

1,1	2,1	3,1	4,1
1,2	**2,2**	3,2	4,2
1,3	2,3	**3,3**	4,3
1,4	2,4	3,4	**4,4**

$$P(\boldsymbol{r} = 1) = \frac{\binom{4}{1}\bullet(1)}{4^2} = \frac{4}{16} \qquad P(\boldsymbol{r} = 2) = \frac{12}{4^2}$$

- $\boldsymbol{s} = 3$

1,1,1	2,1,1	3,1,1	4,1,1
1,1,2	2,1,2	3,1,2*	4,1,2*
1,1,3	2,1,3*	3,1,3	4,1,3*
1,1,4	2,1,4*	3,1,4*	4,1,4
1,2,1	2,2,1	3,2,1*	4,2,1*
1,2,2	**2,2,2**	3,2,2	4,2,2
1,2,3*	2,2,3	3,2,3	4,2,3*
1,2,4*	2,2,4	3,2,4*	4,2,4
1,3,1	2,3,1*	3,3,1	4,3,1*
1,3,2*	2,3,2	3,3,2	4,3,2*
1,3,3	2,3,3	**3,3,3**	4,3,3
1,3,4*	2,3,4*	3,3,4	4,3,4
1,4,1	2,4,1*	3,4,1*	4,4,1
1,4,2*	2,4,2	3,4,2*	4,4,2
1,4,3*	2,4,3*	3,4,3	4,4,3
1,4,4	2,4,4	3,4,4	**4,4,4**

$$P(\boldsymbol{r} = 1) = \frac{\binom{4}{1}\bullet(1)}{4^3} = \frac{4}{64} \qquad P(\boldsymbol{r} = 2) = \frac{\binom{4}{2}\bullet(6)}{4^3} = \frac{36}{64}$$

$$P(\boldsymbol{r^*} = 3) = \frac{\binom{4}{3}\bullet(6)}{4^3} = \frac{24}{64}$$

- s = 4

1,1,1,1	2,1,1,1*	3,1,1,1*	4,1,1,1*
1,1,1,2*	2,1,1,2*	3,1,1,2	4,1,1,2
1,1,1,3*	2,1,1,3	3,1,1,3*	4,1,1,3
1,1,1,4*	2,1,1,4	3,1,1,4	4,1,1,4*
1,1,2,1*	2,1,2,1*	3,1,2,1	4,1,2,1
1,1,2,2*	2,1,2,2*	3,1,2,2	4,1,2,2
1,1,2,3	2,1,2,3	3,1,2,3	4,1,2,3′
1,1,2,4	2,1,2,4	3,1,2,4′	4,1,2,4
1,1,3,1*	2,1,3,1	3,1,3,1*	4,1,3,1
1,1,3,2	2,1,3,2	3,1,3,2	4,1,3,2′
1,1,3,3*	2,1,3,3	3,1,3,3*	4,1,3,3
1,1,3,4	2,1,3,4′	3,1,3,4	4,1,3,4
1,1,4,1*	2,1,4,1	3,1,4,1	4,1,4,1*
1,1,4,2	2,1,4,2	3,1,4,2′	4,1,4,2
1,1,4,3	2,1,4,3′	3,1,4,3	4,1,4,3
1,1,4,4*	2,1,4,4	3,1,4,4	4,1,4,4*
1,2,1,1*	2,2,1,1*	3,2,1,1	4,2,1,1
1,2,1,2*	2,2,1,2*	3,2,1,2	4,2,1,2
1,2,1,3	2,2,1,3	3,2,1,3	4,2,1,3′
1,2,1,4	2,2,1,4	3,2,1,4′	4,2,1,4
1,2,2,1*	2,2,2,1*	3,2,2,1	4,2,2,1
1,2,2,2*	**2,2,2,2**	3,2,2,2*	4,2,2,2*
1,2,2,3	2,2,2,3*	3,2,2,3*	4,2,2,3
1,2,2,4	2,2,2,4*	3,2,2,4	4,2,2,4*
1,2,3,1	2,2,3,1	3,2,3,1	4,2,3,1′
1,2,3,2	2,2,3,2*	3,2,3,2*	4,2,3,2
1,2,3,3	2,2,3,3*	3,2,3,3*	4,2,3,3
1,2,3,4′	2,2,3,4	3,2,3,4	4,2,3,4
1,2,4,1	2,2,4,1	3,2,4,1′	4,2,4,1
1,2,4,2	2,2,4,2*	3,2,4,2	4,2,4,2*
1,2,4,3′	2,2,4,3	3,2,4,3	4,2,4,3
1,2,4,4	2,2,4,4*	3,2,4,4	4,2,4,4*
1,3,1,1*	2,3,1,1	3,3,1,1*	4,3,1,1
1,3,1,2	2,3,1,2	3,3,1,2	4,3,1,2′
1,3,1,3*	2,3,1,3	3,3,1,3*	4,3,1,3
1,3,1,4	2,3,1,4′	3,3,1,4	4,3,1,4

1,3,2,1	2,3,2,1	3,3,2,1	4,3,2,1′
1,3,2,2	2,3,2,2*	3,3,2,2*	4,3,2,2
1,3,2,3	2,3,2,3*	3,3,2,3*	4,3,2,3
1,3,2,4′	2,3,2,4	3,3,2,4	4,3,2,4
1,3,3,1*	2,3,3,1	3,3,3,1*	4,3,3,1
1,3,3,2	2,3,3,2*	3,3,3,2*	4,3,3,2
1,3,3,3*	2,3,3,3*	**3,3,3,3**	4,3,3,3*
1,3,3,4	2,3,3,4	3,3,3,4*	4,3,3,4*
1,3,4,1	2,3,4,1′	3,3,4,1	4,3,4,1
1,3,4,2′	2,3,4,2	3,3,4,2	4,3,4,2
1,3,4,3	2,3,4,3	3,3,4,3*	4,3,4,3*
1,3,4,4	2,3,4,4	3,3,4,4*	4,3,4,4*
1,4,1,1*	2,4,1,1	3,4,1,1	4,4,1,1*
1,4,1,2	2,4,1,2	3,4,1,2′	4,4,1,2
1,4,1,3	2,4,1,3′	3,4,1,3	4,4,1,3
1,4,1,4*	2,4,1,4	3,4,1,4	4,4,1,4*
1,4,2,1	2,4,2,1	3,4,2,1′	4,4,2,1
1,4,2,2	2,4,2,2*	3,4,2,2	4,4,2,2*
1,4,2,3′	2,4,2,3	3,4,2,3	4,4,2,3
1,4,2,4	2,4,2,4*	3,4,2,4	4,4,2,4*
1,4,3,1	2,4,3,1′	3,4,3,1	4,4,3,1
1,4,3,2′	2,4,3,2	3,4,3,2	4,4,3,2
1,4,3,3	2,4,3,3	3,4,3,3*	4,4,3,3*
1,4,3,4	2,4,3,4	3,4,3,4*	4,4,3,4*
1,4,4,1*	2,4,4,1	3,4,4,1	4,4,4,1*
1,4,4,2	2,4,4,2*	3,4,4,2	4,4,4,2*
1,4,4,3	2,4,4,3	3,4,4,3*	4,4,4,3*
1,4,4,4*	2,4,4,4*	3,4,4,4*	**4,4,4,4**

$$P(\boldsymbol{r} = 1) = \frac{\binom{4}{1} \bullet (1)}{4^4} = \frac{4}{256} \qquad P(\boldsymbol{r^*} = 2) = \frac{\binom{4}{2} \bullet (14)}{4^4} = \frac{84}{256}$$

$$P(\boldsymbol{r} = 3) = \frac{\binom{4}{3} \bullet (36)}{4^4} = \frac{144}{256} \qquad P(\boldsymbol{r'} = 4) = \frac{\binom{4}{4} \bullet (24)}{4^4} = \frac{24}{256}$$

- $s = 5$

1,1,1,1,1	1,2,1,1,1*	1,3,1,1,1*	1,4,1,1,1*	2,1,1,1,1*	2,2,1,1,1*	2,3,1,1,1	2,4,1,1,1
1,1,1,1,2*	1,2,1,1,2*	1,3,1,1,2	1,4,1,1,2	2,1,1,1,2*	2,2,1,1,2*	2,3,1,1,2	2,4,1,1,2
1,1,1,1,3*	1,2,1,1,3	1,3,1,1,3*	1,4,1,1,3	2,1,1,1,3	2,2,1,1,3	2,3,1,1,3	2,4,1,1,3′
1,1,1,1,4*	1,2,1,1,4	1,3,1,1,4	1,4,1,1,4*	2,1,1,1,4	2,2,1,1,4	2,3,1,1,4′	2,4,1,1,4
1,1,1,2,1*	1,2,1,2,1*	1,3,1,2,1	1,4,1,2,1	2,1,1,2,1*	2,2,1,2,1*	2,3,1,2,1	2,4,1,2,1
1,1,1,2,2*	1,2,1,2,2*	1,3,1,2,2	1,4,1,2,2	2,1,1,2,2*	2,2,1,2,2*	2,3,1,2,2	2,4,1,2,2
1,1,1,2,3	1,2,1,2,3	1,3,1,2,3	1,4,1,2,3′	2,1,1,2,3	2,2,1,2,3	2,3,1,2,3	2,4,1,2,3′
1,1,1,2,4	1,2,1,2,4	1,3,1,2,4′	1,4,1,2,4	2,1,1,2,4	2,2,1,2,4	2,3,1,2,4′	2,4,1,2,4
1,1,1,3,1*	1,2,1,3,1	1,3,1,3,1*	1,4,1,3,1	2,1,1,3,1	2,2,1,3,1	2,3,1,3,1	2,4,1,3,1′
1,1,1,3,2	1,2,1,3,2	1,3,1,3,2	1,4,1,3,2′	2,1,1,3,2	2,2,1,3,2	2,3,1,3,2	2,4,1,3,2′
1,1,1,3,3*	1,2,1,3,3	1,3,1,3,3*	1,4,1,3,3	2,1,1,3,3	2,2,1,3,3	2,3,1,3,3	2,4,1,3,3′
1,1,1,3,4	1,2,1,3,4′	1,3,1,3,4	1,4,1,3,4	2,1,1,3,4′	2,2,1,3,4′	2,3,1,3,4′	2,4,1,3,4′
1,1,1,4,1*	1,2,1,4,1	1,3,1,4,1	1,4,1,4,1*	2,1,1,4,1	2,2,1,4,1	2,3,1,4,1′	2,4,1,4,1
1,1,1,4,2	1,2,1,4,2	1,3,1,4,2′	1,4,1,4,2	2,1,1,4,2	2,2,1,4,2	2,3,1,4,2′	2,4,1,4,2
1,1,1,4,3	1,2,1,4,3′	1,3,1,4,3	1,4,1,4,3	2,1,1,4,3′	2,2,1,4,3′	2,3,1,4,3′	2,4,1,4,3′
1,1,1,4,4*	1,2,1,4,4	1,3,1,4,4	1,4,1,4,4*	2,1,1,4,4	2,2,1,4,4	2,3,1,4,4′	2,4,1,4,4
1,1,2,1,1*	1,2,2,1,1*	1,3,2,1,1	1,4,2,1,1	2,1,2,1,1*	2,2,2,1,1*	2,3,2,1,1	2,4,2,1,1
1,1,2,1,2*	1,2,2,1,2*	1,3,2,1,2	1,4,2,1,2	2,1,2,1,2*	2,2,2,1,2*	2,3,2,1,2	2,4,2,1,2
1,1,2,1,3	1,2,2,1,3	1,3,2,1,3	1,4,2,1,3′	2,1,2,1,3	2,2,2,1,3	2,3,2,1,3	2,4,2,1,3′
1,1,2,1,4	1,2,2,1,4	1,3,2,1,4′	1,4,2,1,4	2,1,2,1,4	2,2,2,1,4	2,3,2,1,4′	2,4,2,1,4
1,1,2,2,1*	1,2,2,2,1*	1,3,2,2,1	1,4,2,2,1	2,1,2,2,1*	2,2,2,2,1*	2,3,2,2,1	2,4,2,2,1
1,1,2,2,2*	1,2,2,2,2*	1,3,2,2,2	1,4,2,2,2	2,1,2,2,2*	**2,2,2,2,2**	2,3,2,2,2*	2,4,2,2,2*
1,1,2,2,3	1,2,2,2,3	1,3,2,2,3	1,4,2,2,3′	2,1,2,2,3	2,2,2,2,3*	2,3,2,2,3*	2,4,2,2,3
1,1,2,2,4	1,2,2,2,4	1,3,2,2,4′	1,4,2,2,4	2,1,2,2,4	2,2,2,2,4*	2,3,2,2,4	2,4,2,2,4*
1,1,2,3,1	1,2,2,3,1	1,3,2,3,1	1,4,2,3,1′	2,1,2,3,1	2,2,2,3,1	2,3,2,3,1	2,4,2,3,1′
1,1,2,3,2	1,2,2,3,2	1,3,2,3,2	1,4,2,3,2′	2,1,2,3,2	2,2,2,3,2*	2,3,2,3,2*	2,4,2,3,2
1,1,2,3,3	1,2,2,3,3	1,3,2,3,3	1,4,2,3,3′	2,1,2,3,3	2,2,2,3,3*	2,3,2,3,3*	2,4,2,3,3
1,1,2,3,4′	1,2,2,3,4′	1,3,2,3,4′	1,4,2,3,4′	2,1,2,3,4′	2,2,2,3,4	2,3,2,3,4	2,4,2,3,4
1,1,2,4,1	1,2,2,4,1	1,3,2,4,1′	1,4,2,4,1	2,1,2,4,1	2,2,2,4,1	2,3,2,4,1′	2,4,2,4,1
1,1,2,4,2	1,2,2,4,2	1,3,2,4,2′	1,4,2,4,2	2,1,2,4,2	2,2,2,4,2*	2,3,2,4,2	2,4,2,4,2*
1,1,2,4,3′	1,2,2,4,3′	1,3,2,4,3′	1,4,2,4,3′	2,1,2,4,3′	2,2,2,4,3	2,3,2,4,3	2,4,2,4,3
1,1,2,4,4	1,2,2,4,4	1,3,2,4,4′	1,4,2,4,4	2,1,2,4,4	2,2,2,4,4*	2,3,2,4,4	2,4,2,4,4*
1,1,3,1,1*	1,2,3,1,1	1,3,3,1,1*	1,4,3,1,1	2,1,3,1,1	2,2,3,1,1	2,3,3,1,1	2,4,3,1,1′
1,1,3,1,2	1,2,3,1,2	1,3,3,1,2	1,4,3,1,2′	2,1,3,1,2	2,2,3,1,2	2,3,3,1,2	2,4,3,1,2′

