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Einstein's Equations in Vacuum Spacetimes with Two Spacelinke Killing Vectors Using Affine Projection Tensor Geometry

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Einstein's Equations In Vacuum Spacetimes With Two Spacelike Killing Vectors Using Affine Projection Tensor Geometry

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University

by

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This thesis is dedicated to my loving wife Kamela, who has sacrificed so much so that I could do this work.

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Abstract

EINSTEIN'S EQUATIONS IN VACUUM SPACETIMES WITH TWO SPACELIKE KILLING VECTORS USING AFFINE PROJECTION TENSOR GEOMETRY

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Einstein's equations in vacuum spacetimes with two spacelike killing vectors are explored using affine projection tensor geometry. By doing a semi-conformal transformation on the metric, a new "fiducial" geometry is constructed using a projection tensor field. This fiducial geometry provides coordinate independent information about the underlying structure of the spacetime without the use of an explicit form of the metric tensor.

The Background

Introduction

When doing general relativity, one of the hardest tasks is determining exact solutions to Einstein's equations and, once they are found, determining the properties of the spacetimes they describe. This is a difficult task because of the complexity of Einstein's equations, and therefore finding a new solution is uncommon. A typical method used to simplify the problem is to find solutions that exhibit one or more symmetries. In addition, one may try to find solutions representing "empty" universes or "dust filled" universes with very simple matter. Many solutions of these types have been discovered and the process of determining the properties of the underlying spacetimes is the difficult task ahead.

One method is to use projection tensor fields to decompose the Riemann curvature tensor. A projection tensor projects a higher dimensional object onto a lower dimensional one. For example, a 4-vector can be projected into 3-dimensions to get the space component or into 1 dimension to get its time component. This is an excellent method for looking at the hypersurfaces of spacetimes that have symmetries. The problem in the past has been that the projection tensor fields used have assumed a normal projection, with co-dimension one. This produces problems when one tries to project on two-surfaces or lightlike surfaces and other situations where the metric is degenerate.

In 1974, Robert Gowdy wrote the paper entitled *Vacuum Spacetimes with Two-Parameter Spacelike Isometry Groups and Compact Invariant Hypersurfaces: Topologies and Boundary Conditions* [4]. This paper works out a set of exact solutions to Einstein's equations in an empty universe with cylindrical symmetries.

Earlier this year, Gowdy completed a paper on using affine projection tensor geometry to decompose the curvature tensor in Einstein's equations [1]. In his method, the projection can be tilted and the connection, as well as the co-dimension, can be arbitrary. This does away with the problems mentioned before in normal projection tensor techniques. Using this one can decompose the different hypersurfaces of the spacetime and look at them separately.

This thesis will tackle the empty universe with cylindrical symmetries problem using Gowdy's affine projection tensor techniques. It will begin with a brief overview of differential geometry and the geometric objects that will be used in it. A review of the projection tensor techniques and the new geometric objects it produces is then discussed. Assembly notation, another of Gowdy's recent techniques, will be discussed to simplify the number of equations the affine projection tensor techniques produce. The remainder of the thesis will be devoted to the decomposition of the curvature tensor in the proposed spacetime, and then a brief conclusion of the work.

Justification for the Thesis

There are several reasons for wanting to complete this work. First, we will get a better understanding of Gowdy universes. Today, there is still much work being done in Gowdy universes [18,19,20]. The reason for this is that they represent inhomogeneous cosmologies. Our present universe is inhomogeneous. This is pretty obvious; all one has to do is look up in the night sky and see stars and voids. When astronomers study large structures, there seems to be no "largest" object. Stars are in galaxies, galaxies are in galactic clusters, and more recently a "Great Wall" of galactic clusters has been observed. To explain why we have these, the early universe must have contained density fluctuations from which these structures formed. Resent experiments have shown that this is true. The Cosmic Background Explorer (COBE) has shown that the microwave background radiation is anisotropic. It has measured a temperature variation $\Delta T/T$ of 6×10^{-6} in the background[16]. This implies that at the time the radiation dominated universe gave way to the matter dominated universe, it had to be inhomogeneous.

Another benefit from this research is that it provides a tool for doing stellar dynamics. In rotating systems, such as a rotating star, there is no longer a spherical symmetry. These systems then have only two symmetries, or two Killing vector fields, a timelike one and a spacelike. The problem is then of the same type as in this thesis, since the projection is arbitrary and can include either space or time components.

In the real universe, the solutions of Einstein's equations are very complex. By finding solutions of easy systems by making simplifications in spacetimes, we can gain an insight into the machinery of general relativity. These simple and exact solutions also provide a testing ground for computer codes. Complex programs that model the universe need to be tested againts known solutions so that their results in modeling real spacetimes may be correct.

Differential Geometry

In order to understand the methods and equations contained is this thesis, some basic notations for our geometric objects must first be defined. We will start with a world line. A *world line* is a curve that a particle follows in four dimensional spacetime. It is just the history of a particle, giving the positions of the particle at various times. We will

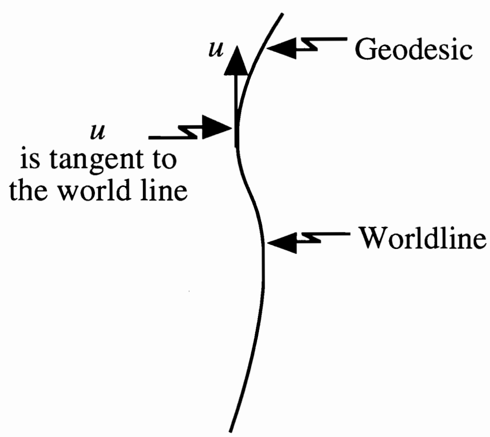

Figure 1. A world line.

often talk about the tangent vector to the world line, or the 4-velocity vector u (See Figure 1). If we divide the world line up by a parameter λ, then u is the derivative along the world line with respect to λ, or

$$u = \frac{\partial}{\partial \lambda} = \partial_\lambda = u^\alpha e_\alpha , \qquad (1)$$

where the last notation uses the Einstein summation convention for repeated indices. u^α is the vector component and e_α is a basis vector. If we let λ be the "ticks" of a clock moving along the curve, it is an affine parameter. That is the type used in this thesis.

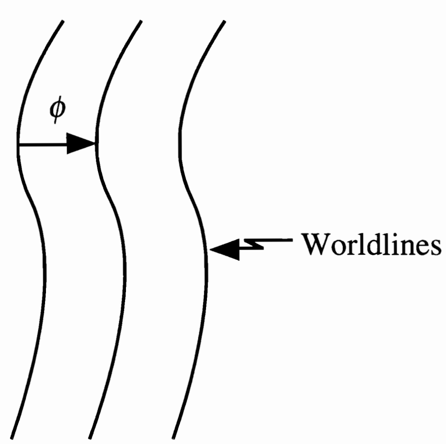

Figure 2. The one-form, ϕ, dual to the tangent space of the world lines.

Vectors live in the *tangent space* of the world line, T_p.

A one-form, ϕ, is the dual to the tangent vectors (See Figure 2). It describes what is happening between adjacent world lines. ϕ can be written as

$$\phi = \phi_\alpha \omega^\alpha , \qquad (2)$$

where ω is the basis one-form such that

$$\omega^\alpha e_\beta = \delta^\alpha{}_\beta .$$

One-forms live in the *cotangent space* to the world line, $\hat{T}_p$.

There is also a need to determine distances in spacetime. For this a metric is needed. The *metric*, g, is a "machine" that "eats" vectors. By putting in two vectors u and v, it gives the inner or dot product of the two vectors.

$$u \cdot v = g(u,v)$$

One can see here that g is a tensor. By using the fact that it is linear and the component form of the vectors, we get the more familiar notation,

$$g(u,v) = g\left(u^\alpha e_\alpha, v^\beta e_\beta\right) = u^\alpha v^\beta g\left(e_\alpha, e_\beta\right) = u^\alpha v^\beta g_{\alpha\beta}. \tag{3}$$

The metric is what we use to find distances since it describes the geometry of the spacetime. It is a *map* from the tangent space, T_p to the real numbers.

$$g : T_p \times T_p \to \Re$$

The inner product of the one-forms is given by the inverse metric, g^{-1}, where

$$\phi \cdot \xi = g^{-1}(\phi,\xi).$$

Using the linearity of the tensor, we get

$$g^{-1}(\phi,\xi) = g^{-1}\left(\phi_\alpha \omega^\alpha, \xi_\beta \omega^\beta\right) = \phi_\alpha \xi_\beta g^{-1}\left(\omega^\alpha, \omega^\beta\right) = \phi_\alpha \xi_\beta g^{\alpha\beta}. \tag{4}$$

The inverse metric maps from the cotangent space, $\hat{T}_p$ to the real numbers, i.e.,

$$g^{-1} : \hat{T}_p \times \hat{T}_p \to \Re.$$

I must now discuss how to take a derivative. The directional derivative of the vector v in the direction of u in spacetime is given by

$$\nabla_u v = \nabla_{u^\alpha e_\alpha} v = u^\alpha \nabla_{e_\alpha} v.$$

Notice that the derivative is linear in the direction that it is taken. Using the chain rule, we get the following

$$u^{\alpha}\nabla_{e_\alpha}v = u^{\alpha}\left(\nabla_{e_\alpha}v^{\beta}\right)e_{\beta} + u^{\alpha}v^{\beta}\nabla_{e_\alpha}e_{\beta},$$

or, using the usual notation,

$$u^{\alpha}\nabla_{\alpha}v = u^{\alpha}\left(\nabla_{\alpha}v^{\beta}\right)e_{\beta} + u^{\alpha}v^{\beta}\nabla_{\alpha}e_{\beta}.$$

For the second term, we need the *connection*. The connection tells us how to take a derivative in the spacetime:

$$u^{\alpha}v^{\beta}\nabla_{\alpha}e_{\beta} = u^{\alpha}v^{\gamma}\Gamma^{\beta}{}_{\gamma\alpha}e_{\beta}$$

where Γ is the *connection coefficient*. This yields

$$u^{\alpha}\nabla_{\alpha}v = u^{\alpha}\left(\nabla_{\alpha}v^{\beta}\right)e_{\beta} + u^{\alpha}v^{\gamma}\Gamma^{\beta}{}_{\gamma\alpha}e_{\beta} \tag{5}$$

Similarly, the covariant derivative of a one-form is given by

$$\nabla_{\alpha}\phi = \nabla_{\alpha}\left(\phi_{\beta}\omega^{\beta}\right) = \left(\nabla_{\alpha}\phi_{\beta}\right)\omega^{\beta} - \phi_{\gamma}\Gamma^{\gamma}{}_{\beta\alpha}\omega^{\beta}. \tag{6}$$

Notice that the all that is different is the sign of the connection coefficient and we contract with the other index. The last index always refers to the differentiating index.

The *metricity*, Q, is the covariant derivative of the metric tensor and describes the relationship between the metric and the connection.

$$Q_{\alpha\beta\gamma} = \nabla_{\gamma}g_{\alpha\beta} = e_{\gamma}g_{\alpha\beta} - g_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\gamma} - g_{\sigma\beta}\Gamma^{\sigma}{}_{\alpha\gamma} \tag{7}$$

or

$$Q^{\alpha\beta}{}_{\gamma} = -\nabla_{\gamma}g^{\alpha\beta} = -e_{\gamma}g^{\alpha\beta} - g^{\alpha\sigma}\Gamma^{\beta}{}_{\sigma\gamma} - g^{\sigma\beta}\Gamma^{\alpha}{}_{\sigma\gamma}. \tag{8}$$

If the connection is *metric compatible*, then we say that

$$Q_{\alpha\beta\gamma} = 0. \tag{9}$$

The *commutation coefficients*, $c^{\gamma}{}_{\alpha\beta}$, are defined by

$$\left[e_{\alpha}, e_{\beta}\right] = c^{\gamma}{}_{\alpha\beta}e_{\gamma}.$$

If we now take the commutation between two covariant derivative operators acting on a function *f*, we get

$$\left[\nabla_\alpha, \nabla_\beta\right] f = \left(c^\gamma{}_{\alpha\beta} e_\gamma - 2\Gamma^\gamma{}_{[\alpha\beta]} e_\gamma\right) f = S^\gamma{}_{\alpha\beta} e_\gamma f = S^\gamma{}_{\alpha\beta} \nabla_\gamma f, \tag{10}$$

where $S^\gamma{}_{\alpha\beta}$ is the *torsion* of the spacetime. This is just taking the derivative of a function at a point in one direction and then another, and then subtracting off the derivatives taken in reverse order. In general the torsion is set to zero in general relativity to simplify Einstein's equations.

