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Haar Measures for Groupoids

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A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

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Abstract

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The definition of a groupoid is presented as well as examples of common structures from which a groupoid can be formed. Haar measure existence and uniqueness theorems for topological groups are used for the construction of Haar systems on groupoids. Some Haar systems are presented in addition to an example of a groupoid which admits no Haar system.

Introduction

In this paper, we begin with some history and use of groupoids. Next, we discuss Haar measure on groups. We incude the definition and several properties of topological groups, and the main results of the section are the existence and uniqueness of a Haar measure for a topological group. In the following section we introduce the formal definition of a groupoid and subsequently talk about properties of groupoids, a cross over between disciplines to music composition, and examples of structures which produce groupoids. In the last section, we define the analogous Haar system for a groupoid, and finish with a counterexample which highlights a groupoid that does not admit a Haar system.

1.1 History

In a 1926 paper on the composition of quadratic forms in four variables, H. Brandt [2] introduced groupoids. Further structures which were topological, differentiable, and algebraic were added to groupoids by C. Ehresmann [4], resulting in groupoids holding value as a tool in differentiable topology and geometry.

1.2 Uses and applications

Groupoids can also be used to generate C^*-algebras as seen in [3], and have also found use in applications in other fields. In particular, music composition. In [6], David Lewin presents the definition of a Generalized Interval System (GIS) which we include in Chapter 3.

Haar measure on Groups

Here we provide several definitions from topology which help introduce the idea of a topological group. We provide some useful properties of topological groups and introduce the space of complex-valued continuous functions with compact support, in which we work for the duration of the paper. We also use several measure theory definitions to build to the definition and idea of a Haar measure on a topoogical group. The remainder of the chapter is devoted to the rigorous construction of the proofs for the two main results in this chapter, that of existence and uniqness concerning Haar measures on topological groups.

2.1 Fundamentals

DEFINITION 2.1. If $A \subset X$, the union of all open sets contained in A is called the *interior* of A and is denoted as A'.

DEFINITION 2.2. If $x \in X$ (or $E \subset X$), a *neighborhood* of x (or E) is a set $A \subset X$ such that $x \in A'$ (or $E \subset A'$).

DEFINITION 2.3. A collection $\mathcal{C}$ of subsets of a space X such that $X = \cup_{C \in \mathcal{C}} C$ is called a *cover* of X. A cover of a space that consists of open sets is called an *open cover*. A topological space X is *compact* if every open cover of X includes a finite subcollection that is also a cover of X.

DEFINITION 2.4. A *locally compact* topological space is one in which every point has a compact neighborhood.

DEFINITION 2.5. A topological space X is a *Hausdorff* space if for every x and y in X, with $x \neq y$, there are disjoint open sets U and V with $x \in U$ and $y \in V$.

DEFINITION 2.6. A set is *precompact* if its closure is compact

DEFINITION 2.7. A *topological group* is a group G endowed with a topology such that the group operations $(x,y) \to xy$ and $x \to x^{-1}$ are continuous from $G \times G$ and G to G.

DEFINITION 2.8. We use the term *locally compact group* to mean a topological group whose topology is locally compact and Hausdorff.

If G is a topological group, we denote the identity element of G by e, and if $A, B \subset G$ and $x \in G$, we define

$$xA = \{xy : y \in A\},$$

$$Ax = \{yx : y \in A\},$$

$$A^{-1} = \{x^{-1} : x \in A\}, \text{ and}$$

$$AB = \{yz : y \in A, z \in B\}$$

We say that $A \subset G$ is *symmetric* if $A = A^{-1}$

Here are some of the basic properties of topological groups:

PROPOSITION 2.9. Let G be a topological group.

1. The topology of G is translation invariant: if U is open and $x \in G$, then xU and Ux are open.

2. For every neighborhood U of e there is a symmetric neighborhood V of e with $V \subset U$.

3. For every neighborhood U of e there is a neighborhood V of e with $VV \subset U$.

4. If H is a subgroup of G, then so is its closure $\overline{H}$.

5. Every open subgroup of G is also closed.

6. If K_1, K_2 are compact subsets of G, so is K_1K_2.

Proof. (1) is equivalent to the continuity in each variable of the map $(x,y) \to xy$, and (2) and (3) are equivalent to the continuity of $x \to x^{-1}$ and $(x,y) \to xy$ at the identity. To illustrate (4), if $x,y \in \overline{H}$ there exists nets x_α, y_β in H which converge to x and y. Then $x_\alpha^{-1} \to x^{-1}$ and $x_\alpha y_\beta \to xy$ (where $A \times B$ is directed as a product space), so x^{-1} and xy belong to $\overline{H}$. For (5), if H is an open subgroup the cosets xH are open for all x, so that $\cup_{x \notin H} xH$ is open and H is closed. Finally, (6) is true because K_1K_2 is the image of the compact set $K_1 \times K_2$ under the continuous map $(x,y) \to xy$. $\square$

If f is a continuous function on the topological group G and $y \in G$, we define the left and right translates of f through y by

$$L_y f(x) = f(y^{-1}x), \ R_y f(x) = f(xy).$$

DEFINITION 2.10. Suppose X is a topological space, and $f : X \to \mathbb{C}$ is a function. Then the *support* of f, written supp (f), is the set $\mathrm{supp}\,(f) = \overline{\{x \in X : f(x) \neq 0\}}$. In other words, supp (f) is the closure of the set where f does not vanish.

If supp(f) is compact, we say that f is *compactly supported* , and we define $C_c(X) = \{f \in C(X) : \mathrm{supp}\,(f) \text{ is compact}\}$, where $C(X)$ denotes the collection of complex-valued continuous functions on X. For example, every continuous function on a compact topological space has compact support since every closed subset of a compact space is compact.

We also employ the following notation $C_c^+ = \{f \in C_c(G) : f(x) \geq 0 \text{ and } ||f||_u > 0\}$, where $||f||_u = \sup\{|f(x)| : x \in X\}$.

DEFINITION 2.11. f is called *left uniformly continuous* if for every $\varepsilon > 0$ there is a neighborhood V of e such that $||L_y f - f||_u < \varepsilon$ for $y \in V$. Similiarily, f is *right uniformly continuous* if for every $\varepsilon > 0$ there is a neighborhood V of e such that $||R_y f - f||_u < \varepsilon$ for $y \in V$.

PROPOSITION 2.12. If $f \in C_c(G)$, then f is left and right uniformly continuous.

Proof. We shall consider right uniform continuity; the proof on the left is the same. Let $K = \operatorname{supp}(f)$ and suppose $\varepsilon > 0$. For each $x \in K$ consider $R_x \circ f$ and $R_x(f(e)) = f(x)$. So, by continuity, there exists a neighborhood U_x of e such that $R_x f(U_x) \subseteq B_{\frac{1}{2}\varepsilon}(f(x))$. Also $R_y(f(x)) = f(xy)$ for all $x \in G$. Then, for all y in the neighborhood U_x of e, $|f(xy) - f(x)| < \frac{1}{2}\varepsilon$. Now by Proposition 2.9(3), there is a neighborhood W so that $WW \subseteq U_x$. Combining this with Proposition 2.9(2), there exists a symmetric neighborhood V_x with $V_x \subseteq W$. Now we can say that $V_x V_x \subseteq WW \subseteq U_x$. Then for $x \in K, x = xe \in xV_x$. Therefore, $\{xV_x\}_{x \in K}$ covers K, and since K is compact, there are $x_1, \ldots, x_n \in K$ such that $K \subset \cup_1^n x_j V_{x_j}$. Then for $x \in K$, there exists j so that $x \in x_j V_{x_j} \Rightarrow x_j^{-1} x \in V_{x_j}$. Let $V = \cap_1^n V_{x_j}$.

