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ABSTRACT

Ordinal indexing of the class of strictly singular operators

Craig Stevenson

The Schreier families are a collection of finite subsets of \mathbb{N} and have been used to provide refinements of the following Banach space notions: unconditional basic sequences, convergent sequences, spreading model and strictly singular operators to name a few. We use the Schreier families to study subclasses of strictly singular operators on Banach spaces. We also provide a sufficient condition on the strictly singular operators implying every operator falls into one of these subclasses.

Dissertation Director: Dr. Kevin Beanland

Ordinal indexing of the class of strictly singular operators

by

Craig Stevenson

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St. Mary's College of Maryland

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Department of Mathematics
Virginia Commonwealth University

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ABSTRACT

The Schreier families are a collection of finite subsets of \mathbb{N} and have been used to provide refinements of the following Banach space notions: unconditional basic sequences, convergent sequences, spreading model and strictly singular operators to name a few. We use the Schreier families to study subclasses of strictly singular operators on Banach spaces. We also provide a sufficient condition on the strictly singular operators implying every operator falls into one of these subclasses.

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INTRODUCTION

Banach space theory was born in 1932 with the publication *Operations Lineaires* by Stefan Banach. The first active period in this field was from 1950 to 1980 with much of the important work done by R.C. James, J. Lindenstrauss, A. Pełczyński, H.P. Rosenthal, Jean Bourgain, B. Maurey, P. Enflo and W.B. Johnson (to name a few). The second active period was in the mid-1990s in which W.T. Gowers solved many long-standing open problems and researchers such as: E. Odell, Th. Schlumprecht and S.A. Argyros made many important contributions. During these periods many well-known classical problems were solved and deep connections between Banach space theory and other areas of mathematics were established. The most influential monographs in the area are the works of Dunford and Schwartz, in the late 1950s [6] and Lindenstrauss and Tzafriri, in the late 1970s [10]. Important recent developments in Banach space theory have integrated different areas of mathematics including combinatorial Ramsey theory and descriptive set theory.

Descriptive set theory began with the work of Borel, Baire and Lebesgue at the turn of the twentieth century. This theory came out of the study of the abstract notion of a function introduced by Dirichlet and Riemann. The first notion of indexing classes of functions by the ordinal numbers was by Baire in 1899. In 1907 Suslin called the projections of Borel sets analytic and showed that there are analytic sets which are not Borel. This initiated a categorical study of definable sets.

One goal of this manuscript is to reveal how powerful techniques in descriptive theory can be used to penetrate the structure of Banach spaces. One way this occurs is by providing refinements, using ordinal indexing, of commonly used notions in

Banach space theory. Using methods in descriptive set theory to study Banach spaces originated in the works of Choquet in the 1960s. Important recent developments have been made by V. Ferenczi and C. Rosendal, on the classification of the isomorphism relationship [7]; and S.A. Argyros and P. Dodos, on the study of universal spaces and tree amalgamations [4]. These seminal works have revolutionized how researchers view this connection and how they attack problems. The Schreier families were introduced by S.A. Argyros and D. Alspach [1] and have played a critical role in these developments. The Schreier families are a collection of finite subsets of \mathbb{N} and have been used to provide refinements of the following Banach space notions: unconditional basic sequences, convergent sequences, spreading model and strictly singular operators to name a few. In this paper we focus on using Schreier families to study subclasses of strictly singular operators on Banach spaces.

This thesis is an exposition of the main results of [2, 3, 5]. Chapter 2 introduces several basic concepts in Banach space theory which will be important in later sections: the definition of a Banach space, Schauder basis, Schauder basic sequences and the definition of a separable Banach space. Chapter 3 introduces the theory of linear operators between Banach spaces: the definition of linear operator, the collection of all continuous linear operators $\mathcal{L}(X, Y)$ and the strictly singular operators.

Given this background, chapter 4 introduces the Schreier families which allow us to define subclasses of the strictly singular operators, the S_ξ - strictly singular operators (ω_1 denotes the first uncountable ordinal). In chapter 5 we discuss the basic concepts in the theory of trees on the natural numbers, define well-founded trees, show that each Schreier family is a well founded tree, and define the order of a tree. Chapter 6 introduces basic concepts of Polish spaces: the definition of Polish space, Borel subsets of Polish spaces, Borel functions, analytic subsets of Polish spaces and give three important example of Polish spaces.

Chapter 7 contains the statement of the main results in [2, 3] and the proof of the main result from [5]. In these papers they use descriptive set theory to demonstrate the class of strictly singular operators between certain Banach spaces can be written as a union on ω_1 subclasses defined in a natural way. In particular, the main result of [5] provides a sufficient condition on the strictly singular operators implying *every* operator falls into one of these subclasses.

LIST OF SYMBOLS

\mathbb{N}	The natural numbers.
$2\mathbb{N}$	The even natural numbers
\mathbb{Q}	The rational numbers.
\mathbb{R}	The real numbers.
\mathbb{C}	The complex numbers.
X^*	The dual space of the Banach space X .
$\mathcal{L}(X, Y)$	The space of bounded linear operators from X to Y .
c_0	The sequences of scalars that converge to 0 endowed. with the $\ \cdot\ _\infty$ norm.
c_{00}	The (dense) subspace of c_0 of finitely nonzero sequences.
ℓ_∞	The collection of bounded $x = (x_n)_{n=1}^\infty$. with the norm $\ x\ _\infty = \sup_n x_n $
ℓ_p	The sequences of scalars that are p summable.
\mathcal{L}_p	The functions which are p integrable.

CHAPTER 1

BANACH SPACE THEORY

This section serves as a brief introduction to the structure of Banach spaces. There are two fundamental structures present in every Banach space. The first is a linear structure and the second is a topological structure. We assume the reader is familiar with the notion of a vector space (the linear structure) so we begin with the definition of a norm, the topological structure. In the sequel, all vector spaces will be considered over the reals.

DEFINITION 1.0.1. *Let X be a vector space. A norm $\|\cdot\|$ on X is a real-valued function, with domain X , such that the following conditions are satisfied by all members x and y of X and each scalar α :*

- (1) $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$;
- (2) $\|\alpha x\| = |\alpha| \|x\|$;
- (3) $\|x + y\| \leq \|x\| + \|y\|$

The pair $(X, \|\cdot\|)$ is called a normed space or normed vector space or normed linear space.

In this manuscript we consider only infinite dimensional linear spaces. A linear space is infinite dimensional if it contains infinitely many linearly independent vectors. An easy example of such a space is the vector space of all real polynomials with integer coefficients. In this space, the set of vectors $\{1, x, x^2, \dots\}$ is clearly a linearly independent set.

The examples of Banach spaces we focus on are in the class of spaces referred to as sequence spaces. The underlying vector space for the sequence spaces consists of sequences of real numbers with coordinate-wise addition and scalar multiplication defined in the usual way.

These two fundamental spaces are in the little-ell family, they are “little-ell-one” and “little-ell-infinity” respectively.

EXAMPLE 1.0.2. *The sequence spaces ℓ_1 and ℓ_∞ :*

(ℓ_1) Let $\ell_1 = \{(a_i)_{i=1}^\infty : \sum_{i=1}^\infty |a_i| < \infty\}$ and let $\|(a_i)\|_1 = \sum_{i=1}^\infty |a_i|$.

(ℓ_∞) Let $\ell_\infty = \{(a_i)_{i=1}^\infty : \sup_{i \in \mathbb{N}} |a_i| < \infty\}$ and let $\|(a_i)\|_\infty = \sup\{|a_i| : i \in \mathbb{N}\}$.

Let us see that the sequence spaces ℓ_1 and ℓ_∞ are in fact normed spaces, equipped with their respective norms. Since the methods used to verify ℓ_1 and ℓ_∞ are normed spaces are so similar, we do these verifications simultaneously.

The first condition of being a normed space is that the norm of any vector is greater than or equal to zero and that the only vector with norm equal to zero is the zero vector. For both ℓ_1 and ℓ_∞ it is clear that the only vector with norm zero is the zero vector. Since both the ℓ_1 and ℓ_∞ norms are defined in terms of the absolute value, it is clear that they are always positive.