Using these, it can be shown that the connection coefficients can be written as

$$\Gamma^\beta{}_{\gamma\alpha} = \frac{1}{2} g^{\beta\sigma}\left(e_\alpha g_{\sigma\gamma} + e_\gamma g_{\alpha\sigma} - e_\sigma g_{\gamma\alpha} - g_{\gamma\rho} c^\rho{}_{\sigma\alpha} - g_{\sigma\rho} c^\rho{}_{\alpha\gamma} + g_{\alpha\rho} c^\rho{}_{\gamma\sigma} + S_{\gamma\sigma\alpha} + S_{\sigma\alpha\gamma} - S_{\alpha\gamma\sigma}\right) \tag{11}$$

where the c's are the commutation coefficients of the basis and the S's are torsion. In most cases, $c^\rho{}_{\gamma\sigma}$ is zero. From this we can see that the connection depends greatly on the metric being used.

The *Riemann curvature tensor*, $R_\rho{}^\sigma{}_{\alpha\beta}$, is defined as

$$u^\rho R_\rho{}^\sigma{}_{\alpha\beta} = \left\{\left[\nabla_\beta, \nabla_\alpha\right] - S^\gamma{}_{\alpha\beta} \nabla_\gamma\right\} u^\sigma. \tag{12}$$

Physically, curvature tells us what happens when a vector is parallel transported around a closed loop on the manifold. If we contract the first two indicies, we get the *Ricci curvature tensor.*

$$R_\sigma{}^\sigma{}_{\alpha\beta} = R_{\alpha\beta} \tag{13}$$

These are the basic geometric objects that will be used in this thesis.

Lie Derivatives

The Lie derivative tells what happens when we drag one vector field along another. The Lie derivative of a vector field u with respect to a vector field N, is obtained from

$$\left(\pounds_N u\right) f = [N, u] f \tag{14}$$

for any function f. We will need to relate the Lie derivative to the covariant derivative, so we write the commutator as

$$\left(\pounds_N u\right)^\alpha \nabla_\alpha f = \left[\left(N^\beta \nabla_\beta\right)\left(u^\gamma \nabla_\gamma\right) - \left(u^\gamma \nabla_\gamma\right)\left(N^\beta \nabla_\beta\right)\right] f .$$

From Eq. (10), the definition of the torsion tensor, we can define the following quantity

$$\nabla'_\alpha N^\beta = \nabla_\alpha N^\beta - S^\beta{}_{\alpha\gamma} N^\gamma .$$

then the Lie derivative is given by

$$\left(\pounds_N u\right)^\alpha = N^\beta \nabla_\beta u^\alpha - u^\beta \nabla'_\beta N^\alpha .$$

The Lie derivative of an arbitrary rank tensor T is then related to the covariant derivatives of the tensor by

$$\begin{aligned}
\left(\pounds_N T\right)^{\alpha_1 \alpha_2 \cdots \alpha_n}{}_{\beta_1 \beta_2 \cdots \beta_m} = {} & N^\gamma \nabla_\gamma T^{\alpha_1 \alpha_2 \cdots \alpha_n}{}_{\beta_1 \beta_2 \cdots \beta_m} - T^{\gamma \alpha_2 \cdots \alpha_n}{}_{\beta_1 \beta_2 \cdots \beta_m} \nabla'_\gamma N^{\alpha_1} \\
& - T^{\alpha_1 \gamma \cdots \alpha_n}{}_{\beta_1 \beta_2 \cdots \beta_m} \nabla'_\gamma N^{\alpha_2} - \cdots - T^{\alpha_1 \alpha_2 \cdots \gamma}{}_{\beta_1 \beta_2 \cdots \beta_m} \nabla'_\gamma N^{\alpha_n} \\
& + T^{\alpha_1 \alpha_2 \cdots \alpha_n}{}_{\gamma \beta_2 \cdots \beta_m} \nabla'_{\beta_1} N^\gamma + T^{\alpha_1 \alpha_2 \cdots \alpha_n}{}_{\beta_1 \gamma \cdots \beta_m} \nabla'_{\beta_2} N^\gamma \\
& + \cdots + T^{\alpha_1 \alpha_2 \cdots \alpha_n}{}_{\beta_1 \beta_2 \cdots \gamma} \nabla'_{\beta_m} N^\gamma .
\end{aligned} \tag{15}$$

Affine Projection Tensors

Using the methods in Gowdy's new projection tensor techniques [1], certain geometric objects can be formed that exist without ever defining what the metric of the spacetime is. A brief overview follows.

I will start with defining the projection tensors. $H^{\alpha}{}_{\beta}$ is a projection tensor field that assigns each point P of the spacetime manifold a linear map into the tangent space $H(P): T_p \to T_p$ such that

$$H^2 = H. \tag{16}$$

What this means is that *H* takes things that live in the four-dimensional spacetime manifold and projects them into the subspace HT_p. *H* also act as the identity operator on HT_p. The complementary projection tensor, $V^{\alpha}{}_{\beta}$, is to $H^{\alpha}{}_{\beta}$ such that

$$H + V = I, \tag{17}$$

where *I* is the identity map. Also note that

$$HV = 0. \tag{18}$$

Take for example a tensor $T_{\alpha}{}^{\beta}{}_{\delta\gamma}$. We can project each of its indices into either HT_p, $H^*\hat{T}_p$, VT_p or $V^*\hat{T}_p$.

$$T\left[{}_{H}{}^{V}{}_{HH}\right]_{\alpha}{}^{\beta}{}_{\delta\gamma} = H^{\mu}{}_{\alpha} V^{\beta}{}_{\nu} H^{\sigma}{}_{\delta} H^{\rho}{}_{\gamma} T_{\mu}{}^{\nu}{}_{\sigma\rho}.$$

This is called a *projected* tensor. The notation in the brackets tells what projection tensor is used to project the corresponding index. The projected metric tensors are

$$\begin{aligned} g_{HH\alpha\beta} &= H^{\rho}{}_{\alpha} H^{\sigma}{}_{\beta} g_{\rho\sigma} \\ g_{VV\alpha\beta} &= V^{\rho}{}_{\alpha} V^{\sigma}{}_{\beta} g_{\rho\sigma} \end{aligned} \tag{19}$$

and the cross projected metric tensors are

$$\begin{aligned} g_{HV\alpha\beta} &= H^{\rho}{}_{\alpha} V^{\sigma}{}_{\beta} g_{\rho\sigma} \\ g_{VH\alpha\beta} &= V^{\rho}{}_{\alpha} H^{\sigma}{}_{\beta} g_{\rho\sigma}. \end{aligned}$$

The full metric in terms of these *restricted* or projected metrics is

$$g_{\alpha\beta} = g_{HH\alpha\beta} + g_{VV\alpha\beta} + g_{HV\alpha\beta} + g_{VH\alpha\beta}. \tag{20}$$

Similarly, we have the restricted inverse or form-metrics

$$\begin{aligned}
g^{HH\alpha\beta} &= H^{\alpha}{}_{\rho} H^{\beta}{}_{\sigma} g^{\rho\sigma} \\
g^{VV\alpha\beta} &= V^{\alpha}{}_{\rho} V^{\beta}{}_{\sigma} g^{\rho\sigma} \\
g^{HV\alpha\beta} &= H^{\alpha}{}_{\rho} V^{\beta}{}_{\sigma} g^{\rho\sigma} \\
g^{VH\alpha\beta} &= V^{\alpha}{}_{\rho} H^{\beta}{}_{\sigma} g^{\rho\sigma}.
\end{aligned}$$

Letting X, Y and Z represent either H or V, the general form of the form-metrics becomes

$$g^{XY\alpha\beta} = X^{\alpha}{}_{\rho} Y^{\beta}{}_{\sigma} g^{\rho\sigma}. \tag{21}$$

There will also be times when we need to know the derivative of the projection tensor. In order to do this, we will use a set of objects called the *projection curvature tensors*.

$$\begin{aligned}
h_H{}^{\alpha}{}_{\beta\gamma} &= H^{\rho}{}_{\beta} H^{\sigma}{}_{\gamma} \nabla_{\sigma} H^{\alpha}{}_{\rho} \\
h^T_H{}_{\beta}{}^{\alpha}{}_{\gamma} &= H^{\alpha}{}_{\rho} H^{\sigma}{}_{\gamma} \nabla_{\sigma} H^{\rho}{}_{\beta} \\
h_V{}^{\alpha}{}_{\beta\gamma} &= V^{\rho}{}_{\beta} V^{\sigma}{}_{\gamma} \nabla_{\sigma} V^{\alpha}{}_{\rho} \\
h^T_V{}_{\beta}{}^{\alpha}{}_{\gamma} &= V^{\alpha}{}_{\rho} V^{\sigma}{}_{\gamma} \nabla_{\sigma} V^{\rho}{}_{\beta}.
\end{aligned} \tag{22}$$

The projection curvatures obey the following projection identities:

$$\begin{aligned}
h_H\left[{}^{V}{}_{H\,H}\right]^{\alpha}{}_{\beta\gamma} &= V^{\alpha}{}_{\delta} h_H{}^{\delta}{}_{\sigma\rho} H^{\sigma}{}_{\beta} H^{\rho}{}_{\gamma} = h_H{}^{\alpha}{}_{\beta\gamma} \\
h^T_H\left[{}_{H}{}^{V}{}_{H}\right]_{\beta}{}^{\alpha}{}_{\gamma} &= H^{\sigma}{}_{\beta} V^{\alpha}{}_{\delta} h^T_H{}_{\sigma}{}^{\delta}{}_{\rho} H^{\rho}{}_{\gamma} = h^T_H{}_{\beta}{}^{\alpha}{}_{\gamma} \\
h_V\left[{}^{H}{}_{V\,V}\right]^{\alpha}{}_{\beta\gamma} &= H^{\alpha}{}_{\delta} h_V{}^{\delta}{}_{\sigma\rho} V^{\sigma}{}_{\beta} V^{\rho}{}_{\gamma} = h_V{}^{\alpha}{}_{\beta\gamma} \\
h^T_V\left[{}_{V}{}^{H}{}_{V}\right]_{\beta}{}^{\alpha}{}_{\gamma} &= V^{\sigma}{}_{\beta} H^{\alpha}{}_{\delta} h^T_V{}_{\sigma}{}^{\delta}{}_{\rho} V^{\rho}{}_{\gamma} = h^T_V{}_{\beta}{}^{\alpha}{}_{\gamma}.
\end{aligned}$$

Using these, it can be shown that the derivative of the projection tensor is given by

$$\nabla_{\gamma} H^{\alpha}{}_{\beta} = h_H{}^{\alpha}{}_{\beta\gamma} - h_V{}^{\alpha}{}_{\beta\gamma} + h^T_H{}_{\beta}{}^{\alpha}{}_{\gamma} - h^T_V{}_{\beta}{}^{\alpha}{}_{\gamma} \tag{23}$$

and

$$\nabla_\gamma V^\alpha{}_\beta = h_V{}^\alpha{}_{\beta\gamma} - h_H{}^\alpha{}_{\beta\gamma} + h^T_{V\,\beta}{}^\alpha{}_\gamma - h^T_{H\,\beta}{}^\alpha{}_\gamma . \tag{24}$$

Notice that the derivative of the projection tensors are similar, the *H* and *V* are just interchanged.