1,1,3,1,3*	1,2,3,1,3	1,3,3,1,3*	1,4,3,1,3	2,1,3,1,3	2,2,3,1,3	2,3,3,1,3	2 ,4,3,1,3′
1,1,3,1,4	1,2,3,1,4′	1,3,3,1,4	1,4,3,1,4	2,1,3,1,4′	2,2,3,1,4′	2,3,3,1,4′	2,4,3,1,4′
1,1,3,2,1	1,2,3,2,1	1,3,3,2,1	1,4,3,2,1′	2,1,3,2,1	2,2,3,2,1	2,3,3,2,1	2,4,3,2,1′
1,1,3,2,2	1,2,3,2,2	1,3,3,2,2	1,4,3,2,2′	2,1,3,2,2	2,2,3,2,2*	2,3,3,2,2*	2,4,3,2,2
1,1,3,2,3	1,2,3,2,3	1,3,3,2,3	1,4,3,2,3′	2,1,3,2,3	2,2,3,2,3*	2,3,3,2,3*	2,4,3,2,3
1,1,3,2,4′	1,2,3,2,4′	1,3,3,2,4′	1,4,3,2,4′	2,1,3,2,4′	2,2,3,2,4	2,3,3,2,4	2,4,3,2,4
1,1,3,3,1*	1,2,3,3,1	1,3,3,3,1*	1,4,3,3,1	2,1,3,3,1	2,2,3,3,1	2,3,3,3,1	2,4,3,3,1′
1,1,3,3,2	1,2,3,3,2	1,3,3,3,2	1,4,3,3,2′	2,1,3,3,2	2,2,3,3,2*	2,3,3,3,2*	2,4,3,3,2
1,1,3,3,3*	1,2,3,3,3	1,3,3,3,3*	1,4,3,3,3	2,1,3,3,3	2,2,3,3,3*	2,3,3,3,3*	2,4,3,3,3
1,1,3,3,4	1,2,3,3,4′	1,3,3,3,4	1,4,3,3,4	2,1,3,3,4′	2,2,3,3,4	2,3,3,3,4	2,4,3,3,4
1,1,3,4,1	1,2,3,4,1′	1,3,3,4,1	1,4,3,4,1	2,1,3,4,1′	2,2,3,4,1′	2,3,3,4,1′	2,4,3,4,1′
1,1,3,4,2′	1,2,3,4,2′	1,3,3,4,2′	1,4,3,4,2′	2,1,3,4,2′	2,2,3,4,2	2,3,3,4,2	2,4,3,4,2
1,1,3,4,3	1,2,3,4,3′	1,3,3,4,3	1,4,3,4,3	2,1,3,4,3′	2,2,3,4,3	2,3,3,4,3	2,4,3,4,3
1,1,3,4,4	1,2,3,4,4′	1,3,3,4,4	1,4,3,4,4	2,1,3,4,4′	2,2,3,4,4	2,3,3,4,4	2,4,3,4,4
1,1,4,1,1*	1,2,4,1,1	1,3,4,1,1	1,4,4,1,1*	2,1,4,1,1	2,2,4,1,1	2,3,4,1,1′	2,4,4,1,1
1,1,4,1,2	1,2,4,1,2	1,3,4,1,2′	1,4,4,1,2	2,1,4,1,2	2,2,4,1,2	2,3,4,1,2′	2,4,4,1,2
1,1,4,1,3	1,2,4,1,3′	1,3,4,1,3	1,4,4,1,3	2,1,4,1,3′	2,2,4,1,3′	2,3,4,1,3′	2,4,4,1,3′
1,1,4,1,4*	1,2,4,1,4	1,3,4,1,4	1,4,4,1,4*	2,1,4,1,4	2,2,4,1,4	2,3,4,1,4′	2,4,4,1,4
1,1,4,2,1	1,2,4,2,1	1,3,4,2,1′	1,4,4,2,1	2,1,4,2,1	2,2,4,2,1	2,3,4,2,1′	2,4,4,2,1
1,1,4,2,2	1,2,4,2,2	1,3,4,2,2′	1,4,4,2,2	2,1,4,2,2	2,2,4,2,2*	2,3,4,2,2	2,4,4,2,2*
1,1,4,2,3′	1,2,4,2,3′	1,3,4,2,3′	1,4,4,2,3′	2,1,4,2,3′	2,2,4,2,3	2,3,4,2,3	2,4,4,2,3
1,1,4,2,4	1,2,4,2,4	1,3,4,2,4′	1,4,4,2,4	2,1,4,2,4	2,2,4,2,4*	2,3,4,2,4	2,4,4,2,4*
1,1,4,3,1	1,2,4,3,1′	1,3,4,3,1	1,4,4,3,1	2,1,4,3,1′	2,2,4,3,1′	2,3,4,3,1′	2,4,4,3,1′
1,1,4,3,2′	1,2,4,3,2′	1,3,4,3,2′	1,4,4,3,2′	2,1,4,3,2′	2,2,4,3,2	2,3,4,3,2	2,4,4,3,2
1,1,4,3,3	1,2,4,3,3′	1,3,4,3,3	1,4,4,3,3	2,1,4,3,3′	2,2,4,3,3	2,3,4,3,3	2,4,4,3,3
1,1,4,3,4	1,2,4,3,4′	1,3,4,3,4	1,4,4,3,4	2,1,4,3,4′	2,2,4,3,4	2,3,4,3,4	2,4,4,3,4
1,1,4,4,1*	1,2,4,4,1	1,3,4,4,1	1,4,4,4,1*	2,1,4,4,1	2,2,4,4,1	2,3,4,4,1′	2,4,4,4,1
1,1,4,4,2	1,2,4,4,2	1,3,4,4,2′	1,4,4,4,2	2,1,4,4,2	2,2,4,4,2*	2,3,4,4,2	2,4,4,4,2*
1,1,4,4,3	1,2,4,4,3′	1,3,4,4,3	1,4,4,4,3	2,1,4,4,3′	2,2,4,4,3	2,3,4,4,3	2,4,4,4,3
1,1,4,4,4*	1,2,4,4,4	1,3,4,4,4	1,4,4,4,4*	2,1,4,4,4	2,2,4,4,4*	2,3,4,4,4	2,4,4,4,4*
3,1,1,1,1*	3,2,1,1,1	3,3,1,1,1*	3,4,1,1,1	4,1,1,1,1*	4,2,1,1,1	4,3,1,1,1	4,4,1,1,1*
3,1,1,1,2	3,2,1,1,2	3,3,1,1,2	3,4,1,1,2′	4,1,1,1,2	4,2,1,1,2	4,3,1,1,2′	4,4,1,1,2
3,1,1,1,3*	3,2,1,1,3	3,3,1,1,3*	3,4,1,1,3	4,1,1,1,3	4,2,1,1,3′	4,3,1,1,3	4,4,1,1,3
3,1,1,1,4	3,2,1,1,4′	3,3,1,1,4	3,4,1,1,4	4,1,1,1,4*	4,2,1,1,4	4,3,1,1,4	4,4,1,1,4*
3,1,1,2,1	3,2,1,2,1	3,3,1,2,1	3,4,1,2,1′	4,1,1,2,1	4,2,1,2,1	4,3,1,2,1′	4,4,1,2,1
3,1,1,2,2	3,2,1,2,2	3,3,1,2,2	3,4,1,2,2′	4,1,1,2,2	4,2,1,2,2	4,3,1,2,2′	4,4,1,2,2

3,1,1,2,3	3,2,1,2,3	3,3,1,2,3	3,4,1,2,3′	4,1,1,2,3′	4,2,1,2,3′	4,3,1,2,3′	4,4,1,2,3′
3,1,1,2,4′	3,2,1,2,4′	3,3,1,2,4′	3,4,1,2,4′	4,1,1,2,4	4,2,1,2,4	4,3,1,2,4′	4,4,1,2,4
3,1,1,3,1*	3,2,1,3,1	3,3,1,3,1*	3,4,1,3,1	4,1,1,3,1	4,2,1,3,1′	4,3,1,3,1	4,4,1,3,1
3,1,1,3,2	3,2,1,3,2	3,3,1,3,2	3,4,1,3,2′	4,1,1,3,2′	4,2,1,3,2′	4,3,1,3,2′	4,4,1,3,2′
3,1,1,3,3*	3,2,1,3,3	3,3,1,3,3*	3,4,1,3,3	4,1,1,3,3	4,2,1,3,3′	4,3,1,3,3	4,4,1,3,3
3,1,1,3,4	3,2,1,3,4′	3,3,1,3,4	3,4,1,3,4	4,1,1,3,4	4,2,1,3,4′	4,3,1,3,4	4,4,1,3,4
3,1,1,4,1	3,2,1,4,1′	3,3,1,4,1	3,4,1,4,1	4,1,1,4,1*	4,2,1,4,1	4,3,1,4,1	4,4,1,4,1*
3,1,1,4,2′	3,2,1,4,2′	3,3,1,4,2′	3,4,1,4,2′	4,1,1,4,2	4,2,1,4,2	4,3,1,4,2′	4,4,1,4,2
3,1,1,4,3	3,2,1,4,3′	3,3,1,4,3	3,4,1,4,3	4,1,1,4,3	4,2,1,4,3′	4,3,1,4,3	4,4,1,4,3
3,1,1,4,4	3,2,1,4,4′	3,3,1,4,4	3,4,1,4,4	4,1,1,4,4*	4,2,1,4,4	4,3,1,4,4	4,4,1,4,4*
3,1,2,1,1	3,2,2,1,1	3,3,2,1,1	3,4,2,1,1′	4,1,2,1,1	4,2,2,1,1	4,3,2,1,1′	4,4,2,1,1
3,1,2,1,2	3,2,2,1,2	3,3,2,1,2	3,4,2,1,2′	4,1,2,1,2	4,2,2,1,2	4,3,2,1,2′	4,4,2,1,2
3,1,2,1,3	3,2,2,1,3	3,3,2,1,3	3,4,2,1,3′	4,1,2,1,3′	4,2,2,1,3′	4,3,2,1,3′	4,4,2,1,3′
3,1,2,1,4′	3,2,2,1,4′	3,3,2,1,4′	3,4,2,1,4′	4,1,2,1,4	4,2,2,1,4	4,3,2,1,4′	4,4,2,1,4
3,1,2,2,1	3,2,2,2,1	3,3,2,2,1	3,4,2,2,1′	4,1,2,2,1	4,2,2,2,1	4,3,2,2,1′	4,4,2,2,1
3,1,2,2,2	3,2,2,2,2*	3,3,2,2,2*	3,4,2,2,2	4,1,2,2,2	4,2,2,2,2*	4,3,2,2,2	4,4,2,2,2*
3,1,2,2,3	3,2,2,2,3*	3,3,2,2,3*	3,4,2,2,3	4,1,2,2,3′	4,2,2,2,3	4,3,2,2,3	4,4,2,2,3
3,1,2,2,4′	3,2,2,2,4	3,3,2,2,4	3,4,2,2,4	4,1,2,2,4	4,2,2,2,4*	4,3,2,2,4	4,4,2,2,4*
3,1,2,3,1	3,2,2,3,1	3,3,2,3,1	3,4,2,3,1′	4,1,2,3,1′	4,2,2,3,1′	4,3,2,3,1′	4,4,2,3,1′
3,1,2,3,2	3,2,2,3,2*	3,3,2,3,2*	3,4,2,3,2	4,1,2,3,2′	4,2,2,3,2	4,3,2,3,2	4,4,2,3,2
3,1,2,3,3	3,2,2,3,3*	3,3,2,3,3*	3,4,2,3,3	4,1,2,3,3′	4,2,2,3,3	4,3,2,3,3	4,4,2,3,3
3,1,2,3,4′	3,2,2,3,4	3,3,2,3,4	3,4,2,3,4	4,1,2,3,4′	4,2,2,3,4	4,3,2,3,4	4,4,2,3,4
3,1,2,4,1′	3,2,2,4,1′	3,3,2,4,1′	3,4,2,4,1′	4,1,2,4,1	4,2,2,4,1	4,3,2,4,1′	4,4,2,4,1
3,1,2,4,2′	3,2,2,4,2	3,3,2,4,2	3,4,2,4,2	4,1,2,4,2	4,2,2,4,2*	4,3,2,4,2	4,4,2,4,2*
3,1,2,4,3′	3,2,2,4,3	3,3,2,4,3	3,4,2,4,3	4,1,2,4,3′	4,2,2,4,3	4,3,2,4,3	4,4,2,4,3
3,1,2,4,4′	3,2,2,4,4	3,3,2,4,4	3,4,2,4,4	4,1,2,4,4	4,2,2,4,4*	4,3,2,4,4	4,4,2,4,4*
3,1,3,1,1*	3,2,3,1,1	3,3,3,1,1*	3,4,3,1,1	4,1,3,1,1	4,2,3,1,1′	4,3,3,1,1	4,4,3,1,1
3,1,3,1,2	3,2,3,1,2	3,3,3,1,2	3,4,3,1,2′	4,1,3,1,2′	4,2,3,1,2′	4,3,3,1,2′	4,4,3,1,2′
3,1,3,1,3*	3,2,3,1,3	3,3,3,1,3*	3,4,3,1,3	4,1,3,1,3	4,2,3,1,3′	4,3,3,1,3	4,4,3,1,3
3,1,3,1,4	3,2,3,1,4′	3,3,3,1,4	3,4,3,1,4	4,1,3,1,4	4,2,3,1,4′	4,3,3,1,4	4,4,3,1,4
3,1,3,2,1	3,2,3,2,1	3,3,3,2,1	3,4,3,2,1′	4,1,3,2,1′	4,2,3,2,1′	4,3,3,2,1′	4,4,3,2,1′
3,1,3,2,2	3,2,3,2,2*	3,3,3,2,2*	3,4,3,2,2	4,1,3,2,2′	4,2,3,2,2	4,3,3,2,2	4,4,3,2,2
3,1,3,2,3	3,2,3,2,3*	3,3,3,2,3*	3,4,3,2,3	4,1,3,2,3′	4,2,3,2,3	4,3,3,2,3	4,4,3,2,3
3,1,3,2,4′	3,2,3,2,4	3,3,3,2,4	3,4,3,2,4	4,1,3,2,4′	4,2,3,2,4	4,3,3,2,4	4,4,3,2,4
3,1,3,3,1*	3,2,3,3,1	3,3,3,3,1*	3,4,3,3,1	4,1,3,3,1	4,2,3,3,1′	4,3,3,3,1	4,4,3,3,1
3,1,3,3,2	3,2,3,3,2*	3,3,3,3,2*	3,4,3,3,2	4,1,3,3,2′	4,2,3,3,2	4,3,3,3,2	4,4,3,3,2

3,1,3,3,3*	3,2,3,3,3*	**3,3,3,3,3**	3,4,3,3,3*	4,1,3,3,3	4,2,3,3,3	4,3,3,3,3*	4,4,3,3,3*
3,1,3,3,4	3,2,3,3,4	3,3,3,3,4*	3,4,3,3,4*	4,1,3,3,4	4,2,3,3,4	4,3,3,3,4*	4,4,3,3,4*
3,1,3,4,1	3,2,3,4,1′	3,3,3,4,1	3,4,3,4,1	4,1,3,4,1	4,2,3,4,1′	4,3,3,4,1	4,4,3,4,1
3,1,3,4,2′	3,2,3,4,2	3,3,3,4,2	3,4,3,4,2	4,1,3,4,2′	4,2,3,4,2	4,3,3,4,2	4,4,3,4,2
3,1,3,4,3	3,2,3,4,3	3,3,3,4,3*	3,4,3,4,3*	4,1,3,4,3	4,2,3,4,3	4,3,3,4,3*	4,4,3,4,3*
3,1,3,4,4	3,2,3,4,4	3,3,3,4,4*	3,4,3,4,4*	4,1,3,4,4	4,2,3,4,4	4,3,3,4,4*	4,4,3,4,4*
3,1,4,1,1	3,2,4,1,1′	3,3,4,1,1	3,4,4,1,1	4,1,4,1,1*	4,2,4,1,1	4,3,4,1,1	4,4,4,1,1*
3,1,4,1,2′	3,2,4,1,2′	3,3,4,1,2′	3,4,4,1,2′	4,1,4,1,2	4,2,4,1,2	4,3,4,1,2′	4,4,4,1,2
3,1,4,1,3	3,2,4,1,3′	3,3,4,1,3	3,4,4,1,3	4,1,4,1,3	4,2,4,1,3′	4,3,4,1,3	4,4,4,1,3
3,1,4,1,4	3,2,4,1,4′	3,3,4,1,4	3,4,4,1,4	4,1,4,1,4*	4,2,4,1,4	4,3,4,1,4	4,4,4,1,4*
3,1,4,2,1′	3,2,4,2,1′	3,3,4,2,1′	3,4,4,2,1′	4,1,4,2,1	4,2,4,2,1	4,3,4,2,1′	4,4,4,2,1
3,1,4,2,2′	3,2,4,2,2	3,3,4,2,2	3,4,4,2,2	4,1,4,2,2	4,2,4,2,2*	4,3,4,2,2	4,4,4,2,2*
3,1,4,2,3′	3,2,4,2,3	3,3,4,2,3	3,4,4,2,3	4,1,4,2,3′	4,2,4,2,3	4,3,4,2,3	4,4,4,2,3
3,1,4,2,4′	3,2,4,2,4	3,3,4,2,4	3,4,4,2,4	4,1,4,2,4	4,2,4,2,4*	4,3,4,2,4	4,4,4,2,4*
3,1,4,3,1	3,2,4,3,1′	3,3,4,3,1	3,4,4,3,1	4,1,4,3,1	4,2,4,3,1′	4,3,4,3,1	4,4,4,3,1
3,1,4,3,2′	3,2,4,3,2	3,3,4,3,2	3,4,4,3,2	4,1,4,3,2′	4,2,4,3,2	4,3,4,3,2	4,4,4,3,2
3,1,4,3,3	3,2,4,3,3	3,3,4,3,3*	3,4,4,3,3*	4,1,4,3,3	4,2,4,3,3	4,3,4,3,3*	4,4,4,3,3*
3,1,4,3,4	3,2,4,3,4	3,3,4,3,4*	3,4,4,3,4*	4,1,4,3,4	4,2,4,3,4	4,3,4,3,4*	4,4,4,3,4*
3,1,4,4,1	3,2,4,4,1′	3,3,4,4,1	3,4,4,4,1	4,1,4,4,1*	4,2,4,4,1	4,3,4,4,1	4,4,4,4,1*
3,1,4,4,2′	3,2,4,4,2	3,3,4,4,2	3,4,4,4,2	4,1,4,4,2	4,2,4,4,2*	4,3,4,4,2	4,4,4,4,2*
3,1,4,4,3	3,2,4,4,3	3,3,4,4,3*	3,4,4,4,3*	4,1,4,4,3	4,2,4,4,3	4,3,4,4,3*	4,4,4,4,3*
3,1,4,4,4	3,2,4,4,4	3,3,4,4,4*	3,4,4,4,4*	4,1,4,4,4*	4,2,4,4,4*	4,3,4,4,4*	**4,4,4,4,4**

$$P(\boldsymbol{r} = 1) = \frac{\binom{4}{1}\bullet(1)}{4^5} = \frac{4}{1024} \qquad P(\boldsymbol{r^*} = 2) = \frac{\binom{4}{2}\bullet(30)}{4^5} = \frac{180}{1024}$$

$$P(\boldsymbol{r} = 3) = \frac{\binom{4}{3}\bullet(150)}{4^5} = \frac{600}{1024} \qquad P(\boldsymbol{r'} = 4) = \frac{\binom{4}{4}\bullet(240)}{4^5} = \frac{240}{1024}$$

3.5 Illustration Summary Chart

	$C = 2$		$C = 3$			$C = 4$			
	r		r			r			
	1	**2**	**1**	**2**	**3**	**1**	**2**	**3**	**4**
$s= 2$	A=1	A=2	A=1	A=2	A=0	A=1	A=2	A=0	A=0
$s= 3$	A=1	A=6	A=1	A=6	A=6	A=1	A=6	A=6	A=0
$s= 4$	A=1	A=14	A=1	A=14	A= 36	A=1	A= 14	A=36	A=24
$s= 5$	A=1	A=30	A=1	A=30	A=150	A=1	A=30	A=150	A=240

From the above illustration and its summary chart, we can definitely see several clear patterns for A.

3.6 Pattern Outlines for $\boldsymbol{A}$

- $\boldsymbol{A} = 1$, when $\boldsymbol{r} = 1$. This is an intuitive result. Since we want the outcome to have the same element in every one of its positions, and we know that no matter how you arrange the identical elements over and over again, you will always get one and only one outcome.
- $\boldsymbol{A} = \boldsymbol{s}!$, when $\boldsymbol{r} = \boldsymbol{s}$. Since we have $\boldsymbol{s}$ positions to fill and we want each of the $\boldsymbol{s}$ positions to have an unique element, according to the rule of permutations, we would have $\boldsymbol{s}$ ways to fill the first position, $\boldsymbol{s} - 1$ ways to fill the second position, $\boldsymbol{s} - 2$ ways to fill the third position, and this process continues until all of the $\boldsymbol{s}$ positions are filled. Therefore, we will ultimately end up with a total of $\boldsymbol{s}!$ ways.