The second condition of being a normed space is that the norm is linear with respect to multiplication by scalars. For any vector $(a_i)_{i=1}^\infty \in \ell_1$, and scalar, $\lambda \in \mathbb{R}$,

$$\|\lambda(a_i)_{i=1}^\infty\|_1 = \sum_{i=1}^\infty |\lambda a_i| = |\lambda| \sum_{i=1}^\infty |a_i| = |\lambda| \|(a_i)_{i=1}^\infty\|_1.$$

Similarly, for any vector $(b_i)_{i=1}^\infty \in \ell_\infty$ and scalar, $\lambda \in \mathbb{R}$,

$$\|\lambda(b_i)_{i=1}^\infty\|_\infty = \sup\{|\lambda b_i| : i \in \mathbb{N}\} = |\lambda| \sup\{|b_i| : i \in \mathbb{N}\} = |\lambda| \|(b_i)_{i=1}^\infty\|.$$

The final condition of being a normed space is that any two vectors satisfy the triangle inequality. For any pair of vectors, $(a_i)_{i=1}^\infty, (b_i)_{i=1}^\infty \in \ell_1$,

$$\|(a_i)_{i=1}^\infty + (b_i)_{i=1}^\infty\|_1 = \sum_{i=1}^{\infty} |a_i + b_i| \leq \sum_{i=1}^{\infty} |a_i| + \sum_{i=1}^{\infty} |b_i| = \|(a_i)_{i=1}^\infty\|_1 + \|(b_i)_{i=1}^\infty\|_1.$$

Similarly, for any pair of vectors $(c_i)_{i=1}^\infty, (d_i)_{i=1}^\infty \in \ell_\infty$

$$\begin{aligned} \|(a_i)_{i=1}^\infty + (b_i)_{i=1}^\infty\| &= \sup\{|a_i + b_i| : i \in \mathbb{N}\} \leq \sup\{|a_i| : i \in \mathbb{N}\} + \sup\{|b_i| : i \in \mathbb{N}\} \\ &\leq \|(a_i)_{i=1}^\infty\| + \|(b_i)_{i=1}^\infty\|. \end{aligned}$$

Our next example is considered to be the motivation for the definition of Banach spaces. Let $C[0, 1]$ be the space of continuous real valued functions on the interval $[0, 1]$. This is clearly a vector space with addition and scalar multiplication defined in the standard way. The norm we put on this space is called the sup-norm.

EXAMPLE 1.0.3. *Let $C[0, 1] = \{f : [0, 1] \rightarrow \mathbb{R} : f \text{ is continuous}\}$ and let $\|f\|_\infty = \sup\{|f(x)| : x \in [0, 1]\}$. Then $(C[0, 1], \|\cdot\|_\infty)$ is a normed space. We verify the three properties of a norm in the same order they are given in the definition.*

- (1) $\|f\|_\infty \geq 0$ since $\sup\{|f(x)| : x \in [0, 1]\} \geq 0$.
- (2) $\|\lambda f\|_\infty = \sup\{|\lambda f(x)| : x \in [0, 1]\} = |\lambda| \sup\{|f(x)| : x \in [0, 1]\}$.
- (3) $\|f + g\|_\infty = \sup\{|f(x) + g(x)| : x \in [0, 1]\} \leq \sup\{|f(x)| : x \in [0, 1]\} + \sup\{|g(x)| : x \in [0, 1]\} \leq \|f\|_\infty + \|g\|_\infty$.

Given any normed space there is a natural way to define a metric on that space using the norm.

DEFINITION 1.0.4. *Let X be a normed space. The metric induced by the norm of X is the metric d on X defined by the formula $d(x, y) = \|x - y\|$. The norm topology on X is the topology obtained from this metric.*

Now we are ready to define Banach spaces.

DEFINITION 1.0.5. X is a Banach space if it is a normed space and that norm induces a complete metric. That is, for every Cauchy sequence (x_n) in X , there is an $x \in X$ such that $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$.

The proofs that the previously mentioned normed spaces are actually Banach spaces all share a common structure. We start by considering an arbitrary Cauchy sequence in the normed space. Using the completeness of the reals we then define a vector to which this Cauchy sequence must converge to, if it converges. Then we show that the Cauchy sequence converges in norm to this vector and show that the vector is an element of the space.

EXAMPLE 1.0.6. The normed space ℓ_1 is a Banach Space.

Let $((a_i^j)_{i=1}^\infty)_{j=1}^\infty$ be a Cauchy sequence in ℓ_1 . This means that for all $\varepsilon > 0$ there exists a $J \in \mathbb{N}$ such that for all $j, k \geq J$

$$\|(a_i^j)_{i=1}^\infty - (a_i^k)_{i=1}^\infty\|_1 = \sum_{i=1}^{\infty} |a_i^j - a_i^k| < \varepsilon.$$

Note that for all $i_0, j, k \in \mathbb{N}$

$$|a_{i_0}^j - a_{i_0}^k| \leq \sum_{i=1}^{\infty} |a_i^j - a_i^k|.$$

Therefore, for each $i \in \mathbb{N}$, $(a_i^j)_{j=1}^\infty$ is a Cauchy sequence in \mathbb{R} . By invoking the completeness of \mathbb{R} we define a sequence $(a_i)_{i=1}^\infty$ coordinatewise by $\lim_{j \rightarrow \infty} a_i^j = a_i$. We show the sequence $(a_i)_{i=1}^\infty$ satisfies the following:

- (1) For all $\varepsilon > 0$ there exists $J \in \mathbb{N}$ such that for all $j \geq J$ and for all $n \in \mathbb{N}$

$$\sum_{i=1}^n |a_i^j - a_i| < \varepsilon;$$
- (2) $\|(a_i)_{i=1}^\infty\|_1 = \sum_{i=1}^\infty |a_i| < \infty$.

Notice that (1) is equivalent to $\lim_{j \rightarrow \infty} \|(a_i^j)_{i=1}^\infty - (a_i)_{i=1}^\infty\| = 0$ and (2) implies that $(a_i)_{i=1}^\infty \in \ell_1$.

Proof of (1). Let $\varepsilon > 0$. Since $((a_i^j)_{i=1}^\infty)_{j=1}^\infty$ is a Cauchy sequence in ℓ_1 there exists $J \in \mathbb{N}$ such that for all $j, k \geq J$, $\sum_{i=1}^\infty |a_i^j - a_i^k| < \frac{\varepsilon}{2}$. Let $j \geq J$ and $n \in \mathbb{N}$.

Find $k \geq J$ such that $\sum_{i=1}^n |a_i^k - a_i| < \frac{\varepsilon}{2}$. It follows that

$$\sum_{i=1}^n |a_i^j - a_i| \leq \sum_{i=1}^n |a_i^j - a_i^k| + \sum_{i=1}^n |a_i^k - a_i| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Proof of (2). Using (1) we can find $j \in \mathbb{N}$ such that

$$\|(a_i)_{i=1}^\infty\|_1 \leq \|(a_i^j)_{i=1}^\infty - (a_i)_{i=1}^\infty\|_1 + \|(a_i^j)_{i=1}^\infty\|_1 < 1 + \sum_{i=1}^\infty |a_i^j| < \infty.$$

EXAMPLE 1.0.7. The normed space $C[0, 1]$ is a Banach Space. Let $(f_i)_{i=1}^\infty$ be a Cauchy sequence in $C[0, 1]$. This means that for all $\varepsilon > 0$ there exists J such that for all $j, k \geq J$

$$\|f_j - f_k\| = \sup_{x \in [0, 1]} \{|f_j(x) - f_k(x)|\} < \varepsilon.$$

Note that for all $x_0 \in [0, 1]$ and $j, k \in \mathbb{N}$

$$|f_j(x_0) - f_k(x_0)| \leq \sup_{x \in [0, 1]} \{|f_j(x) - f_k(x)|\}.$$

Therefore for each $x \in [0, 1]$, $(f_i(x))_{i=1}^\infty$ is a Cauchy sequence in \mathbb{R} . By invoking the completeness of \mathbb{R} we define a function f pointwise by $f(x) = \lim_{i \rightarrow \infty} f_i(x)$. The function $f(x)$ satisfies the following

(1) For all $\varepsilon > 0$ there exists $J \in \mathbb{N}$ such that for all $i \geq J$ and for all $x \in [0, 1]$

$$|f_i(x) - f(x)| < \varepsilon.$$

(2) $\|f\| = \sup_{x \in [0, 1]} \{f(x)\} < \infty$.

Notice that (1) is equivalent to $\lim_{i \rightarrow \infty} \|f_i - f\| = 0$ and (2) implies that $f \in C[0, 1]$.