Since the projection curvature tensors each have two indices that project into the same subspace, we can contract on those and define the *divergence form* as

$$\theta^T_{H\,\beta} = h^T_{H\,\beta}{}^\alpha{}_\alpha . \tag{25}$$

The *twist tensor* and the *expansion rate tensor* can be obtained by using the antisymmetric and symmetric parts and are defined as

$$\omega_H{}^\alpha{}_{\beta\gamma} = \frac{1}{2}\left(h_H{}^\alpha{}_{\beta\gamma} - h_H{}^\alpha{}_{\gamma\beta}\right) = h_H{}^\alpha{}_{[\beta\gamma]} \tag{26}$$

and

$$\theta_H{}^\alpha{}_{\beta\gamma} = \frac{1}{2}\left(h_H{}^\alpha{}_{\beta\gamma} + h_H{}^\alpha{}_{\gamma\beta}\right) = h_H{}^\alpha{}_{(\beta\gamma)} \tag{27}$$

respectively. Notice that the square brackets around the indexes represent the anti-symmetric parts of the tensor and the parenthesis represent the symmetric part of the tensor.

Now that we have these projected tensors, we need to discuss how to take the covariant derivative in the subspace that it exists in. We can do this by defining the *projected* or *restricted derivative*. These objects act *only* on restricted objects, so

$$D_\gamma T = O \nabla_\gamma T$$

where

$$OT = T.$$

Then

$$\begin{aligned} D_{H\,\gamma} T &= H^\gamma{}_\alpha D_\gamma T \\ D_{V\,\gamma} T &= V^\gamma{}_\alpha D_\gamma T \end{aligned} \tag{28}$$

This is the directional derivative projected into the subspace. The full covariant derivative on a projected object is given by the restricted derivative plus correction terms, which are the projection curvatures. Given a vector $u \in HT_p$, the full covariant derivative is

$$\nabla_{H\,\gamma} u^{\alpha} = D_{H\,\gamma} u^{\alpha} + h_H{}^{\alpha}{}_{\beta\gamma} u^{\beta} \tag{29}$$

and for $u \in VT_p$

$$\nabla_{V\,\gamma} u^{\alpha} = D_{V\,\gamma} u^{\alpha} - h^T_{V\,\beta}{}^{\alpha}{}_{\gamma} u^{\beta}. \tag{30}$$

Given a one-form $\phi \in H^*T_p$, the full covariant derivative is

$$\nabla_{H\,\gamma} u^{\alpha} = D_{H\,\gamma} u^{\alpha} + h^T_{H\,\beta}{}^{\alpha}{}_{\gamma} u^{\beta} \tag{31}$$

and for $\phi \in V^*T_p$

$$\nabla_{V\,\gamma} u^{\alpha} = D_{V\,\gamma} u^{\alpha} - h_V{}^{\alpha}{}_{\beta\gamma} u^{\beta}. \tag{32}$$

The covariant derivative of higher rank tensors will include combinations of the projected derivative and corrections for each tensor index.

We can now discuss the projected and restricted metricity, which is

$$\begin{aligned} Q[{}^{H\,H}{}_{H}]^{\alpha\,\beta}{}_{\gamma} &= -H^{\rho}{}_{\gamma} H^{\alpha}{}_{\delta} H^{\beta}{}_{\sigma} \nabla_{\rho}\, g^{\delta\,\sigma} \\ &= -H^{\rho}{}_{\gamma} \nabla_{\rho}\left(H^{\alpha}{}_{\delta} H^{\beta}{}_{\sigma} g^{\delta\,\sigma}\right) + H^{\rho}{}_{\gamma}\left(\nabla_{\rho} H^{\alpha}{}_{\delta}\right) H^{\beta}{}_{\sigma} g^{\delta\,\sigma} \\ &\quad + H^{\rho}{}_{\gamma} H^{\alpha}{}_{\delta}\left(\nabla_{\rho} H^{\beta}\right)_{\sigma} g^{\delta\,\sigma} \\ &= -D_{H\,\gamma} g^{HH\alpha\beta} + + H^{\rho}{}_{\gamma}\left(\nabla_{\rho} H^{\alpha}{}_{\delta}\right) H^{\beta}{}_{\sigma} g^{\delta\,\sigma} \\ &\quad + H^{\rho}{}_{\gamma} H^{\alpha}{}_{\delta}\left(\nabla_{\rho} H^{\beta}{}_{\sigma}\right) g^{\delta\,\sigma}. \end{aligned}$$

But,

$$\nabla_{\gamma} H^{\alpha}{}_{\beta} = h_H{}^{\alpha}{}_{\beta\gamma} - h_V{}^{\alpha}{}_{\beta\gamma} + h^T_{H\,\beta}{}^{\alpha}{}_{\gamma} - h^T_{V\,\beta}{}^{\alpha}{}_{\gamma}.$$

Putting this in we get

$$\begin{aligned} Q[{}^{H\,H}{}_{H}]^{\alpha\,\beta}{}_{\gamma} = {}& -D_{H\,\gamma} g^{HH\alpha\beta} + H^{\rho}{}_{\gamma}\left(h_H{}^{\alpha}{}_{\delta\rho} - h_V{}^{\alpha}{}_{\delta\rho} + h^T_{H\,\delta}{}^{\alpha}{}_{\rho} - h^T_{V\,\delta}{}^{\alpha}{}_{\rho}\right) H^{\beta}{}_{\sigma} g^{\delta\,\sigma} \\ & + H^{\rho}{}_{\gamma} H^{\alpha}{}_{\delta}\left(h_H{}^{\beta}{}_{\sigma\rho} - h_V{}^{\beta}{}_{\sigma\rho} + h^T_{H\,\sigma}{}^{\beta}{}_{\rho} - h^T_{V\,\sigma}{}^{\beta}{}_{\rho}\right) g^{\delta\,\sigma}. \end{aligned}$$

If we now take two H projections

$$\begin{aligned}H^{\alpha}{}_{\mu} H^{\beta}{}_{\nu} Q\left[{}^{H\,H}{}_{H}\right]^{\mu\,\nu}{}_{\gamma} = Q\left[{}^{H\,H}{}_{H}\right]^{\alpha\,\beta}{}_{\gamma} \\ = -D_{H\,\gamma}\, g^{HH\alpha\beta} \\ + H^{\alpha}{}_{\mu} H^{\beta}{}_{\nu} H^{\rho}{}_{\gamma}\left(h_{H}{}^{\mu}{}_{\delta\rho} - h_{V}{}^{\mu}{}_{\delta\rho} + h^{T}_{H\,\delta}{}^{\mu}{}_{\rho} - h^{T}_{V\,\delta}{}^{\mu}{}_{\rho}\right) H^{\nu}{}_{\sigma}\, g^{\delta\,\sigma} \\ + H^{\alpha}{}_{\mu} H^{\beta}{}_{\nu} H^{\rho}{}_{\gamma} H^{\mu}{}_{\delta}\left(h_{H}{}^{\nu}{}_{\sigma\rho} - h_{V}{}^{\nu}{}_{\sigma\rho} + h^{T}_{H\,\sigma}{}^{\nu}{}_{\rho} - h^{T}_{V\,\sigma}{}^{\nu}{}_{\rho}\right) g^{\delta\,\sigma}.\end{aligned}$$

Many of these terms go to zero because of the projection identities. We end up with

$$\begin{aligned}Q\left[{}^{H\,H}{}_{H}\right]^{\alpha\,\beta}{}_{\gamma} &= -D_{H\,\gamma}\, g^{HH\alpha\beta} + h^{T}_{H\,\delta}{}^{\alpha}{}_{\gamma} H^{\nu}{}_{\sigma}\, g^{\delta\,\sigma} + H^{\alpha}{}_{\delta}\, h^{T}_{H\,\sigma}{}^{\beta}{}_{\gamma}\, g^{\delta\,\sigma} \\ &= -D_{H\,\gamma}\, g^{HH\alpha\beta} + h^{T}_{H\,\delta}{}^{\alpha}{}_{\gamma}\, g^{VH\,\delta\,\beta} + h^{T}_{H\,\sigma}{}^{\beta}{}_{\gamma}\, g^{HV\,\alpha\,\sigma}.\end{aligned} \tag{33}$$

The first term is what we will define as the restricted metricity

$$-D_{Z\,\gamma}\, g^{XY\alpha\beta} = Q_{Z}^{XY\,\alpha\,\beta}{}_{\gamma}. \tag{34}$$

We then obtain

$$Q\left[{}^{H\,H}{}_{H}\right]^{\alpha\,\beta}{}_{\gamma} = Q_{H}^{HH\,\alpha\,\beta}{}_{\gamma} + h^{T}_{H\,\delta}{}^{\alpha}{}_{\gamma}\, g^{VH\,\delta\,\beta} + h^{T}_{H\,\sigma}{}^{\beta}{}_{\gamma}\, g^{HV\,\alpha\,\sigma}. \tag{35}$$

The other projections are given by Gowdy [2] and are

$$\begin{aligned}Q\left[{}^{H\,H}{}_{V}\right]^{\alpha\,\beta}{}_{\gamma} &= Q_{V}^{HH\,\alpha\,\beta}{}_{\gamma} - h_{V}{}^{\beta}{}_{\mu\,\gamma}\, g^{HV\,\alpha\,\mu} - h_{V}{}^{\alpha}{}_{\delta\,\gamma}\, g^{HV\,\delta\,\beta} \\ Q\left[{}^{H\,V}{}_{H}\right]^{\alpha\,\beta}{}_{\gamma} &= Q_{H}^{HV\,\alpha\,\beta}{}_{\gamma} - h_{V}{}^{\beta}{}_{\mu\,\gamma}\, g^{HH\,\alpha\,\mu} + h^{T}_{H\,\delta}{}^{\alpha}{}_{\gamma}\, g^{VV\,\delta\,\beta} \\ Q\left[{}^{H\,V}{}_{V}\right]^{\alpha\,\beta}{}_{\gamma} &= Q_{V}^{HV\,\alpha\,\beta}{}_{\gamma} + h_{V\,\mu}{}^{\beta}{}_{\gamma}\, g^{HH\,\alpha\,\mu} - h^{T}_{H\,\delta}{}^{\alpha}{}_{\gamma}\, g^{VV\,\delta\,\beta}.\end{aligned} \tag{36}$$

The term $Q_{H}^{HH\,\alpha\,\beta}{}_{\gamma}$ is called the *intrinsic metricity* associated with the subspace HT_p, $Q_{V}^{VV\,\alpha\,\beta}{}_{\gamma}$ is the intrinsic metricity associated with the subspace VT_p, and the other combinations are called the *cross-projected metricities*.