- The value of $\boldsymbol{A}$ principally depends on the values of $\boldsymbol{r}$ and $\boldsymbol{s}$, although $1 \leq \boldsymbol{r} \leq \min(\boldsymbol{C}, \boldsymbol{s})$. This result is expected, because the value of $\boldsymbol{C}$ only determines the outcomes of the combinations and once a certain combination is determined, all we need is to figure out the ways of arranging all the elements for this particular combination. Therefore, the process of determining number of arrangements principally depends on the given values of $\boldsymbol{r}$ and $\boldsymbol{s}$.

We are now able to determine the value of $\boldsymbol{A}$ for $\boldsymbol{r} = 1$ and $\boldsymbol{r} = \boldsymbol{s}$, the following analysis will demonstrate techniques to compute the value of $\boldsymbol{A}$ for $1 < \boldsymbol{r} < \min(\boldsymbol{C}, \boldsymbol{s})$.

3.7 Determine Value of $\boldsymbol{A}$ for $1 < \boldsymbol{r} < \min(\boldsymbol{C}, \boldsymbol{s})$

We will again start with simple cases. For $\boldsymbol{s} = 2$, $\boldsymbol{r} = 2$, we have

1,2
2,1

To find $\boldsymbol{s} = 3$, $\boldsymbol{r} = 2$, we would add one additional slot in front of all outcomes for $\boldsymbol{s} = 2$, $\boldsymbol{r} = 2$. Since we have two possible choices, we can either fill the additional slot with 1 or with 2, we will have

1,1,2	2,1,2
1,2,1	2,2,1

Notice that there are two possible outcomes left out from the above list: (1, 2, 2) and (2, 1, 1). In the case of $\boldsymbol{s} = 3$, $\boldsymbol{r} = 2$, two identical elements are allowed, but it is not tolerable for $\boldsymbol{s} = 2$, $\boldsymbol{r} = 2$. Therefore, the final result is

1,1,2	2,1,2
1,2,1	2,2,1
1,2,2	2,1,1

The key point here is that the two outcomes with identical elements: (1, 1) and (2, 2), are actually the outcomes for $\boldsymbol{s} = 2$, $\boldsymbol{r} = 1$. Since the value of $\boldsymbol{A}$ only depends on the values of $\boldsymbol{s}$ and $\boldsymbol{r}$, let's denote $\boldsymbol{A}$ by ${}_{s}A_{r}$.. We would have ${}_{3}A_{2} = 2 \bullet (2 + 1) = 6$. We multiply by two, because $\boldsymbol{r} = 2$, implying there are two options to fill the one additional slot. We use 2 + 1, because for each of the options (in this case, either 1 or 2), the number of arrangements for $\boldsymbol{s} = 3$, $\boldsymbol{r} = 2$ equals the number of arrangements for $\boldsymbol{s} = 2$, $\boldsymbol{r} = 2$ plus the number of arrangements for $\boldsymbol{s} = 2$, $\boldsymbol{r} = 1$.

What about $\boldsymbol{s} = 4$, $\boldsymbol{r} = 2$? Following the same approach above, we would add one additional slot in front of all outcomes for $\boldsymbol{s} = 3$, $\boldsymbol{r} = 2$. Since we again have two possible choices, we can either fill the additional slot with 1 or with 2, we will have

1,1,1,2	2,1,1,2
1,1,2,1	2,1,2,1
1,1,2,2	2,1,2,2
1,2,1,2	2,2,1,2
1,2,2,1	2,2,2,1
1,2,1,1	2,2,1,1

Notice again that there are two possible outcomes are left out from the above list: (1, 2, 2, 2) and (2, 1, 1, 1). In the case of $\boldsymbol{s} = 4$, $\boldsymbol{r} = 2$, three identical elements are allowed, but

it is not tolerable for $\boldsymbol{s}$ = 3, $\boldsymbol{r}$ = 2. Therefore, the final result is

1,1,1,2	2,1,1,2
1,1,2,1	2,1,2,1
1,1,2,2	2,1,2,2
1,2,1,2	2,2,1,2
1,2,2,1	2,2,2,1
1,2,1,1	2,2,1,1
1,2,2,2	2,1,1,1

We have ${}_4\boldsymbol{A}_2 = 2 \bullet (6 + 1) = 14$. By multiplying by two, we take care of the options for filling up the one additional slot; 6 + 1 states the number of arrangements for $\boldsymbol{s}$ = 4, $\boldsymbol{r}$ = 2 equals the number of arrangements for $\boldsymbol{s}$ = 3, $\boldsymbol{r}$ = 2 plus the number of arrangements for $\boldsymbol{s}$ = 3, $\boldsymbol{r}$ = 1 under each option.

Applying the same technique for $\boldsymbol{s}$ = 5, $\boldsymbol{r}$ = 2, we would get outcomes

1,1,1,1,2	2,1,1,1,2
1,1,1,2,1	2,1,1,2,1
1,1,1,2,2	2,1,1,2,2
1,1,2,1,2	2,1,2,1,2
1,1,2,2,1	2,1,2,2,1
1,1,2,1,1	2,1,2,1,1
1,1,2,2,2	2,1,2,2,2
1,2,1,1,2	2,2,1,1,2
1,2,1,2,1	2,2,1,2,1
1,2,1,2,2	2,2,1,2,2
1,2,2,1,2	2,2,2,1,2
1,2,2,2,1	2,2,2,2,1
1,2,2,1,1	2,2,2,1,1
1,2,1,1,1	2,2,1,1,1
1,2,2,2,2	2,1,1,1,1

Then ${}_{5}\boldsymbol{A}_{2} = 2 \bullet (14 + 1) = 30$, since within each combination, there are 14 ways of arrangement for $\boldsymbol{s} = 4$, $\boldsymbol{r} = 2$ plus an additional one for $\boldsymbol{s} = 4$, $\boldsymbol{r} = 1$.

Now, we can write a general formula for $1 < \boldsymbol{r} < \min(\boldsymbol{C}, \boldsymbol{s})$, that is

$${}_{s}\boldsymbol{A}_{r.} = \boldsymbol{r} \bullet ({}_{s-1}\boldsymbol{A}_{r} + {}_{s-1}\boldsymbol{A}_{r-1}).$$

Note: Appendix D proofs mathematically that the above formula works in general.

3.8 Formula Validation

We will pick several cases to validate the appropriateness of the above formula for $1 < \boldsymbol{r} < \min(\boldsymbol{C}, \boldsymbol{s})$.

- $\boldsymbol{s} = 4$, $\boldsymbol{r} = 3$. According to the formula, we would have ${}_{4}\boldsymbol{A}_{3} = 3 \bullet ({}_{3}\boldsymbol{A}_{3} + {}_{3}\boldsymbol{A}_{2})$.

  To solve ${}_{3}\boldsymbol{A}_{3}$, we know that since $\boldsymbol{r} = \boldsymbol{s} = 3$, ${}_{3}\boldsymbol{A}_{3} = \boldsymbol{s}! = 3! = 6$. We have already computed the value for ${}_{3}\boldsymbol{A}_{2} = 6$ from the above illustrations. Therefore, ${}_{4}\boldsymbol{A}_{3} = 3 \bullet (6 + 6) = 36$, which matches the result on page twenty. To have a sense of what really is going on with this technique, here is the explanation.

  For ${}_{3}\boldsymbol{A}_{3}$, we have

1,2,3
1,3,2
2,1,3
2,3,1
3,1,2
3,2,1

To find $\boldsymbol{s} = 4$, $\boldsymbol{r} = 3$, we would add one additional slot in front of all outcomes for $\boldsymbol{s} = 3$, $\boldsymbol{r} = 3$. And since we have three possible choices, we can either fill the addition slot with 1, 2 or 3, we will have

1,1,2,3	2,1,2,3	3,1,2,3
1,1,3,2	2,1,3,2	3,1,3,2
1,2,1,3	2,2,1,3	3,2,1,3
1,2,3,1	2,2,3,1	3,2,3,1
1,3,1,2	2,3,1,2	3,3,1,2
1,3,2,1	2,3,2,1	3,3,2,1

However, there are six possible outcomes left out for each of the options from the above list. For instance, if we place 1 in the slot, the six missing outcomes are (1, 2, 2, 3), (1, 2, 3, 2), (1, 2, 3, 3), (1, 3, 2, 3), (1, 3, 3, 2) and (1, 3, 2, 2). In the case of $\boldsymbol{s} = 4$, $\boldsymbol{r} = 3$, two identical elements are allowed, but it is not tolerable for $\boldsymbol{s} = 3$, $\boldsymbol{r} = 3$. Therefore, the final result is

1,1,2,3	2,1,2,3	3,1,2,3
1,1,3,2	2,1,3,2	3,1,3,2
1,2,1,3	2,2,1,3	3,2,1,3
1,2,3,1	2,2,3,1	3,2,3,1
1,3,1,2	2,3,1,2	3,3,1,2
1,3,2,1	2,3,2,1	3,3,2,1
1,2,2,3	2,1,1,3	3,1,1,2
1,2,3,2	2,1,3,1	3,1,2,1
1,2,3,3	2,1,3,3	3,1,2,2
1,3,2,2	2,3,1,1	3,2,1,1
1,3,2,3	2,3,1,3	3,2,1,2
1,3,3,2	2,3,3,1	3,2,2,1

By looking at the eighteen outcomes that have two identical elements,

2,2,3	1,1,3	1,1,2
2,3,2	1,3,1	1,2,1
2,3,3	1,3,3	1,2,2
3,2,2	3,1,1	2,1,1
3,2,3	3,1,3	2,1,2
3,3,2	3,3,1	2,2,1

as we expected, these are the outcomes for $\boldsymbol{s} = 3$, $\boldsymbol{r} = 2$. This further demonstrates the correctness of the formula.

- $\boldsymbol{s} = 5$, $\boldsymbol{r} = 4$. According to the formula, ${}_5\boldsymbol{A}_4 = 4 \bullet ({}_4\boldsymbol{A}_4 + {}_4\boldsymbol{A}_3)$. Since ${}_4\boldsymbol{A}_4 = 4! = 24$ and ${}_4\boldsymbol{A}_3 = 36$, ${}_5\boldsymbol{A}_4 = 4 \bullet (24 + 36) = 240$, which matches the result on page 28.
- It turns out that the formula is also appropriate for $\boldsymbol{r} = 1$ and $\boldsymbol{r} = \boldsymbol{s}$. For instance, according to the formula, ${}_5\boldsymbol{A}_1 = 1 \bullet ({}_4\boldsymbol{A}_1 + {}_4\boldsymbol{A}_0)$. We know that ${}_4\boldsymbol{A}_1 = 1$ and outcome for ${}_4\boldsymbol{A}_0$ does not exist (i.e., there is zero outcome), therefore, ${}_5\boldsymbol{A}_1 = 1 \bullet (1 + 0) = 1$. In the case of $\boldsymbol{r} = \boldsymbol{s}$, for instance, ${}_3\boldsymbol{A}_3 = 3 \bullet ({}_2\boldsymbol{A}_3 + {}_2\boldsymbol{A}_2)$. Since ${}_2\boldsymbol{A}_3$ does not exist (i.e., there is zero outcome) and we know that ${}_2\boldsymbol{A}_2 = 2$, ${}_3\boldsymbol{A}_3 = 3 \bullet (0 + 2) = 6$.

CHAPTER 4

Conclusion

We now conclude that if we have:

$\boldsymbol{C}$ ordered possible elements that are equally likely

$\boldsymbol{s}$ independent sampled subjects

$\boldsymbol{r}$ distinct elements, where $\boldsymbol{r} = 1, 2, 3, \ldots, \min(\boldsymbol{C}, \boldsymbol{s})$, then

P($\boldsymbol{s}$ subjects utilize exactly $\boldsymbol{r}$ distinct elements)

$$= \frac{\binom{C}{r} \bullet ({}_s\mathrm{A}_\mathrm{r})}{C^s} \qquad \text{where } {}_s\boldsymbol{A}_{\boldsymbol{r}}. = \boldsymbol{r} \bullet ({}_{s\text{-}1}\boldsymbol{A}_{\boldsymbol{r}} + {}_{s\text{-}1}\boldsymbol{A}_{\boldsymbol{r}\text{-}1}).$$

Using the SAS code in Appendix E, by plugging in any values of $\boldsymbol{s}$ and $\boldsymbol{r}$ of interest, we will be able to compute the value of ${}_s\boldsymbol{A}_{\boldsymbol{r}}$. Appendix F also gives the values of ${}_s\boldsymbol{A}_{\boldsymbol{r}}$ for $\boldsymbol{r} = 1, 2, 3, \ldots, 40$ and $\boldsymbol{s} = 1, 2, 3, \ldots, 120$.

As a matter of fact, we can define $\boldsymbol{r}$ as a random variable as following:

$$f\ (\boldsymbol{r}; \boldsymbol{C}, \boldsymbol{s}) = \frac{\binom{C}{r} \bullet ({}_s\mathrm{A}_\mathrm{r})}{C^s}$$

where $\boldsymbol{r} = 1, 2, 3, \ldots, \min(\boldsymbol{C}, \boldsymbol{s})$

$${}_{s}\boldsymbol{A}_{r.} = \boldsymbol{r} \cdot ({}_{s-1}\boldsymbol{A}_{r} + {}_{s-1}\boldsymbol{A}_{r-1}).$$

4.1 Real World Applications

Application 1: Suppose five people rolling a die, each of them roll the die exactly once, we want to know what is the probability that all of the 5 people roll a same number?

Solution: We have $\boldsymbol{C} = 6$ (since there are six sides on a die), $\boldsymbol{s} = 5$ (since there are 5 people) and $\boldsymbol{r} = 1$ (since we want all the 5 people roll a same number), therefore, according to the formula, we have:

$$\text{P(all the five people roll a same \#)} = \frac{\binom{C}{r} \cdot ({}_{s}A_{r})}{C^{s}} = \frac{\binom{6}{1} \cdot 1}{6^{5}} = \frac{6}{7776} = 0.000772$$

We also want to know what is the probability that all of the five people roll a different number?

Solution: In this case, we still have $\boldsymbol{C} = 6$, $\boldsymbol{s} = 5$, but $\boldsymbol{r} = 5$ (five people can only roll at most five different numbers of a die), according to the formula, we have:

P(all the five people roll a different number)

$$= \frac{\binom{C}{r} \cdot ({}_{s}A_{r})}{C^{s}} = \frac{\binom{6}{5} \cdot 5!}{6^{5}} = \frac{720}{7776} = 0.0926$$

To find out the probability that these five people roll 4 different numbers, i.e., $\boldsymbol{r}$ = 4, we have:

P(the five people roll 4 different numbers)

$$= \frac{\binom{C}{r} \bullet ({}_s\mathrm{A}_\mathrm{r})}{C^s} = \frac{\binom{6}{4} \bullet 240}{6^5} = \frac{3600}{7776} = 0.463$$

Application 2: Suppose there are 100 people attending a conference, we want to know the probability that these 100 people all have the same birthday.

Solution: We have $\boldsymbol{C}$ = 365, $\boldsymbol{s}$ = 100, and $\boldsymbol{r}$ = 1, therefore, according to the formula:

P(all the 100 people have exactly the same birthday)

$$= \frac{\binom{C}{r} \bullet ({}_s\mathrm{A}_\mathrm{r})}{C^s} = \frac{\binom{365}{1} \bullet 1}{365^{100}} = \frac{1}{365^{99}} \approx 0$$

As expected, the probability is extremely slim. As a matter of fact, it is close to 0.

To find the probability that these 100 people have 50 different birthdays, we have:

P(these 100 people have 50 different birthdays)

$$= \frac{\binom{C}{r} \bullet ({}_s\mathrm{A}_\mathrm{r})}{C^s} = \frac{\binom{365}{50} \bullet 1.3108\mathrm{E}{+}166}{365^{100}} = \frac{1.31\mathrm{E}{+}228}{1.69545569\mathrm{E}{+}256} = 7.73\mathrm{E}\text{-}29$$

We can see that this probability is also very small.

Application 3: Going back to the problem we referred in Chapter 1, suppose there are 100 people attending a conference, we want to know the probability that these 100 people all have the same initial for their last names.

Solution: We have $\boldsymbol{C}$ = 26, $\boldsymbol{s}$ = 100, and $\boldsymbol{r}$ = 1, therefore, according to the formula:

P(all the 100 people have exactly the same initial for their last names)

$$= \frac{\binom{C}{r} \bullet ({}_{s}\mathrm{A}_{\mathrm{r}})}{C^{s}} = \frac{\binom{26}{1} \bullet 1}{26^{100}} = \frac{1}{26^{99}} \approx 0$$

As expected, the probability is extremely slim. As a matter of fact, it is close to 0.

To find the probability that these 100 people have 26 different initials for their last names, we have:

P(these 100 people have 26 different initials for their last names)

$$= \frac{\binom{C}{r} \bullet ({}_{s}\mathrm{A}_{\mathrm{r}})}{C^{s}} = \frac{\binom{26}{26} \bullet \text{ 18E+140}}{26^{100}} = 0.5822420031.$$

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APPENDIX A

When the order matters and each object can be chosen only once, then the number of permutations = $\frac{n!}{(n-r)!}$.