Proof of (1). Let $\varepsilon > 0$. Since $(f_i)_{i=1}^\infty$ is a Cauchy sequence in $C[0, 1]$ there exists $J \in \mathbb{N}$ such that for all $i, j \geq J$, $\|f_i - f_j\| < \frac{\varepsilon}{2}$. Let $i \geq J$ and let $x \in [0, 1]$. Find $j \in \mathbb{N}$ such that $j \geq J$ and $|f_j(x) - f(x)| < \frac{\varepsilon}{2}$. It follows that

$$|f_i(x) - f(x)| \leq |f_i(x) - f_j(x)| + |f_j(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Proof of (2). Using (1) we can find $i \in \mathbb{N}$ such that

$$\|f\| \leq \|f_i - f\| + \|f_i\| \leq 1 + \|f_i\| < \infty$$

For completeness, we give a brief list of examples of Banach spaces and their norms. The interested reader may refer to [11] for the proofs that these spaces are Banach spaces.

$$C[0, 1] = \{f : [0, 1] \rightarrow \mathbb{R} : f \text{ is continuous}\}, \|f\| = \sup\{|f(x)| : x \in [0, 1]\}$$

$$\ell_p = \{(a_i)_{i=1}^\infty : \sum_{i=1}^\infty |a_i|^p < \infty\}, \|(a_i)\|_p = \left(\sum_{i=1}^\infty |a_i|^p\right)^{1/p}$$

$$\ell_\infty = \{(a_i)_{i=1}^\infty : \sup\{|a_i| : i \in \mathbb{N}\} < \infty\}, \|(a_i)\|_\infty = \sup\{|a_i| : i \in \mathbb{N}\}$$

$$c_0 = \{(a_i)_{i=1}^\infty : \lim_{i \rightarrow \infty} a_i = 0\}, \|(a_i)\| = \sup\{|a_i| : i \in \mathbb{N}\}$$

$$L_p[0, 1] = \{f : [0, 1] \rightarrow \mathbb{R} : f \text{ is measurable, } |f|^p \text{ is integrable}\}, \text{ for } 1 \leq p < \infty,$$

$$\|f\|_p = \left(\int_0^1 |f|^p\right)^{\frac{1}{p}}$$

Some Banach spaces contain a special type of sequence called a Schauder basis. The properties of this sequence are similar to the properties of a basis for a finite dimensional vector space.

DEFINITION 1.0.8. *Let X be a Banach space. A sequence $(x_n)_{n=1}^\infty \subset X$ is called a Schauder basis for X if for each $x \in X$ there is a unique sequence of scalars (a_i) such that $x = \sum_{i=1}^\infty a_i x_i$. In other words,*

$$\lim_{n \rightarrow \infty} \left\| x - \sum_{i=1}^n a_i x_i \right\| = 0$$

Note that if a sequence $\sum_{i=1}^\infty a_i x_i$ converges we have that for all $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $n \geq N$, $\left\| \sum_{i=n}^\infty a_i x_i \right\| < \varepsilon$. For the rest of this paper, whenever we refer to a basis for a Banach space we mean a Schauder basis. Sequence spaces provide the easiest examples of Schauder basis. For each $i \in \mathbb{N}$ let,

$$e_i = (0, \dots, 0, 1, 0, \dots)$$

with 1 in the i^{th} coordinate. The sequence (e_i) is a Schauder basis for for each of the ℓ_p spaces and c_0 . As an example, we show that (e_i) is a Schauder basis for c_0 : Let $x = (a_1, a_2, \dots, a_n, \dots) \in c_0$. Then $\|x - \sum_{i=1}^n a_i e_i\| = \sup_{i \geq n+1} |a_i|$ which converges to 0 as n goes off to infinity.

There are many Banach spaces which do not have a Schauder basis. For example at the end of this section we show ℓ_∞ does not have such a basis. The closed linear span of a Schauder basis must equal the whole space. Next we give a weaker definition. Namely, of a sequence in a Banach space that is a Schauder basis for its closed linear span but its closed linear span may not be the whole space.

DEFINITION 1.0.9. *A sequence $(x_n)_{i=1}^\infty \subset X$ is called a Schauder basic sequence if it is a basis for its closed linear span, denote this $[x_n]$. A basic sequence is normalized if $\|x_n\| = 1$ for all $n \in \mathbb{N}$. Let \mathfrak{B}_X be the set of all normalized basic sequences in X .*

Note that if (x_n) is a basic sequence then $\frac{x_n}{\|x_n\|}$ is a normalized basic sequence. The next criteria is extremely useful in verifying that a given sequence is basic. From this it is easy to see that every basic sequence consists of linear independent vectors. Let us note that it is not the case that every linearly independent sequence in a space is a basic sequence. This criteria does not tell us what the closed linear span of the sequence of vectors is. In particular, you cannot use this criteria alone to show that a given sequence forms a basis for the the underlying space. We omit the proof this theorem and refer the reader to [11].

THEOREM 1.0.10 (Grunblum's Criteria). *$(x_n)_{n=1}^\infty$ is a basic sequence if and only if there exists $C \geq 1$ such that for all $(a_i) \in c_{00}$ and for all $m \leq n$*

$$\left\| \sum_{i=1}^m a_i x_i \right\| \leq C \left\| \sum_{i=1}^n a_i x_i \right\|.$$

The smallest such C for which this holds is called the basic constant.

As mentioned before it is not possible to find a basis for every Banach space. The next important theorem, due to Mazur (a student of Banach), states that no matter which Banach space you are given this space contains a subspace with a basis. To put it another way, every Banach space contains a basic sequence. The proof is rather technical and involved and requires the inductive construction of the sequence. It can be found in [11].

THEOREM 1.0.11. *Every Banach space contains a basic sequence with constant $C < 1 + \varepsilon$ for every $\varepsilon > 0$.*

The next definition is of critical importance in Banach space theory. This notion is topological and therefore not unique to Banach space.

DEFINITION 1.0.12. *A Banach space is separable if it contains a countable dense subsequence.*

For a long time (40 years) it was not known whether every separable Banach space has a basis. In 1973 P. Enflo found a subspace of the separable space c_0 that has no basis. This example is extremely complicated. Since we will be focusing on separable spaces throughout this paper we state and prove the next proposition which is the converse of the above statement.

PROPOSITION 1.0.13. *If X has a basis then X is separable.*

PROOF. Let (x_n) be a basis for X . The set

$$D = \left\{ \sum_i a_i x_i : (a_i) \in c_{00} \text{ and } a_i \in \mathbb{Q} \text{ for all } i \right\}$$

is countable and dense. □

Since (e_i) is basis for all ℓ_p and c_0 it follows that these spaces are separable. The above proposition may also be used to show that a space does not have a basis. For

example, we can now show that ℓ_∞ is non-separable. Although it may seem like (e_i) is a basis for ℓ_∞ , the obvious representation of each sequence will not be convergent in the ℓ_∞ norm. (e.g $(1, 1, \dots) = \sum_i e_i$ is not convergent.)

EXAMPLE 1.0.14. ℓ_∞ is not separable.

PROOF. Consider the set of all sequences of 0's and 1's. There are an uncountable number of such sequences yet any pair of them are distance 1 apart, since different sequences will differ in at least one coordinate. Thus ℓ_∞ is not separable. \square

CHAPTER 2

BOUNDED LINEAR OPERATORS

In this section we introduce the theory of linear operators between two normed linear spaces. In elementary linear algebra it is shown that every linear operator between two finite dimensional vector spaces has a matrix representation. For infinite dimensional spaces this is not the case. We start by recalling the definition of linear operator between two vector spaces.

DEFINITION 2.0.15. *Let X and Y be vector spaces. A linear operator T from X into Y is a function $T : X \rightarrow Y$ such that the following two conditions are satisfied whenever $x, y \in X$ and $\alpha \in \mathbb{R}$:*

- (1) $T(x + y) = T(x) + T(y)$;
- (2) $T(\alpha x) = \alpha T(x)$.

If X and Y are Banach spaces it is natural to consider the space of *continuous* linear operators from X to Y . We will denote this space $\mathcal{L}(X, Y)$. The next proposition allows us to define a norm on $\mathcal{L}(X, Y)$. We omit the proof of this proposition and send the reader to [12] for a thorough treatment of this topic.

PROPOSITION 2.0.16. *Let X and Y be Banach spaces. The following are equivalent*

- (1) $T \in \mathcal{L}(X, Y)$.
- (2) *There exists a $C > 0$ such that $\sup_{x \in X} \frac{\|Tx\|}{\|x\|} \leq C$.*
- (3) *There exists a $C > 0$ such that $\sup_{x \in X, \|x\|=1} \|Tx\| \leq C$.*
- (4) *There exists a C for all $x \in X$, $\|Tx\| \leq C\|x\|$*

If an operator satisfies (2), (3), and (4) we say it is bounded. This proposition states that $T \in \mathcal{L}(X, Y)$ if and only if T is bounded.

From the previous proposition, there is a natural norm, called the operator norm, that one may put on this space which makes $\mathcal{L}(X, Y)$ a Banach space (i.e. it is complete with respect to this norm).

DEFINITION 2.0.17. Let X and Y be Banach spaces. For each T in $\mathcal{L}(X, Y)$, the norm or operator norm is defined by,

$$\|T\| = \sup_{x \in X} \frac{\|Tx\|}{\|x\|}$$

The routine proof that this defines a norm and that $\mathcal{L}(X, Y)$ in this norm is complete follows from arguments similar to those in [12].