The torsion tensor $S^{\gamma}{}_{\alpha\beta}$ can also be projected. The *restricted torsion tensors* $S^{\,Z}_{\,XY}{}^{\gamma}{}_{\alpha\beta}$ are defined in general by

$$\left[D_{X\beta}, D_{Y\alpha}\right] f = S^{\,H}_{\,XY}{}^{\gamma}{}_{\alpha\beta}\, D_{H\gamma} f + S^{\,V}_{\,XY}{}^{\gamma}{}_{\alpha\beta}\, D_{V\gamma} f. \tag{37}$$

By working through the full projections of the torsion tensor, we get the following

$$
\begin{aligned}
&S\left[{}^{H}{}_{H\,H}\right]^{\gamma}{}_{\alpha\beta} = S^{\;H}_{HH}{}^{\gamma}{}_{\alpha\beta}\\
&S\left[{}^{V}{}_{H\,H}\right]^{\gamma}{}_{\alpha\beta} = S^{\;V}_{HH}{}^{\gamma}{}_{\alpha\beta} - 2\,h_{H}{}^{\gamma}{}_{[\alpha\beta]}\\
&S\left[{}^{H}{}_{H\,V}\right]^{\gamma}{}_{\alpha\beta} = S^{\;H}_{HV}{}^{\gamma}{}_{\alpha\beta} - h^{T}_{H}{}_{\beta}{}^{\gamma}{}_{\alpha}\\
&S\left[{}^{H}{}_{V\,V}\right]^{\gamma}{}_{\alpha\beta} = S^{\;H}_{VV}{}^{\gamma}{}_{\alpha\beta} - 2\,h_{V}{}^{\gamma}{}_{[\alpha\beta]}\\
&S\left[{}^{V}{}_{V\,H}\right]^{\gamma}{}_{\alpha\beta} = S^{\;V}_{VH}{}^{\gamma}{}_{\alpha\beta} - h^{T}_{V}{}_{\beta}{}^{\gamma}{}_{\alpha}\\
&S\left[{}^{V}{}_{V\,V}\right]^{\gamma}{}_{\alpha\beta} = S^{\;V}_{VV}{}^{\gamma}{}_{\alpha\beta}.
\end{aligned}
\tag{38}
$$

Assembly Notation

When using this projection technique, one ends up with many more equations, each involving *H*'s, *V*'s or combinations of both. These equations will become more compact if we use assembly notation [2]. The projection labels on the projected tensor always correspond to one of the indices on the tensor. So $g^{XY\alpha\beta}$ could be written as $g^{\langle X\alpha\rangle\langle Y\beta\rangle}$.

The restricted tensors will have the following form

$$Q_Z^{XY\alpha\beta}{}_{\gamma} = Q^{\langle X\alpha\rangle\langle Y\beta\rangle}{}_{\langle Z\gamma\rangle} \tag{39}$$

$$S_{XY}^{Z}{}^{\gamma}{}_{\alpha\beta} = S^{\langle Z\gamma\rangle}{}_{\langle X\alpha\rangle\langle Y\beta\rangle}. \tag{40}$$

The fully projected tensors will be given as

$$Q\left[{}^{XY}{}_{Z}\right]^{\alpha\beta}{}_{\gamma} = Q[\;]^{\langle X\alpha\rangle\langle Y\beta\rangle}{}_{\langle Z\gamma\rangle}, \tag{41}$$

$$S\left[{}^{Z}{}_{XY}\right]^{\gamma}{}_{\alpha\beta} = S[\;]^{\langle Z\gamma\rangle}{}_{\langle X\alpha\rangle\langle Y\beta\rangle}. \tag{42}$$

If we now define an index inside pointed brackets *without* a projection label to mean that we use the summation convention for the projections as well as the usual index, we get an *assembly*. For example, the metric and form-metric assemblies are

$$g_{\langle\alpha\rangle\langle\beta\rangle}$$

and

$$g^{\langle\alpha\rangle\langle\beta\rangle}.$$

Using this notation the definition of the restricted metric becomes

$$Q^{\langle\alpha\rangle\langle\beta\rangle}{}_{\langle\gamma\rangle} = -D_{\langle\gamma\rangle}\, g^{\langle\alpha\rangle\langle\beta\rangle} \tag{43}$$

where $Q^{\langle\alpha\rangle\langle\beta\rangle}{}_{\langle\gamma\rangle}$ is now the *restricted metric assembly.*

The definition of the restricted torsion assembly becomes

$$\left[D_{\langle\beta\rangle}, D_{\langle\alpha\rangle}\right] f = S^{\langle\gamma\rangle}{}_{\langle\alpha\rangle\langle\beta\rangle}\, D_{\langle\gamma\rangle}\, f, \tag{44}$$

where here there is a projection label associated with α and β, and the index γ is summed over both H and V. $S^{\langle\gamma\rangle}{}_{\langle\alpha\rangle\langle\beta\rangle}$ is called the *restricted torsion assembly*. Similar expression can be obtained for $Q[\;]^{\langle\alpha\rangle\langle\beta\rangle}{}_{\langle\gamma\rangle}$, the *projected metric assembly*, and $S[\;]^{\langle\gamma\rangle}{}_{\langle\alpha\rangle\langle\beta\rangle}$, the *projected torsion assembly*.

In order to work in this assembly notation, it is necessary to define the *projection gradient assembly*. Using this, Eq. (23) becomes

$$\nabla_{\langle\gamma\rangle}H^{\langle\alpha\rangle}{}_{\langle\delta\rangle}= \nabla H[\;]^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle}= \nabla H^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle}. \tag{45}$$

The covariant derivative of a vector field u where $u\in HT_p$ is

$$\nabla_{\langle\gamma\rangle}u^{\langle\alpha\rangle} = D_{\langle\gamma\rangle}u^{\langle\alpha\rangle} + \nabla H^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle}u^{\langle\beta\rangle}, \tag{46}$$

and the covariant derivative of a form field ϕ where $\phi\in H^*\hat{T}_p$ is

$$\nabla_{\langle\gamma\rangle}\phi_{\langle\alpha\rangle} = D_{\langle\gamma\rangle}\phi_{\langle\alpha\rangle} + \phi_{\langle\beta\rangle}\nabla H^{\langle\beta\rangle}{}_{\langle\alpha\rangle\langle\gamma\rangle}\,. \tag{47}$$

For vectors v and forms ω such that $v\in VT_p$ and $\omega\in V^*\hat{T}_p$, these equations become

$$\nabla_{\langle\gamma\rangle}v^{\langle\alpha\rangle} = D_{\langle\gamma\rangle}v^{\langle\alpha\rangle} - \nabla H^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle}v^{\langle\beta\rangle} \tag{48}$$

and

$$\nabla_{\langle\gamma\rangle}\omega_{\langle\alpha\rangle} = D_{\langle\gamma\rangle}\omega_{\langle\alpha\rangle} - \omega_{\langle\beta\rangle}\nabla H^{\langle\beta\rangle}{}_{\langle\alpha\rangle\langle\gamma\rangle} \tag{49}$$

since the complement of the assembly $\nabla H^{\langle\beta\rangle}{}_{\langle\alpha\rangle\langle\gamma\rangle}$ is $-\nabla H^{\langle\beta\rangle}{}_{\langle\alpha\rangle\langle\gamma\rangle}$. To get the covariant derivative of an arbitrary vector u, we can define the *complementation tensor*

$$C^{\alpha}{}_{\beta}= H^{\alpha}{}_{\beta}- V^{\alpha}{}_{\beta} \tag{50}$$

and

$$C^{\langle\alpha\rangle}{}_{\langle\beta\rangle}= H^{\langle\alpha\rangle}{}_{\langle\beta\rangle}- V^{\langle\alpha\rangle}{}_{\langle\beta\rangle}. \tag{51}$$

Then for any arbitrary vector field u and arbitrary form field ϕ, we have

$$\nabla_{\langle\gamma\rangle}u^{\langle\alpha\rangle} = D_{\langle\gamma\rangle}u^{\langle\alpha\rangle} + \nabla H^{\langle\alpha\rangle}{}_{\langle\delta\rangle\langle\gamma\rangle}C^{\langle\delta\rangle}{}_{\langle\beta\rangle}u^{\langle\beta\rangle} \tag{52}$$

and

$$\nabla_{\langle\gamma\rangle}\phi_{\langle\alpha\rangle} = D_{\langle\gamma\rangle}\phi_{\langle\alpha\rangle} + \phi_{\langle\beta\rangle}C^{\langle\beta\rangle}{}_{\langle\delta\rangle}\nabla H^{\langle\delta\rangle}{}_{\langle\alpha\rangle\,\langle\gamma\rangle}. \tag{53}$$

The projection gradient assembly and the complementation assembly obey the projection identity

$$\nabla H^{\langle\alpha\rangle}{}_{\langle\delta\rangle\,\langle\gamma\rangle}C^{\langle\delta\rangle}{}_{\langle\beta\rangle} = -C^{\langle\alpha\rangle}{}_{\langle\delta\rangle}\nabla H^{\langle\delta\rangle}{}_{\langle\beta\rangle\,\langle\gamma\rangle}. \tag{54}$$

The derivatives of the vector and form fields can be made to look the same as their unprojected form by defining the *projection curvature assembly* $K^{\langle\alpha\rangle}{}_{\langle\beta\rangle\,\langle\gamma\rangle}$ where

$$K^{\langle\alpha\rangle}{}_{\langle\beta\rangle\,\langle\gamma\rangle} = \nabla H^{\langle\alpha\rangle}{}_{\langle\delta\rangle\,\langle\gamma\rangle}C^{\langle\delta\rangle}{}_{\langle\beta\rangle} = -C^{\langle\alpha\rangle}{}_{\langle\delta\rangle}\nabla H^{\langle\delta\rangle}{}_{\langle\beta\rangle\,\langle\gamma\rangle} \tag{55}$$

and the components of K are

$$\begin{array}{ll} K^{\langle H\alpha\rangle}{}_{\langle H\beta\rangle\,\langle H\gamma\rangle} = 0 & K^{\langle V\alpha\rangle}{}_{\langle H\beta\rangle\,\langle H\gamma\rangle} = h_H{}^{\alpha}{}_{\beta\gamma} \\ K^{\langle H\alpha\rangle}{}_{\langle H\beta\rangle\,\langle V\gamma\rangle} = 0 & K^{\langle H\alpha\rangle}{}_{\langle V\beta\rangle\,\langle H\gamma\rangle} = -h_H^T{}_{\beta}{}^{\alpha}{}_{\gamma} \\ K^{\langle V\alpha\rangle}{}_{\langle V\beta\rangle\,\langle H\gamma\rangle} = 0 & K^{\langle H\alpha\rangle}{}_{\langle V\beta\rangle\,\langle V\gamma\rangle} = h_V{}^{\alpha}{}_{\beta\gamma} \\ K^{\langle V\alpha\rangle}{}_{\langle V\beta\rangle\,\langle V\gamma\rangle} = 0 & K^{\langle V\alpha\rangle}{}_{\langle H\beta\rangle\,\langle V\gamma\rangle} = -h_V^T{}_{\beta}{}^{\alpha}{}_{\gamma} \end{array} \tag{56}$$

Then

$$\nabla_{\langle\gamma\rangle}u^{\langle\alpha\rangle} = D_{\langle\gamma\rangle}u^{\langle\alpha\rangle} + K^{\langle\alpha\rangle}{}_{\langle\beta\rangle\,\langle\gamma\rangle}u^{\langle\beta\rangle} \tag{57}$$

and

$$\nabla_{\langle\gamma\rangle}\phi_{\langle\alpha\rangle} = D_{\langle\gamma\rangle}\phi_{\langle\alpha\rangle} - \phi_{\langle\beta\rangle}K^{\langle\beta\rangle}{}_{\langle\alpha\rangle\,\langle\gamma\rangle}. \tag{58}$$

The gradient of an arbitrary rank tensor T decomposed into its restricted parts is then given by

$$\nabla_{\langle\gamma\rangle}T[\ ]^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle} = D_{\langle\gamma\rangle}T^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle}$$
$$+T^{\langle\delta\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle}K^{\langle\alpha_1\rangle}{}_{\langle\delta\rangle\langle\gamma\rangle}$$
$$+T^{\langle\alpha_1\rangle\langle\delta\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle}K^{\langle\alpha_2\rangle}{}_{\langle\delta\rangle\langle\gamma\rangle}$$
$$+\cdots+T^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\delta\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle}K^{\langle\alpha_n\rangle}{}_{\langle\delta\rangle\langle\gamma\rangle}$$
$$-T^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\delta\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle}K^{\langle\delta\rangle}{}_{\langle\beta_1\rangle\langle\gamma\rangle} \tag{59}$$
$$-T^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\delta\rangle\cdots\langle\beta_m\rangle}K^{\langle\delta\rangle}{}_{\langle\beta_2\rangle\langle\gamma\rangle}$$
$$-\cdots-T^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\delta\rangle}K^{\langle\delta\rangle}{}_{\langle\beta_m\rangle\langle\gamma\rangle}.$$