If $x \in K$, so that $x_j^{-1} x \in V_{x_j}$ for some j, and $y \in V$, then $|f(x_j(x_j^{-1} y)) - f(x_j)| < \varepsilon/2$. This is true because $(x_j^{-1} y)$ is an element of U_{x_j}, the above defined neighborhood. Also, $|f(x_j(x_j^{-1} x)) - f(x_j)| < \varepsilon/2$ because $x_j(x_j^{-1} x) = x$ where $(x_j^{-1} x)$ is in $V_{x_j} \subset U_{x_j}$. Now the following application of the triangle inequality will hold true.

$$|f(xy) - f(x)| \leq |f(xy) - f(x_j)| + |f(x_j) - f(x)| < \varepsilon$$

If $x \notin K$ and $y \in V$ then $f(x) = 0$, and either $f(xy) = 0$ or $x_j^{-1} xy \in V_{x_j}$ for some j; in the latter case $x_j^{-1} x = x_j^{-1} xyy^{-1} \in U_{x_j}$, so that $|f(x_j)| < \varepsilon/2$ and hence $|f(xy)| < \varepsilon$. $\square$

DEFINITION 2.13. Let Ω be a set. A nonempty collection $\mathcal{A}$ of subsets of Ω is called a σ *-algebra* if the following two conditions are satisfied:

1. $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$,

2. $\{A_n\}_n \subset \mathcal{A}$ implies $\cup_n A_n \in \mathcal{A}$.

DEFINITION 2.14. Let Ω be a set and $\mathcal{A}$ be a σ-algebra of subsets of Ω. A *measure* on $\mathcal{A}$ is an extended real-valued function satisfying the following conditions:

1. $\mu(A) \geq 0$ for all $A \in \mathcal{A}$.

2. $\mu(\emptyset) = 0$

3. If $A_1, A_2, \ldots$ are in $\mathcal{A}$, with $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$\mu(\bigcup_n A_n) = \sum_n \mu(A_n).$$

If X is any topological space, the σ-algebra generated by the family of open sets in X is called the *Borel σ-algebra* on X and will be denoted by $\mathcal{B}_X$. Its members are called *Borel sets*. We add that a measure, μ, on the Borel sets is called a *Borel measure* . We are now prepared to build to the definition of a left Haar measure.

DEFINITION 2.15. Let μ be a Borel Measure on a topological space X and E a Borel subset of X. μ is called *outer regular* on E if $\mu(E) = \inf\{\mu(U) : U \supset E, U \text{ open}\}$.

μ is called *inner regular* on E if $\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$.

DEFINITION 2.16. A *Radon Measure* on a topological space X is a Borel measure which is finite on compact sets, outer regular on all Borel sets, and inner regular on all open sets.

DEFINITION 2.17. Suppose that G is a locally compact group. A Borel measure μ on G is called *left-invariant* (resp. *right-invariant*) if $\mu(xE) = \mu(E)$ (or $\mu(Ex) = \mu(E)$) for all $x \in G$ and $E \in \mathcal{B}_G$. Similarly, a linear functional I on $C_c(G)$ is called left- or right-invariant if $I(L_x f) = I(f)$ or $I(R_x f) = I(f)$ for all f.

DEFINITION 2.18. If G is a locally compact group, a *left (right) Haar measure* on G is a nonzero left-invariant (right-invariant) Radon measure μ on G.

EXAMPLE 2.19. Lebesgue measure is a (left and right) Haar measure on the topological group $(\mathbb{R},+)$. To illustrate, given a set $E \subset \mathbb{R}$ and a point $x \in R$, xE denotes the set of real numbers of the form $x+y$ with $y \in E$. Lebesgue measure, μ is invariant in $\mathbb{R}$ under translations in the sense that, for measurable E, $\mu(E) = \mu(xE)$. Counting measure is a (left and right) Haar measure on any group with the discrete topology, where X is a set and T is the collection of all subsets of X, and T is the discrete topology of X. Counting measure of E = $N(E)$ where $N(E)$ denotes the number of elements of E if E is finite and ∞ if E is infinite. Counting measure is a (left or right) Haar measure because multiplying E on the left or right by an element results in the same number of elements in E, thus $N(E) = N(xE) = N(Ex)$.

2.2 Existence and Uniqueness Theorems

Before we present the existence theorem, we give some motivation. If $E \in \mathcal{B}_G$ and V is open and non-empty, let $(E:V)$ denote the smallest number of left translates of V that cover E, that is,

$$(E:V) = \inf\{|A| : E \subset \bigcup_{x\in A} xV\}$$

where $|A|$ = cardinality of A if A is finite and ∞ otherwise. $(E:V)$ is thus a rough measure of the relative sizes of E and V. If we fix a precompact open set E_0, then when the size of E_0 is normalized to be 1 the ratio $(E:V)/(E_0:V)$ gives a rough estimate of the size of E. This estimate becomes more accurate the smaller V is, and it is left-invariant as a function of E, since $(E:V) = \inf\{|A| : E \subset \bigcup_{x\in A} xV\}$ which is the same thing as $(xE:V) = \inf\{|A| : xE \subset \bigcup_{x\in A} xV\}$. We would then be interested in the limit of the ratio $E \to (E:V)/(E_0:V)$, for fixed E_0, as V shrinks to $\{e\}$, hoping to obtain a Haar measure.

This idea can be made to work as it stands, but it is simpler to carry out if we think of integrals of functions instead of measures of sets. If $f, \varphi \in C_c^+$, then $\{x : \varphi(x) > \frac{1}{2} \parallel \varphi \parallel_u\}$ is open and nonempty. To see that the set is open, consider the following. If φ is continuous, then given $\varepsilon > 0$, there exists a neighborhood U_x of x with $\varphi(U_x) \subseteq B_\varepsilon(\varphi(x))$. Define $S = \{y : \varphi(y) > \frac{1}{2} \parallel \varphi \parallel_u\}$ and select $y \in S$, $\varphi(x) > \frac{1}{2} \parallel \varphi \parallel_u$. Then $|\varphi(x) - \varphi(y)| < \varepsilon$, and $|\varphi(x)| < \varepsilon + |\varphi(y)|$. Pick ε to be equal to $\frac{1}{2}(|\varphi(x)| - \frac{1}{2} \parallel \varphi \parallel_u)$. Then it is seen that $\varphi(y) > \frac{1}{2} \parallel \varphi \parallel_u$. We now show it is nonempty. If $\parallel \varphi \parallel_u > 0$, for $\varepsilon > 0$ there exists x such that $\parallel \varphi \parallel_u - \varphi(x) \leq \varepsilon$. So then, $\parallel \varphi \parallel_u \leq \varphi(x) + \varepsilon$. Let $\varepsilon = \frac{1}{2} \parallel \varphi \parallel_u$, and subtracting ε from both sides results in $\frac{1}{2} \parallel \varphi \parallel_u \leq \varphi(x)$. Also, since $f \in C_c^+$, it has compact support and finitely many left translates of $\{x : \varphi(x) > \frac{1}{2} \parallel \varphi \parallel_u\}$ cover supp(f), i.e. $\sum L_{x_j}\varphi \geq \frac{1}{2} \parallel \varphi \parallel_u$. Now it follows that for some $x_1, \ldots, x_n \in G$,

$$f \leq \parallel f \parallel_u = \frac{2 \parallel f \parallel_u}{\parallel \varphi \parallel_u} \frac{1}{2} \parallel \varphi \parallel_u \leq \frac{2 \parallel f \parallel_u}{\parallel \varphi \parallel_u} \sum_1^n L_{x_j}\varphi$$

It therefore makes sense to define the "Haar covering number" of f with respect to φ as:

$$(f : \varphi) = \inf\{\sum_1^n c_j : f \leq \sum_1^n c_j L_{x_j}\varphi\}$$

which is in $(0, \infty]$ for some $n \in N$ and $x_1, \ldots, x_n \in G$.

We now present a Lemma with properties of the quantity $(f : \varphi)$.