Banach spaces X and Y are said to be linearly isomorphic if there is a $T \in \mathcal{L}(X, Y)$ such that T is continuous, one-to-one, onto, with continuous inverse. In this case, T is called a linear isomorphism. This differs from the purely topologic notion of *homeomorphism* where the map is not required to be linear. For the rest of this paper, when we say isomorphism it is understood that we mean linear isomorphism.

In linear algebra a matrix is called *singular* if it is not invertible. Singular matrices are not isomorphisms the same goes for singular operators. An operator in $\mathcal{L}(X, Y)$ is called strictly singular if its restriction to any infinite dimensional subspace of the domain is not an isomorphism (it can't avoid being singular!). Let $SS(X, Y)$ denote the subspace of strictly singular operators from X to Y .

The next remark characterizes the notion of linear isomorphism.

REMARK 2.0.18. $T \in \mathcal{L}(X, Y)$ is an isomorphism if and only if there exists $0 < c \leq C < \infty$ such that $c\|x\| \leq \|Tx\| \leq C\|x\|$. X is isomorphic to Y if and only if there exists an onto isomorphism $T : X \rightarrow Y$.

This allows us to make the following remark giving a quantifiable version of the definition of strictly singular.

DEFINITION 2.0.19. *An operator $T \in \mathcal{L}(X, Y)$ is strictly singular if and only if for any $\varepsilon > 0$ and any infinite dimensional subspace $Z \subset X$ there is a $z \in Z$ such that $\|Tz\| < \varepsilon\|z\|$. If this is the case we say $T \in SS(X, Y)$.*

In other words for every infinite dimensional subspace and every $\varepsilon > 0$ there is a vector that witnesses the fact that T is not a ‘ ε ’-isomorphism. The main result of this section is a characterization of the strictly singular operators in terms of their behavior when restricted to basic sequences in the domain space. This characterization will be extremely important when defining subclasses of strictly singular operators. As previously mentioned, the main result of this manuscript is a theorem relating to subclasses of strictly singular operators. The final proposition of this section roughly states that in order to show that a given operator is strictly singular, it suffices to consider only subspaces spanned by basic sequences, and restrict ourselves to finitely supported vectors on these sequences.

PROPOSITION 2.0.20. *The following are equivalent.*

- (1) $T \in SS(X, Y)$
- (2) *For all normalized basic sequences, $(x_n) \subset X$, T restricted to $[x_n]$ is not an isomorphism*
- (3) *For all normalized basic sequences $(x_n) \subset X$, for all $\varepsilon > 0$ there exists an F , finite subset of \mathbb{N} , and there exists a $x \in [x_n]_{n \in F} \setminus \{0\}$ such that $\|Tx\| < \varepsilon\|x\|$.*

PROOF. (1) implies (2) follows immediately from the definition of a strictly singular operator. We will now show that (2) implies (1). Let $Z \subset X$, by Mazur there exists a sequence, $(x_n) \subset Z$, which is basic and by (2), T restricted to $[x_n]$ is not an isomorphism. Thus T restricted to Z is not an isomorphism so (2) implies (1). (3) implies (2) is obvious so all that is left is to show that (2) implies (3). Let $(x_n) \subset X$

be a normalized basic sequence and let $\varepsilon > 0$, by (2) we can choose a vector, x , in the closed linear span of (x_n) such that $\|Tx\| < \frac{\varepsilon}{2}\|x\|$. Since x is an element of $[x_n]$, x can be written $x = \sum_{i=1}^{\infty} a_i x_i$ with $\|x\| = 1$. Thus we can choose a $N \in \mathbb{N}$ such that the following two conditions hold

- (1) $\|\sum_{i=N+1}^{\infty} a_i x_i\| < \frac{\varepsilon(1-2\varepsilon)}{2\|T\|}$
- (2) $1 - \varepsilon < \|\sum_{i=1}^N a_i x_i\|$.

We will now show that our finitely supported vector, $\sum_{i=1}^N a_i x_i$, witnesses the condition that $\|T \sum_{i=1}^N a_i x_i\| < \varepsilon \|\sum_{i=1}^N a_i x_i\|$. It now follows that

$$\|T \sum_{i=1}^N a_i x_i\| = \|T \left(\sum_{i=1}^N a_i x_i + \sum_{i=N+1}^{\infty} a_i x_i - \sum_{i=N+1}^{\infty} a_i x_i \right)\| \leq \|T \sum_{i=1}^N a_i x_i\| + \|T\| \|\sum_{i=N+1}^{\infty} a_i x_i\|.$$

By replacing $\|T \sum_{i=1}^N a_i x_i\|$ with ε and $\|\sum_{i=N+1}^{\infty} a_i x_i\|$ with $\frac{\varepsilon(1-2\varepsilon)}{2\|T\|}$ we have

$$\|T \sum_{i=1}^N a_i x_i\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon(1-2\varepsilon)}{2} = \varepsilon \left(\frac{1}{2} + \frac{1}{2} - \varepsilon \right) = \varepsilon(1 - \varepsilon) \leq \varepsilon \|\sum_{i=1}^N a_i x_i\|$$

□

CHAPTER 3

SCHREIER FAMILIES

In this section we define a subclass of the strictly singular operators. Recall the following equivalent definition of a strictly singular operator.

DEFINITION 3.0.21. *$T \in SS(X, Y)$ if and only if for all normalized basic sequences $(x_n) \subset X$ and for all $\varepsilon > 0$ there exists an F , finite subset of \mathbb{N} , and $x \in [x_n]_{n \in F} \setminus \{0\}$ such that $\|Tx\| < \varepsilon\|x\|$.*

Note that this definition requires us to find a finite set $F \subset \mathbb{N}$. If we place some additional restrictions on the finite set we can define a new type of operator.

DEFINITION 3.0.22. *Let \mathcal{A} be a collection of finite subsets of \mathbb{N} . An operator, T , is \mathcal{A} -strictly if and only if for all normalized basic sequences $(x_n)_n \subset X$ and $\varepsilon > 0$ there exists $F \in \mathcal{A}$ and $x \in [x_n]_{n \in F} \setminus \{0\}$ such that $\|Tx\| < \varepsilon\|x\|$. Denote the set of all \mathcal{A} -strictly singular operators $T \in SS_{\mathcal{A}}(X, Y)$. Note, $SS_{\mathcal{A}}(X, Y) \subseteq SS(X, Y)$.*

With this in mind now introduce a collection of finite subsets of \mathbb{N} called the Schreier families. The Schreier families were introduced by D. Alspach and S.A. Argyros [1]. This collection of subsets has several important properties which we will make use of.

However, before we can define the Schreier families we must define the ordinal numbers and state several of their properties. Consult [8] for a detailed treatment.

Ordinal Numbers

For each well-ordered set, we assign to that set a number, called an ordinal number. Two well-ordered sets are assigned the same ordinal number if there exists a map between them which is one-to-one, onto, and order preserving. The finite ordinals are denoted

$$0, 1, 2, 3, \dots, n, \dots$$

With 0 representing the empty set and n representing the set containing n elements. The first infinite ordinal is denoted by ω and is used to represent the well-ordered set \mathbb{N} . The first uncountable ordinal is denoted ω_1 . We must now define the concepts of limit ordinal and successor ordinal.

A limit ordinal is an ordinal with no immediate predecessor. The first limit ordinal is ω , since it is defined to be the first ordinal with no immediate predecessor. Other examples of limit ordinals include: $\omega \cdot 2, \omega^2, \omega^\omega, \omega^{\omega^\omega}$.

A successor ordinal is an ordinal with an immediate predecessor. The next ordinal number after ω is the ordinal which is assigned to the well-ordered set

$$\{1, 2, \dots, n, \dots, \omega\}$$

. It is denoted $\omega + 1$. Therefore $\omega + 1$ is a successor ordinal since it has an immediate predecessor, ω . Other examples include:

$$1, 2, \dots, \omega + 2, \dots, \omega + \omega + 1$$

If A and B are two finite subsets of \mathbb{N} , we say $A \leq B$ if $\max A \leq \min B$. Similarly, for $n \in \mathbb{N}$ and $A \subset \mathbb{N}$, $n \leq A$ means $\{n\} \leq \min A$. Let $\mathbb{N}^{<\mathbb{N}}$ be the set of all finite sequences of natural numbers. By convention let $\emptyset < A$ and $A < \emptyset$ for all finite A . By $[\mathbb{N}]^{<\mathbb{N}}$ we denote the subset of $\mathbb{N}^{<\mathbb{N}}$ consisting of all strictly increasing finite sequences.