Proof: The formula $P_n^r = n(n-1)(n-2)\ldots(n-r+1)$ cannot be used for $r = 0$, but we do have

$$P_n^0 = \frac{n!}{(n-0)!} = 1$$

For $r = 1, 2, \ldots, n$, we have

$$P_n^r = n(n-1)(n-2)\ldots(n-r+1)$$

$$= \frac{n(n-1)(n-2)\cdot\ldots\cdot(n-r+1)(n-r)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

APPENDIX B

One method of deriving a formula for $\binom{n}{r}$ proceeds as follows:

1. Count the number of ways in which one can make an ordered list of *r* different elements from the set of *n*. This is equivalent to calculating the number of *r*-permutations.
2. Recognizing that we have listed every subset many times, we correct the calculation by dividing by the number of different lists containing the same *r* elements: $\binom{n}{r} = \frac{P_n^r}{P_r^r}$, and since $P_n^r = \frac{n!}{(n-r)!}$, we find

$$\binom{n}{r} = \frac{P_n^r}{P_r^r} = \frac{\frac{n!}{(n-r)!}}{\frac{r!}{(r-r)!}} = \frac{n!0!}{r!(n-r)!} = \frac{n!}{r!(n-r)!}$$

APPENDIX C

The number of ways in which a set of *n* distinct objects can be partitioned into *k* subsets with n_1 objects in the first subset, n_2 objects in the second subset, …, and n_k objects in the *k*th subset is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}$$

Proof: Since the n_1 objects going into the first subset can be chosen in $\binom{n}{n_1}$ ways, the n_2 objects going into the second subset can then be chosen in $\binom{n-n_1}{n_2}$ ways, the n_3 objects going into the third subset can then be chosen in $\binom{n-n_1-n_2}{n_3}$ ways, and so forth, thus the total number of partition is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \binom{n}{n_1} \bullet \binom{n-n_1}{n_2} \bullet \ldots \bullet \binom{n-n_1-n_2-\ldots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1! \cdot (n-n_1)!} \bullet \frac{(n-n_1)!}{n_2! \cdot (n-n_1-n_2)!}$$

$$\bullet \ldots \bullet \frac{(n-n_1-n_2-\ldots-n_{k-1})!}{n_k! \cdot 0!}$$

$$= \frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}$$

APPENDIX D

To show ${}_{s}A_{r.} = r \cdot ({}_{s-1}A_r + {}_{s-1}A_{r-1})$ works in general, we will proof it by using the method of induction.

Let r be fixed, we will induct on s. From the previous analysis we identified the following results:

i. If $r > s$, then ${}_{s}A_{r.} = 0$.

ii. If $r = 1$, then ${}_{s}A_{r.} = 1$.

iii. If $r = s$, then ${}_{s}A_{r.} = s!$.

In the case of $r < s$, we assume ${}_{s}A_{r.} = r \cdot ({}_{s-1}A_r + {}_{s-1}A_{r-1})$ is true for $s = 1, 2, 3, \ldots, k.$

We will proof the fact that ${}_{k+1}A_r = r \cdot ({}_{k}A_r + {}_{k}A_{r-1})$ is also true.

Let's first write ${}_{k+1}A_r = r \cdot ({}_{k}A_r + {}_{k}A_{r-1}) = r \cdot {}_{k}A_r + r \cdot {}_{k}A_{r-1}$. We know that in order to fill $k + 1$ slots with r distinct elements, we need to add one additional slot in front of all the outcomes of ${}_{k}A_r$ and then include outcomes that are left out. From previous analysis we know that there are r ways we can fill the additional slot, therefore we will have a total of $r \cdot {}_{k}A_r$ outcomes. For the outcomes left out, they are actually the

outcomes by adding one additional slot to the outcomes in the event of filling $\boldsymbol{k}$ slots with $\boldsymbol{r}$ - 1distinct elements. Depending on which element is excluded from the $\boldsymbol{r}$ distinct elements, there are again $\boldsymbol{r}$ ways we can fill the one additional slot, so we will have a total of $\boldsymbol{r} \cdot {}_{k}A_{r-1}$ outcomes. Therefore, by adding these two parts together, we could have

$${}_{k+1}A_{r} = r \cdot {}_{k}A_{r} + r \cdot {}_{k}A_{r-1} = r \cdot ({}_{k}A_{r} + {}_{k}A_{r-1})$$

And therefore, we could conclude that if $\boldsymbol{r} < \boldsymbol{s}$, then

$${}_{s}A_{r.} = r \cdot ({}_{s-1}A_{r} + {}_{s-1}A_{r-1}), \qquad \text{where } 1 < r < \min (C, s).$$

APPENDIX E

Note: User needs to specify the values of C, S and R of interest by deleting symbol “**X**” and inserting the actual values.

```
/*  calculate total # of combinations and total # of possible outcomes,
    and then calculate the division of the two  */

data Non_A;
C = X;
R = X;
S = X;
comb = gamma (C+1)/(gamma(R+1)*gamma(C-R+1));
C_s = C**s;
Division = comb/C_s;
output;
run;


data Out_Table ;
set Non_A;

%let S = X;
%let R = X;
keep AA1 - AA&R;

Array AP (&R) AP1 - AP&R;
Array AA (&R) AA1 - AA&R;


/*    initialize AA     */

AA(1) = 1;
Do i=2 to &R; AA(i)=0; end; output;
```

```
/*    calculate values AA for each row and set
      current values of AA as previous value of AA     */

Do SS = 2 to &S;
        Do j=1 to &R; AP(j)=AA(j); end;
        Do RR = 2 to &R;
                AA(RR) = RR * ( AP(RR) + AP(RR-1) );
        end;
output;
end;

S=&S;
R=&R;
A = AA(&R);
Prob=Division*A;

Put '  Answer:     S= ' S '  R= ' R '  P = ' Prob '  A= ' A comma6.0;
run;
```

APPENDIX F

This Appendix gives the values of ${}_{s}A_{r}$ for r = 1, 2, 3, …, 40 and s = 1, 2, 3, …, 120.

Each row represents each s value and each column represents each r value.

According to the formula ${}_{s}A_{r}$. = $r \cdot ({}_{s-1}A_{r} + {}_{s-1}A_{r-1})$ and by looking at the table, if s = 2 and r =2, we would have ${}_{2}A_{2} = 2 \cdot ({}_{1}A_{2} + {}_{1}A_{1}) = 2 \cdot (0 + 1) = 2$.

__Note:__ By going to the following website, you will be able to find the values of any combinations $\binom{C}{r}$ and power function C^{S}. And then you will be able to find the probability by simply plugging in the value of ${}_{s}A_{r}$ into to the formula.

" http://www.ciphersbyritter.com/JAVASCRP/PERMCOMB.HTM#Combinations"

S / R	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0
2	1	2	0	0	0	0	0
3	1	6	6	0	0	0	0
4	1	14	36	24	0	0	0
5	1	30	150	240	120	0	0
6	1	62	540	1560	1800	720	0
7	1	126	1806	8400	16800	15120	5040
8	1	254	5796	40824	126000	191520	141120
9	1	510	18150	186480	834120	1905120	2328480
10	1	1022	55980	818520	5103000	16435440	29635200
11	1	2046	171006	3498000	29607600	129230640	322494480
12	1	4094	519156	14676024	165528000	953029440	3162075840
13	1	8190	1569750	60780720	901020120	6711344640	28805736960
14	1	16382	4733820	249401880	4809004200	45674188560	2.4862E+11
15	1	32766	14250606	1016542800	25292030400	3.02899E+11	2.06006E+12
16	1	65534	42850116	4123173624	1.31543E+11	1.96915E+12	1.65407E+13
17	1	131070	128746950	16664094960	6.7833E+11	1.26041E+13	1.29569E+14
18	1	262142	386634060	67171367640	3.47497E+12	7.96948E+13	9.95211E+14
19	1	524286	1160688606	2.70232E+11	1.77107E+13	4.99019E+14	7.52434E+15
20	1	1048574	3483638676	1.08557E+12	8.99047E+13	3.10038E+15	5.61635E+16
21	1	2097150	10454061750	4.35622E+12	4.54952E+14	1.91417E+16	4.14847E+17
22	1	4194302	31368476700	1.74667E+13	2.29654E+15	1.1758E+17	3.03792E+18
23	1	8388606	94118013006	6.99922E+13	1.157E+16	7.19258E+17	2.20885E+19
24	1	16777214	2.82379E+11	2.80345E+14	5.82001E+16	4.38497E+18	1.59654E+20
25	1	33554430	8.47188E+11	1.12251E+15	2.92402E+17	2.6659E+19	1.14828E+21
26	1	67108862	2.54166E+12	4.49343E+15	1.46762E+18	1.61709E+20	8.22454E+21
27	1	134217726	7.62519E+12	1.79839E+16	7.36058E+18	9.79057E+20	5.87038E+22
28	1	268435454	2.2876E+13	7.19661E+16	3.68928E+19	5.91851E+21	4.1778E+23
29	1	536870910	6.86288E+13	2.87956E+17	1.84824E+20	3.57324E+22	2.96589E+24
30	1	1073741822	2.05888E+14	1.1521E+18	9.2556E+20	2.15503E+23	2.10113E+25
31	1	2147483646	6.17667E+14	4.60922E+18	4.63356E+21	1.29857E+24	1.48588E+26
32	1	4294967294	1.85301E+15	1.84393E+19	2.31908E+22	7.81924E+24	1.04921E+27
33	1	8589934590	5.55903E+15	7.37647E+19	1.16046E+23	4.70546E+25	7.39917E+27
34	1	17179869182	1.66771E+16	2.95081E+20	5.80601E+23	2.83024E+26	5.21236E+28
35	1	34359738366	5.00314E+16	1.18039E+21	2.90448E+24	1.70163E+27	3.66846E+29
36	1	68719476734	1.50094E+17	4.72177E+21	1.45283E+25	1.02272E+28	2.57984E+30
37	1	1.37439E+11	4.50283E+17	1.88877E+22	7.26651E+25	6.14503E+28	1.81304E+31
38	1	2.74878E+11	1.35085E+18	7.55525E+22	3.6342E+26	3.69138E+29	1.27343E+32
39	1	5.49756E+11	4.05255E+18	3.02215E+23	1.81748E+27	2.21701E+30	8.93986E+32
40	1	1.09951E+12	1.21577E+19	1.20888E+24	9.0889E+27	1.33129E+31	6.27342E+33

S / R	8	9	10	11	12	13
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	0	0	0	0	0	0
7	0	0	0	0	0	0
8	40320	0	0	0	0	0
9	1451520	362880	0	0	0	0
10	30240000	16329600	3628800	0	0	0
11	479001600	419126400	199584000	39916800	0	0
12	6411968640	8083152000	6187104000	2634508800	479001600	0
13	76592355840	1.30456E+11	1.42703E+11	97037740800	37362124800	6227020800
14	8.43185E+11	1.86344E+12	2.73159E+12	2.63714E+12	1.6128E+12	5.66659E+11
15	8.73443E+12	2.43596E+13	4.59502E+13	5.9056E+13	5.09993E+13	2.83329E+13
16	8.63559E+13	2.97846E+14	7.03098E+14	1.15507E+15	1.32066E+15	1.03132E+15
17	8.23173E+14	3.45782E+15	1.00094E+16	2.04398E+16	2.97088E+16	3.05758E+16
18	7.62193E+15	3.85289E+16	1.34673E+17	3.34942E+17	6.01784E+17	7.83699E+17
19	6.89372E+16	4.15358E+17	1.73202E+18	5.16576E+18	1.12407E+19	1.80113E+19
20	6.11692E+17	4.35865E+18	2.14737E+19	7.58755E+19	1.96878E+20	3.80276E+20
21	5.34284E+18	4.47331E+19	2.58324E+20	1.07084E+21	3.27304E+21	7.50299E+21
22	4.60615E+19	4.50684E+20	3.03057E+21	1.46208E+22	5.21266E+22	1.40088E+23
23	3.92796E+20	4.47071E+21	3.48125E+22	1.94165E+23	8.00969E+23	2.49879E+24
24	3.31907E+21	4.37715E+22	3.92832E+23	2.51876E+24	1.19416E+25	4.28969E+25
25	2.78298E+22	4.23815E+23	4.36604E+24	3.20275E+25	1.73524E+26	7.12901E+26
26	2.31825E+23	4.06481E+24	4.78985E+25	4.00329E+26	2.46662E+27	1.15235E+28
27	1.92039E+24	3.86697E+25	5.19634E+26	4.9305E+27	3.44034E+28	1.81872E+29
28	1.58328E+25	3.65311E+26	5.58303E+27	5.99515E+28	4.72007E+29	2.81158E+30
29	1.30005E+26	3.43029E+27	5.94834E+28	7.20879E+29	6.3835E+30	4.26866E+31
30	1.06376E+27	3.20427E+28	6.29137E+29	8.58399E+30	8.52526E+31	6.37912E+32
31	8.6782E+27	2.97958E+29	6.6118E+30	1.01344E+32	1.12604E+33	9.40114E+33
32	7.06143E+28	2.75972E+30	6.90976E+31	1.18752E+33	1.47286E+34	1.36853E+35
33	5.73308E+29	2.5473E+31	7.18573E+32	1.38228E+34	1.90993E+35	1.97056E+36
34	4.64566E+30	2.34417E+32	7.44046E+33	1.59955E+35	2.45779E+36	2.81002E+37
35	3.75822E+31	2.15157E+33	7.67488E+34	1.84135E+36	3.1413E+37	3.97255E+38
36	3.03593E+32	1.97023E+34	7.89003E+35	2.10991E+37	3.99052E+38	5.57268E+39
37	2.44938E+33	1.80053E+35	8.08706E+36	2.40769E+38	5.04181E+39	7.76325E+40
38	1.97401E+34	1.64252E+36	8.26711E+37	2.73741E+39	6.3391E+40	1.07477E+42
39	1.58939E+35	1.49604E+37	8.43136E+38	3.10209E+40	7.93541E+41	1.4796E+43
40	1.27867E+36	1.36074E+38	8.58097E+39	3.50505E+41	9.89474E+42	2.02665E+44

S / R	14	15	16	17	18	19
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	0	0	0	0	0	0
7	0	0	0	0	0	0
8	0	0	0	0	0	0
9	0	0	0	0	0	0
10	0	0	0	0	0	0
11	0	0	0	0	0	0
12	0	0	0	0	0	0
13	0	0	0	0	0	0
14	87178291200	0	0	0	0	0
15	9.15372E+12	1.30767E+12	0	0	0	0
16	5.24813E+14	1.56921E+14	2.09228E+13	0	0	0
17	2.17859E+16	1.0226E+16	2.8455E+15	3.55687E+14	0	0
18	7.33063E+17	4.80178E+17	2.09144E+17	5.44202E+16	6.40237E+15	0
19	2.12347E+19	1.81986E+19	1.10292E+19	4.48059E+18	1.09481E+18	1.21645E+17
20	5.49443E+20	5.91499E+20	4.67644E+20	2.63666E+20	1.00357E+20	2.31126E+19
21	1.30161E+22	1.71141E+22	1.69463E+22	1.24323E+22	6.55241E+21	2.34593E+21
22	2.87267E+23	4.51953E+23	5.44967E+23	4.99436E+23	3.41724E+23	1.69068E+23
23	5.98297E+24	1.10883E+25	1.59507E+25	1.77548E+25	1.51409E+25	9.70506E+24
24	1.18745E+26	2.56069E+26	4.32624E+26	5.72995E+26	5.92123E+26	4.72073E+26
25	2.26298E+27	5.62221E+27	1.10191E+28	1.70955E+28	2.09721E+28	2.02197E+28
26	4.16624E+28	1.18278E+29	2.66261E+29	4.77949E+29	6.85218E+29	7.82645E+29
27	7.44603E+29	2.3991E+30	6.15262E+30	1.26516E+31	2.0937E+31	2.78894E+31
28	1.29706E+31	4.71556E+31	1.36828E+32	3.19671E+32	6.04594E+32	9.27701E+32
29	2.20951E+32	9.01894E+32	2.94373E+33	7.76048E+33	1.66368E+34	2.91136E+34
30	3.69093E+33	1.68427E+34	6.153E+34	1.81972E+35	4.3915E+35	8.69257E+35
31	6.06038E+34	3.08004E+35	1.25396E+36	4.13953E+36	1.11802E+37	2.48597E+37
32	9.80069E+35	5.52912E+36	2.49915E+37	9.16893E+37	2.75755E+38	6.84759E+38
33	1.56369E+37	9.76378E+37	4.88329E+38	1.98357E+39	6.614E+39	1.82498E+40
34	2.46505E+38	1.69912E+39	9.37547E+39	4.20223E+40	1.54756E+41	4.72411E+41
35	3.84447E+39	2.91844E+40	1.77194E+41	8.73763E+41	3.54201E+42	1.19162E+43
36	5.93841E+40	4.95433E+41	3.30205E+42	1.78663E+43	7.9484E+43	2.93706E+44
37	9.09395E+41	8.32225E+42	6.07597E+43	3.59861E+44	1.7523E+45	7.09061E+45
38	1.38184E+43	1.38475E+44	1.10531E+45	7.15055E+45	3.8019E+46	1.68015E+47
39	2.08504E+44	2.2844E+45	1.99006E+46	1.4035E+47	8.13052E+47	3.91465E+48
40	3.1262E+45	3.73935E+46	3.54959E+47	2.72425E+48	1.71612E+49	8.98264E+49