LEMMA 2.20. Suppose that $f, g, \varphi \in C_c^+$.

1. $(f : \varphi) = (L_x f : \varphi)$ for any $x \in G$.
2. $(cf : \varphi) = c(f : \varphi)$ for any $c > 0$.
3. $(f + g : \varphi) \leq (f : \varphi) + (g : \varphi)$.
4. $(f : \varphi) \leq (f : g)(g : \varphi)$.
5. $(f : \varphi) \leq (g : \varphi)$ whenever $f < g$.

Proof. 1. $(f:\varphi) = f(y) \le \sum c_j L_{x_j}\varphi(y) = f(y) \le \sum c_j \varphi(x_j^{-1}y)$.

$(L_x f:\varphi) = f(x^{-1}y) \le \sum c_j \varphi(x_j^{-1}x^{-1}y) = L_{x_j} f(y) \le \sum c_j L_{xx_j}\varphi(y)$.

2. $(cf:\varphi) = (cf(y) \le \sum c_j L_{x_j} c\varphi(y)) = c(f(y) \le \sum c_j L_{x_j}\varphi(y)) = c(f:\varphi)$.

3. $f \le \sum_1^m c_j L_{x_j}\varphi$ and $g \le \sum_{m+1}^n c_j L_{x_j}\varphi$, then $f+g \le \sum_1^n c_j L_{x_j}\varphi$
$\le (\inf \sum_1^m c_j\varphi) + (\inf \sum_{m+1}^n c_j\varphi)$ and $(f+g:\varphi) \le (f:\varphi)+(g:\varphi)$.

4. if $f \le \sum c_i L_{x_i} g$ and $g \le \sum d_j L_{y_j}\varphi$ then $f \le \sum_{i,j} c_i d_j L_{x_i y_j}\varphi$. Then,
$(f:\varphi) \le \inf \sum_{i,j} c_i d j = (\inf \sum c_i)(\inf \sum d_j) = (f:g)(g:\varphi)$.

5. If $g \le \sum c_j L_{x_j}\varphi$ and $f < g$ then $f \le \sum c_j L_{x_j}\varphi$.

□

In a similiar manner to the measure on sets motivation which began the section, at this point we make a normalization by choosing $f_0 \in C_c^+$ once and for all to give a rough estimate of the size of f, and define for $f,\varphi \in C_c^+$,

$$I_\varphi(f) = \frac{(f:\varphi)}{(f_0:\varphi)}$$

Now, we can also say that

PROPOSITION 2.21. $(f_0:f)^{-1} \le I_\varphi(f) \le (f:f_0)$.

Proof. By 2.20 (4), $(f_0:f)^{-1}(f_0:\varphi) \le (f:\varphi) \le (f:f_0)(f_0:\varphi)$. □

For fixed φ, then, the functional I_φ is left-invariant (by Lemma 2.20 (1)). We now show that, in a certain sense, I_φ is approximately additive when $\mathrm{supp}(\varphi)$ is small.

LEMMA 2.22. If $f_1, f_2 \in C_c^+$ and $\varepsilon > 0$, there is a neighborhood V of e such that $I_\varphi(f_1) + I_\varphi(f_2) \le I_\varphi(f_1+f_2) + \varepsilon$ whenever $\mathrm{supp}(\varphi) \subset V$.

Proof. Fix $g \in C_c^+$ such that g = 1 on $\mathrm{supp}(f_1 + f_2)$ and let δ be a positive number. Set $h = f_1 + f_2 + \delta g$ and $h_i = f_i/h\,(i = 1,2)$, where it is understood that $h_i = 0$ outside of $\mathrm{supp}(f_i)$. Then $h_i \in C_c^+$, so by Proposition 2.12 there is a neighborhood V of e such that $|h_i(x) - h_i(y)| < \delta$ if i = 1,2 and $y^{-1}x \in V$. If $\varphi \in C_c^+$, $\mathrm{supp}(\varphi) \subset V$, and $h \leq \sum_1^n c_j L_{x_j}\varphi$, then $|h_i(x) - h_i(x_j)| < \delta$ whenever $x_j^{-1}x \in \mathrm{supp}(\varphi)$, so

$$f_i(x) = h(x)h_i(x) \leq \sum_j c_j \varphi(x_j^{-1}x)h_i(x) \leq \sum_j c_j \varphi(x_j^{-1}x)[h_i(x_j) + \delta].$$

But then,

$$(f_i : \varphi) \leq \sum_j c_j [h_i(x_j) + \delta],$$

and since $h_1 + h_2 \leq 1$,

$$(f_1 : \varphi) + (f_2 : \varphi) \leq \sum_j c_j [1 + 2\delta].$$

Now, $\sum_j c_j$ can be made arbitrarily close to $(h : \varphi)$, so by Lemma 2.20(3),

$$I_\varphi(f_1) + I_\varphi(f_2) \leq (1 + 2\delta)I_\varphi(h) \leq (1 + 2\delta)[I_\varphi(f_1 + f_2) + \delta I_\varphi(g)].$$

In view of Lemma 2.20(6), therefore, if we choose δ small enough so that

$$2\delta(f_1 + f_2 : f_0) + \delta(1 + 2\delta)(g : f_0) < \varepsilon$$

we are done. $\square$

For the following proposition we remind the reader of the finite intersection property, which states any finite subcollection of a family of sets has a non-empty intersection.

PROPOSITION 2.23. A topological space X is compact if and only if for every family $\{F_\alpha\}_{\alpha \in A}$ of closed sets with the finite intersection property, $\bigcap_{\alpha \in A} F_\alpha \neq \emptyset$.

Proof. ($\Rightarrow$) Suppose X is compact, i.e. any collection of open subsets that cover X has a finite collection that also cover X. Further, suppose $\{F_i\}_{i\in I}$ is an arbitrary collection of closed subsets with the finite intersection property. We claim that $\cap_{i\in I}F_i$ is non-empty. Suppose otherwise, that is suppose $\cap_{i\in I}F_i = \emptyset$. Then, $X = (\cap_{i\in I}F_i)^c = \cup_{i\in I}F_i^c$. Since each F_i is closed, the collection $\{F_i^c\}_{i\in I}$ is an open cover for X. By compactness, there is a finite subset $J \subset I$ such that $X = \cup_{i\in J}F_i^c$. But then, $X = (\cap_{i\in J}F_i)^c$, so $\cap_{i\in J}F_i = \emptyset$ which is a contradiction by the finite intersection property of $\{F_i\}_{i\in I}$.

($\Leftarrow$) Assume X has the finite intersection property. To prove that X is compact, let $\{F_i\}_{i\in I}$ be a collection of open sets in X that cover X. We claim that this collection contains a finite subcollection of sets that also cover X. We go by contradiction. Suppose that $X \neq \cup_{i\in I}F_i$ holds for all finite $J \subset I$. Let us first show that the collection of closed subsets $\{F_i^c\}_{i\in I}$ has the finite intersection property. If J is a finite subset of I, then $\cap_{i\in I}F_i^c = (\cup_{i\in I}F_i)^c \neq \emptyset$, where the last assertion follows since J was finite. Now, since X has the finite intersection property, $\emptyset \neq \cap_{i\in I}F_i^c = (\cup_{i\in I}F_i)^c$. This contradicts the assumption that $\{F_i\}_{i\in I}$ is a cover for X. $\square$

An additional result which we will call upon to prove the main theorem in this section is the Riesz Representation Theorem , which follows. A reference for the proof can be found in [5]. In the Riesz Representation Theorem, we need the following definitions.

DEFINITION 2.24. Let X be a vector space over K, where K = the real or complex numbers. A linear map from X to K is called a *linear functional*.

DEFINITION 2.25. A linear functional I on $C_c(X)$ will be called *positive* if $I(f) \geq 0$ whenever $f \geq 0$.