For any ordinal number $0 \leq \xi < \omega_1$, the Schreier family S_ξ is a subset of $[\mathbb{N}]^{<\mathbb{N}}$ defined by the following transfinite recursive process. Let, $S_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$. Assuming ξ is a successor ordinal and S_ζ has been defined for $\zeta + 1 = \xi$ let,

$$S_\xi = \left\{ \bigcup_{i=1}^n F_i : n \geq 1, n \leq F_1 < \cdots < F_n, \text{ and } F_i \in S_\zeta \text{ for } 1 \leq i \leq n \right\} \cup \{\emptyset\}.$$

For each $n \in \mathbb{N}$ and $\alpha < \omega_1$. Let $S_\alpha([n, \infty)) = \{F \in S_\alpha : n \leq F\}$. If $\xi < \omega_1$ is a limit ordinal and S_α has been defined for all $\alpha < \xi$ then fix an increasing sequence $(\xi_n)_{n=1}^\infty$ such that $\lim_{n \rightarrow \infty} \xi_n = \xi$ and define,

$$S_\xi = \bigcup_{n=1}^\infty S_{\xi_n}([n, \infty)).$$

Before proceeding further we give some examples. Note that the collection S_1 is the collection of all finite sets whose number of elements is less than or equal to the minimum element of the set. For example,

$$\{1\}, \{2, 3\}, \{2, 4\}, \{4, 5, 6, 7\} \in S_1 \text{ and } \{1, 2\}, \{3, 4, 5, 6\} \notin S_1.$$

Let F be a finite subset of \mathbb{N} and $n = \min F$. For a successor ordinal $\xi < \omega_1$ $F \in S_\xi$ if F can be written as the disjoint union of n sets each of which is in $S_{\xi-1}$. Using this condition it is easy to see that $S_\zeta \subset S_{\zeta+1}$ for all $\zeta < \omega_1$. For example, $F = \{2, 3, 4, 5, 6, 7\} \in S_2$ since $\min F = 2$ and $F = \{2, 3\} \cup \{4, 5, 6, 7\}$ the union of two S_1 sets. In addition, $\{2, 3, 4, 5, 6, 7, 8\} \notin S_2$, since it cannot be written as the union of two S_1 sets.

For limit ordinals $\xi < \omega_1$ the situation is a bit different. At the limit ordinal step we do not simply union all of the previous sets. Let F be a finite subset of \mathbb{N} , $\min F = n$ and fix an increasing sequence (ξ_k) such that $\lim_{k \rightarrow \infty} \xi_k = \xi$. Then $F \in S_\xi$ exactly when $F \in S_{\xi_n}$. In the case of $\xi = \omega$, suppose $\xi_n = n$ for all $n \in \mathbb{N}$. Then

$$F = \{2, 3, 4, 5, 6, 7\} \in S_\omega \text{ since } \min F = 2 \text{ and } F \in S_2.$$

Likewise, $\{2, 3, 4, 5, 6, 7, 8\} \notin S_\omega$. However, $\{2, 3, 4, 5, 6, 7, 8\} \in S_3$. We summarize the properties of these families which we have already described.

REMARK 3.0.23. (1) $S_\xi \subseteq S_{\xi+1}$ for every ξ . However, $\xi < \zeta$ does not generally imply $S_\xi \subseteq S_\zeta$.

(2) Let $1 \leq \xi < \zeta < \omega_1$. Then there exists a natural number, n , depending on ξ and ζ , such that $S_\xi[n, \infty) \subset S_\zeta$.

Although we will not be using all of the following properties directly, in the interest of completeness we record them below.

REMARK 3.0.24. Let $0 \leq \xi < \omega_1$.

(1) S_ξ is hereditary. A collection \mathcal{F} is called hereditary if whenever $G \subset F \in \mathcal{F}$ then $G \in \mathcal{F}$.

(2) S_ξ is spreading. A collection \mathcal{F} is called spreading if whenever $\{n_1, n_2, \dots, n_k\} \in \mathcal{F}$ with $n_1 < n_2 < \dots < n_k$ and $m_1 < m_2 < \dots < m_k$ satisfies $n_i \leq m_i$, for $i \leq k$ then $\{m_1, m_2, \dots, m_k\} \in \mathcal{F}$.

(3) S_ξ is pointwise closed. A collection \mathcal{F} is called pointwise closed if \mathcal{F} is closed in the topology of pointwise convergence in $2^{\mathbb{N}^{<\mathbb{N}}}$.

Since S_ξ satisfies all of the above properties it is called a regular family. Having defined the Schreier families for each ordinal $1 \leq \xi < \omega_1$ we may define ω_1 subsets of $SS(X, Y)$ indexed by each $1 \leq \xi < \omega_1$. For a Banach space X let,

DEFINITION 3.0.25. $T \in SS_\xi(X, Y)$ if and only if $(x_n)_n \in \mathfrak{B}_X$ and $\varepsilon > 0$ there exists $F \in S_\xi$ and $x \in [x_n]_{n \in F} \setminus \{0\}$ such that $\|Tx\| < \varepsilon\|x\|$

This definition is more restrictive than that of $SS(X, Y)$ since we require that the finite set be chosen from the particular Schreier family S_ξ . It follows that $SS_\xi(X, Y) \subset SS(X, Y)$ for all $\xi < \omega_1$. The following proposition collects a number of properties of these operators. Notice that in (1) we observe that $SS_\xi(X, Y) \subset SS_\zeta(X, Y)$ for $\xi < \zeta$ even though for $\xi < \zeta$, $S_\xi \not\subseteq S_\zeta$. This is the final proposition of this section.

PROPOSITION 3.0.26. *Suppose that X and Y are two Banach spaces and $1 \leq \xi, \zeta < \omega_1$. Then*

- (1) *If $1 \leq \xi < \zeta < \omega_1$ then $SS_\xi(X, Y) \subset SS_\zeta(X, Y)$*
- (2) *$SS_\xi(X, Y)$ is norm-closed.*
- (3) *If $S \in SS_\xi(X)$ and $T \in \mathcal{L}(X)$ then TS and ST belong to $SS_\xi(X)$.*
- (4) *If $S \in SS_\xi(X)$ and $T \in SS_\zeta(X)$ then $S + T \in SS_{\xi+\zeta}(X)$. In particular, if $S, T \in SS_\xi(X)$ then $S + T \in SS_{2\xi}(X)$.*

PROOF. We will prove the first two items and half of the third item. The proofs we omit require machinery from general Banach space theory which we have not included in this manuscript. Consult [3] for the detailed proofs.

(1) For $1 \leq \xi < \zeta < \omega_1$, by 3.0.23 there exists $N \in \mathbb{N}$ such that $S_\xi([n, \infty)) \subset S_\zeta$. Let $T \in SS_\xi(X, Y)$, $\varepsilon > 0$, and $(x_n)_n$ be a normalized basic sequence. Consider the basic sequence $(y_n)_n$ where $y_i = x_{N+i}$. There exists $F \in S_\xi$ and $z \in [y_i]_{i \in F} \setminus \{0\}$ such that $\|Tz\| \leq \varepsilon\|z\|$. Since $F \in S_\xi$ and $F \subseteq [N, \infty)$ we have $F \in S_\zeta$.

(2) Let $(T_n)_n \subset SS_\xi(X)$ and $T \in \mathcal{L}(X)$ such that $\lim_n \|T_n - T\| = 0$. It remains to show that T is an element of $SS_\xi(X, Y)$. To show that $T \in SS_\xi(X, Y)$ we will verify that for all basic sequences $(x_n)_n \subset X$ and for all $\varepsilon > 0$, there is a finite set $F \in S_\xi$ and a vector z in the closed linear span of $(x_n)_{n \in F} \setminus \{0\}$, which satisfies the inequality $\|Tz\| < \varepsilon\|z\|$. Let $(x_n)_n$ be a normalized basic sequence and let $\varepsilon > 0$. Since T_n converges to T in norm, we can choose $n_0 \in \mathbb{N}$ such that $\|T_{n_0} - T\| < \varepsilon/2$. Since $T_{n_0} \in SS_\xi(X)$, there exists a finite set $F \in S_\xi$ and a vector z in the closed linear span of $(x_n)_{n \in F} \setminus \{0\}$ such that $\|T_{n_0}z\| < (\varepsilon/2)\|z\|$. Thus

$$\|Tz\| = \|Tz + T_{n_0}z - T_{n_0}z\| \leq \|(T_{n_0} - T)z\| + \|T_{n_0}z\| < \frac{\varepsilon}{2}\|z\| + \frac{\varepsilon}{2}\|z\| = \varepsilon\|z\|.$$

Thus $T \in SS_\xi(X, Y)$ showing that $SS_\xi(X, Y)$ is closed.