We can use this equation to obtain expressions for the fully projected metricity and torsion. They are

$$Q[\ ]^{\langle\alpha\rangle\langle\beta\rangle}{}_{\langle\gamma\rangle} = Q^{\langle\alpha\rangle\langle\beta\rangle}{}_{\langle\gamma\rangle} - 2g^{\langle\delta\rangle(\langle\alpha\rangle}K^{\langle\beta\rangle)}{}_{\langle\delta\rangle\langle\gamma\rangle} \tag{60}$$

and

$$S[\ ]^{\langle\gamma\rangle}{}_{\langle\alpha\rangle\langle\beta\rangle} = S^{\langle\gamma\rangle}{}_{\langle\alpha\rangle\langle\beta\rangle} - 2K^{\langle\gamma\rangle}{}_{[\langle\alpha\rangle\langle\beta\rangle]}. \tag{61}$$

The next assembly needed is the *restricted curvature assembly*. It is defined by how restricted covariant derivatives act on restricted vector fields, which is given by the following equation

$$u^{\langle\rho\rangle}R_{\langle\rho\rangle}{}^{\langle\sigma\rangle}{}_{\langle\alpha\rangle\langle\beta\rangle} = \left\{\left[D_{\langle\beta\rangle}, D_{\langle\alpha\rangle}\right] - S^{\langle\gamma\rangle}{}_{\langle\alpha\rangle\langle\beta\rangle}D_{\langle\gamma\rangle}\right\}u^{\langle\sigma\rangle}. \tag{62}$$

Since the restricted derivative always projects itself back into the subspace that it lives in, the first two indices *must* have the same projection label. This gives the notation of the restricted curvature assembly as

$$R_{\langle Z\rho\rangle}{}^{\langle Z\sigma\rangle}{}_{\langle X\alpha\rangle\langle Y\beta\rangle} = R^{Z}_{XY\,\rho}{}^{\sigma}{}_{\alpha\beta} \tag{63}$$

and also gives

$$R_{\langle H\rho\rangle}{}^{\langle V\sigma\rangle}{}_{\langle\alpha\rangle\langle\beta\rangle} = R_{\langle V\rho\rangle}{}^{\langle H\sigma\rangle}{}_{\langle\alpha\rangle\langle\beta\rangle} = 0. \tag{64}$$

The fully projected curvature assembly is given by

$$R[\ ]_{\langle\rho\rangle}{}^{\langle\sigma\rangle}{}_{\langle\alpha\rangle\langle\beta\rangle} = R_{\langle\rho\rangle}{}^{\langle\sigma\rangle}{}_{\langle\alpha\rangle\langle\beta\rangle} + 2D_{[\langle\beta\rangle}K^{\langle\sigma\rangle}{}_{|\langle\rho\rangle|\langle\alpha\rangle]} - S^{\langle\delta\rangle}{}_{\langle\alpha\rangle\langle\beta\rangle}K^{\langle\sigma\rangle}{}_{\langle\rho\rangle\langle\delta\rangle}$$
$$-2K^{\langle\sigma\rangle}{}_{\langle\delta\rangle[\langle\alpha\rangle}K^{\langle\delta\rangle}{}_{|\langle\rho\rangle|\langle\beta\rangle]} \tag{65}$$

This can be expanded to give

$$R[\ ]_{\langle H\rho\rangle}{}^{\langle H\sigma\rangle}{}_{\langle\alpha\rangle\langle\beta\rangle} = R_{\langle H\rho\rangle}{}^{\langle H\sigma\rangle}{}_{\langle\alpha\rangle\langle\beta\rangle} - 2K^{\langle\delta\rangle}{}_{\langle H\rho\rangle[\langle\alpha\rangle}K^{\langle H\sigma\rangle}{}_{|\langle\delta\rangle|\langle\beta\rangle]} \tag{66}$$

and

$$R[\ ]_{\langle H\rho\rangle}{}^{\langle V\sigma\rangle}{}_{\langle\alpha\rangle\langle\beta\rangle} = 2D_{[\langle\beta\rangle}K^{\langle V\sigma\rangle}{}_{|\langle H\rho\rangle|\langle\alpha\rangle]} - S^{\langle\delta\rangle}{}_{\langle\alpha\rangle\langle\beta\rangle}K^{\langle V\sigma\rangle}{}_{\langle H\rho\rangle\langle\delta\rangle} \tag{67}$$

which are the Gauss-Codazzi equations for surface imbedding [2].

Restricted Lie Derivatives

The restricted Lie derivative is defined in the same manner as the restricted covariant derivative. If we have a restricted tensor T that is characterized by the projection

$$OT = T$$

where O represents a combination of the projections H or V for each index of T, the restricted Lie derivative of T is given by

$$L_N T = O \pounds_N T.$$

The restricted Lie derivative of an assembly of restricted tensor fields is given by

$$\begin{aligned}
\left(L_N T\right)^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle} = {} & N^{\langle\gamma\rangle} D_{\langle\gamma\rangle} T^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle} \\
& - T^{\langle\gamma\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle} D'_{\langle\gamma\rangle} N^{\langle\alpha_1\rangle} \\
& - T^{\langle\alpha_1\rangle\langle\gamma\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle} D'_{\langle\gamma\rangle} N^{\langle\alpha_3\rangle} \\
& - \cdots - T^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\gamma\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle} D'_{\langle\gamma\rangle} N^{\langle\alpha_n\rangle} \\
& + T^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\gamma\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle} D'_{\langle\beta_1\rangle} N^{\langle\gamma\rangle} \\
& + T^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\gamma\rangle\cdots\langle\beta_m\rangle} D'_{\langle\beta_2\rangle} N^{\langle\gamma\rangle} \\
& + \cdots + T^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\gamma\rangle} D'_{\langle\beta_m\rangle} N^{\langle\gamma\rangle}.
\end{aligned} \tag{68}$$

Using this result, we can obtain the decomposition of the Lie derivative of the projection tensor.

$$\pounds_N H^{\langle\alpha\rangle}{}_{\langle\beta\rangle} = N^{\langle\gamma\rangle} \nabla_{\langle\gamma\rangle} H^{\langle\alpha\rangle}{}_{\langle\beta\rangle} - H^{\langle\gamma\rangle}{}_{\langle\beta\rangle} \nabla'_{\langle\gamma\rangle} N^{\langle\alpha\rangle} + H^{\langle\alpha\rangle}{}_{\langle\gamma\rangle} \nabla'_{\langle\beta\rangle} N^{\langle\gamma\rangle}.$$

Using Eq. (59) and Eq. (61) we get

$$\begin{aligned}
\pounds_N H^{\langle\alpha\rangle}{}_{\langle\beta\rangle} = {} & H^{\langle\alpha\rangle}{}_{\langle\gamma\rangle} D_{\langle\beta\rangle} N^{\langle\gamma\rangle} - H^{\langle\gamma\rangle}{}_{\langle\beta\rangle} D_{\langle\gamma\rangle} N^{\langle\alpha\rangle} \\
& + N^{\langle\gamma\rangle}\left(H^{\langle\delta\rangle}{}_{\langle\beta\rangle} S^{\langle\alpha\rangle}{}_{\langle\delta\rangle\langle\gamma\rangle} - H^{\langle\alpha\rangle}{}_{\langle\delta\rangle} S^{\langle\delta\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle}\right).
\end{aligned}$$

This equation is antisymmetric under complementation, i.e.,

$$\pounds_N H^{\langle\alpha\rangle}{}_{\langle\beta\rangle} = -\pounds_N V^{\langle\alpha\rangle}{}_{\langle\beta\rangle}.$$

This leads to the expression,

$$\pounds_N H^{\langle\alpha\rangle}{}_{\langle\beta\rangle} = \tfrac{1}{2}\Big(C^{\langle\alpha\rangle}{}_{\langle\gamma\rangle}\, D_{\langle\beta\rangle}\, N^{\langle\gamma\rangle} - C^{\langle\gamma\rangle}{}_{\langle\beta\rangle}\, D_{\langle\gamma\rangle}\, N^{\langle\alpha\rangle}\Big) + \tfrac{1}{2} N^{\langle\gamma\rangle}\Big(C^{\langle\delta\rangle}{}_{\langle\beta\rangle}\, S^{\langle\alpha\rangle}{}_{\langle\delta\rangle\,\langle\gamma\rangle} - C^{\langle\alpha\rangle}{}_{\langle\delta\rangle}\, S^{\langle\delta\rangle}{}_{\langle\beta\rangle\,\langle\gamma\rangle}\Big). \tag{69}$$

Killing Vectors and Isometry Group Orbits

A method often used in general relativity is to impose certain symmetries. When we have a motion that preserves the metric tensor, it is called an *isometry*. One way to describe this is to use a Killing vector.

A *Killing vector* is defined by *Killing's equation*

$$\begin{aligned} \pounds_{\xi}\, g_{\alpha\beta} &= g_{\sigma\beta}\nabla_{\alpha}\xi^{\sigma} + g_{\alpha\sigma}\nabla_{\beta}\xi^{\sigma} \\ &= \nabla_{\alpha}\xi_{\beta} + \nabla_{\beta}\xi_{\alpha} = 0. \end{aligned} \tag{70}$$

This equation says that the spacetime along a Killing vector field is unchanging. Thus the geometry of the manifold is unchanged by a translation of all the points in the manifold by $\kappa\xi$, where ξ is the Killing vector and κ is a constant [8]. When there are several Killing vectors, they will form *group orbits*. A group orbit is the set of all points that can be reached from a given point on the manifold along the Killing vector field. These are surfaces where the geometry does not change. One example is the surface of an infinitely long cylinder. The two Killing vectors in cylindrical coordinates are θ and z. If we place two points on the surface and move them through an angle θ, it does not change the distance between the points. Similarly, the distance between two points on the surface does not change by sliding them up or down the cylinder in the direction of the z-axis. However, moving them along the radial direction will change the distance between two points on the surface; thus it is *not* a Killing vector.