THEOREM 2.26. The Riesz Representation Theorem. If I is a positive linear functional on $C_c(X)$, there is a unique Radon measure μ on X such that $I(f) = \int f\, d\mu$ for all $f \in C_c(X)$.

THEOREM 2.27. Every locally compact group G possesses a left Haar Measure.

Proof. For each $f \in C_c^+$ let X_f be the interval $[(f_0 : f)^{-1}, (f : f_0)]$, and let

$$X = \prod_{f \in C_c^+} X_f.$$

Then X is a compact Hausdorff space by Tychonoff's theorem, and by Lemma 2.20(6), every I_φ is an element of X. For each compact neighborhood V of e, let K_v be the closure in X of $\{I_\varphi : \mathrm{supp}(\varphi) \subset V\}$. We can write,

$$\bigcap_1^n K_j \supset K_{\bigcap_1^n K_j}.$$

To show this let $x \in K_{\bigcap_1^n K_j}$. That means that x is in the closure of $\{I_\varphi : \mathrm{supp}(\varphi) \subset K_{\bigcap_1^n K_j}\}$ in X. Since $\mathrm{supp}\,(\varphi)$ is in the intersection of the K_j's, it is in each K_j, i.e. $\mathrm{supp}\,(\varphi) \subset K_j$ for all $1 \leq j \leq n$. Therefore x is in the closure of $\{I_\varphi : \mathrm{supp}\,(\varphi) \subset K_j$, for all $1 \leq j \leq n$. $x \in \bigcap_1^n K_j$. Now, by Proposition 2.23 there is an element I in the intersection of the K_v's. Every neighborhood of I in X intersects $\{I\varphi : \mathrm{supp}(\varphi) \subset V\}$ for all V. In other words, for any neighborhood v of e and any $f_1, \ldots, f_n \in C_c^+$ and $\varepsilon > 0$ there exists $\varphi \in C_c^+$ with $\mathrm{supp}(\varphi) \subset V$ such that $|I(f_j) - I_\varphi(f_j)| < \varepsilon$ for $j = 1, \ldots, n$. Therefore in view of Lemma 2.20(1)-(3) and Lemma 2.22, I is left-invariant and satisfies $I(c_1 f_1 + c_2 f_2) = c_1 I(f_1) + c_2 I(f_2)$ for any $c_1, c_2 \geq 0$, and it follows easily that if we define $I(f) = I(f^+) - I(f^-)$ for $f \in C_c(G)$, where $f^+(x) = \max(f(x), 0)$ and $f^-(x) = \max(-f(x), 0)$, then I is a left-invariant positive linear functional on $C_c(G)$. Moreover, $I(f) > 0$ for all $f \in C_c^+$ by Lemma 2.20(6). The proof is then completed by invoking the Riesz representation theorem. Therefore μ is a left invariant Haar measure on G. $\square$

Having established the existence of Haar measure, we now examine its uniqueness.

THEOREM 2.28. If μ and ν are left Haar measures on G, there exists $c > 0$ such that $\mu = c\nu$.

Proof. The assertion that $\mu = c\nu$ is equivalent to the assertion that $\int f d\mu / \int f d\nu$ is independent of $f \in C_c^+$. To explain this further, when $c = \frac{\mu}{v}$ and the measures μ and v are written in their integral forms of $\int f d\mu$ and $\int f dv$, then the ratio of $\frac{\int f d\mu}{\int f dv}$ is the same for any f chosen in C_c^+. Suppose then that $f, g \in C_c^+$; we shall show that $\frac{\int f d\mu}{\int f d\nu} = \frac{\int g d\mu}{\int g d\nu}$. Fix a symmetric compact neighborhood V_0 of e and set

$$A = (\operatorname{supp}(f))V_0 \cup V_0(\operatorname{supp}(f))$$

$$B = (\operatorname{supp}(g))V_0 \cup V_0(\operatorname{supp}(g))$$

Then A and B are compact by Proposition 2.9(6), and for $y \in V_0$ the functions $x \to f(xy) - f(yx)$ and $x \to g(xy) - g(yx)$ are supported in A and B respectively. Next given $\varepsilon > 0$, by Proposition 2.12 there is a symmetric compact neighborhood $V \subset V_0$ of e such that $\sup_x |f(xy) - f(yx)| < \varepsilon$ and $\sup_x |g(xy) - g(yx)| < \varepsilon$ for $y \in V$.

Pick $h \in C_c^+$ with supp(h)$\in V$ and $h(x) = h(x^{-1})$. Since a group is a subset of a permutation group, any element x can be rewritten as yx. Then we have,

$$\left(\int h d\nu\right)\left(\int f d\mu\right) = \iint h(y) f(x)\, d\mu(x)\, d\nu(y)$$

$$= \iint h(y) f(yx)\, d\mu(x)\, d\nu(y)$$

Now by substitution, application of Fubini's theorem, and $h(x) = h(x^{-1})$, we have

$$\left(\int h d\mu\right)\left(\int f d\nu\right) = \iint h(x) f(y)\, d\mu(x)\, d\nu(y)$$

$$= \iint h(y^{-1}x) f(y)\, d\mu(x)\, d\nu(y)$$

$$= \iint h(x^{-1}y) f(y)\, d\nu(y)\, d\mu(x)$$

$$= \iint h(y) f(xy)\, d\nu(y)\, d\mu(x)$$

$$= \iint h(y) f(xy)\, d\mu(x)\, d\nu(y).$$

Since the function $x \to f(xy) - f(yx)$ is supported in A and $\sup_x |f(xy) - f(yx)| < \varepsilon$, we establish that

$$\left| \left(\int h\, d\mu \right) \left(\int f\, d\nu \right) - \left(\int h\, d\nu \right) \left(\int f\, d\mu \right) \right|$$

$$= \left| \iint h(y) \left[f(xy) - f(yx) \right] d\mu(x)\, d\nu(y) \right|$$

$$\leq \varepsilon \mu(A) \int h\, d\nu.$$

By the same approach,

$$\left| \left(\int h\, d\mu \right) \left(\int g\, d\nu \right) - \left(\int h\, d\nu \right) \left(\int g\, d\mu \right) \right| \leq \varepsilon \mu(B) \int h\, d\nu.$$

Applying the triangle inequality, we get

$$\left| \frac{\int f\, d\mu}{\int f\, d\nu} - \frac{\int g\, d\mu}{\int g\, d\nu} \right| \leq \left| \frac{\int f\, d\mu}{\int f\, d\nu} - \frac{\int h\, d\mu}{\int h\, d\nu} \right| + \left| \frac{\int h\, d\mu}{\int h\, d\nu} - \frac{\int g\, d\mu}{\int g\, d\nu} \right|$$

Dividing the two above inequalities by $(\int h\, d\nu)(\int f\, d\nu)$ and $(\int h\, d\nu)(\int g\, d\nu)$, respectively, we get

$$\left| \frac{(\int h\, d\mu)(\int f\, d\nu)}{(\int h\, d\nu)(\int f\, d\nu)} - \frac{(\int h\, d\nu)(\int f\, d\mu)}{(\int h\, d\nu)(\int f\, d\nu)} \right| \leq \frac{\varepsilon \mu(A) \int h\, d\nu}{(\int h\, d\nu)(\int f\, d\nu)}$$

$$\left| \frac{(\int h\, d\mu)(\int g\, d\nu)}{(\int h\, d\nu)(\int g\, d\nu)} - \frac{(\int h\, d\nu)(\int g\, d\mu)}{(\int h\, d\nu)(\int g\, d\nu)} \right| \leq \frac{\varepsilon \mu(B) \int h\, d\nu}{(\int h\, d\nu)(\int g\, d\nu)}$$

After cancellation we are left with the following

$$\left|\frac{(\int h\,d\mu)}{\int h\,d\nu} - \frac{(\int f\,d\mu)}{\int f\,d\nu}\right| \leq \frac{\varepsilon\,\mu(A)}{\int f\,d\nu}$$

$$\left|\frac{(\int h\,d\mu)}{\int h\,d\nu} - \frac{(\int g\,d\mu)}{\int g\,d\nu}\right| \leq \frac{\varepsilon\,\mu(B)}{\int g\,d\nu}$$

Then we can pull the ε out to get

$$\left|\frac{\int f\,d\mu}{\int f\,d\nu} - \frac{\int g\,d\mu}{\int g\,d\nu}\right| \leq \varepsilon\left(\frac{\mu(A)}{\int f\,d\nu} + \frac{\mu(B)}{\int g\,d\nu}\right).$$

Since we can pick an ε as close to 0 as we want, we can say that

$$\left|\frac{\int f\,d\mu}{\int f\,d\nu}\right| = \left|\frac{\int g\,d\mu}{\int g\,d\nu}\right|,$$

and we are done. $\square$

Groupoids

In this chapter we introduce the definition of a groupoid and provide several short but useful propositions concerning groupoid behavior. We also present a cross-field example of application in the area of music composition. We conclude this chapter with examples of common structures which can be manipulated into groupoids.