(3) Let $S \in SS_\xi(X)$ and $T \in \mathcal{L}(X)$. We show that $TS \in SS_\xi(X)$. Let $(x_n)_n$ be a normalized basic sequence in X and let $\varepsilon > 0$. If $T = 0$ the claim is trivially

verified. Suppose that $T \neq 0$, then there exists $F \in S_\xi$ and $z \in [x_n]_{n \in F} \setminus \{0\}$ such that $\|Sz\| < (\varepsilon/\|T\|)\|z\|$. Thus, $\|TSz\| \leq \|T\|\|Sz\| < \varepsilon\|z\|$. (**3**). \square

CHAPTER 4

TREES ON \mathbb{N}

A *tree* on \mathbb{N} is a collection of finite subsets of \mathbb{N} closed under the partial order of initial segment inclusion. For example if \mathcal{T} is a tree on \mathbb{N} and $(2, 3, 4, 5) \in \mathcal{T}$ then $(2), (2, 3), (2, 3, 4) \in \mathcal{T}$. More precisely, if $(n_1, \dots, n_k) \in \mathcal{T}$ then for all $\ell \leq k$, $(n_1, \dots, n_\ell) \in \mathcal{T}$. Let \prec denote this partial ordering. If $\alpha, \beta \in \mathcal{T}$ with $\beta \prec \alpha$ we say β is an initial segment of α . Assume further that if $(n_1, \dots, n_k) \in \mathcal{T}$ then $n_1 < \dots < n_k$.

Let $\mathcal{T}r$ denote the set of all trees on \mathbb{N} . Each $\mathcal{T} \in \mathcal{T}r$, can be uniquely represented as the usual characteristic function from $[\mathbb{N}]^{<\mathbb{N}} \rightarrow \{0, 1\}$. For example, the tree $\mathcal{T} = \{(2, 3), (2), \emptyset\}$ is represented by the function which takes the value 1 on $(2, 3), (2) \in [\mathbb{N}]^{<\mathbb{N}}$ and zero on every other element of $[\mathbb{N}]^{<\mathbb{N}}$. Let,

$$2^{[\mathbb{N}]^{<\mathbb{N}}} = \{ \text{all functions } f : [\mathbb{N}]^{<\mathbb{N}} \rightarrow \{0, 1\} \}.$$

Identify the set $\mathcal{T}r$ as a subset of $2^{[\mathbb{N}]^{<\mathbb{N}}}$ in the following way.

$$\mathcal{T}r = \{ f \in 2^{[\mathbb{N}]^{<\mathbb{N}}} : \text{if } f(\beta) = 1 \text{ and } \alpha \prec \beta \text{ then } f(\alpha) = 1 \}$$

We now define an important subset of $\mathcal{T}r$ called the well founded trees. Each tree $\mathcal{T} \in \mathcal{T}r$ is said to be well-founded if there does not exist an infinitely ascending sequence $\alpha_1 \prec \alpha_2 \prec \alpha_3 \prec \dots$ where $\alpha_i \in \mathcal{T}$ for all $i \in \mathbb{N}$. Denote by \mathcal{WF} the subset of $\mathcal{T}r$ consisting of all well-founded trees.

We now define the order of a tree. For every $\mathcal{T} \in \mathcal{T}r$ we let

$$\mathcal{T}' = \{ s \in \mathcal{T} : \text{there exists } t \in \mathcal{T} \text{ with } s \prec t \}.$$

Observe that $\mathcal{T}' \in \mathcal{T}r$. For every $\mathcal{T} \in \mathcal{T}r$ we define $(\mathcal{T}^{(\xi)})_{\xi < \omega_1}$ as follows:

$$\mathcal{T}^{(0)} = \mathcal{T}, \mathcal{T}^{(\xi+1)} = (\mathcal{T}^{(\xi)})' \text{ and } \mathcal{T}^{(\lambda)} = \bigcap_{\xi < \lambda} \mathcal{T}^{(\xi)}$$

whenever λ is a limit ordinal. $\mathcal{T} \in \mathcal{WF}$ if and only if the sequence $(\mathcal{T}^{(\xi)})_{\xi < \omega_1}$ is eventually empty. For every $\mathcal{T} \in \mathcal{WF}$, the order of \mathcal{T} , $o(\mathcal{T})$, is defined to be the least countable ordinal ξ such that $\mathcal{T}^{(\xi)} = \emptyset$.

For our purposes the most important examples of trees on \mathbb{N} are the Schreier families \mathcal{S}_ξ for each $1 \leq \xi < \omega_1$. It can be shown that for all $1 \leq \xi < \omega_1$, $\mathcal{S}_\xi \in \mathcal{WF}$. The following proposition, found in [1], precisely defines the order of the Schreier families.

PROPOSITION 4.0.27. *For each $\xi < \omega_1$, $o(\mathcal{S}_\xi) = \omega^\xi$.*

We will prove the initial case; namely that, $o(\mathcal{S}_1) = \omega$.

Since $(n+1, \dots, 2(n+1)) \in \mathcal{S}_1$ it follows that $(n+1) \in \mathcal{S}_1^{(n)}$ for each $n \in \mathcal{S}_1$. Therefore for each $n \in \mathbb{N}$, $\mathcal{S}_1^{(n)} \neq \emptyset$. Whence, $o(\mathcal{S}_1) \leq \omega$.

Conversely, for every $\alpha \in \mathcal{S}_1$, by the definition of \mathcal{S}_1 , each successor of α has length at most $\min \alpha$. Thus $\alpha \notin \mathcal{S}_1^{(\min \alpha + 1)}$. Therefore $o(\mathcal{S}_1) \geq \omega$.

CHAPTER 5

POLISH SPACES

The primary objects of study in descriptive set theory are *Polish* spaces. A topological space P is called a Polish space if it is homeomorphic to a separable complete metric space. Two topological spaces S and P are homeomorphic if there is a one to one, onto continuous map $f : P \rightarrow S$ with continuous inverse.

Given any space P and its topology \mathfrak{T} , denote by $B(\mathfrak{T})$ the smallest collection of subsets of P containing \mathfrak{T} and closed under the operations of complementation and countable unions. The collection $B(\mathfrak{T})$ is called the *Borel* subsets of P generated by the topology \mathfrak{T} .

By definition a closed subset of a Polish space is Polish. The next proposition, which we state without proof, says that any Borel subspace of a Polish space is itself a Polish space with a new topology.

PROPOSITION 5.0.28. *Let P be a Polish space and $S \in B(\mathfrak{T})$. There exist a finer topology \mathcal{U} (i.e. $\mathfrak{T} \subset \mathcal{U}$) on P such that,*

- (1) S is clopen (both closed and open) in \mathcal{U} ;
- (2) (P, \mathcal{U}) is a Polish space;
- (3) $B(\mathfrak{T}) = B(\mathcal{U})$.

Moreover, S with the subspace topology generated by \mathcal{U} is a Polish space.

There exist subsets of certain Polish spaces that are not Borel. The nonmeasurable subset of \mathbb{R} is perhaps the most ‘well’ know example of such a set. In fact, the continuous image of a Polish space need not be Borel! The sets obtained as continuous

images of Polish spaces are called *analytic* subsets. Definition 5.0.29 gives an equivalent formulation of the notion of analytic set. This is just one of the many equivalent formulations (see [9, page 196]). Before stating it we need to define of Borel functions. A function $f : (P, \mathfrak{T}) \rightarrow (S, \mathcal{U})$ for Polish spaces P and S is called a *Borel function* if $f^{-1}(A) \in B(\mathfrak{T})$ for all $A \in B(\mathcal{U})$.

DEFINITION 5.0.29. *A subset \mathcal{A} of a Polish space P is an analytic set if it is a Borel image of a Polish space. In other words a subset \mathcal{A} of P is analytic if there is a Polish space S and an map $\phi : S \rightarrow P$ such that $\phi(S) = \mathcal{A}$.*

A set $\mathcal{C} \subset P$ is called *coanalytic* if it is the complement of an analytic set. Although we will not be using this fact, we point out that a set $\mathcal{A} \subset P$ is Borel if and only if \mathcal{A} is both analytic and coanalytic. We will use the next proposition in the proof of our main theorem. Due to the fact that, with our definition of analytic, the proof is quite involved, we omit it. Refer to [9] for a full account.

PROPOSITION 5.0.30. *The countable union of analytic sets is analytic.*

The remainder of this chapter is devoted to giving examples of Polish spaces we need to prove our main result in Chapter 6.