The Problem

Cylindrical Spacetimes

This thesis begins by assuming a vacuum spacetime with two-parameter isometry groups or G_2 spacetimes. Let us also restrict this thesis to those spacetimes that have two spacelike Killing vector fields. Now define H as the projection onto the group orbits produced by the Killing vector fields ξ. Then

$$H^{\alpha}{}_{\beta}\xi^{\beta} = \xi^{\alpha} \tag{71}$$

and

$$V^{\alpha}{}_{\beta}\xi^{\beta} = 0\,. \tag{72}$$

We can now choose a reference frame adapted to our spacetimes. Let x^A label the orbits where $A = 0,1$. These functions are then constant on each orbit. For the coordinates on the orbits, we choose x^a to be the Killing vectors where $a = 2,3$. We then have

$$\xi_a = \partial_a = \frac{\partial}{\partial x^a}\,. \tag{73}$$

Notice that all capital Latin indices will now sum from 0 to 1 and all lower case Latin indices will sum over 2 and 3. This frame is now adapted to the orbit projections and we have the following projection identities.

$$\begin{aligned} H\partial_a &= \partial_a \\ V\partial_a &= 0 \\ H\partial_A &= 0 \\ V\partial_A &= \partial_A \end{aligned} \tag{74}$$

This frame also gives the following pullback identities.

$$\begin{aligned} H^* dx^a &= dx^a \\ V^* dx^a &= 0 \end{aligned} \tag{75}$$

$$H^{*}dx^{A} = 0$$

$$V^{*}dx^{A} = dx^{A}$$

Notice that these projections are now normal to each other. Using these, we get the metric tensor

$$g^{\alpha\beta} = g^{HHab} + g^{VVAB} = dx^{a} \bullet dx^{b} + dx^{A} \bullet dx^{B}. \qquad (76)$$

Since the projections are normal to each other and are onto group orbits, several nice conditions arise. The normality of these projections imply that

$$h_{H}{}^{\alpha}{}_{\beta\gamma} = h^{T}_{H\,\beta}{}^{\alpha}{}_{\gamma}$$

and

$$h_{V}{}^{\alpha}{}_{\beta\gamma} = h^{T}_{V\,\beta}{}^{\alpha}{}_{\gamma}.$$

The fact that we are projecting onto group orbits also tells us that

$$h_{V}{}^{\alpha}{}_{\beta\gamma} = 0, \qquad (77)$$

which is to say that the surfaces orthogonal to the group orbits are extrinsically flat and that the curves on these orbits must be geodesics. We can now set the only surviving projection tensor curvature to

$$h_{H}{}^{\alpha}{}_{\beta\gamma} = h^{\alpha}{}_{\beta\gamma}, \qquad (78)$$

the *H* subscript no longer necessary. We can decompose this into

$$h^{\alpha}{}_{\beta\gamma} = \sigma^{\alpha}{}_{\beta\gamma} + \tfrac{1}{2}\theta^{\alpha}\, g_{HH\beta\gamma} \qquad (79)$$

where θ^{α} is the divergence given by

$$\theta^{\alpha} = h^{\alpha\sigma}{}_{\sigma} \qquad (80)$$

and $\sigma^{\alpha}{}_{\beta\gamma}$ is the shear tensor given by

$$\sigma^{\alpha}{}_{\beta\gamma} = h^{\alpha}{}_{\beta\gamma} - \tfrac{1}{2} g_{HH\beta\gamma}\, h^{\alpha\sigma}{}_{\sigma}. \qquad (81)$$

The Semi-Conformal Transformation

We now assume that we have a metric compatible connection, i.e.,

$$Q^{\alpha\beta}{}_{\gamma} = -\nabla_{\gamma} g^{\alpha\beta} = 0. \tag{82}$$

Consider another form-metric, $\tilde{g}^{\alpha\beta}$, the fiducial metric, that differs from the regular metric by only a semi-conformal factor Ω. A conformal transformation is when the metric that solves Einstein's equations is multiplied by a scalar function, the conformal factor, that depends on the position. This conformal factor does not change the angles between the coordinates, but only changes lengths. The new metric is called the "fiducial" metric. By defining objects such as derivatives in terms of the fiducial metric plus corrections, the conformal factor gives information about the topology of the spacetime. This method is often used in doing initial value problems in general relativity. Here, we will let the conformal factor be different in the two hypersurfaces *H* and *V*. This is done by defining

$$\Omega^{\alpha}{}_{\beta} = \Omega^{H} H^{\alpha}{}_{\beta} + \Omega^{V} V^{\alpha}{}_{\beta}, \tag{83}$$

where Ω^{H} and Ω^{V} are just functions. It will further simplify the solution if we let these functions be

$$\Omega^{H} = 1 \tag{84}$$

$$\Omega^{V} = e^{a} = exp(a) \tag{85}$$

where a is a function, **not an index**. We then have

$$\tilde{g}_{\alpha\beta} = \Omega^{\rho}{}_{\alpha} \Omega^{\sigma}{}_{\beta} g_{\rho\sigma}. \tag{86}$$

Remembering that the cross projected terms are zero since we have normal projections, we then have

$$\tilde{g}_{\alpha\beta} = g_{HH\alpha\beta} + \left(e^{2a}\right) g_{VV\alpha\beta}. \tag{87}$$

The fiducial form-metric is just

$$\tilde{g}^{\alpha\beta} = g^{HH\alpha\beta} + \left(e^{-2a}\right) g^{VV\alpha\beta} \tag{88}$$

since $\tilde{g}_{\alpha\beta}$ is block diagonalized. We will also assume that the metricity for the fiducial connection vanishes,

$$\tilde{Q}^{\alpha\beta}{}_{\gamma} = -\tilde{\nabla}_{\gamma} \tilde{g}^{\alpha\beta} = 0\,, \tag{89}$$

and that the torsion in both cases is zero.

By determining what a is, it will tell us about the topology of the regular spacetime.

Fiducial Derivatives

In order to continue, we must first consider how to take a derivative in terms of the fiducial framework. In the fiducial metric, the covariant derivative of a vector is given by

$$\tilde{\nabla}_\alpha u = e_\alpha u^\beta e_\beta + u^\gamma \tilde{\Gamma}^\beta{}_{\gamma\alpha} e_\beta. \tag{90}$$

Subtracting Eq. (90) from Eq. (6) gives

$$\nabla_\alpha u - \tilde{\nabla}_\alpha u = e_\alpha u^\beta e_\beta + u^\gamma \Gamma^\beta{}_{\gamma\alpha} e_\beta - e_\alpha u^\beta e_\beta - u^\gamma \tilde{\Gamma}^\beta{}_{\gamma\alpha} e_\beta,$$

or

$$\nabla_\alpha u - \tilde{\nabla}_\alpha u = u^\gamma \left(\Gamma^\beta{}_{\gamma\alpha} - \tilde{\Gamma}^\beta{}_{\gamma\alpha} \right) e_\beta$$

The term $\Gamma^\beta{}_{\gamma\alpha} - \tilde{\Gamma}^\beta{}_{\gamma\alpha}$ will occur often in this thesis, so we define it as

$$\Gamma^\beta{}_{\gamma\alpha} - \tilde{\Gamma}^\beta{}_{\gamma\alpha} \equiv \gamma^\beta{}_{\gamma\alpha}. \tag{91}$$

We can get gamma in terms of gradients of the conformal factor:

$$\begin{aligned} \gamma^\beta{}_{\gamma\alpha} &= \Gamma^\beta{}_{\gamma\alpha} - \tilde{\Gamma}^\beta{}_{\gamma\alpha} \\ &= \frac{1}{2} g^{\beta\sigma} \left(e_\alpha g_{\sigma\gamma} + e_\gamma g_{\alpha\sigma} - e_\sigma g_{\gamma\alpha} \right) - \frac{1}{2} \tilde{g}^{\beta\sigma} \left(e_\alpha \tilde{g}_{\sigma\gamma} + e_\gamma \tilde{g}_{\alpha\sigma} - e_\sigma \tilde{g}_{\gamma\alpha} \right). \end{aligned}$$

By putting the fiducial metric into this equation and much algebra, it can be shown that

$$\gamma^\beta{}_{\gamma\alpha} = - \left(V^\beta{}_\gamma \left(e_\alpha a \right) + V^\beta{}_\alpha \left(e_\gamma a \right) - g^{VV\beta\sigma} \left(e_\sigma a \right) g_{VV\gamma\alpha} \right) \tag{92}$$

and in assembly notation

$$\gamma^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle} = - V^{\langle\alpha\rangle}{}_{\langle\beta\rangle} \left(e_{\langle\gamma\rangle} a \right) - V^{\langle\alpha\rangle}{}_{\langle\gamma\rangle} \left(e_{\langle\beta\rangle} a \right) + g^{VV\langle\alpha\rangle\langle\sigma\rangle} \left(e_{\langle\sigma\rangle} a \right) g_{VV\langle\beta\rangle\langle\gamma\rangle}. \tag{93}$$

The regular covariant derivative of a vector in terms of the fiducial covariant derivative in component form is then

$$\begin{aligned} \nabla_\alpha u^\beta &= \tilde{\nabla}_\alpha u^\beta + u^\gamma \gamma^\beta{}_{\gamma\alpha} \\ &= \tilde{\nabla}_\alpha u^\beta - u^{V\beta} \left(e_\alpha a \right) - u^\gamma V^\beta{}_\alpha \left(e_\gamma a \right) + g^{VV\beta\sigma} \left(e_{V\sigma} a \right) u_{V\alpha} \end{aligned} \tag{94}$$

As before, the regular covariant derivative of a one-form in terms of the fiducial covariant derivative in component form is given with a sign change

$$\begin{aligned}\nabla_\alpha \phi_\beta &= \tilde{\nabla}_\alpha \phi_\beta - \phi_\gamma \gamma^\gamma{}_{\beta\alpha} \\ &= \tilde{\nabla}_\alpha \phi_\beta + \phi_{V\beta}\left(e_\alpha a\right) + \phi_{V\alpha}\left(e_\beta a\right) - \phi^{V\sigma}\left(e_{V\sigma} a\right) g_{VV\beta\alpha}\end{aligned} \tag{95}$$

Using these results, the covariant derivative of an arbitrary rank tensor T is then related to the fiducial covariant derivatives of the tensor by

$$\begin{aligned}\nabla_\gamma T^{\alpha_1 \alpha_2 \cdots \alpha_n}{}_{\beta_1 \beta_2 \cdots \beta_m} &= \tilde{\nabla}_\gamma T^{\alpha_1 \alpha_2 \cdots \alpha_n}{}_{\beta_1 \beta_2 \cdots \beta_m} + T^{\delta \alpha_2 \cdots \alpha_n}{}_{\beta_1 \beta_2 \cdots \beta_m} \gamma^{\alpha_1}{}_{\delta\gamma} \\ &\quad + T^{\alpha_1 \delta \cdots \alpha_n}{}_{\beta_1 \beta_2 \cdots \beta_m} \gamma^{\alpha_2}{}_{\delta\gamma} + \cdots + T^{\alpha_1 \alpha_2 \cdots \delta}{}_{\beta_1 \beta_2 \cdots \beta_m} \gamma^{\alpha_n}{}_{\delta\gamma} \\ &\quad - T^{\alpha_1 \alpha_2 \cdots \alpha_n}{}_{\gamma \beta_2 \cdots \beta_m} \gamma^{\delta}{}_{\beta_1 \gamma} - T^{\alpha_1 \alpha_2 \cdots \alpha_n}{}_{\beta_1 \gamma \cdots \beta_m} \gamma^{\delta}{}_{\beta_2 \gamma} \\ &\quad - \cdots - T^{\alpha_1 \alpha_2 \cdots \alpha_n}{}_{\beta_1 \beta_2 \cdots \gamma} \gamma^{\delta}{}_{\beta_m \gamma}.\end{aligned} \tag{96}$$

From Eq. (57) we can obtain restricted covariant derivative of a vector in terms of the restricted fiducial derivative

$$\tilde{\nabla}_{\langle\gamma\rangle} u^{\langle\alpha\rangle} = \tilde{D}_{\langle\gamma\rangle} u^{\langle\alpha\rangle} + \tilde{K}^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle} u^{\langle\beta\rangle}.$$

We can then get the restricted derivatives in terms of the fiducial framework

$$D_{\langle\gamma\rangle} u^{\langle\alpha\rangle} + K^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle} u^{\langle\beta\rangle} = \tilde{D}_{\langle\gamma\rangle} u^{\langle\alpha\rangle} + \tilde{K}^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle} u^{\langle\beta\rangle} + u^{\langle\beta\rangle} \gamma^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle}$$

or

$$D_{\langle\gamma\rangle} u^{\langle\alpha\rangle} = \tilde{D}_{\langle\gamma\rangle} u^{\langle\alpha\rangle} - k^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle} u^{\langle\beta\rangle} + u^{\langle\beta\rangle} \gamma^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle},$$

where

$$k^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle} = K^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle} - \tilde{K}^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle}\ . \tag{97}$$