S / R	20	21	22	23	24	25
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	0	0	0	0	0	0
7	0	0	0	0	0	0
8	0	0	0	0	0	0
9	0	0	0	0	0	0
10	0	0	0	0	0	0
11	0	0	0	0	0	0
12	0	0	0	0	0	0
13	0	0	0	0	0	0
14	0	0	0	0	0	0
15	0	0	0	0	0	0
16	0	0	0	0	0	0
17	0	0	0	0	0	0
18	0	0	0	0	0	0
19	0	0	0	0	0	0
20	2.4329E+18	0	0	0	0	0
21	5.10909E+20	5.10909E+19	0	0	0	0
22	5.71367E+22	1.1802E+22	1.124E+21	0	0	0
23	4.5241E+24	1.44771E+24	2.84372E+23	2.5852E+22	0	0
24	2.84583E+26	1.25408E+26	3.81059E+25	7.13516E+24	6.20448E+23	0
25	1.51331E+28	8.60982E+27	3.59731E+27	1.04054E+27	1.86135E+26	1.55112E+25
26	7.07057E+29	4.98602E+29	2.68557E+29	1.06671E+29	2.94403E+28	5.04114E+27
27	2.9794E+31	2.53188E+31	1.68775E+31	8.63023E+30	3.26666E+30	8.62035E+29
28	1.15367E+33	1.15737E+33	9.28319E+32	5.86678E+32	2.85525E+32	1.03217E+32
29	4.16274E+34	4.85318E+34	4.58852E+34	3.48449E+34	2.09329E+34	9.71857E+33
30	1.41482E+36	1.89334E+36	2.07717E+36	1.85679E+36	1.33867E+36	7.66286E+35
31	4.56815E+37	6.94714E+37	8.73514E+37	9.04812E+37	7.6691E+37	5.26238E+37
32	1.41083E+39	2.41821E+39	3.4501E+39	4.09015E+39	4.01213E+39	3.23287E+39
33	4.19117E+40	8.04098E+40	1.29103E+41	1.73426E+41	1.94455E+41	1.81125E+41
34	1.20323E+42	2.56875E+42	4.60928E+42	6.95816E+42	8.82913E+42	9.3895E+42
35	3.35128E+43	7.92116E+43	1.57917E+44	2.66051E+44	3.78895E+44	4.55466E+44
36	9.0858E+44	2.36721E+45	5.21682E+45	9.75126E+45	1.54787E+46	2.0859E+46
37	2.40457E+46	6.87916E+46	1.66849E+47	3.44266E+47	6.05519E+47	9.08443E+47
38	6.22726E+47	1.94958E+48	5.18409E+48	1.17556E+49	2.27948E+49	3.78491E+49
39	1.58148E+49	5.40185E+49	1.56941E+50	3.89614E+50	8.29211E+50	1.5161E+51
40	3.9459E+50	1.4665E+51	4.64111E+51	1.25708E+52	2.92518E+52	5.86327E+52

S / R	26	27	28	29	30	31
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	0	0	0	0	0	0
7	0	0	0	0	0	0
8	0	0	0	0	0	0
9	0	0	0	0	0	0
10	0	0	0	0	0	0
11	0	0	0	0	0	0
12	0	0	0	0	0	0
13	0	0	0	0	0	0
14	0	0	0	0	0	0
15	0	0	0	0	0	0
16	0	0	0	0	0	0
17	0	0	0	0	0	0
18	0	0	0	0	0	0
19	0	0	0	0	0	0
20	0	0	0	0	0	0
21	0	0	0	0	0	0
22	0	0	0	0	0	0
23	0	0	0	0	0	0
24	0	0	0	0	0	0
25	0	0	0	0	0	0
26	4.03291E+26	0	0	0	0	0
27	1.41555E+29	1.08889E+28	0	0	0	0
28	2.60934E+31	4.11599E+30	3.04888E+29	0	0	0
29	3.36208E+33	8.15653E+32	1.23785E+32	8.84176E+30	0	0
30	3.40097E+35	1.12799E+35	2.63042E+34	3.84617E+33	2.65253E+32	0
31	2.8766E+37	1.22282E+37	3.89488E+36	8.74362E+35	1.23343E+35	8.22284E+33
32	2.11613E+39	1.10684E+39	4.51446E+38	1.38308E+38	2.99311E+37	4.07853E+36
33	1.39074E+41	8.70204E+40	4.36321E+40	1.71029E+40	5.04718E+39	1.0543E+39
34	8.32518E+42	6.10455E+42	3.65827E+42	1.76131E+42	6.64501E+41	1.89146E+41
35	4.60582E+44	3.89603E+44	2.73359E+44	1.57168E+44	7.27744E+43	2.64631E+43
36	2.38172E+46	2.2955E+46	1.85629E+46	1.24853E+46	6.89827E+45	3.07636E+45
37	1.16158E+48	1.26285E+48	1.1625E+48	9.00398E+47	5.81506E+47	3.09214E+47
38	5.38207E+49	6.54597E+49	6.79098E+49	5.98241E+49	4.44571E+49	2.76123E+49
39	2.38341E+51	3.22057E+51	3.73435E+51	3.70428E+51	3.12844E+51	2.23415E+51
40	1.01387E+53	1.51308E+53	1.94738E+53	2.1572E+53	2.04982E+53	1.6624E+53

S / R	32	33	34	35	36	37
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	0	0	0	0	0	0
7	0	0	0	0	0	0
8	0	0	0	0	0	0
9	0	0	0	0	0	0
10	0	0	0	0	0	0
11	0	0	0	0	0	0
12	0	0	0	0	0	0
13	0	0	0	0	0	0
14	0	0	0	0	0	0
15	0	0	0	0	0	0
16	0	0	0	0	0	0
17	0	0	0	0	0	0
18	0	0	0	0	0	0
19	0	0	0	0	0	0
20	0	0	0	0	0	0
21	0	0	0	0	0	0
22	0	0	0	0	0	0
23	0	0	0	0	0	0
24	0	0	0	0	0	0
25	0	0	0	0	0	0
26	0	0	0	0	0	0
27	0	0	0	0	0	0
28	0	0	0	0	0	0
29	0	0	0	0	0	0
30	0	0	0	0	0	0
31	0	0	0	0	0	0
32	2.63131E+35	0	0	0	0	0
33	1.38933E+38	8.68332E+36	0	0	0	0
34	3.81834E+40	4.87134E+39	2.95233E+38	0	0	0
35	7.27454E+42	1.42081E+42	1.75664E+41	1.03331E+40	0	0
36	1.0796E+45	2.86946E+44	5.428E+43	6.50988E+42	3.71993E+41	0
37	1.32991E+47	4.50961E+46	1.16017E+46	2.12765E+45	2.47748E+44	1.37638E+43
38	1.41505E+49	5.87687E+48	1.92773E+48	4.80527E+47	8.55142E+46	9.67592E+45
39	1.33641E+51	6.60905E+50	2.65356E+50	8.42889E+49	2.03775E+49	3.52203E+48
40	1.14258E+53	6.59114E+52	3.14929E+52	1.22376E+52	3.76799E+51	8.84282E+50

S / R	38	39	40
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	0	0	0
7	0	0	0
8	0	0	0
9	0	0	0
10	0	0	0
11	0	0	0
12	0	0	0
13	0	0	0
14	0	0	0
15	0	0	0
16	0	0	0
17	0	0	0
18	0	0	0
19	0	0	0
20	0	0	0
21	0	0	0
22	0	0	0
23	0	0	0
24	0	0	0
25	0	0	0
26	0	0	0
27	0	0	0
28	0	0	0
29	0	0	0
30	0	0	0
31	0	0	0
32	0	0	0
33	0	0	0
34	0	0	0
35	0	0	0
36	0	0	0
37	0	0	0
38	5.23023E+44	0	0
39	3.8756E+47	2.03979E+46	0
40	1.48565E+50	1.59103E+49	8.15915E+47

S / R	1	2	3	4	5	6
41	1	2.19902E+12	3.6473E+19	4.83556E+24	4.54506E+28	7.99322E+31
42	1	4.39805E+12	1.09419E+20	1.93424E+25	2.27277E+29	4.79866E+32
43	1	8.79609E+12	3.28257E+20	7.73699E+25	1.13648E+30	2.88056E+33
44	1	1.75922E+13	9.84771E+20	3.09481E+26	5.68279E+30	1.72902E+34
45	1	3.51844E+13	2.95431E+21	1.23793E+27	2.84155E+31	1.03775E+35
46	1	7.03687E+13	8.86294E+21	4.95172E+27	1.42084E+32	6.22821E+35
47	1	1.40737E+14	2.65888E+22	1.98069E+28	7.10444E+32	3.73778E+36
48	1	2.81475E+14	7.97664E+22	7.92278E+28	3.55232E+33	2.24309E+37
49	1	5.6295E+14	2.39299E+23	3.16912E+29	1.7762E+34	1.34607E+38
50	1	1.1259E+15	7.17898E+23	1.26765E+30	8.88115E+34	8.07748E+38
51	1	2.2518E+15	2.15369E+24	5.07059E+30	4.44064E+35	4.84702E+39
52	1	4.5036E+15	6.46108E+24	2.02824E+31	2.22034E+36	2.90848E+40
53	1	9.0072E+15	1.93832E+25	8.11296E+31	1.11018E+37	1.74522E+41
54	1	1.80144E+16	5.81497E+25	3.24518E+32	5.55095E+37	1.0472E+42
55	1	3.60288E+16	1.74449E+26	1.29807E+33	2.77549E+38	6.28353E+42
56	1	7.20576E+16	5.23348E+26	5.19229E+33	1.38775E+39	3.77028E+43
57	1	1.44115E+17	1.57004E+27	2.07692E+34	6.93879E+39	2.26225E+44
58	1	2.8823E+17	4.71013E+27	8.30767E+34	3.46941E+40	1.35739E+45
59	1	5.76461E+17	1.41304E+28	3.32307E+35	1.73471E+41	8.14457E+45
60	1	1.15292E+18	4.23912E+28	1.32923E+36	8.67355E+41	4.88685E+46
61	1	2.30584E+18	1.27173E+29	5.31691E+36	4.33678E+42	2.93216E+47
62	1	4.61169E+18	3.8152E+29	2.12676E+37	2.16839E+43	1.75932E+48
63	1	9.22337E+18	1.14456E+30	8.50706E+37	1.0842E+44	1.05561E+49
64	1	1.84467E+19	3.43368E+30	3.40282E+38	5.42099E+44	6.3337E+49
65	1	3.68935E+19	1.03011E+31	1.36113E+39	2.7105E+45	3.80025E+50
66	1	7.3787E+19	3.09032E+31	5.44452E+39	1.35525E+46	2.28017E+51
67	1	1.47574E+20	9.27095E+31	2.17781E+40	6.77625E+46	1.36811E+52
68	1	2.95148E+20	2.78128E+32	8.71123E+40	3.38813E+47	8.2087E+52
69	1	5.90296E+20	8.34385E+32	3.48449E+41	1.69406E+48	4.92524E+53
70	1	1.18059E+21	2.50316E+33	1.3938E+42	8.47032E+48	2.95515E+54
71	1	2.36118E+21	7.50947E+33	5.57519E+42	4.23516E+49	1.7731E+55
72	1	4.72237E+21	2.25284E+34	2.23007E+43	2.11758E+50	1.06386E+56
73	1	9.44473E+21	6.75852E+34	8.9203E+43	1.05879E+51	6.38318E+56
74	1	1.88895E+22	2.02756E+35	3.56812E+44	5.29395E+51	3.82991E+57
75	1	3.77789E+22	6.08267E+35	1.42725E+45	2.64698E+52	2.29795E+58
76	1	7.55579E+22	1.8248E+36	5.70899E+45	1.32349E+53	1.37877E+59
77	1	1.51116E+23	5.4744E+36	2.2836E+46	6.61744E+53	8.27264E+59
78	1	3.02231E+23	1.64232E+37	9.13439E+46	3.30872E+54	4.96359E+60
79	1	6.04463E+23	4.92696E+37	3.65375E+47	1.65436E+55	2.97816E+61
80	1	1.20893E+24	1.47809E+38	1.4615E+48	8.27181E+55	1.78689E+62

S / R	7	8	9	10	11	12
41	4.40072E+34	1.02795E+37	1.23617E+39	8.71704E+40	3.94994E+42	1.22943E+44
42	3.0861E+35	8.25883E+37	1.12181E+40	8.84066E+41	4.44083E+43	1.52271E+45
43	2.16363E+36	6.63175E+38	1.01706E+41	8.95284E+42	4.98216E+44	1.88055E+46
44	1.51656E+37	5.32271E+39	9.21322E+41	9.05454E+43	5.57885E+45	2.31644E+47
45	1.0628E+38	4.2703E+40	8.3398E+42	9.14668E+44	6.23634E+46	2.84668E+48
46	7.44686E+38	3.42474E+41	7.54425E+43	9.23007E+45	6.96058E+47	3.49085E+49
47	5.21716E+39	2.74575E+42	6.82065E+44	9.30552E+46	7.75817E+48	4.27255E+50
48	3.65463E+40	2.20077E+43	6.1633E+45	9.37372E+47	8.63635E+49	5.22015E+51
49	2.55981E+41	1.76354E+44	5.56678E+46	9.43536E+48	9.6031E+50	6.36782E+52
50	1.79281E+42	1.41288E+45	5.02597E+47	9.49102E+49	1.06672E+52	7.75662E+53
51	1.25553E+43	1.13174E+46	4.53609E+48	9.54128E+50	1.18383E+53	9.43595E+54
52	8.79211E+43	9.06397E+46	4.09267E+49	9.58664E+51	1.31271E+54	1.14652E+56
53	6.15652E+44	7.25821E+47	3.69156E+50	9.62757E+52	1.45453E+55	1.39158E+57
54	4.31078E+45	5.81149E+48	3.32893E+51	9.66449E+53	1.61057E+56	1.68735E+58
55	3.01828E+46	4.65264E+49	3.00127E+52	9.69778E+54	1.78226E+57	2.04414E+59
56	2.11324E+47	3.72453E+50	2.70533E+53	9.72779E+55	1.97115E+58	2.47436E+60
57	1.47953E+48	2.98131E+51	2.43815E+54	9.75484E+56	2.17897E+59	2.99288E+61
58	1.03583E+49	2.38623E+52	2.19702E+55	9.77922E+57	2.40759E+60	3.61761E+62
59	7.25175E+49	1.90982E+53	1.97946E+56	9.80119E+58	2.65911E+61	4.37002E+63
60	5.0768E+50	1.52843E+54	1.78324E+57	9.82099E+59	2.9358E+62	5.27593E+64
61	3.5541E+51	1.22315E+55	1.60629E+58	9.83882E+60	3.24019E+63	6.36635E+65
62	2.48808E+52	9.78806E+55	1.44676E+59	9.85488E+61	3.57503E+64	7.6785E+66
63	1.74178E+53	7.83244E+56	1.30297E+60	9.86935E+62	3.94337E+65	9.2571E+67
64	1.21932E+54	6.26735E+57	1.17337E+61	9.88238E+63	4.34856E+66	1.11558E+69
65	8.53566E+54	5.01485E+58	1.0566E+62	9.89411E+64	4.79429E+67	1.34392E+70
66	5.97523E+55	4.01256E+59	9.51391E+62	9.90468E+65	5.2846E+68	1.61846E+71
67	4.18282E+56	3.21053E+60	8.56613E+63	9.91419E+66	5.82396E+69	1.94849E+72
68	2.92807E+57	2.56876E+61	7.71241E+64	9.92276E+67	6.41726E+70	2.34518E+73
69	2.04971E+58	2.05524E+62	6.94348E+65	9.93047E+68	7.0699E+71	2.82191E+74
70	1.43483E+59	1.64436E+63	6.25098E+66	9.93742E+69	7.78781E+72	3.39478E+75
71	1.0044E+60	1.3156E+64	5.62736E+67	9.94367E+70	8.57753E+73	4.08308E+76
72	7.03093E+60	1.05256E+65	5.06581E+68	9.94929E+71	9.44622E+74	4.90999E+77
73	4.92173E+61	8.42105E+65	4.56018E+69	9.95436E+72	1.04018E+76	5.90332E+78
74	3.44525E+62	6.73723E+66	4.10492E+70	9.95892E+73	1.14529E+77	7.09647E+79
75	2.4117E+63	5.39006E+67	3.69503E+71	9.96303E+74	1.26092E+78	8.52951E+80
76	1.68821E+64	4.31224E+68	3.32601E+72	9.96672E+75	1.3881E+79	1.02505E+82
77	1.18176E+65	3.44993E+69	2.9938E+73	9.97005E+76	1.52801E+80	1.23173E+83
78	8.27235E+65	2.76004E+70	2.69473E+74	9.97304E+77	1.68191E+81	1.47991E+84
79	5.79068E+66	2.2081E+71	2.42551E+75	9.97573E+78	1.8512E+82	1.77791E+85
80	4.0535E+67	1.76652E+72	2.18315E+76	9.97816E+79	2.03741E+83	2.13571E+86