1. Our first example of a Polish space has already been given in Chapter 4,

$$2^{[\mathbb{N}]^{<\mathbb{N}}} = \{ \text{all functions } f : [\mathbb{N}]^{<\mathbb{N}} \rightarrow \{0, 1\} \}.$$

Let $(\alpha_n)_{n=1}^{\infty}$ be an enumeration of $[\mathbb{N}]^{<\mathbb{N}}$. For $f, g \in 2^{[\mathbb{N}]^{<\mathbb{N}}}$, a metric which induces a separable topology $2^{[\mathbb{N}]^{<\mathbb{N}}}$ is given by:

$$d(f, g) = \frac{1}{n} \text{ where } n \text{ is the first natural number such that } f(\alpha_n) \neq g(\alpha_n).$$

It is easily verified that d is a metric and that,

$$d(f_k, f) \rightarrow 0 \text{ if and only if } f_k(\alpha_n) \rightarrow f(\alpha_n) \text{ for all } n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, $f_k(\alpha_n) \rightarrow f(\alpha_n)$ means that there exists a $K \in \mathbb{N}$ such that for all $k \geq K$, $f_k(\alpha_n) = f(\alpha_n)$. Recall that $\mathcal{T} \in \mathcal{T}r$ corresponds to the function $f_{\mathcal{T}} : [\mathbb{N}]^{<\mathbb{N}} \rightarrow \{0, 1\}$ where $f(\alpha) = 1$ for $\alpha \in \mathcal{T}$ and $f(\alpha) = 0$ for $\alpha \notin \mathcal{T}$. Note that $f \in \mathcal{T}r$ if and only if $f(\alpha) = 1$ implies $f(\beta) = 1$ for all $\beta \prec \alpha$. Since we have placed a topology on $2^{[\mathbb{N}]^{<\mathbb{N}}}$ we may now make the following remark.

REMARK 5.0.31. *The space $\mathcal{T}r$ is a Polish space, with the separable metrizable topology being the one it inherits as a subspace of $2^{[\mathbb{N}]^{<\mathbb{N}}}$.*

PROOF. It suffices to show that $\mathcal{T}r$ is closed in $2^{[\mathbb{N}]^{<\mathbb{N}}}$. Let $(f_n)_{n=1}^{\infty} \subset \mathcal{T}r$ and suppose $f \in 2^{[\mathbb{N}]^{<\mathbb{N}}}$ such that $d(f_n, f) \rightarrow 0$. We must show that $f \in \mathcal{T}r$. Let $\alpha \in [\mathbb{N}]^{<\mathbb{N}}$ such that $f(\alpha) = 1$. It suffices to prove that for all $\beta \prec \alpha$, $f(\beta) = 1$. To this end, fix $\beta \prec \alpha$. Since $f_n(\alpha) \rightarrow f(\alpha) = 1$ we know that for sufficiently large n , $f_n(\alpha) = 1$. Since $f_n \in \mathcal{T}r$ for all $n \in \mathbb{N}$ it follows that $f_n(\beta) = 1$. Thus, $f_n(\beta) \rightarrow 1 = f(\beta)$. \square

Since $\mathcal{T}r$ is a Polish space, we may consider Borel, analytic and coanalytic subsets of it. A fundamental theorem in this area, and an important one for our purposes, is the following, *Boundedness Principle for Well-Founded Trees*.

THEOREM 5.0.32. *If $\mathcal{A} \subset \mathcal{WF}$ is analytic as a subset of $\mathcal{T}r$, then*

$$\sup\{o(\mathcal{T}) : \mathcal{T} \in \mathcal{A}\} < \omega_1.$$

2. Every separable Banach space is an example of a Polish space. Let X be a separable Banach space and define,

$$X^{\mathbb{N}} = \{(x_n)_{n=1}^{\infty} : x_n \in X \text{ for all } n \in \mathbb{N}\}.$$

$X^{\mathbb{N}}$ is a Polish space in the topology induced by the following metric,

$$\rho((x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x_n - y_n\|}{1 + \|x_n - y_n\|}.$$

Note that convergence in this metric is just coordinatewise convergence of the sequences. Let D be a dense subset of X . The space $X^{\mathbb{N}}$ is separable in this topology since,

$$D^{<\mathbb{N}} = \{(x_1, x_2, \dots, x_N, 0, 0, \dots) : \text{where } N \in \mathbb{N} \text{ and } x_i \in D\}$$

is dense in $X^{\mathbb{N}}$.

We will consider the subspace \mathfrak{B}_X of $X^{\mathbb{N}}$ containing all normalized basic sequences. Recall the definition of \mathfrak{B}_X from Chapter 1,

$$\mathfrak{B}_X = \{(x_i)_{i=1}^\infty \in X^{\mathbb{N}} : \exists k \in \mathbb{N} \text{ such that } \forall (a_i) \in c_{00} \text{ and } m < n,$$

$$\left\| \sum_{i=1}^m a_i x_i \right\| \leq k \left\| \sum_{i=1}^n a_i x_i \right\| \}$$

Combining Proposition 5.0.28 and the following proposition we may conclude that \mathfrak{B}_X is a Polish space.

PROPOSITION 5.0.33. *\mathfrak{B}_X is a Borel subset of $X^{\mathbb{N}}$.*

PROOF. For each $k \in \mathbb{N}$ let,

$$\mathfrak{B}_k = \{(x_i)_{i=1}^\infty \in \mathfrak{B}_X : \text{for all } (a_i) \text{ and } m < n,$$

$$\left\| \sum_{i=1}^m a_i x_i \right\| \leq k \left\| \sum_{i=1}^n a_i x_i \right\| \}.$$

\mathfrak{B}_k is the set of all normalized basis sequences with basis constant less than k . Note that $\cup_{k=1}^\infty \mathfrak{B}_k = \mathfrak{B}_X$. It suffices to show that for each $k \in \mathbb{N}$, \mathfrak{B}_k is a closed subset of $X^{\mathbb{N}}$.

To this end suppose, $\rho((x_i^j)_i, (x_i)_i) \rightarrow 0$ as $j \rightarrow \infty$ and $(x_i^j)_i \in \mathfrak{B}_k$ for all $j \in \mathbb{N}$. We wish to show that $(x_i)_i \in \mathfrak{B}_k$. Fix $(a_i)_i \in c_{00}$ and $m < n \in \mathbb{N}$.

We know that for each fixed $i \in \mathbb{N}$, $\|x_i^j - x_i\| \rightarrow 0$ as $j \rightarrow \infty$. Let $\varepsilon > 0$ and find $J \in \mathbb{N}$ such that for all $j \geq J$ and for all $i \leq n$,

$$\|x_i^j - x_i\| < \frac{\varepsilon}{2k \sum_{i=1}^n |a_i|}.$$

Letting $j \geq J$,

$$\begin{aligned} \left\| \sum_{i=1}^m a_i x_i \right\| &\leq \left\| \sum_{i=1}^m a_i x_i - x_i^j \right\| + \left\| \sum_{i=1}^m a_i x_i^j \right\| \\ &\leq \sum_{i=1}^m |a_i| \|x_i - x_i^j\| + \left\| \sum_{i=1}^m a_i x_i^j \right\| < \frac{\varepsilon}{2} + k \left\| \sum_{i=1}^n a_i x_i^j \right\| \\ &\leq \frac{\varepsilon}{2} + k \left\| \sum_{i=1}^n a_i x_i - x_i^j \right\| + k \left\| \sum_{i=1}^n a_i x_i \right\| \\ &\leq \frac{\varepsilon}{2} + k \sum_{i=1}^n |a_i| \|x_i - x_i^j\| + k \left\| \sum_{i=1}^n a_i x_i \right\| \\ &\leq \varepsilon + k \left\| \sum_{i=1}^n a_i x_i \right\| \end{aligned}$$

It follows that,

$$\left\| \sum_{i=1}^m a_i x_i \right\| < \varepsilon + k \left\| \sum_{i=1}^n a_i x_i \right\|$$

for any arbitrary $\varepsilon > 0$. This implies \mathfrak{B}_k is closed. \square

3. Let X and Y be separable Banach spaces. Define,

$$\mathcal{L}^1(X, Y) = \{T \in \mathcal{L}(X, Y) : \|T\| \leq 1\}.$$

In general the space $\mathcal{L}^1(X, Y)$ will not be separable in the topology induced the operator norm. However there is another topology called the “strong-operator” topology, denoted \mathfrak{T}_{sot} , such that $(\mathcal{L}^1(X, Y), \mathfrak{T}_{sot})$ is a Polish space. The metric which induces the topology \mathfrak{T}_{sot} is given by,

$$d(S, T) = \sum_{n=1}^{\infty} \frac{1}{2^n} \|(T - S)d_n\|.$$

Where (d_n) is a countable dense subset of $Ba(X)$. As before, we have that

$T_n \rightarrow T$ in $\mathfrak{T}_{\text{tot}}$ if and only if $T_n(x) \rightarrow T(x)$ for all $x \in X$.

We omit the proof that $\mathfrak{T}_{\text{tot}}$ is a separable topology on $\mathcal{L}^1(X, Y)$. See [9] for a detailed account of this proof.

CHAPTER 6

MAIN RESULTS

In this section we state three main theorems concerning strictly singular operators. The first can be found in [3] and was originally proved by Pandelis Dodos.