We would like to get k in terms of γ. We start with Eq. (55)

$$k^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle} = \nabla_{\langle\gamma\rangle} H^{\langle\alpha\rangle}{}_{\langle\delta\rangle} C^{\langle\delta\rangle}{}_{\langle\beta\rangle} - \tilde{\nabla}_{\langle\gamma\rangle} H^{\langle\alpha\rangle}{}_{\langle\delta\rangle} C^{\langle\delta\rangle}{}_{\langle\beta\rangle}$$

Using Eqs. (94) and (95), we know

$$\tilde{\nabla}_{\langle\alpha\rangle}u^{\langle\beta\rangle} = \nabla_{\langle\alpha\rangle}u^{\langle\beta\rangle} - u^{\langle\gamma\rangle}\gamma^{\langle\beta\rangle}{}_{\langle\gamma\rangle\langle\alpha\rangle}$$

$$\tilde{\nabla}_{\langle\alpha\rangle}\phi_{\langle\beta\rangle} = \nabla_{\langle\alpha\rangle}\phi_{\langle\beta\rangle} + \phi_{\langle\gamma\rangle}\gamma^{\langle\gamma\rangle}{}_{\langle\beta\rangle\langle\alpha\rangle}.$$

We then have

$$\tilde{\nabla}_{\langle\gamma\rangle}H^{\langle\alpha\rangle}{}_{\langle\delta\rangle} = \nabla_{\langle\gamma\rangle}H^{\langle\alpha\rangle}{}_{\langle\delta\rangle} - H^{\langle\rho\rangle}{}_{\langle\delta\rangle}\gamma^{\langle\alpha\rangle}{}_{\langle\rho\rangle\langle\gamma\rangle} + H^{\langle\alpha\rangle}{}_{\langle\rho\rangle}\gamma^{\langle\rho\rangle}{}_{\langle\delta\rangle\langle\gamma\rangle}.$$

The above equation for k then becomes

$$k^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle} = H^{\langle\rho\rangle}{}_{\langle\beta\rangle}\gamma^{\langle\alpha\rangle}{}_{\langle\rho\rangle\langle\gamma\rangle} - H^{\langle\alpha\rangle}{}_{\langle\rho\rangle}C^{\langle\delta\rangle}{}_{\langle\beta\rangle}\gamma^{\langle\rho\rangle}{}_{\langle\delta\rangle\langle\gamma\rangle}\ . \tag{98}$$

The equation for the restricted derivative is then

$$D_{\langle\gamma\rangle}u^{\langle\alpha\rangle} = \tilde{D}_{\langle\gamma\rangle}u^{\langle\alpha\rangle} - k^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle}u^{\langle\beta\rangle} + u^{\langle\beta\rangle}\gamma^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle}$$

$$D_{\langle\gamma\rangle}u^{\langle\alpha\rangle} = \tilde{D}_{\langle\gamma\rangle}u^{\langle\alpha\rangle} - H^{\langle\rho\rangle}{}_{\langle\beta\rangle}u^{\langle\beta\rangle}\gamma^{\langle\alpha\rangle}{}_{\langle\rho\rangle\langle\gamma\rangle} + H^{\langle\alpha\rangle}{}_{\langle\rho\rangle}C^{\langle\delta\rangle}{}_{\langle\beta\rangle}u^{\langle\beta\rangle}\gamma^{\langle\rho\rangle}{}_{\langle\delta\rangle\langle\gamma\rangle} + u^{\langle\beta\rangle}\gamma^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle}.$$

Substituting in Eq. (93) yields

$$\begin{aligned} D_{\langle\gamma\rangle}u^{\langle\alpha\rangle} &= \tilde{D}_{\langle\gamma\rangle}u^{\langle\alpha\rangle} + H^{\langle\rho\rangle}{}_{\langle\beta\rangle}u^{\langle\beta\rangle}V^{\langle\alpha\rangle}{}_{\langle\gamma\rangle}\left(e_{\langle\rho\rangle}a\right) \\ &\quad - V^{\langle\alpha\rangle}{}_{\langle\beta\rangle}u^{\langle\beta\rangle}\left(e_{\langle\gamma\rangle}a\right) - V^{\langle\alpha\rangle}{}_{\langle\gamma\rangle}u^{\langle\beta\rangle}\left(e_{\langle\beta\rangle}a\right) + g^{VV\langle\alpha\rangle\langle\sigma\rangle}\left(e_{\langle\sigma\rangle}a\right)g_{VV\langle\beta\rangle\langle\gamma\rangle}u^{\langle\beta\rangle}. \end{aligned}$$

The components of this assembly are

$$\begin{aligned} D_{H\gamma}u^{H\alpha} &= \tilde{D}_{H\gamma}u^{H\alpha} \\ D_{V\gamma}u^{H\alpha} &= \tilde{D}_{V\gamma}u^{H\alpha} \\ D_{H\gamma}u^{V\alpha} &= \tilde{D}_{H\gamma}u^{V\alpha} - \left(e_{H\gamma}a\right)u^{V\alpha} \\ D_{V\gamma}u^{V\alpha} &= \tilde{D}_{V\gamma}u^{V\alpha} - \left(e_{V\gamma}a\right)u^{V\alpha} - V^{\alpha}{}_{\gamma}\left(e_{V\rho}a\right)u^{V\rho} + g^{VV\alpha\rho}\left(e_{V\rho}a\right)g_{VV\sigma\gamma}u^{V\sigma}. \end{aligned} \tag{99}$$

The restricted derivative of a one-form can also be obtained by using this method. Starting with Eq. (58) we have

$$\tilde{\nabla}_{\langle\gamma\rangle}\phi_{\langle\beta\rangle} = \tilde{D}_{\langle\gamma\rangle}\phi_{\langle\beta\rangle} - \tilde{K}^{\langle\alpha\rangle}{}_{\langle\beta\rangle\langle\gamma\rangle}\phi_{\langle\alpha\rangle}$$

then

$$D_{\langle\gamma\rangle}\,\phi_{\langle\beta\rangle} = \tilde{D}_{\langle\gamma\rangle}\,\phi_{\langle\beta\rangle} + k^{\langle\alpha\rangle}{}_{\langle\beta\rangle\,\langle\gamma\rangle}\,\phi_{\langle\alpha\rangle} - \phi_{\langle\alpha\rangle}\,\gamma^{\langle\alpha\rangle}{}_{\langle\beta\rangle\,\langle\gamma\rangle}.$$

Substituting in $k^{\langle\alpha\rangle}{}_{\langle\beta\rangle\,\langle\gamma\rangle}$ and $\gamma^{\langle\alpha\rangle}{}_{\langle\beta\rangle\,\langle\gamma\rangle}$ yields

$$\begin{aligned} D_{\langle\gamma\rangle}\,\phi_{\langle\beta\rangle} = \tilde{D}_{\langle\gamma\rangle}\,\phi_{\langle\beta\rangle} &- H^{\langle\rho\rangle}{}_{\langle\beta\rangle}\,V^{\langle\alpha\rangle}{}_{\langle\gamma\rangle}\left(e_{\langle\rho\rangle}a\right)\phi_{\langle\alpha\rangle} \\ &+ V^{\langle\alpha\rangle}{}_{\langle\beta\rangle}\left(e_{\langle\gamma\rangle}a\right)\phi_{\langle\alpha\rangle} + V^{\langle\alpha\rangle}{}_{\langle\gamma\rangle}\left(e_{\langle\beta\rangle}a\right)\phi_{\langle\alpha\rangle} - g^{VV\langle\alpha\rangle\langle\sigma\rangle}\left(e_{\langle\sigma\rangle}a\right)\,g_{VV\langle\beta\rangle\langle\gamma\rangle}\,\phi_{\langle\alpha\rangle}. \end{aligned}$$

The components of this assembly are

$$\begin{aligned} D_{H\gamma}\,\phi_{H\beta} &= \tilde{D}_{H\gamma}\,\phi_{H\beta} \\ D_{V\gamma}\,\phi_{H\beta} &= \tilde{D}_{V\gamma}\,\phi_{H\beta} \\ D_{H\gamma}\,\phi_{V\beta} &= \tilde{D}_{H\gamma}\,\phi_{V\beta} + \left(e_{H\gamma}a\right)\phi_{V\beta} \\ D_{V\gamma}\,\phi_{V\beta} &= \tilde{D}_{V\gamma}\,\phi_{V\beta} + \left(e_{V\gamma}a\right)\phi_{V\beta} + \left(e_{V\beta}a\right)\phi_{V\gamma} - g^{VV\sigma\rho}\left(e_{V\rho}a\right)\,g_{VV\beta\gamma}\,\phi_{V\sigma} \end{aligned} \tag{100}$$

The restricted derivatives can easily be written by defining a *restricted derivative correction assembly*, $d^{\langle\alpha\rangle}{}_{\langle\beta\rangle\,\langle\gamma\rangle}$, whose components are

$$\begin{aligned} d^{\langle H\alpha\rangle}{}_{\langle H\beta\rangle\,\langle H\gamma\rangle} &= 0 \\ d^{\langle H\alpha\rangle}{}_{\langle H\beta\rangle\,\langle V\gamma\rangle} &= 0 \\ d^{\langle H\alpha\rangle}{}_{\langle V\beta\rangle\,\langle H\gamma\rangle} &= 0 \\ d^{\langle H\alpha\rangle}{}_{\langle V\beta\rangle\,\langle V\gamma\rangle} &= 0 \\ d^{\langle V\alpha\rangle}{}_{\langle H\beta\rangle\,\langle H\gamma\rangle} &= 0 \\ d^{\langle V\alpha\rangle}{}_{\langle H\beta\rangle\,\langle V\gamma\rangle} &= 0 \\ d^{\langle V\alpha\rangle}{}_{\langle V\beta\rangle\,\langle H\gamma\rangle} &= -\left(e_{H\gamma}\,a\right)V^{\alpha}{}_{\beta} \\ d^{\langle V\alpha\rangle}{}_{\langle V\beta\rangle\,\langle V\gamma\rangle} &= -\left(e_{V\gamma}a\right)V^{\beta}{}_{\alpha} - V^{\alpha}{}_{\gamma}\left(e_{V\rho}a\right)V^{\rho}{}_{\beta} + g^{VV\alpha\rho}\left(e_{V\rho}a\right)g_{VV\sigma\gamma}\,V^{\sigma}{}_{\beta}. \end{aligned} \tag{101}$$

The restricted derivative of an arbitrary rank tensor T is then given by

$$
\begin{aligned}
D_{\langle\gamma\rangle} T^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle} = \tilde{D}_{\langle\gamma\rangle} & T^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle} \\
& + T^{\langle\delta\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle} d^{\langle\alpha_1\rangle}{}_{\langle\delta\rangle\,\langle\gamma\rangle} \\
& + T^{\langle\alpha_1\rangle\langle\delta\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle} d^{\langle\alpha_2\rangle}{}_{\langle\delta\rangle\,\langle\gamma\rangle} \\
& + \cdots + T^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\delta\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle} d^{\langle\alpha_n\rangle}{}_{\langle\delta\rangle\,\langle\gamma\rangle} \\
& - T^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\delta\rangle\langle\beta_2\rangle\cdots\langle\beta_m\rangle} d^{\langle\delta\rangle}{}_{\langle\beta_1\rangle\,\langle\gamma\rangle} \\
& - T^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\delta\rangle\cdots\langle\beta_m\rangle} d^{\langle\delta\rangle}{}_{\langle\beta_2\rangle\,\langle\gamma\rangle} \\
& - \cdots - T^{\langle\alpha_1\rangle\langle\alpha_2\rangle\cdots\langle\alpha_n\rangle}{}_{\langle\beta_1\rangle\langle\beta_2\rangle\cdots\langle\delta\rangle} d^{\langle\delta\rangle}{}_{\langle\beta_m\rangle\,\langle\gamma\rangle}.
\end{aligned}
\tag{103}
$$