A *groupoid* is a set G endowed with a *product map* $(x,y) \to xy : G^{(2)} \to G$ where $G^{(2)}$ is a subset of $G \times G$ called the *set of composable pairs*, and an *inverse map* $x \to x^{-1} : G \to G$ such that the following conditions hold:

1. If $(x,y) \in G^{(2)}$ and $(y,z) \in G^{(2)}$, then $(xy,z) \in G^{(2)}$, $(x,yz) \in G^{(2)}$ and $(xy)z = x(yz)$;

2. $(x^{-1})^{-1} = x$ for all $x \in G$;

3. For all $x \in G$, $(x,x^{-1}) \in G^{(2)}$, and if $(z,x) \in G^{(2)}$, then $(zx)x^{-1} = z$;

4. For all $x \in G$, $(x^{-1},x) \in G^{(2)}$, and if $(x,y) \in G^{(2)}$, then $x^{-1}(xy) = y$.

The maps r and d on G, defined by the formulae $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$, are called the *range* and the *domain* maps. They have a common image, i.e. $xx^{-1} = u = x^{-1}x$, called the *unit space* of G, which is denoted $G^{(0)}$. Its elements are *units* in the sense of the following proposition.

PROPOSITION 3.1. For $x \in G$, $xd(x) = r(x)x = x$.

Proof. This is shown by $xd(x) = xx^{-1}x = x = xx^{-1}x = r(x)x$. $\square$

PROPOSITION 3.2. For $x, y \in G$, it is useful to note that a pair (x,y) lies in $G^{(2)}$ precisely when $d(x) = r(y)$.

Proof. To prove this assertion, assume that $d(x) = r(y)$, then $x^{-1}x = yy^{-1}$. Multiplying on the left by x, we get $xx^{-1}x = xyy^{-1}$. This means that $(x, x^{-1}x) \in G^{(2)}$ as well as $(x, yy^{-1}) \in G^{(2)}$. Then it follows that, $(x^{-1}, x) \in G^{(2)}$ and $(x, yy^{-1}) \in G^{(2)}$, and by properties (3) and (4) of groupoid definition $(x, y) \in G^{(2)}$.

The other direction of the proof follows. If $(x, y) \in G^{(2)}$ then $(y, y^{-1}) \in G^{(2)}$ by (3) and $(x^{-1}, x) \in G^{(2)}$ by (4). Then by (1), (x, yy^{-1}) and $(x, x^{-1}x)$ are both elements of $G^{(2)}$. Now, $xyy^{-1} = xx^{-1}x$ multiplying on the left by x^{-1} results in $x^{-1}xyy^{-1} = x^{-1}xx^{-1}x$ which cancels to leave $yy^{-1} = x^{-1}x$. Thus $r(y) = d(x)$. □

PROPOSITION 3.3. For $x \in G$, the cancellation laws hold (e.g. $xy = xz$ if and only if $y = z$).

Proof. To illustrate one direction, $xy = xz \Rightarrow x^{-1}(xy) = x^{-1}(xz)$ then $y = z$. The other direction is complete after $y = z \Rightarrow xy = xz$. □

DEFINITION 3.4. A *fibre* of the range map is denoted $G^u = r^{-1}(\{u\})$ and likewise a fibre of the domain map is denoted $G_v = d^{-1}(\{v\})$. Also for $u, v \in G^{(0)}$, $G^u_v = G^u \cap G_v$. More generally, given the subsets $A, B \subset G^{(0)}$, we define $G^A = r^{-1}(A)$ and $G_B = d^{-1}(B)$ and $G^A_B = r^{-1}(A) \cap d^{-1}(B)$.

DEFINITION 3.5. G^A_A is called the *reduction* of G to A with the unit space A.

PROPOSITION 3.6. If we define $(G^A_A)^{(2)} = G^{(2)} \cap (G^A_A \times G^A_A)$, then G^A_A becomes a groupoid.

Proof. To show that G^A_A is a groupoid, consider property (1). If $(x, y) \in (G^A_A)^{(2)}$ then $(x, y) \in G^{(2)}$, and $(x, y) \in G^A_A \times G^A_A$ that is $x, y \in G^A_A$. Likewise, if $(y, z) \in (G^A_A)^{(2)}$ then $(y, z) \in G^{(2)}$ and $y, z \in G^A_A$. Now, $xy \in G$ so $d(xy) = d(y) \in A$ and $r(xy) = r(x) \in A$, then $xy \in G^A_A$, as well as, $yz \in G$ implies $d(yz) = d(y) \in A$ and $r(yz) = r(z) \in A$ so that $yz \in G^A_A$.

Also, $(xy, z) \in G^{(2)}$, $(x, yz) \in G^{(2)}$ so therfore, $(xy, z) \in (G_A^A)^{(2)}$ and $(x, yz) \in (G_A^A)^{(2)}$. Since $z \in G_A^A$, $z \in G$ and $(xy)z = x(yz)$. To illustrate property (2), we want to show that $(x^{-1})^{-1} = x$ for all $x \in G_A^A$. For $x \in G_A^A$, $d(x^{-1}) = r(x)$ and $r(x^{-1}) = d(x)$. Then $d((x^{-1}))^{-1} = r(x^{-1}) = d(x)$ and $r((x^{-1}))^{-1} = d(x^{-1}) = r(x)$. Since $((x^{-1}))^{-1} = x$ in G, $((x^{-1}))^{-1} = x$ in G_A^A. Looking at property (3), $x \in G_A^A$ implies $d(x) \in A$ and $r(x) \in A$. Since $d(x^{-1}) = r(x) \in A$ and $r(x^{-1}) = d(x) \in A$ and it is shown that $x^{-1} \in G_A^A$. Then
$(x, x^{-1}) \in G^{(2)}$ because $d(x) = r(x^{-1})$. Therefore $(x, x^{-1}) \in (G_A^A)^{(2)}$. Now, assume that $(z, x) \in (G_A^A)^{(2)}$. Then, $(z, x) \in G^{(2)}$ and $z, x \in G_A^A$. Since $x \in G_A^A$, $x^{-1} \in G_A^A$ as well, and by containment, $x^{-1} \in G$. Then it seen that $(zx)x^{-1} = z$. Property (4) follows in a similiar manner. For all $x \in G_A^A$, $x^{-1} \in G_A^A$ and $(x^{-1}, x) \in (G_A^A)^{(2)}$. Then assume that $(x, y) \in (G_A^A)^{(2)}$. As shown above, $x^{-1} \in G$ and $x^{-1}(xy) = y$. Therefore G_A^A, the reduction of G to A is a groupoid. □

If A and B are subsets of G, the following subsets of G can be formed:

$$A^{-1} = \{x \in G : x^{-1} \in A\}$$
$$AB = \{xy : (x,y) \in G^{(2)} \cap (A \times B)\}$$

DEFINITION 3.7. A *homomorphism of groupoids* is a map $\phi : G \to H$ (with G, H groupoids) satisfying the following condition: if $(x,y) \in G^{(2)}$, then $(\phi(x), \phi(y)) \in H^{(2)}$ and $\phi(xy) = \phi(x)\phi(y)$.