S / R	13	14	15	16	17	18
41	2.76327E+45	4.66041E+46	6.07796E+47	6.27765E+48	5.23466E+49	3.57939E+50
42	3.75208E+46	6.91144E+47	9.816E+48	1.10167E+50	9.96613E+50	7.38514E+51
43	5.07565E+47	1.02013E+49	1.57607E+50	1.91973E+51	1.88153E+52	1.50871E+53
44	6.84282E+48	1.49924E+50	2.51713E+51	3.32374E+52	3.52495E+53	3.05436E+54
45	9.19681E+49	2.19474E+51	4.00058E+52	5.72072E+53	6.55745E+54	6.13234E+55
46	1.23259E+51	3.20139E+52	6.33007E+53	9.79325E+54	1.21202E+56	1.22186E+57
47	1.64775E+52	4.65451E+53	9.97532E+54	1.6682E+56	2.22692E+57	2.4175E+58
48	2.19762E+53	6.74699E+54	1.56612E+56	2.82873E+57	4.06935E+58	4.75235E+59
49	2.92477E+54	9.75346E+55	2.45038E+57	4.77654E+58	7.39878E+59	9.28671E+60
50	3.88498E+55	1.40643E+57	3.82187E+58	8.03452E+59	1.33899E+61	1.80479E+62
51	5.15131E+56	2.02339E+58	5.94377E+59	1.34667E+61	2.41288E+62	3.48963E+63
52	6.81937E+57	2.90487E+59	9.21916E+60	2.24978E+62	4.33082E+63	6.71566E+64
53	9.01422E+58	4.16229E+60	1.42645E+62	3.74715E+63	7.74486E+64	1.28677E+66
54	1.18994E+60	5.9534E+61	2.20211E+63	6.22367E+64	1.38033E+66	2.4556E+67
55	1.56886E+61	8.50135E+62	3.39246E+64	1.03102E+66	2.45236E+67	4.66854E+68
56	2.06609E+62	1.21215E+64	5.21621E+65	1.70391E+67	4.34429E+68	8.8448E+69
57	2.71808E+63	1.72594E+65	8.00614E+66	2.80972E+68	7.67495E+69	1.67026E+71
58	3.57241E+64	2.45437E+66	1.22681E+68	4.62365E+69	1.35251E+71	3.14462E+72
59	4.69116E+65	3.48613E+67	1.87703E+69	7.59413E+70	2.37786E+72	5.90376E+73
60	6.15532E+66	4.94626E+68	2.86784E+70	1.24509E+72	4.17147E+73	1.10548E+75
61	8.07051E+67	7.01094E+69	4.37595E+71	2.03804E+73	7.30317E+74	2.06495E+76
62	1.05744E+69	9.9283E+70	6.66909E+72	3.33087E+74	1.27618E+76	3.84836E+77
63	1.38466E+70	1.40477E+72	1.01526E+74	5.4361E+75	2.22614E+77	7.15677E+78
64	1.81209E+71	1.98606E+73	1.54395E+75	8.8602E+76	3.87685E+78	1.32829E+80
65	2.37022E+72	2.80585E+74	2.34572E+76	1.44234E+78	6.74127E+79	2.4607E+81
66	3.09875E+73	3.96137E+75	3.56067E+77	2.34527E+79	1.17054E+81	4.55061E+82
67	4.04942E+74	5.5893E+76	5.40043E+78	3.8094E+80	2.02978E+82	8.40179E+83
68	5.28958E+75	7.88172E+77	8.18448E+79	6.18145E+81	3.51538E+83	1.54886E+85
69	6.90694E+76	1.11085E+79	1.23949E+81	1.00213E+83	6.08124E+84	2.85122E+86
70	9.0157E+77	1.56485E+80	1.87591E+82	1.62323E+84	1.05085E+86	5.24166E+87
71	1.17645E+79	2.20342E+81	2.83733E+83	2.62719E+85	1.81403E+87	9.62415E+88
72	1.5347E+80	3.10125E+82	4.28905E+84	4.2489E+86	3.12852E+88	1.765E+90
73	2.00149E+81	4.36324E+83	6.48009E+85	6.86687E+87	5.39072E+89	3.23331E+91
74	2.60961E+82	6.13656E+84	9.78558E+86	1.10907E+89	9.28096E+90	5.91699E+92
75	3.40172E+83	8.62772E+85	1.47704E+88	1.79016E+90	1.59662E+92	1.08176E+94
76	4.43333E+84	1.21264E+87	2.2285E+89	2.88789E+91	2.74468E+93	1.97592E+95
77	5.77665E+85	1.70391E+88	3.36095E+90	4.65629E+92	4.71505E+94	3.60605E+96
78	7.52566E+86	2.39356E+89	5.06698E+91	7.50384E+93	8.09475E+95	6.57576E+97
79	9.8026E+87	3.36152E+90	7.63637E+92	1.20872E+95	1.38886E+97	1.1982E+99
80	1.27665E+89	4.71985E+91	1.1505E+94	1.94617E+96	2.38162E+98	2.1818E+100

S / R	19	20	21	22	23	24
41	2.03276E+51	9.68832E+51	3.90829E+52	1.34367E+53	3.95873E+53	1.00374E+54
42	4.54234E+52	2.34422E+53	1.0242E+54	3.8159E+54	1.21955E+55	3.35907E+55
43	1.00336E+54	5.5969E+54	2.6431E+55	1.06482E+56	3.68263E+56	1.09887E+57
44	2.19304E+55	1.32005E+56	6.72585E+56	2.92409E+57	1.09191E+58	3.52112E+58
45	4.74711E+56	3.07871E+57	1.68964E+58	7.91269E+58	3.18394E+59	1.10713E+60
46	1.01847E+58	7.10685E+58	4.19477E+59	2.11251E+60	9.14298E+60	3.42125E+61
47	2.16724E+59	1.62506E+60	1.03015E+61	5.57038E+61	2.58876E+62	1.04053E+63
48	4.57707E+60	3.68357E+61	2.50457E+62	1.45211E+63	7.23534E+63	3.11858E+64
49	9.59939E+61	8.28256E+62	6.03315E+63	3.74566E+64	1.99812E+65	9.22108E+65
50	2.00033E+63	1.8485E+64	1.44089E+65	9.56774E+65	5.45717E+66	2.69261E+67
51	4.14354E+64	4.09707E+65	3.41406E+66	2.4219E+67	1.47521E+68	7.77197E+68
52	8.53575E+65	9.02284E+66	8.02992E+67	6.07927E+68	3.95001E+69	2.21932E+70
53	1.74939E+67	1.97528E+68	1.87576E+69	1.5141E+70	1.04833E+71	6.27438E+71
54	3.56833E+68	4.30044E+69	4.35391E+70	3.74368E+71	2.75939E+72	1.75745E+73
55	7.24639E+69	9.31455E+70	1.00463E+72	9.19397E+72	7.20765E+73	4.88013E+74
56	1.46552E+71	2.00784E+72	2.30533E+73	2.24369E+74	1.86922E+75	1.34422E+76
57	2.95253E+72	4.30878E+73	5.26284E+74	5.44329E+75	4.81526E+76	3.67473E+77
58	5.92716E+73	9.20807E+74	1.19568E+76	1.31331E+77	1.2327E+78	9.97501E+78
59	1.18591E+75	1.96016E+76	2.7043E+77	3.15233E+78	3.13728E+79	2.68985E+80
60	2.3654E+76	4.1575E+77	6.09066E+78	7.53006E+79	7.94078E+80	7.20859E+81
61	4.7043E+77	8.78807E+78	1.36635E+80	1.79061E+81	1.99957E+82	1.92064E+83
62	9.3305E+78	1.8517E+80	3.05388E+81	4.23993E+82	5.01085E+83	5.08944E+84
63	1.84591E+80	3.89001E+81	6.802E+82	9.99971E+83	1.25001E+85	1.34173E+86
64	3.64322E+81	8.1492E+82	1.51011E+84	2.34958E+85	3.10503E+86	3.52014E+87
65	7.17449E+82	1.7027E+84	3.34236E+85	5.5013E+86	7.68197E+87	9.19355E+88
66	1.40991E+84	3.5489E+85	7.37653E+86	1.28382E+88	1.89338E+89	2.39082E+90
67	2.76528E+85	7.37978E+86	1.6236E+88	2.98668E+89	4.65006E+90	6.19238E+91
68	5.41367E+86	1.53126E+88	3.56453E+89	6.92789E+90	1.13821E+92	1.59777E+93
69	1.05803E+88	3.1708E+89	7.80708E+90	1.60256E+92	2.77722E+93	4.10782E+94
70	2.06442E+89	6.5532E+90	1.70607E+92	3.69738E+93	6.75619E+94	1.05253E+96
71	4.02199E+90	1.35193E+92	3.72037E+93	8.50957E+94	1.63896E+96	2.68822E+97
72	7.82465E+91	2.7843E+93	8.09669E+94	1.95395E+96	3.96533E+97	6.84508E+98
73	1.52022E+93	5.72509E+94	1.75877E+96	4.47683E+97	9.56968E+98	1.738E+100
74	2.94985E+94	1.17542E+96	3.81365E+97	1.0236E+99	2.304E+100	4.4008E+101
75	5.71713E+95	2.40984E+97	8.25551E+98	2.3358E+100	5.5346E+101	1.1115E+103
76	1.10681E+97	4.93402E+98	1.7843E+100	5.3204E+101	1.3267E+103	2.8004E+104
77	2.14048E+98	1.0089E+100	3.8506E+101	1.2097E+103	3.1737E+104	7.0394E+105
78	4.1354E+99	2.0607E+101	8.2981E+102	2.7461E+104	7.5778E+105	1.7656E+107
79	7.9822E+100	4.2041E+102	1.7859E+104	6.2241E+105	1.8061E+107	4.4194E+108
80	1.5394E+102	8.5678E+103	3.8386E+105	1.4086E+107	4.2971E+108	1.104E+110

S / R	25	26	27	28	29	30
41	2.19711E+54	4.16052E+54	6.82276E+54	9.68926E+54	1.19033E+55	1.26211E+55
42	8.00214E+55	1.65298E+56	2.96549E+56	4.62337E+56	6.26184E+56	7.3573E+56
43	2.8403E+57	6.37832E+57	1.24699E+58	2.12488E+58	3.15671E+58	4.08574E+58
44	9.84793E+58	2.39684E+59	5.08901E+59	9.44122E+59	1.53166E+60	2.17274E+60
45	3.34226E+60	8.79225E+60	2.02118E+61	4.06847E+61	7.17977E+61	1.11132E+62
46	1.11235E+62	3.15497E+62	7.83109E+62	1.7051E+63	3.26199E+63	5.48789E+63
47	3.63618E+63	1.1095E+64	2.96624E+64	6.96699E+64	1.44046E+65	2.62496E+65
48	1.16918E+65	3.83012E+65	1.10045E+66	2.7813E+66	6.19775E+66	1.21963E+67
49	3.70259E+66	1.29982E+67	4.00535E+67	1.08689E+68	2.60392E+68	5.5182E+68
50	1.15617E+68	4.3422E+68	1.43239E+69	4.16479E+69	1.07034E+70	2.43664E+70
51	3.56359E+69	1.42958E+70	5.03986E+70	1.56721E+71	4.31177E+71	1.05209E+72
52	1.0852E+71	4.64343E+71	1.74675E+72	5.79935E+72	1.7049E+73	4.44981E+73
53	3.26782E+72	1.48944E+73	5.96994E+73	2.11291E+74	6.62603E+74	1.84641E+75
54	9.73815E+73	4.72219E+74	2.01403E+75	7.58773E+75	2.53429E+76	7.52705E+76
55	2.8739E+75	1.48096E+76	6.71288E+76	2.68849E+77	9.54989E+77	3.0184E+78
56	8.40478E+76	4.59771E+77	2.21234E+78	9.40739E+78	3.54913E+79	1.19202E+80
57	2.43725E+78	1.41393E+79	7.21469E+79	3.25352E+80	1.30206E+81	4.64079E+81
58	7.01181E+79	4.3099E+80	2.32973E+81	1.113E+82	4.7195E+82	1.78286E+83
59	2.00233E+81	1.30288E+82	7.45394E+82	3.76872E+83	1.69143E+84	6.76442E+84
60	5.67828E+82	3.9081E+83	2.36434E+84	1.26395E+85	5.99806E+85	2.53675E+86
61	1.59978E+84	1.16374E+85	7.43891E+85	4.20108E+86	2.10598E+87	9.40968E+87
62	4.47962E+85	3.44167E+86	2.32272E+87	1.38459E+88	7.32567E+88	3.4547E+89
63	1.24714E+87	1.0113E+88	7.20058E+88	4.52722E+89	2.52597E+90	1.25618E+91
64	3.45329E+88	2.95365E+89	2.21721E+90	1.46924E+91	8.63822E+91	4.52633E+92
65	9.51325E+89	8.57734E+90	6.78395E+91	4.73468E+92	2.93116E+93	1.61705E+94
66	2.60815E+91	2.47745E+92	2.06325E+93	1.51566E+94	9.87343E+94	5.73049E+95
67	7.11808E+92	7.11949E+93	6.2397E+94	4.82156E+95	3.30284E+96	2.01535E+97
68	1.93433E+94	2.03614E+95	1.87695E+96	1.52475E+97	1.09765E+98	7.0369E+98
69	5.23527E+95	5.79689E+96	5.61751E+97	4.79484E+98	3.6254E+99	2.4404E+100
70	1.41151E+97	1.64331E+98	1.6732E+99	1.4998E+100	1.1904E+101	8.4087E+101
71	3.79191E+98	4.6396E+99	4.9615E+100	4.6681E+101	3.8871E+102	2.8797E+103
72	1.0152E+100	1.3049E+101	1.4649E+102	1.446E+103	1.2626E+104	9.8053E+104
73	2.7091E+101	3.6566E+102	4.3074E+103	4.4589E+104	4.081E+105	3.3204E+106
74	7.2072E+102	1.0212E+104	1.2617E+105	1.3691E+106	1.3128E+107	1.1185E+108
75	1.9118E+104	2.8424E+105	3.6824E+106	4.1868E+107	4.2042E+108	3.7495E+109
76	5.0574E+105	7.8874E+106	1.071E+108	1.2754E+109	1.3406E+110	1.251E+111
77	1.3344E+107	2.1822E+108	3.1046E+109	3.871E+110	4.2577E+111	4.1551E+112
78	3.5119E+108	6.0207E+109	8.9717E+110	1.1708E+112	1.347E+113	1.3743E+114
79	9.2212E+109	1.6567E+111	2.5849E+112	3.5295E+113	4.2458E+114	4.5269E+115
80	2.4158E+111	4.5471E+112	7.4266E+113	1.0606E+115	1.3336E+116	1.4854E+117

S / R	31	32	33	34	35	36
41	1.15079E+55	8.97595E+54	5.94559E+54	3.31175E+54	1.53057E+54	5.76201E+53
42	7.47997E+56	6.55482E+56	4.92411E+56	3.1475E+56	1.69481E+56	7.58436E+55
43	4.59955E+58	4.49113E+58	3.78805E+58	2.74435E+58	1.69481E+58	8.83168E+57
44	2.69244E+60	2.90902E+60	2.73213E+60	2.22101E+60	1.5537E+60	9.28071E+59
45	1.5082E+62	1.79247E+62	1.86158E+62	1.68407E+62	1.32115E+62	8.93439E+61
46	8.12052E+63	1.05622E+64	1.20584E+64	1.20552E+64	1.05183E+64	7.97252E+63
47	4.21861E+65	5.97846E+65	7.46477E+65	8.19861E+65	7.90071E+65	6.65668E+65
48	2.12151E+67	3.26306E+67	4.43626E+67	5.32555E+67	5.63476E+67	5.24066E+67
49	1.03575E+69	1.72306E+69	2.54078E+69	3.31902E+69	3.83611E+69	3.91515E+69
50	4.92147E+70	8.8282E+70	1.40707E+71	1.99233E+71	2.50429E+71	2.79045E+71
51	2.28101E+72	4.39989E+72	7.55663E+72	1.15579E+73	1.57382E+73	1.90611E+73
52	1.03326E+74	2.13789E+74	3.94565E+74	6.49895E+74	9.55365E+74	1.25277E+75
53	4.58255E+75	1.01477E+76	2.00757E+76	3.55117E+76	5.61841E+76	7.9493E+76
54	1.99298E+77	4.71368E+77	9.97372E+77	1.88997E+78	3.20935E+78	4.88438E+78
55	8.51162E+78	2.14613E+79	4.84684E+79	9.81696E+79	1.78476E+80	2.91374E+80
56	3.57431E+80	9.59134E+80	2.30768E+81	4.98569E+81	9.68261E+81	1.69146E+82
57	1.47756E+82	4.21301E+82	1.07805E+83	2.47975E+83	5.1339E+83	9.575E+83
58	6.01908E+83	1.82098E+84	4.94785E+84	1.20965E+85	2.66478E+85	5.29521E+85
59	2.4186E+85	7.75325E+85	2.23372E+86	5.79508E+86	1.35605E+87	2.86559E+87
60	9.59463E+86	3.25499E+87	9.92983E+87	2.72979E+88	6.77445E+88	1.51979E+89
61	3.76073E+88	1.34863E+89	4.35099E+89	1.26574E+90	3.32649E+90	7.91005E+90
62	1.45753E+90	5.51904E+90	1.88087E+91	5.78286E+91	1.60728E+92	4.04515E+92
63	5.58929E+91	2.2325E+92	8.02817E+92	2.60567E+93	7.64948E+93	2.03488E+94
64	2.1221E+93	8.93257E+93	3.38602E+94	1.15889E+95	3.5893E+95	1.00794E+96
65	7.98166E+94	3.53749E+95	1.41216E+96	5.09146E+96	1.66187E+97	4.92072E+97
66	2.9756E+96	1.38741E+97	5.82751E+97	2.21123E+98	7.59854E+98	2.3697E+99
67	1.10008E+98	5.39191E+98	2.3809E+99	9.4995E+99	3.4334E+100	1.1267E+101
68	4.035E+99	2.0774E+100	9.6364E+100	4.0394E+101	1.5342E+102	5.292E+102
69	1.469E+101	7.939E+101	3.8656E+102	1.701E+103	6.7834E+103	2.4574E+104
70	5.3104E+102	3.0106E+103	1.5376E+104	7.0977E+104	2.9695E+105	1.1289E+106
71	1.9069E+104	1.1333E+105	6.0676E+105	2.936E+106	1.2878E+107	5.133E+107
72	6.8041E+105	4.2368E+106	2.3763E+107	1.2045E+108	5.5348E+108	2.3115E+109
73	2.4132E+107	1.5735E+108	9.24E+108	4.9034E+109	2.3588E+110	1.0314E+111
74	8.5103E+108	5.8075E+109	3.5684E+110	1.9813E+111	9.9719E+111	4.5621E+112
75	2.985E+110	2.1307E+111	1.3692E+112	7.9498E+112	4.1836E+113	2.0014E+114
76	1.0416E+112	7.7735E+112	5.2216E+113	3.1685E+114	1.7425E+115	8.711E+115
77	3.6167E+113	2.8208E+114	1.9797E+115	1.2548E+116	7.2077E+116	3.7633E+117
78	1.25E+115	1.0184E+116	7.4637E+116	4.9394E+117	2.9619E+118	1.6142E+119
79	4.3009E+116	3.6589E+117	2.7991E+118	1.9332E+119	1.2095E+120	6.8776E+120
80	1.4736E+118	1.3085E+119	1.0444E+120	7.5245E+120	4.91E+121	2.9114E+122