Generalized Area Change of a Group Orbit

The generalized area element will give us a connection between the divergence and the rate of change of the projected area element [2]. Consider a vector field N^{α} that is in the VT_P subspace. If we Lie-drag the logarithmic rate of change of an area element dA in the HT_P subspace we have

$$\frac{\pounds_N\, dA}{dA} = \frac{1}{2} g_{HH\alpha\beta}\, L_N\, g^{HH\alpha\beta}. \tag{104}$$

From Eq. (68), we know that

$$\begin{aligned} L_N\, g^{HH\alpha\beta} &= -Q_N\left[{}^{HH}\right]^{\alpha\beta} - 2g^{HH\rho(\alpha} D_{H\rho} N^{H\beta)} + 2g^{VH\rho(\beta} h^{T}_{H\,\rho}{}^{\alpha)}{}_{\sigma} N^{H\sigma} \\ &+2g^{HH\rho(\alpha} h^{T}_{H\,\sigma}{}^{\beta)}{}_{\rho} N^{V\sigma} - 2g^{VH\rho(\alpha} h_V{}^{\beta)}{}_{\rho\sigma} N^{V\sigma} \end{aligned}$$

or, in our case

$$L_N\, g^{HH\alpha\beta} = 2g^{HH\rho(\alpha} h^{T}_{H\,\sigma}{}^{\beta)}{}_{\rho} N^{V\sigma}.$$

We then have

$$\frac{1}{2} g_{HH\alpha\beta}\, L_N\, g^{HH\alpha\beta} = \frac{1}{2} g_{HH\alpha\beta} 2g^{HH\rho(\alpha} h^{T}_{H\,\sigma}{}^{\beta)}{}_{\rho} N^{V\sigma} = N^{\sigma}\theta_{H\sigma}.$$

The connection between the divergence and the rate of change of the projected area element is then

$$N^{\sigma}\theta_{\sigma} = -\frac{\pounds_N\, dA}{dA}. \tag{105}$$

Einstein's Equations

Using the normal projections on group orbits, it can be shown that the normally projected Ricci curvature tensor is given by

$$R\Big[_{HH}\Big]_{\alpha\beta} = R^{H}_{HH\,\alpha\beta} + D_{V\sigma}\, h^{\sigma}{}_{\alpha\beta} - \theta_{\sigma} h^{\sigma}{}_{\alpha\beta}$$

$$R\Big[_{VV}\Big]_{\alpha\beta} = R^{V}_{VV\,\alpha\beta} + D_{V\sigma}\,\theta_{\alpha} - h_{\alpha}{}^{\rho}{}_{\sigma}\, h_{\beta}{}^{\sigma}{}_{\rho} \tag{106}$$

$$R\Big[_{HV}\Big]_{\alpha\beta} = R^{H}_{VH\,\alpha\beta}$$

$$R\Big[_{VH}\Big]_{\alpha\beta} = R^{V}_{HV\,\alpha\beta} + D_{H\sigma}\, h_{\alpha}{}^{\sigma}{}_{\beta} - D_{H\beta}\,\theta_{\alpha}.$$

Using these, Einstein's equations can be given in trace and trace-free parts by the following set of equations [3]:

$$R^{V}_{VV} = \tfrac{2-s}{s} R^{H}_{HH} + \tfrac{1-s}{s}\left(2 D_{V\alpha}\theta^{\alpha} - \theta_{\alpha}\theta^{\alpha}\right) + \sigma_{\alpha}{}^{\gamma}{}_{\beta}\,\sigma^{\alpha\beta}{}_{\gamma} - \tfrac{16\pi\kappa}{s} T_{HH}$$

$$D_{V\alpha}\theta^{\alpha} - \theta_{\alpha}\theta^{\alpha} + R^{H}_{HH} = \tfrac{8\pi\kappa}{2-d}\left[s T_{VV} + (2-d+s) T_{HH}\right]$$

$$\mathrm{TF}\, R^{H}_{HH\alpha\beta} + D_{V\gamma}\,\sigma^{\gamma}{}_{\alpha\beta} - \theta_{\gamma}\,\sigma^{\gamma}{}_{\alpha\beta} = 8\pi\kappa\,\mathrm{TF}\, T_{HH\alpha\beta} \tag{107}$$

$$\mathrm{TF}\, R^{V}_{VV\,\alpha\beta} + \mathrm{TF}\, D_{V\beta}\,\theta_{\alpha} - \mathrm{TF}\,\sigma_{\alpha}{}^{\rho}{}_{\delta}\,\sigma_{\beta}{}^{\delta}{}_{\rho} - \tfrac{1}{s}\mathrm{TF}\,\theta_{\alpha}\theta_{\beta} = 8\pi\kappa\;\mathrm{TF}\, T_{VV\alpha\beta}$$

$$\left(1-\tfrac{1}{s}\right) D_{H\beta}\,\theta_{\alpha} - D_{H\delta}\,\sigma_{\alpha}{}^{\delta}{}_{\beta} = 8\pi\kappa\; T_{VH\alpha\beta}$$

where d is the number of dimensions in the spacetime, and s is the number of dimensions in the *H* subspace. In our case, d=4 and s=2. Also, R^{H}_{HH} is zero here since there is no curvature in the group orbits, i.e., all points in the group orbit are the same. The stress-energy tensor *T* is zero since we are working with vacuum spacetimes. Einstein's equations then become

$$R^{V}_{VV} = -\tfrac{1}{2}\left(2 D_{V\alpha}\theta^{\alpha} - \theta_{\alpha}\theta^{\alpha}\right) + \sigma_{\alpha}{}^{\gamma}{}_{\beta}\,\sigma^{\alpha\beta}{}_{\gamma} \tag{108}$$

$$D_{V\alpha}\theta^{\alpha} - \theta_{\alpha}\theta^{\alpha} = 0 \tag{109}$$

$$D_{V\gamma}\sigma^{\gamma}{}_{\alpha\beta} - \theta_{\gamma}\sigma^{\gamma}{}_{\alpha\beta} = 0 \tag{110}$$

$$\mathrm{TF}\,R^{V}_{VV\,\alpha\beta} + \mathrm{TF}\,D_{V\beta}\theta_{\alpha} - \mathrm{TF}\sigma_{\alpha}{}^{\rho}{}_{\delta}\sigma_{\beta}{}^{\delta}{}_{\rho} - \tfrac{1}{2}\mathrm{TF}\theta_{\alpha}\theta_{\beta} = 0 \tag{111}$$

$$-\tfrac{1}{2}D_{H\beta}\theta_{\alpha} - D_{H\delta}\sigma_{\alpha}{}^{\delta}{}_{\beta} = 0 \tag{112}$$

The divergence-integrability equation [1, 2] is given by

$$D_{V[\beta}\theta_{\alpha]} = 0. \tag{113}$$

From this we see that θ_{α} is the gradient of some potential function *y* that depends only on the group orbit. From Eq. (105), we know that the divergence is related to the group orbit area. We can define *y* such that

$$\theta_{\alpha} = -R^{-1} D_{V\alpha} R, \tag{114}$$

and

$$\theta^{\alpha} = -R^{-1} g^{VV\alpha\beta} D_{V\beta} R, \tag{115}$$

where *R* is the radius of the orbit. Eq. (109) then reduces to

$$g^{VV\alpha\beta} D_{V\alpha} D_{V\beta} R = 0. \tag{116}$$

This equation is the same as Wald's (D.11) [11]. Thus, since we are in two dimensions, it also holds that (116) is conformally invariant, or

$$\tilde{g}^{VV\alpha\beta}\tilde{D}_{V\alpha}\tilde{D}_{V\beta}\tilde{R} = 0. \tag{117}$$

In a holonomic basis, this becomes

$$g^{AB} D_A D_B R = 0$$

or

$$\ddot{R} - R'' = 0 \tag{118}$$

which is the wave equation. *R* is the radius of the group orbit and could be either in the time or space directions. If *R* is time-like, and we impose periodic boundaries on the universe, this gives us a "wave-in-a-box" solution where the box is expanding in time. If

R is space-like, it would represent universes with cylindrical symmetries. R tells us about the topology of the spacetime.

Using Eqs. (113), (114), and (115) we can get the other equations in terms of the restricted derivatives. Equation (108) becomes

$$R^{V}_{VV} = -\tfrac{5}{4} R^{-2} g^{VV\alpha\beta}\left(D_{V\alpha} R\right)\left(D_{V\beta} R\right) + h_{\alpha}{}^{\gamma}{}_{\beta} h^{\alpha\beta}{}_{\gamma}, \tag{119}$$

equation (110) becomes

$$\begin{aligned} &R^{2} D_{V\gamma} h^{\gamma}{}_{\alpha\beta} + \tfrac{1}{2} R D_{V\gamma} g_{HH\alpha\beta} g^{VV\gamma\rho} D_{V\rho} R \\ &+ R\left(D_{V\gamma} R\right) h^{\gamma}{}_{\alpha\beta} + \tfrac{1}{2} g_{HH\alpha\beta} g^{VV\gamma\rho}\left(D_{V\gamma} R\right)\left(D_{V\rho} R\right) = 0, \end{aligned} \tag{120}$$

equation (111) becomes

$$-\mathrm{TF}\left[R^{-1} D_{V\beta} D_{V\alpha} R\right] - \mathrm{TF}\tfrac{1}{4}\left[R^{-2}\left(D_{V\alpha} R\right)\left(D_{V\beta} R\right)\right] = \mathrm{TF}\, h_{\alpha}{}^{\rho}{}_{\delta} h_{\beta}{}^{\delta}{}_{\rho}, \tag{121}$$

and equation (112) becomes

$$R^{-2}\left(D_{H\beta} R\right)\left(D_{V\alpha} R\right) + D_{H\delta} h_{\alpha}{}^{\delta}{}_{\beta} = 0. \tag{122}$$

Conclusion

Einstein's equations in the projection tensor field formalism have been sorted out in this thesis. All of them have been reduced to equations involving only the restricted derivatives. By substituting in the connection between the regular and fiducial derivatives, and solving for the conformal factor, I hope to learn more about the topology of cylindrical spacetimes. However, considerable work still needs to be completed on them.

A great deal of this work has been done in a general enough method that many of the formulas derived, such as the restricted derivative assembly, that they could be applied to other situations as well. By working out all the equations without defining the explicit form of the metric tensor until the very end, we could apply these methods to any other situation that exhibits symmetries in two or more directions.

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Vita

Miles Daniel Lawrence was born on October 18, 1969, in Fort Rucker, Alabama, and is a U.S. citizen. After graduating from Skyline High School in Idaho Falls, Idaho in 1988, he moved to Nampa, ID where he earned his Bachelor of Science in physics from Northwest Nazarene College in 1992. While there, he was a teaching assistant for the physics and math departments beginning his freshman year. In the 1991-1992 academic year he was the president of the Society of Physics Students. He graduated cum laude and was given science honors, by vote of the faculty of the division of Mathematics and Natural Sciences.

After graduating, he married Kamela Lawrence and the two of them moved to Richmond, Virginia where he attended Virginia Commonwealth University in the M.S. program of the physics department. While there he worked with Dr. Robert Gowdy and fellow graduate student Robert W. Helber, the culmination of that work being this thesis. He was inducted into the Sigma Pi Sigma Honor Society in Physics, VCU chapter, in the spring of 1993, and in the spring of 1994, he was given the Billy Sloope Service Award and the Physics Department Award for Academic Achievements by unanimous vote of the faculty. During the summer of 1994, his book, "Laboratory Manual for General Physics" was published by Kendall-Hunt Publishing Co. and was used the following year at VCU

In 1994, after completion of his Master of Science degree in physics from Virginia Commonwealth University, he moved on to Purdue University in West Lafayette, Indiana to pursue his Ph.D. in physics.