DEFINITION 3.8. If $R = X \times X$, then R is called the *trivial groupoid* on X.

DEFINITION 3.9. A groupoid G is said to be *principal* if the map $(r(x), d(x)) : G \to G^0 \times G^0$ is one-to-one.

Now we present David Lewin's definition of a Generalized Interval System (GIS).

DEFINITION 3.10. A *Generalized Interval System* (GIS) is an ordered triple $(S, IVLS, int)$, where S, the space of the GIS, is a family of elements, $IVLS$, the group of intervals for the GIS, is a mathematical group, and int is a function mapping $S \times S$ into $IVLS$, all subject to the two conditions following.

1. For all $r, s,$ and $t \in S$, $int(r,s)int(s,t) = int(r,t)$,
2. For every $s \in S$ and every $i \in IVLS$, there is a unique $t \in S$ which lies in the interval i from s, that is a unique t which satisfies the equation $int(s,t) = i$.

This concept from music composition can be related back to mathematics when a (GIS) is compared to a groupoid. Property (1) of (GIS) definition refers to closure of composition in the system, much like property (1) of the groupoid definition which requires that if $(x,y) \in G^{(2)}$ and $(y,z) \in G^{(2)}$, then $(xy,z) \in G^{(2)}$, $(x,yz) \in G^{(2)}$.

3.1 Examples of structures

1. Groups.

   DEFINITION 3.11. A *group* is an ordered pair $(G,*)$ where G is a set and $*$ is a binary operation on G satisfying the following:

   (a) $(a*b)*c = a*(b*c) \forall a, b, c \in G$ (Associative)

   (b) $\exists$ element $e \in G$, called an identity of G, such that $\forall a \in G$ we have $a*e = e*a = a$

   (c) $\forall a \in G$ there is an element $a^{-1} \in G$, called the inverse of a such that $a*a^{-1} = a^{-1}*a = e$.

   A group G is a groupoid with $G^{(2)} = G \times G$ and $G^{(0)} = \{e\}$, the unit element. To show this, we begin with groupoid property (1). Assume that $(x,y) \in G \times G$ and

$(y,z) \in G \times G$. Since groups are closed under their binary operation, $(xy,z) \in G \times G$ and $(x,yz) \in G \times G$. Also the associatve property of groups indicates that $(xy)z = x(yz)$ for $x,y,z \in G$. Property (2) is shown by the following. $(x^{-1})^{-1}x^{-1} = e$ then multiply on the right by x resulting in $(x^{-1})^{-1}x^{-1}x = ex = (x^{-1})^{-1} = x$. To demonstrate properties (3) and (4), G as a group has an inverse for all elements of G. Then, $(x,x^{-1}) \in G \times G$ and $(x^{-1},x) \in G \times G$. Now assume $(z,x) \in G \times G$, then by associativity $(zx)x^{-1} = z(xx^{-1}) = z$. Likewise, $(x,y) \in G \times G$ implies $x^{-1}(xy) = (x^{-1}x)y = y$.

2. Group Bundles. Let $\{G_u\}_{u \in U}$ be a family of groups indexed by a set U. Here, two elements may be composed if and only if they belong to the same group G_u and the inverse of an element $x \in G_u$ is its inverse in the group G_u. The disjoint union of the family $\{G_u\}_{u \in U}$ is a groupoid. To illustrate property (1) of the groupoid definition, recall that $G^{(2)} \subset \bigcup\{G_u\}_{u \in U} \times \bigcup\{G_u\}_{u \in U}$ and that $G^{(2)}$ is the set of composable pairs. Assume that $(x,y) \in G^{(2)}$ and $(y,z) \in G^{(2)}$. Because two elements may composed only if they belong to the same group, x,y,z are in the same group G_u; and the closure property of G_u indicates that $(xy,z) \in G^{(2)}$ and $(x,yz) \in G^{(2)}$. By associativity of G_u, $(xy)z = x(yz)$. Property (2) is satisfied because any $x \in \bigcup\{G_u\}_{u \in U}$ is in one distinct group G_u, and x^{-1} is its inverse element in G_u. In the group G_u, the following holds: $(x^{-1})^{-1}x^{-1} = e$ and so multiplying on the right by x results in $(x^{-1})^{-1}x^{-1}x = ex = (x^{-1})^{-1} = x$. To show properties (3) and (4), we indicate that every $x \in \cup\{G_u\}_{u \in U}$ is in a distinct group and hence has a distinct inverse x^{-1}. Then $\forall x \in \cup\{G_u\}_{u \in U}$, $(x,x^{-1}) \in G^{(2)}$ and similiarily $(x^{-1},x) \in G^{(2)}$. Now, assume that $(z,x) \in G^{(2)}$ and $(x,y) \in G^{(2)}$. Then z,y,x,x^{-1} are in the same group and by associativity $(zx)x^{-1} = z(xx^{-1}) = z$ and $x^{-1}(xy) = (x^{-1}x)y = y$.

3. Spaces. A space X is a groupoid letting $X^{(2)} = \operatorname{diag}(X) = \{(x,x), x \in X\}$ and defining the operations by $xx = x$ and $x^{-1} = x$. Property (1) is shown by letting $(x,x) \in X^{(2)}$

and $(x,x) \in X^{(2)}$. Since $xx = x$, then $(xx,x) \in X^{(2)}$ and $(x,xx) \in X^{(2)}$. Also, $(xx)x = xx = x = xx = x(xx)$. Property 2 is illustrated naturally by the inverse map defined as $x^{-1} = x$, in which case $(x^{-1})^{-1} = x$. To show property (3), recall that $x^{-1} = x$ and then for $(x,x) \in X^{(2)}, (x,x^{-1}) \in X^{(2)}$ as well. Finally, if $(x,x) \in X^{(2)}$, then $(xx)x^{-1} = xx^{-1} = xx = x$. Property (4) follows in the same manner.

4. Transformation Groups. Let Γ be a group acting on a set X such that for $x \in X$ and $g \in \Gamma$, xg denotes the transform of x by g.

   Let $G = X \times \Gamma, G^{(2)} = \{((x,g),(y,h)) : y = xg\}$. With the product $(x,g)(xg,h) = (x,gh)$, the inverse $(x,g)^{-1} = (xg,g^{-1})$, and the unit space of G identified with X, G becomes a groupoid. To show this, consider groupoid property (1). If $(m,n) \in G^{(2)}$ and $(n,p) \in G^{(2)}$ then $m = (x,g), n = (y,h), p = (z,j)$ such that $y = xg$, $z = yh$. Then the product $(mn,p) = ((x,gh),(z,j)) \in G^{(2)}$ since $y = xg, z = yh \Rightarrow z = xgh$. In a similiar manner, the product $(m,np) = ((x,g)(y,hj)) \in G^{(2)}$ because $y = xg$. This indicates that $(mn)p = (x,gh)(xgh,j) = (x,ghj)$ as well as $m(np) = (x,g)(xg,hj) = (x,ghj)$. To show property (2), let $m \in G = (x,g)$. Then $m^{-1} = (xg,g^{-1})$ and $(m^{-1})^{-1} = (xgg^{-1},g) = (x,g) = m$. Now for property (3), with $G = X \times \Gamma$, $m \in G = (x,g)$ and $m^{-1} = (xg,g^{-1})$. Therefore, $(m,m^{-1}) \in G^{(2)}$ because $xg = xg$. Now assume that $(p,m) \in G^{(2)}$, then $p = (z,j)$ and $m = (x,g)$ such that $x = zj$. The product $(pm)m^{-1}$ becomes $((z,j)(zj,g))(xg,g^{-1})$ which is computed as follows. $(z,jg)(xg,g^{-1}) = (z,jg)(zjg,g^{-1}) = (z,jgg^{-1}) = (z,j)$, which is equal to p, therefore $(pm)m^{-1} = p$. Now for property (4), $(m^{-1},m) \in G^{(2)}$ because $xgg^{-1} = x$ since $g \in \Gamma$, a group. In a similiar manner as above, the assumption of $(m,n) \in G^{(2)}$ assigns $m = (x,g), n = (y,h)$ such that $y = xg$. We want to show that $m^{-1}(mn) = n$. $(mn) = (x,gh)$, then $(xg,g^{-1})(x,gh) = (xg,g^{-1}gh) = (xg,h) = (y,h) = n$ and we are done.