S / R	37	38	39	40
41	1.72134E+53	3.92482E+52	6.41452E+51	6.69051E+50
42	2.76884E+55	8.03252E+54	1.78085E+54	2.83343E+53
43	3.83068E+57	1.35739E+57	3.82721E+56	8.25675E+55
44	4.68508E+59	1.97147E+59	6.78645E+58	1.86116E+58
45	5.16734E+61	2.52949E+61	1.03354E+61	3.45904E+60
46	5.21764E+63	2.92479E+63	1.38958E+63	5.5178E+62
47	4.88036E+65	3.09412E+65	1.68261E+65	7.76545E+64
48	4.26871E+67	3.0303E+67	1.86293E+67	9.83661E+66
49	3.51847E+69	2.77362E+69	1.90836E+69	1.13863E+69
50	2.75044E+71	2.39099E+71	1.82597E+71	1.2188E+71
51	2.05013E+73	1.95374E+73	1.64462E+73	1.21791E+73
52	1.46381E+75	1.52147E+75	1.40336E+75	1.14501E+75
53	1.00514E+77	1.13441E+77	1.14069E+77	1.01935E+77
54	6.66024E+78	8.13026E+78	8.87286E+78	8.64014E+78
55	4.27151E+80	5.62039E+80	6.63122E+80	7.0052E+80
56	2.65854E+82	3.75892E+82	4.77813E+82	5.45457E+82
57	1.6095E+84	2.43864E+84	3.32945E+84	4.09308E+84
58	9.49791E+85	1.53829E+86	2.24955E+86	2.96901E+86
59	5.47345E+87	9.45472E+87	1.47726E+88	2.08743E+88
60	3.08545E+89	5.6727E+89	9.44865E+89	1.42587E+90
61	1.70394E+91	3.3281E+91	5.89733E+91	9.48296E+91
62	9.23129E+92	1.91217E+93	3.59792E+93	6.15211E+93
63	4.91228E+94	1.07741E+95	2.14893E+95	3.90001E+95
64	2.57045E+96	5.96084E+96	1.25828E+97	2.41958E+97
65	1.324E+98	3.24189E+98	7.23201E+98	1.4711E+99
66	6.7195E+99	1.735E+100	4.0848E+100	8.7774E+100
67	3.363E+101	9.1466E+101	2.2697E+102	5.1449E+102
68	1.6612E+103	4.7536E+103	1.2419E+104	2.9658E+104
69	8.1044E+104	2.4376E+105	6.6974E+105	1.6831E+106
70	3.9079E+106	1.2343E+107	3.5627E+107	9.4114E+107
71	1.8636E+108	6.1752E+108	1.8708E+109	5.1896E+109
72	8.7945E+109	3.0547E+110	9.7044E+110	2.8242E+111
73	4.1092E+111	1.495E+112	4.9761E+112	1.5178E+113
74	1.902E+113	7.2425E+113	2.5237E+114	8.0618E+114
75	8.7254E+114	3.4749E+115	1.2667E+116	4.2342E+116
76	3.9689E+116	1.652E+117	6.2954E+117	2.2004E+118
77	1.7908E+118	7.7859E+118	3.0995E+119	1.132E+120
78	8.0184E+119	3.6391E+120	1.5124E+121	5.7676E+121
79	3.5641E+121	1.6876E+122	7.3178E+122	2.912E+123
80	1.5732E+123	7.7671E+123	3.5121E+124	1.4575E+125

S / R	1	2	3	4	5	6
81	1	2.41785E+24	4.43426E+38	5.84601E+48	4.1359E+56	1.07214E+63
82	1	4.8357E+24	1.33028E+39	2.3384E+49	2.06795E+57	6.43282E+63
83	1	9.67141E+24	3.99084E+39	9.35361E+49	1.03398E+58	3.8597E+64
84	1	1.93428E+25	1.19725E+40	3.74144E+50	5.16988E+58	2.31582E+65
85	1	3.86856E+25	3.59175E+40	1.49658E+51	2.58494E+59	1.38949E+66
86	1	7.73713E+25	1.07753E+41	5.98631E+51	1.29247E+60	8.33695E+66
87	1	1.54743E+26	3.23258E+41	2.39452E+52	6.46235E+60	5.00217E+67
88	1	3.09485E+26	9.69774E+41	9.5781E+52	3.23117E+61	3.0013E+68
89	1	6.1897E+26	2.90932E+42	3.83124E+53	1.61559E+62	1.80078E+69
90	1	1.23794E+27	8.72796E+42	1.5325E+54	8.07794E+62	1.08047E+70
91	1	2.47588E+27	2.61839E+43	6.12998E+54	4.03897E+63	6.48281E+70
92	1	4.95176E+27	7.85517E+43	2.45199E+55	2.01948E+64	3.88969E+71
93	1	9.90352E+27	2.35655E+44	9.80797E+55	1.00974E+65	2.33381E+72
94	1	1.9807E+28	7.06965E+44	3.92319E+56	5.04871E+65	1.40029E+73
95	1	3.96141E+28	2.1209E+45	1.56928E+57	2.52435E+66	8.40173E+73
96	1	7.92282E+28	6.36269E+45	6.2771E+57	1.26218E+67	5.04104E+74
97	1	1.58456E+29	1.90881E+46	2.51084E+58	6.31089E+67	3.02462E+75
98	1	3.16913E+29	5.72642E+46	1.00434E+59	3.15544E+68	1.81477E+76
99	1	6.33825E+29	1.71793E+47	4.01735E+59	1.57772E+69	1.08886E+77
100	1	1.26765E+30	5.15378E+47	1.60694E+60	7.88861E+69	6.53319E+77
101	1	2.5353E+30	1.54613E+48	6.42775E+60	3.9443E+70	3.91991E+78
102	1	5.0706E+30	4.6384E+48	2.5711E+61	1.97215E+71	2.35195E+79
103	1	1.01412E+31	1.39152E+49	1.02844E+62	9.86076E+71	1.41117E+80
104	1	2.02824E+31	4.17456E+49	4.11376E+62	4.93038E+72	8.46701E+80
105	1	4.05648E+31	1.25237E+50	1.6455E+63	2.46519E+73	5.08021E+81
106	1	8.11296E+31	3.7571E+50	6.58202E+63	1.2326E+74	3.04812E+82
107	1	1.62259E+32	1.12713E+51	2.63281E+64	6.16298E+74	1.82887E+83
108	1	3.24519E+32	3.38139E+51	1.05312E+65	3.08149E+75	1.09732E+84
109	1	6.49037E+32	1.01442E+52	4.21249E+65	1.54074E+76	6.58395E+84
110	1	1.29807E+33	3.04325E+52	1.685E+66	7.70372E+76	3.95037E+85
111	1	2.59615E+33	9.12976E+52	6.73999E+66	3.85186E+77	2.37022E+86
112	1	5.1923E+33	2.73893E+53	2.69599E+67	1.92593E+78	1.42213E+87
113	1	1.03846E+34	8.21678E+53	1.0784E+68	9.62965E+78	8.53279E+87
114	1	2.07692E+34	2.46503E+54	4.31359E+68	4.81482E+79	5.11968E+88
115	1	4.15384E+34	7.3951E+54	1.72544E+69	2.40741E+80	3.07181E+89
116	1	8.30767E+34	2.21853E+55	6.90175E+69	1.20371E+81	1.84308E+90
117	1	1.66153E+35	6.65559E+55	2.7607E+70	6.01853E+81	1.10585E+91
118	1	3.32307E+35	1.99668E+56	1.10428E+71	3.00927E+82	6.6351E+91
119	1	6.64614E+35	5.99003E+56	4.41712E+71	1.50463E+83	3.98106E+92
120	1	1.32923E+36	1.79701E+57	1.76685E+72	7.52316E+83	2.38864E+93

S / R	7	8	9	10	11	12
81	2.83746E+68	1.41325E+73	1.965E+77	9.98034E+80	2.24225E+84	2.5653E+87
82	1.98623E+69	1.13062E+74	1.76863E+78	9.98231E+81	2.46757E+85	3.08105E+88
83	1.39037E+70	9.04514E+74	1.59186E+79	9.98408E+82	2.71543E+86	3.70022E+89
84	9.73258E+70	7.23623E+75	1.43276E+80	9.98567E+83	2.98807E+87	4.44353E+90
85	6.81282E+71	5.78906E+76	1.28955E+81	9.9871E+84	3.28798E+88	5.33582E+91
86	4.76899E+72	4.6313E+77	1.16065E+82	9.98839E+85	3.61787E+89	6.40693E+92
87	3.3383E+73	3.70508E+78	1.04462E+83	9.98955E+86	3.98076E+90	7.69265E+93
88	2.33681E+74	2.96409E+79	9.40194E+83	9.9906E+87	4.37993E+91	9.23596E+94
89	1.63577E+75	2.37129E+80	8.46202E+84	9.99154E+88	4.81903E+92	1.10884E+96
90	1.14504E+76	1.89705E+81	7.61603E+85	9.99238E+89	5.30203E+93	1.33119E+97
91	8.01529E+76	1.51765E+82	6.8546E+86	9.99314E+90	5.83333E+94	1.59806E+98
92	5.61071E+77	1.21412E+83	6.16927E+87	9.99383E+91	6.41776E+95	1.9184E+99
93	3.9275E+78	9.71303E+83	5.55245E+88	9.99445E+92	7.06064E+96	2.3028E+100
94	2.74925E+79	7.77046E+84	4.9973E+89	9.995E+93	7.7678E+97	2.7642E+101
95	1.92448E+80	6.21639E+85	4.49764E+90	9.9955E+94	8.54568E+98	3.318E+102
96	1.34713E+81	4.97312E+86	4.04793E+91	9.99595E+95	9.4013E+99	3.9826E+103
97	9.42994E+81	3.97851E+87	3.64318E+92	9.99636E+96	1.0343E+101	4.7803E+104
98	6.60096E+82	3.18282E+88	3.2789E+93	9.99672E+97	1.1378E+102	5.7376E+105
99	4.62067E+83	2.54626E+89	2.95104E+94	9.99705E+98	1.2517E+103	6.8865E+106
100	3.23447E+84	2.03701E+90	2.65596E+95	9.9973E+99	1.377E+104	8.2653E+107
101	2.26413E+85	1.62961E+91	2.39038E+96	9.9976E+100	1.5148E+105	9.92E+108
102	1.58489E+86	1.30369E+92	2.15136E+97	9.9978E+101	1.6664E+106	1.1906E+110
103	1.10942E+87	1.04295E+93	1.93623E+98	9.9981E+102	1.8331E+107	1.4289E+111
104	7.76597E+87	8.34364E+93	1.7426E+99	9.9983E+103	2.0165E+108	1.7149E+112
105	5.43618E+88	6.67492E+94	1.5684E+100	9.9984E+104	2.2183E+109	2.0581E+113
106	3.80533E+89	5.33994E+95	1.4115E+101	9.9986E+105	2.4402E+110	2.47E+114
107	2.66373E+90	4.27195E+96	1.2704E+102	9.9987E+106	2.6844E+111	2.9643E+115
108	1.86461E+91	3.41756E+97	1.1434E+103	9.9989E+107	2.9529E+112	3.5575E+116
109	1.30523E+92	2.73405E+98	1.029E+104	9.999E+108	3.2483E+113	4.2693E+117
110	9.13659E+92	2.1872E+99	9.2612E+104	9.9991E+109	3.5732E+114	5.1236E+118
111	6.39562E+93	1.7498E+100	8.3351E+105	9.9992E+110	3.9307E+115	6.1487E+119
112	4.47693E+94	1.3998E+101	7.5016E+106	9.9992E+111	4.3238E+116	7.3789E+120
113	3.13385E+95	1.1199E+102	6.7515E+107	9.9993E+112	4.7563E+117	8.8553E+121
114	2.1937E+96	8.959E+102	6.0763E+108	9.9994E+113	5.2321E+118	1.0627E+123
115	1.53559E+97	7.1672E+103	5.4687E+109	9.9995E+114	5.7554E+119	1.2753E+124
116	1.07491E+98	5.7337E+104	4.9218E+110	9.9995E+115	6.3311E+120	1.5304E+125
117	7.52438E+98	4.587E+105	4.4297E+111	9.9996E+116	6.9643E+121	1.8366E+126
118	5.2671E+99	3.6696E+106	3.9867E+112	9.9996E+117	7.6608E+122	2.204E+127
119	3.6869E+100	2.9357E+107	3.588E+113	9.9996E+118	8.427E+123	2.6449E+128
120	2.5809E+101	2.3485E+108	3.2292E+114	9.9997E+119	9.2698E+124	3.1739E+129

S / R	13	14	15	16	17	18
81	1.66242E+90	6.62566E+92	1.73283E+95	3.13228E+97	4.0818E+99	3.9701E+101
82	2.16448E+91	9.29919E+93	2.60918E+96	5.03938E+98	6.9924E+100	7.2196E+102
83	2.81783E+92	1.30492E+95	3.92772E+97	8.1048E+99	1.1973E+102	1.3121E+104
84	3.66799E+93	1.83083E+96	5.91115E+98	1.303E+101	2.0491E+103	2.3834E+105
85	4.77416E+94	2.5683E+97	8.8942E+99	2.0943E+102	3.5057E+104	4.3269E+106
86	6.21335E+95	3.6023E+98	1.338E+101	3.3652E+103	5.9953E+105	7.8516E+107
87	8.08568E+96	5.0519E+99	2.0124E+102	5.4057E+104	1.0249E+107	1.4241E+109
88	1.05214E+98	7.084E+100	3.0261E+103	8.6813E+105	1.7515E+108	2.5818E+110
89	1.369E+99	9.9323E+101	4.5498E+104	1.3938E+107	2.9924E+109	4.6787E+111
90	1.7811E+100	1.3924E+103	6.8397E+105	2.2374E+108	5.1107E+110	8.4756E+112
91	2.3172E+101	1.9519E+104	1.028E+107	3.5908E+109	8.7263E+111	1.5348E+114
92	3.0144E+102	2.7359E+105	1.545E+108	5.7618E+110	1.4896E+113	2.7784E+115
93	3.9212E+103	3.8345E+106	2.3216E+109	9.2436E+111	2.5421E+114	5.0278E+116
94	5.1006E+104	5.3738E+107	3.4881E+110	1.4827E+113	4.3372E+115	9.0959E+117
95	6.6344E+105	7.5305E+108	5.2402E+111	2.3779E+114	7.3985E+116	1.6451E+119
96	8.629E+106	1.0552E+110	7.8717E+112	3.813E+115	1.2618E+118	2.9744E+120
97	1.1223E+108	1.4785E+111	1.1823E+114	6.1134E+116	2.1515E+119	5.3767E+121
98	1.4596E+109	2.0714E+112	1.7757E+115	9.8003E+117	3.668E+120	9.7168E+122
99	1.8982E+110	2.9021E+113	2.6667E+116	1.5709E+119	6.2522E+121	1.7556E+124
100	2.4686E+111	4.0656E+114	4.0044E+117	2.5177E+120	1.0656E+123	3.1714E+125
101	3.2102E+112	5.6952E+115	6.0127E+118	4.0347E+121	1.8157E+124	5.7277E+126
102	4.1746E+113	7.9778E+116	9.0275E+119	6.4652E+122	3.0936E+125	1.0342E+128
103	5.4285E+114	1.1175E+118	1.3553E+121	1.0359E+124	5.2701E+126	1.8672E+129
104	7.0589E+115	1.5652E+119	2.0347E+122	1.6596E+125	8.9767E+127	3.3705E+130
105	9.1788E+116	2.1923E+120	3.0543E+123	2.6586E+126	1.5289E+129	6.083E+131
106	1.1935E+118	3.0705E+121	4.5848E+124	4.2586E+127	2.6036E+130	1.0977E+133
107	1.5519E+119	4.3004E+122	6.8818E+125	6.821E+128	4.4333E+131	1.9805E+134
108	2.0178E+120	6.0227E+123	1.0329E+127	1.0925E+130	7.5483E+132	3.5729E+135
109	2.6237E+121	8.4346E+124	1.5503E+128	1.7496E+131	1.2851E+134	6.4449E+136
110	3.4113E+122	1.1812E+126	2.3267E+129	2.8018E+132	2.1876E+135	1.1624E+138
111	4.4354E+123	1.6542E+127	3.4918E+130	4.4867E+133	3.7237E+136	2.0962E+139
112	5.7668E+124	2.3165E+128	5.2402E+131	7.1843E+134	6.3378E+137	3.7799E+140
113	7.4978E+125	3.2439E+129	7.8637E+132	1.1503E+136	1.0787E+139	6.8153E+141
114	9.7483E+126	4.5425E+130	1.18E+134	1.8418E+137	1.8357E+140	1.2287E+143
115	1.2674E+128	6.3608E+131	1.7708E+135	2.9487E+138	3.1238E+141	2.215E+144
116	1.6478E+129	8.9069E+132	2.6571E+136	4.7208E+139	5.3154E+142	3.9925E+145
117	2.1423E+130	1.2472E+134	3.987E+137	7.5575E+140	9.0442E+143	7.1961E+146
118	2.7853E+131	1.7464E+135	5.9823E+138	1.2098E+142	1.5388E+145	1.2969E+148
119	3.6212E+132	2.4453E+136	8.9761E+139	1.9367E+143	2.618E+146	2.3372E+149
120	4.7078E+133	3.424E+137	1.3468E+141	3.1002E+144	4.4539E+147	4.2118E+150

S / R	19	20	21	22	23	24
81	2.9663E+103	1.7444E+105	8.241E+106	3.1833E+108	1.0207E+110	2.7527E+111
82	5.7114E+104	3.548E+106	1.7672E+108	7.1846E+109	2.4209E+111	6.8515E+112
83	1.0989E+106	7.2103E+107	3.7857E+109	1.6195E+111	5.7333E+112	1.7025E+114
84	2.1128E+107	1.464E+109	8.1014E+110	3.6462E+112	1.3559E+114	4.2235E+115
85	4.0596E+108	2.9703E+110	1.732E+112	8.1998E+113	3.2025E+115	1.0462E+117
86	7.7955E+109	6.0218E+111	3.6997E+113	1.8421E+115	7.5543E+116	2.5877E+118
87	1.4961E+111	1.22E+113	7.8958E+114	4.1339E+116	1.7798E+118	6.3918E+119
88	2.8696E+112	2.4698E+114	1.6837E+116	9.2684E+117	4.1887E+119	1.5767E+121
89	5.5012E+113	4.9971E+115	3.5877E+117	2.0761E+119	9.8472E+120	3.8847E+122
90	1.0541E+115	1.0104E+117	7.6391E+118	4.6463E+120	2.3126E+122	9.5597E+123
91	2.0189E+116	2.0419E+118	1.6254E+120	1.039E+122	5.4259E+123	2.3498E+125
92	3.8652E+117	4.1242E+119	3.4563E+121	2.3216E+123	1.2718E+125	5.7698E+126
93	7.3966E+118	8.3257E+120	7.3448E+122	5.1835E+124	2.9787E+126	1.4153E+128
94	1.4149E+120	1.6799E+122	1.5599E+124	1.1565E+126	6.9701E+127	3.4682E+129
95	2.7056E+121	3.3882E+123	3.3111E+125	2.5787E+127	1.6297E+129	8.4908E+130
96	5.1719E+122	6.8305E+124	7.0244E+126	5.7459E+128	3.8077E+130	2.0769E+132
97	9.8831E+123	1.3764E+126	1.4895E+128	1.2796E+130	8.8898E+131	5.076E+133
98	1.888E+125	2.7726E+127	3.1568E+129	2.8478E+131	2.0741E+133	1.2396E+135
99	3.6057E+126	5.583E+128	6.6875E+130	6.3346E+132	4.8359E+134	3.0247E+136
100	6.8841E+127	1.1238E+130	1.4161E+132	1.4083E+134	1.1268E+136	7.3755E+137
101	1.314E+129	2.2614E+131	2.9974E+133	3.1295E+135	2.6241E+137	1.7972E+139
102	2.5075E+130	4.5491E+132	6.342E+134	6.9507E+136	6.1074E+138	4.3761E+140
103	4.7839E+131	9.1483E+133	1.3414E+136	1.5431E+138	1.4207E+140	1.0649E+142
104	9.1249E+132	1.8392E+135	2.8361E+137	3.4244E+139	3.3031E+141	2.5899E+143
105	1.7401E+134	3.6967E+136	5.9944E+138	7.596E+140	7.6758E+142	6.2951E+144
106	3.3178E+135	7.4282E+137	1.2666E+140	1.6843E+142	1.7829E+144	1.5293E+146
107	6.3247E+136	1.4923E+139	2.6754E+141	3.7333E+143	4.1394E+145	3.713E+147
108	1.2055E+138	2.9972E+140	5.6498E+142	8.2722E+144	9.6066E+146	9.0105E+148
109	2.2972E+139	6.0185E+141	1.1927E+144	1.8323E+146	2.2285E+148	2.1856E+150
110	4.3768E+140	1.2083E+143	2.5174E+145	4.0573E+147	5.1678E+149	5.2989E+151
111	8.3381E+141	2.4254E+144	5.3119E+146	8.9815E+148	1.1979E+151	1.2841E+153
112	1.5882E+143	4.8674E+145	1.1206E+148	1.9876E+150	2.7759E+152	3.1107E+154
113	3.0248E+144	9.7666E+146	2.3635E+149	4.3974E+151	6.4302E+153	7.5322E+155
114	5.7601E+145	1.9594E+148	4.9838E+150	9.7263E+152	1.4891E+155	1.8232E+157
115	1.0967E+147	3.9302E+149	1.0507E+152	2.1508E+154	3.4472E+156	4.4113E+158
116	2.088E+148	7.8824E+150	2.2148E+153	4.7548E+155	7.9781E+157	1.067E+160
117	3.9748E+149	1.5807E+152	4.6675E+154	1.0509E+157	1.8459E+159	2.5799E+161
118	7.5659E+150	3.1693E+153	9.835E+155	2.3223E+158	4.2697E+160	6.2362E+162
119	1.44E+152	6.3537E+154	2.072E+157	5.1307E+159	9.8738E+161	1.5069E+164
120	2.7404E+153	1.2736E+156	4.3646E+158	1.1333E+161	2.2828E+163	3.6403E+165

S / R	25	26	27	28	29	30
81	6.3155E+112	1.2451E+114	2.128E+115	3.1777E+116	4.1751E+117	4.8564E+118
82	1.6477E+114	3.4014E+115	6.0817E+116	9.4934E+117	1.3029E+119	1.5822E+120
83	4.2905E+115	9.272E+116	1.7339E+118	2.8285E+119	4.0538E+120	5.1374E+121
84	1.1152E+117	2.5223E+118	4.9318E+119	8.4052E+120	1.2576E+122	1.6628E+123
85	2.8935E+118	6.8478E+119	1.3997E+121	2.4915E+122	3.8909E+123	5.3658E+124
86	7.4954E+119	1.8557E+121	3.9641E+122	7.3682E+123	1.2006E+125	1.7265E+126
87	1.9385E+121	5.0196E+122	1.1204E+124	2.1741E+125	3.6955E+126	5.5396E+127
88	5.0062E+122	1.3555E+124	3.1606E+125	6.4012E+126	1.1347E+128	1.7727E+129
89	1.291E+124	3.6545E+125	8.8996E+126	1.8808E+128	3.4764E+129	5.6586E+130
90	3.3245E+125	9.8373E+126	2.5016E+128	5.5155E+129	1.0627E+131	1.8019E+132
91	8.5503E+126	2.6441E+128	7.0199E+129	1.6144E+131	3.2417E+132	5.7244E+133
92	2.1963E+128	7.097E+129	1.9668E+131	4.7168E+132	9.8692E+133	1.8146E+135
93	5.635E+129	1.9023E+131	5.5018E+132	1.3758E+134	2.9989E+135	5.7398E+136
94	1.4441E+131	5.0926E+132	1.5369E+134	4.0062E+135	9.0957E+136	1.8119E+138
95	3.6971E+132	1.3616E+134	4.287E+135	1.1648E+137	2.7539E+138	5.7086E+139
96	9.4549E+133	3.6363E+135	1.1943E+137	3.3814E+138	8.3242E+139	1.7952E+141
97	2.4157E+135	9.7003E+136	3.3227E+138	9.8024E+139	2.5121E+141	5.6353E+142
98	6.166E+136	2.5849E+138	9.2332E+139	2.8377E+141	7.5693E+142	1.766E+144
99	1.5725E+138	6.881E+139	2.5627E+141	8.2041E+142	2.2774E+144	5.525E+145
100	4.0069E+139	1.8299E+141	7.1052E+142	2.3689E+144	6.8423E+145	1.7258E+147
101	1.0202E+141	4.862E+142	1.9678E+144	6.8319E+145	2.053E+147	5.3827E+148
102	2.5953E+142	1.2907E+144	5.4444E+145	1.968E+147	6.1518E+148	1.6764E+150
103	6.5977E+143	3.4232E+145	1.5048E+147	5.6629E+148	1.8411E+150	5.2138E+151
104	1.676E+145	9.0718E+146	4.1555E+148	1.6277E+150	5.5034E+151	1.6194E+153
105	4.2549E+146	2.4022E+148	1.1465E+150	4.674E+151	1.6432E+153	5.0232E+154
106	1.0795E+148	6.3565E+149	3.1603E+151	1.3408E+153	4.9008E+154	1.5562E+156
107	2.7369E+149	1.6807E+151	8.7045E+152	3.8428E+154	1.4601E+156	4.8158E+157
108	6.935E+150	4.4411E+152	2.3956E+154	1.1004E+156	4.3457E+157	1.4885E+159
109	1.7563E+152	1.1727E+154	6.588E+155	3.1481E+157	1.2922E+159	4.596E+160
110	4.4453E+153	3.0947E+155	1.8104E+157	8.9991E+158	3.8386E+160	1.4176E+162
111	1.1246E+155	8.1619E+156	4.9717E+158	2.5704E+160	1.1393E+162	4.3678E+163
112	2.8435E+156	2.1513E+158	1.3644E+160	7.3365E+161	3.3785E+163	1.3445E+165
113	7.1866E+157	5.6674E+159	3.742E+161	2.0924E+163	1.001E+165	4.1349E+166
114	1.8155E+159	1.4922E+161	1.0256E+163	5.9635E+164	2.9637E+166	1.2705E+168
115	4.5843E+160	3.9269E+162	2.8095E+164	1.6985E+166	8.7677E+167	3.9004E+169
116	1.1571E+162	1.0329E+164	7.6917E+165	4.8345E+167	2.5919E+169	1.1964E+171
117	2.9194E+163	2.7157E+165	2.1046E+167	1.3752E+169	7.6566E+170	3.6671E+172
118	7.3631E+164	7.1367E+166	5.7558E+168	3.9095E+170	2.2603E+172	1.1231E+174
119	1.8564E+166	1.8747E+168	1.5733E+170	1.1108E+172	6.6683E+173	3.4371E+175
120	4.6786E+167	4.9224E+169	4.2987E+171	3.1542E+173	1.966E+175	1.0511E+177

S / R	31	32	33	34	35	36
81	5.0287E+119	4.6586E+120	3.8785E+121	2.9134E+122	1.9819E+123	1.2249E+124
82	1.7095E+121	1.6517E+122	1.4336E+123	1.1224E+124	7.9562E+124	5.1229E+125
83	5.7898E+122	5.8324E+123	5.276E+124	4.3037E+125	3.1775E+126	2.1307E+127
84	1.9541E+124	2.0516E+125	1.9336E+126	1.6426E+127	1.2628E+128	8.8144E+128
85	6.5732E+125	7.1906E+126	7.0578E+127	6.2424E+128	4.9946E+129	3.6278E+130
86	2.204E+127	2.5113E+128	2.5664E+129	2.3624E+130	1.9666E+131	1.4858E+132
87	7.3677E+128	8.7415E+129	9.2977E+130	8.9047E+131	7.7099E+132	6.0569E+133
88	2.4557E+130	3.0331E+131	3.3567E+132	3.3437E+133	3.0101E+134	2.458E+135
89	8.1622E+131	1.0492E+133	1.2078E+134	1.251E+135	1.1706E+136	9.9325E+136
90	2.7057E+133	3.6185E+134	4.332E+135	4.664E+136	4.5349E+137	3.9971E+138
91	8.9463E+134	1.2445E+136	1.549E+137	1.7331E+138	1.7504E+139	1.6022E+140
92	2.9508E+136	4.2687E+137	5.5223E+138	6.419E+139	6.7331E+140	6.3982E+141
93	9.71E+137	1.4604E+139	1.9632E+140	2.3702E+141	2.5813E+142	2.5457E+143
94	3.188E+139	4.984E+140	6.9606E+141	8.7263E+142	9.864E+143	1.0094E+145
95	1.0445E+141	1.6969E+142	2.4615E+143	3.2036E+144	3.7578E+145	3.9889E+146
96	3.4148E+142	5.7643E+143	8.6828E+144	1.1729E+146	1.4274E+147	1.5713E+148
97	1.1142E+144	1.9539E+145	3.0555E+146	4.2831E+147	5.4063E+148	6.1705E+149
98	3.6288E+145	6.6089E+146	1.0728E+148	1.5601E+149	2.0421E+150	2.416E+151
99	1.1797E+147	2.231E+148	3.7584E+149	5.6693E+150	7.6934E+151	9.4327E+152
100	3.8283E+148	7.5166E+149	1.3139E+151	2.0553E+152	2.8911E+153	3.6728E+154
101	1.2403E+150	2.5278E+151	4.5838E+152	7.4348E+153	1.0838E+155	1.4263E+156
102	4.0117E+151	8.4859E+152	1.5961E+154	2.6837E+155	4.0536E+156	5.5248E+157
103	1.2956E+153	2.8439E+154	5.5471E+155	9.6672E+156	1.5127E+158	2.1348E+159
104	4.178E+154	9.515E+155	1.9244E+157	3.4755E+158	5.6328E+159	8.23E+160
105	1.3454E+156	3.1785E+157	6.6645E+158	1.2471E+160	2.0931E+161	3.1656E+162
106	4.3264E+157	1.0602E+159	2.3042E+160	4.4667E+161	7.7624E+162	1.215E+164
107	1.3894E+159	3.531E+160	7.9536E+161	1.597E+163	2.8732E+164	4.6533E+165
108	4.4565E+160	1.1744E+162	2.7412E+163	5.7003E+164	1.0615E+166	1.7786E+167
109	1.4277E+162	3.9006E+163	9.4336E+164	2.0313E+166	3.9148E+167	6.7852E+168
110	4.5682E+163	1.2939E+165	3.2418E+166	7.2272E+167	1.4413E+169	2.5836E+170
111	1.4601E+165	4.2866E+166	1.1125E+168	2.5675E+169	5.2974E+170	9.8198E+171
112	4.6617E+166	1.4184E+168	3.8127E+169	9.1076E+170	1.9439E+172	3.7258E+173
113	1.4868E+168	4.6882E+169	1.305E+171	3.2262E+172	7.1226E+173	1.4113E+175
114	4.7373E+169	1.5478E+171	4.4612E+172	1.1413E+174	2.6058E+175	5.337E+176
115	1.5079E+171	5.1045E+172	1.5233E+174	4.032E+175	9.5198E+176	2.0151E+178
116	4.7955E+172	1.6817E+174	5.1952E+175	1.4227E+177	3.4731E+178	7.5972E+179
117	1.5237E+174	5.5349E+175	1.7699E+177	5.0138E+178	1.2654E+180	2.86E+181
118	4.8372E+175	1.8199E+177	6.0234E+178	1.7649E+180	4.6043E+181	1.0752E+183
119	1.5343E+177	5.9785E+178	2.0478E+180	6.2053E+181	1.6733E+183	4.0364E+184
120	4.863E+178	1.9622E+180	6.955E+181	2.1794E+183	6.0736E+184	1.5133E+186

S / R	37	38	39	40
81	6.898E+124	3.5493E+125	1.6726E+126	7.2349E+126
82	3.0054E+126	1.6109E+127	7.9075E+127	3.563E+128
83	1.3016E+128	7.2633E+128	3.7122E+129	1.7415E+130
84	5.6041E+129	3.2547E+130	1.731E+131	8.4509E+131
85	2.3997E+131	1.4497E+132	8.0203E+132	4.0728E+133
86	1.0221E+133	6.4208E+133	3.6933E+134	1.9499E+135
87	4.3315E+134	2.8283E+135	1.6908E+136	9.277E+136
88	1.8268E+136	1.2394E+137	7.6972E+137	4.3871E+138
89	7.6685E+137	5.4037E+138	3.4852E+139	2.0627E+140
90	3.2049E+139	2.3448E+140	1.57E+141	9.645E+141
91	1.3337E+141	1.0128E+142	7.0375E+142	4.486E+143
92	5.5275E+142	4.3555E+143	3.1396E+144	2.0759E+145
93	2.2819E+144	1.8651E+145	1.3943E+146	9.5595E+146
94	9.3849E+145	7.9546E+146	6.1652E+147	4.3815E+148
95	3.8459E+147	3.3794E+148	2.7147E+149	1.9992E+150
96	1.5706E+149	1.4303E+150	1.1905E+151	9.0827E+151
97	6.3925E+150	6.032E+151	5.2008E+152	4.1093E+153
98	2.5935E+152	2.5351E+153	2.2636E+154	1.8517E+155
99	1.049E+154	1.0619E+155	9.8166E+155	8.3124E+156
100	4.2303E+155	4.4338E+156	4.2426E+157	3.7176E+158
101	1.7011E+157	1.8456E+158	1.8275E+159	1.6568E+160
102	6.8218E+158	7.6596E+159	7.8472E+160	7.358E+161
103	2.7285E+160	3.1699E+161	3.3591E+162	3.2571E+163
104	1.0885E+162	1.3082E+163	1.4337E+164	1.4372E+165
105	4.3321E+163	5.385E+164	6.1016E+165	6.3223E+166
106	1.72E+165	2.2109E+166	2.5896E+167	2.773E+168
107	6.8135E+166	9.055E+167	1.0962E+169	1.2128E+170
108	2.6932E+168	3.6998E+169	4.6283E+170	5.2896E+171
109	1.0623E+170	1.5083E+171	1.9493E+172	2.301E+173
110	4.1815E+171	6.1351E+172	8.1905E+173	9.9836E+174
111	1.6427E+173	2.4902E+174	3.4336E+175	4.3211E+176
112	6.4415E+174	1.0087E+176	1.4362E+177	1.8658E+178
113	2.5212E+176	4.0779E+177	5.9946E+178	8.0375E+179
114	9.8506E+177	1.6454E+179	2.4969E+180	3.4548E+181
115	3.8422E+179	6.6269E+180	1.038E+182	1.4818E+183
116	1.4962E+181	2.6642E+182	4.3066E+183	6.3424E+184
117	5.817E+182	1.0693E+184	1.7835E+185	2.7092E+186
118	2.2581E+184	4.2842E+185	7.3725E+186	1.155E+188
119	8.7528E+185	1.7138E+187	3.0424E+188	4.915E+189
120	3.3879E+187	6.8451E+188	1.2534E+190	2.0877E+191

VITA

Yingying Yang was born in Qingdao, China and is a Chinese citizen. She received her Bachelor of Art degree in Statistics from Rutgers University – New Brunswick in 2003. She was also minor in Economics. She came to America as a foreign exchange student in 1998. She attended and graduated from Doniphan High School in Missouri in 1999.