5. Equivalence Relations.

DEFINITION 3.12. A relation $R \subset A \times A$ on a set A is an *equivalence relation* if

(a) $\forall x \in A$, $(x,x) \in R$ (reflexive)

(b) if $(x,y) \in R$ then $(y,x) \in R$ (symmetric)

(c) if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$ (transitive).

If G is any groupoid, then $R = \{(r(g),d(g)), g \in G\}$ is an equivalence relation on $G^{(0)}$. To show that R is an equivalence relation, we start with property (a). If we indicate an unit of G as x, then $\exists g \in G$ such that $r(g) = x$ and $d(g) = x$, because every unit gets mapped to itself. Then $(x,x) \in R$. For property (b), we proceed as follows. Assume that $(x,y) \in R$, then $\exists g \in G$ such that $d(g) = y$ and $r(g) = x$. Because G is a groupoid, then $\exists g^{-1} \in G$ such that $d(g^{-1}) = y$ and $r(g^{-1}) = x$. Therefore, $(y,x) \in R$. To finish with property (c), we begin in a similiar manner. Assume that $(x,y) \in R$, then $\exists g \in G$ such that $d(g) = y$ and $r(g) = x$. Also assuming that $(y,z) \in R$ implies that $\exists g' \in G$ such that $d(g') = z$ and $r(g') = y$. Since $d(g) = y = r(g')$, $r(gg') = x$, $d(g'g) = z$, so therefore $(x,z) \in R$.

Let $R \subset X \times X$ be an equivalence relation on the set X.

Let $R^{(2)} = \{((x_1,y_1),(x_2,y_2)) \in R \times R : y_1 = x_2\}$. With the product $(x,y)(y,z) = (x,z)$, $(x,y)^{-1} = (y,x)$, and $R^{(0)}$ identified with X; R is a principal groupoid. To illustrate that R is a groupoid, start with property (1). Assume that $(x,y) \in R^{(2)}$, corresponding to $x = (x_1,y_1)$ and $y = (x_2,y_2)$, therefore $y_1 = x_2$. Also assume that $(y,z) \in R^{(2)}$, with $z = (x_3,y_3)$, so that $y_2 = x_3$. Then rewrite y and z as (y_1,y_2) and (y_2,y_3) respectively. Then, $(xy)z = (x_1,y_2)(y_2,y_3) = (x_1,y_3)$ and likewise, $x(yz) = (x_1,y_1)(y_1,y_3) = (x_1,y_3)$. Property (2) can be easily seen by letting $z = (x,y) \forall z \in R$. Then $(z^{-1})^{-1} = ((x,y)^{-1})^{-1} = (y,x)^{-1} = (x,y) = z$, so $(z^{-1})^{-1} = z$. Working property

(3) and (4) together, because R is an equivalence relation and therefore symmetric, $\forall(x,y) \in R, (x,y)^{-1} = (y,x) \in R$. Then $(x,x^{-1}) \in R^{(2)}$ and $(x^{-1},x) \in R^{(2)}$ as well. Continuing on, if $(z,x) \in R^{(2)}$ then they correspond to (x_1,y_1) and (x_2,y_2) such that $y_1 = x_2$ and we rewrite x as (y_1,y_2). Then $(zx)x^{-1}$ becomes $((x_1,y_1)(y_1,y_2)(y_2,y_1)$ which results in $(x_1,y_2)(y_2,y_1) = (x_1,y_1) = z$. Likewise, $(x,y) \in R^{(2)}$ identifies (x_1,y_1) and (x_2,y_2). Again, using $y_1 = x_2$,
then $x^{-1}(xy) = (y_1,x_1)((x_1,y_1(y_1,y_2)) = (y_1,x_1)(x_1,y_2) = (y_1,y_2) = y$.

To show that R is principal, we want to show that $f(x) = (r(x),d(x)) : G \to G^0 \times G^0$ is one-to-one. Let $f(x) = (r(x),d(x))$ and $f(y) = (r(y),d(y))$. Now to show f is one-to-one, assume that $f(x) = f(y)$, that is $(r(x),d(x)) = (r(y),d(y))$. Then there exists $u \in G^0$ such that $u = r(x) = r(y)$ and there exists $v \in G^0$ such that $v = d(x) = d(y)$. Then both x and y have the same domain and range, and $x = y$. Therefore, f is one-to-one.

Haar systems for Groupoids

For developing an algebraic theory of functions on a locally compact groupoid, one needs an analogue of Haar measure on locally compact groups. We use the definition adopted by Renault in [7].

The analogue of Haar measure in the setting of groupoids is a system of measures, called a Haar system, subject to suitable invariance and smoothness conditions called respectively "left invariance" and "continuity". More precisely,

DEFINITION 4.1. A *(left) Haar system* on a locally compact Hausdorff groupoid G is a family of positive Radon Measures on G, $\nu = \{\nu^u, u \in G^{(0)}\}$, such that

1. For all $u \in G^{(0)}$, $\operatorname{supp}(\nu^u) = G^u$, where $\operatorname{supp}(\nu^u)$ is defined to be the set of all $x \in X$ for which every open neighborhood N_x of x has positive measure.

2. For all $f \in C_c(G)$, $u \mapsto \int f(x)\, d\nu^u(x) : G^{(0)} \to \mathbb{C}$ is continuous.

3. For all $f \in C_c(G)$ and all $x \in G$, $\int f(y)\, d\nu^{r(x)}(y) = \int f(xy)\, d\nu^{d(x)}(y)$.

Unlike the case for locally compact groups, Haar systems on groupoids need not exist. Also, when a Haar system does exist, it need not be unique.

4.1 Examples

1. If G is a locally compact Hausdorff group, then G (as a groupoid) admits an essentially unique (left) Haar system $\{\mu\}$ where μ is a Haar measure on G.

2. If Γ is a locally compact Hausdorff group acting continuously on a locally compact Hausdorff space X, then $X \times \Gamma$ (as a groupoid) admits a distinguished (left) Haar system $\{\varepsilon_x \times \mu, x \in X\}$ where μ is a Haar measure on Γ and ε_x is the unit point mass at x.

3. If X is a locally compact Hausdorff space and if μ is a positive Radon measure on X with full support (i.e. $\operatorname{supp}(\mu) = X$) then $\{\varepsilon_x \times \mu,\, x \in X\}$ is a Haar system on $X \times X$ (as a trivial groupoid) where ε_x is the unit point mass at x. Conversely, any Haar system on $X \times X$ may be written in this form (for a positive Radon measure μ).

4. If G is a locally compact groupoid, then $G^{(2)}$ with the topology induced from $G \times G$ is also a locally compact groupoid. If $\{\lambda^u\}$ is a left Haar system for G, then $\{(\lambda^2)^x\}$ is a left Haar system for $G^{(2)}$ where $\int f d(\lambda^2)^x = \int f(x,z) d\lambda^{d(x)}$ for $f \in C_c(G^{(2)})$. For example, if G is a group, $G^{(2)} = G \times G$. As a groupoid, it is the groupoid associated with the transformation group (G,G) where G acts on itself by translation. Its left Haar system is $\delta_x \times \lambda$, where λ is a left Haar measure for G.

4.2 Existence and Uniqueness Theorems

PROPOSITION 4.2. Let G be locally compact group bundle, that is, a locally compact groupoid which is a group bundle in the sense of previous definition. Then a left Haar system, if it exists, is essentially unique in the sense that two left Haar system $\{\lambda^u\}$ and $\{\nu^u\}$ differ by a continuous positive function h on $G^{(0)} : \lambda^u = h(u)\nu^u$.

Proof. By property (2) of the Haar system definition, if u, v are units, then we can pick a δ small enough such that $|u - v| < \delta$ implies that $|h(u) - h(v)| < \varepsilon$ for all positive ε. This can be written as $|\frac{\lambda^u}{\nu^u} - \frac{\lambda^v}{\nu^v}| < \varepsilon$. Formally, if we define a function $f = 1$, then $\frac{\lambda^u}{\nu^u} = \frac{\int 1 \lambda^u}{\int 1 \nu^u}$, and we will stay with the measure notation. When we get a common denominator, we see

$|\frac{\nu^v\lambda^u-\lambda^v\nu^u}{\nu^u\nu^v}| < \varepsilon$. Adding and subtracting by the same thing we get $|\frac{\nu^v\lambda^u-\nu^v\lambda^v+\nu^v\lambda^v-\lambda^v\nu^u}{\nu^u\nu^v}| < \varepsilon$. Now we can pull out a common term and we are left with the above less than or equal to $\left|\frac{|\nu^v||\lambda^u-\lambda^v|+|\lambda^v||\nu^v-\nu^u|}{|\nu^u||\nu^v|}\right| < \varepsilon$. □

DEFINITION 4.3. A function $f : X \to Y$ between two topological spaces is called a *homeomorphism* if it has the following properties:

1. f is a bijection (1-1 and onto)

2. f is continuous

3. f^{-1} is continuous

DEFINITION 4.4. A function $f : X \to Y$ is a *local homeomorphism* if for every point $x \in X$ there exists an open set U containing x, such that $f(U)$ is open in Y and $f_{|U} : U \to f(U)$ is a homeomorphism.

DEFINITION 4.5. A locally compact groupoid is *r-discrete* if its unit space, $G^{(0)}$ is an open subset of G.

LEMMA 4.6. Let G be a r-discrete groupoid.

1. For any $u \in G^{(0)}, G^u$ and G_u are discrete spaces.

2. If a Haar system exists, it is essentially the counting measure system.

3. If a Haar system exists, r and d are local homeomorphisms.

Proof.

1. An x in G^v_u defines a homeomorphism $y \to xy : G^v \to G^u$; since $\{v\}$ is open in G^v, $\{x\}$ is open in G^u.

2. Let $\{\lambda^u\}$ be a left Haar system. Since G^u is discrete and $\{\lambda^u\}$ has support G^u, every point in G^u has positive λ^u-measure. Let $g = \lambda(x_{G^0})$, where x_{G^0} is the characteristic

function of G^0. It is continuous and positive. Replacing λ^u by $g(u)^{-1}\lambda^u$, we may assume that $\lambda^u(\{u\}) = 1$ for any u. Then by invariance, $\lambda^v(\{x\}) = 1$ for any $x \in G_u^v$.

3. We assume that λ^u is the counting measure on G^u. Let x be a point of G. A compact neighborhood V of x meets G^u in finitely many points $x_i, i = 1, \ldots, n$. If $x_i \neq x$, there exists a compact neighborhood V' of x contained in V, which does not contain x_i. Therefore, we may assume that $G^u \cap V = \{x\}$. Then $\lambda^{r(x)}(V) = 1$. By continuity of the Haar system, we may assume that $\lambda^u(V) = 1$ for any $u \in r(v)$.

□

PROPOSITION 4.7. For a locally compact groupoid G, the following properties are equivalent:

1. G is r-discrete and admits a left Haar system,
2. $r : G \to G^{(0)}$ is a local homeomorphism,
3. the product map $G^{(2)} \to G$ is a local homeomorphism, and
4. G has a base of open G-sets.

Proof. (1) $\to$ (2) This has been shown in [1].

(2) $\to$ (3) If $(x,y) \in G^2$, we may choose a compact neighborhood U of x and compact neighborhood V of y such that $r_{|V}$ and $d_{|V}$ are homeomorphisms onto their images; $U \times V \cap G^2$ is then a compact neighborhood of (x,y) on which the product map is injective. $x'y' = x''y'' \to r(x') = r(x'') \to x' = x''$ and $d(y') = d(y'') \to y' = y''$.

(3) $\to$ (4) If $x \in G$ and U is a neighborhood of x, we may find open sets V and W such that $x \in V \subset U, x^{-1} \in W \subset U^{-1}$ and the restriction of the product map to $V \times W$ is injective. So $V \cap W^{-1}$ is the desired open G-set.

(4) $\to$ (2) Clear.

(4) $\to$ (1) The groupoid G is r-discrete: for any $u \in G^0$, there is an open G-set such that $u \in r(s) = ss^{-1} \subset G^0$ and by (3) ss^{-1} is open in G. Let λ^u be the counting measure on G^u and f be in $C_c(G)$. Using a partition of the identity, one can write f as a finite sum of fractions supported on open G-sets s. Therefore it is enough to consider a function f whose support is contained in an open G-set s. Then $\lambda(f)(u) = \lambda^u(f) = f(us) : \lambda(f)$ is continuous. $\square$

4.3 Counterexample which admits no Haar system

In this section we present a construction of a groupoid which admits no Haar system. Let X denote the closed unit interval [0, 1] in the x-axis and let T denote the unit circle in the yz-plane in $\mathbb{R}^3$. Then $H = X \times T$ is a group bundle and a groupoid. Moreover, the object space of H can be identified with X, and H^u is a copy of T for each u. Let G be the subgroupoid of H obtained by replacing H^u by the trivial group $G^u = \{(x,1,0)\}$ for all $x > \frac{1}{2}$. G is a compact Hausdorff groupoid. We now show that, while G admits many families $\nu = \{\nu^u, u \in G^{(0)}\}$ of measures it admits no Haar measure system.

THEOREM 4.8. The groupoid G defined above does not admit a Haar system.

Proof. Suppose on the contrary that $\nu = \{\nu^u, u \in G^{(0)}\}$ is a Haar system on G. Then ν^u is a Haar measure on the group G^x for each x, and in fact is a weight attached to (x, 1, 0) for all $x > \frac{1}{2}$. Putting $f \equiv 1$ on G, we conclude that the function $x \mapsto \nu^u(G^x)$ is continuous. Dividing by this positive function, we see that $\{\alpha^u\}$ is a Haar measure also, where, for each $u \in G^{(0)}$, $\alpha^u = \nu^u/\nu^u(G^x)$.

Moreover, $\alpha^u(G^x) = 1$ for all x, and so we can and will assume from the outset that the ν^u are normalized. Now select a continuous function f on H with values in [0,1] such that $f(x,1,0) = 1$ and

$$\int_{H^x} f d\lambda_x \leq \frac{1}{2}$$

for all $x \in X$, where λ_x denotes Lebesgue measure on H^x. This is possible since the function $f(\theta) = |cos^{101}(\theta)|$ satisfies this requirements. Now, since v^u coincides with λ_x if $x \leq \frac{1}{2}$, we now have $v^u(f) \leq \frac{1}{2}$ if $x \leq \frac{1}{2}$ and $v^u(f) = 1$ if $x > \frac{1}{2}$. Thus the function $x \mapsto v^u(f)$ is discontinuous at $x = \frac{1}{2}$; and property (2) of the Haar system definition is contradicted. Therefore the initial assumption of the Haar system on G is wrong and the claim is proven. $\square$

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