THEOREM 6.0.34. *Let X and Y be separable Banach spaces. Then,*

$$SS(X, Y) = \cup_{\xi < \omega_1} SS_\xi(X, Y).$$

Since it is easy to see the $SS_\xi(X, Y) \subset SS(X, Y)$ for each $1 \leq \xi < \omega_1$ and any spaces X and Y The above theorem states that given any strictly singular operator between two separable Banach spaces there is an $1 \leq \xi < \omega_1$ such that this operators is strictly singular- ξ .

The second is found in [2].

THEOREM 6.0.35. *If Y is a separable Banach space then*

$$\cup_{\xi < \omega_1} SS_\xi(Y^*, X) = SS(Y^*, X).$$

In [3] it is shown that for certain classes of Banach spaces X and Y there is an $\xi < \omega_1$ such that $SS_\xi(X, Y) = SS(X, Y)$. This conclusion is clearly stronger than the conclusion of Theorem 6.0.34. It is natural to ask whether there is a condition on the Banach spaces X and Y which is equivalent to $SS_\xi(X, Y) = SS(X, Y)$. Our next result gives a sufficient condition. Whether this condition is necessary remains to be seen. The rest of this paper is devoted to proving this theorem.

THEOREM 6.0.36. *Let X and Y be separable Banach spaces and $SS^1(X, Y) = SS(X, Y) \cap \mathcal{L}^1(X, Y)$. Suppose that $SS^1(X, Y)$ is Borel subset of $\mathcal{L}^1(X, Y)$ in $\mathfrak{T}_{\text{tot}}$. then there exists a $\xi < \omega_1$ such that $SS_\xi(X, Y) = SS(X, Y)$.*

PROOF. We shall show that $SS_\xi^1(X, Y) = SS^1(X, Y)$. Which will imply $SS_\xi(X, Y) = SS(X, Y)$.

For each, $R \in SS^1(X, Y), m \in \mathbb{N}, (x_n) \in \mathfrak{B}_X$, define a tree on \mathbb{N} in the following way:

$$\mathcal{T}(R, m, (x_n)_n) = \{(l_1, \dots, l_n) : \forall (a_i)_i \in \mathbb{Q}^{<\mathbb{N}}, \|R(\sum_{i=1}^n a_i x_{l_i})\| \geq \frac{1}{m} \|\sum_{i=1}^n a_i x_{l_i}\|\}.$$

It is easy to see that $\mathcal{T}(R, m, (x_n)_n)$ is a tree. Indeed, $(l_1, \dots, l_n) \in \mathcal{T}(R, m, (x_n)_n)$ implies $(l_1, \dots, l_k) \in \mathcal{T}(R, m, (x_n)_n)$ for all $k < n$.

Let us see that $\mathcal{T}(R, m, (x_n)_n) \in \mathcal{WF}$. Supposing not, find an infinite sequence $(l_i)_{i=1}^\infty$ such that $(l_i)_{i=1}^k \in \mathcal{T}(R, m, (x_n)_n)$ for all $k \in \mathbb{N}$. From this it follows that for each $(a_i) \in \mathbb{Q}^{<\mathbb{N}}$

$$\|\sum_{i=1}^\infty a_i x_{l_i}\| \frac{1}{m} \leq \|R \sum_{i=1}^\infty a_i x_{l_i}\|.$$

This implies that $R|_{[x_{l_i}]_{i=1}^\infty}$ is an isomorphism; contradicting the fact that $R \in SS^1(X, Y)$. Therefore $\mathcal{T}(R, m, (x_n)_n) \in \mathcal{WF}$.

We wish to show that the following collection of well founded trees is analytic as a subset of \mathcal{Tr} :

$$\mathcal{A} = \{\mathcal{T}(R, m, (x_n)_n) : R \in SS^1(X, Y), m \in \mathbb{N}, (x_n)_n \in \mathfrak{B}_X\}.$$

Assume for the moment \mathcal{A} is analytic. In this case we quickly prove the theorem.

Applying theorem 5.0.32 to \mathcal{A} we can find a $\xi < \omega_1$ such that $\sup\{o(\mathcal{T}) : \mathcal{T} \in \mathcal{A}\} < \xi$. We claim now that $SS_\xi^1(X, Y) = SS^1(X, Y)$. Suppose, for the sake of contradiction, that there exists $R \in SS^1(X, Y) \setminus SS_\xi^1(X, Y)$. By the definition of $SS_\xi^1(X, Y)$, there exists $(x_n) \in \mathfrak{B}_X$ and $m \in \mathbb{N}$ such that for all $F \in S_\xi$ and for all

$$(a_i) \in \mathbb{Q}^{<\mathbb{N}},$$

$$\|R \sum_{i \in F} a_i x_i\| \geq \frac{1}{m} \left\| \sum_{i \in F} a_i x_i \right\|.$$

This precisely means that $S_\alpha \subset \mathcal{T}(R, m, (x_n))$. Applying Proposition 5.0.32 we have the following contradiction,

$$\alpha > o(\mathcal{T}(R, m, (x_n))) \geq o(S_\alpha) = \omega^\alpha.$$

Therefore, once we prove that \mathcal{A} is analytic and we are home free. We shall do this by writing \mathcal{A} as a countable union of the following sets: For each $m \in \mathbb{N}$ let,

$$\mathcal{A}_m = \{T(R, m, (x_n)_n) : R \in SS^1(X, Y), (x_n)_n \in \mathfrak{B}_X\}.$$

Clearly $\mathcal{A} = \cup_{m=1}^{\infty} \mathcal{A}_m$. Appealing to Lemma 5.0.30 it remains to show that \mathcal{A}_m is analytic. From the definition of analytic we are tasked with finding a Polish space P and a Borel map $\varphi_m : P \rightarrow \mathcal{T}r$ such that $\varphi_m(P) = \mathcal{A}_m$. Our Polish space will be $SS^1(X, Y) \times \mathfrak{B}_X$ and our map is the following:

$$\varphi_m(R, (x_n)_n) = T(R, m, (x_n)_n)$$

We have already proved that \mathfrak{B}_X is a Borel subset of $X^{\mathbb{N}}$ and we have assumed $SS^1(X, Y)$ is a Borel subset of $\mathcal{L}^1(X, Y)$. Applying Proposition 5.0.28 both spaces may be regarded as Polish spaces where the Polish topologies are finer than the Polish topologies on $X^{\mathbb{N}}$ and $\mathcal{L}^1(X, Y)$ respectively and the Borel subsets are the same.

The final step is to show that φ_m is a Borel map. To do this it suffices to show that the inverse image of a basic open neighborhood is a Borel set in the domain. The basic open neighborhoods of $\mathcal{T}r$ are of the form $U_F = \{T \in \mathcal{T}r : F \in T\}$ where F is a fixed finite subset of \mathbb{N} . Fix F and observe that,

$$\begin{aligned}
\varphi^{-1}(U_F) &= \{(R, (x_n)) : F \in \mathcal{T}(R, m, (x_n))\} \\
&= \{(R, (x_n)) : \text{for all } (a_i) \in \mathbb{Q}^{<\mathbb{N}}, \|R \sum_{i \in F} a_i x_i\| \geq \frac{1}{m} \|\sum_{i \in F} a_i x_i\|\} \\
&= \bigcap_{(a_i) \in \mathbb{Q}^{<\mathbb{N}}} \{(R, (x_n)) : \|R \sum_{i \in F} a_i x_i\| \geq \frac{1}{m} \|\sum_{i \in F} a_i x_i\|\}
\end{aligned}$$

Fix $(a_i)_i \in \mathbb{Q}^{<\mathbb{N}}$. We will show that,

$$\mathcal{C}_{((a_i), F)} = \{(R, (x_n)) : \|R \sum_{i \in F} a_i x_i\| \geq \frac{1}{m} \|\sum_{i \in F} a_i x_i\|\}$$

is a Borel subset of $SS^1(X, Y) \times \mathfrak{B}_X$. Using the previous remark, we can do this by showing it is closed in the topology of pointwise convergence. We shall need the following remark.

REMARK 6.0.37. *Suppose $R_k \rightarrow R$ in $\mathfrak{T}_{\text{tot}}$, $\|R_k\| \leq 1$ for all $k \in \mathbb{N}$ $(y_k)_{k=1}^\infty \subset X$ and $y \in X$ such that $\lim_{k \rightarrow \infty} \|y_k - y\| = 0$. Then $\lim_{k \rightarrow \infty} \|R_k(y_k) - R(y)\| = 0$.*

PROOF. Let $\varepsilon > 0$. Find $K \in \mathbb{N}$ such that for all $k \geq K$,

$$\|R_k(y) - R(y)\| < \frac{\varepsilon}{2} \text{ and } \|y_k - y\| < \frac{\varepsilon}{2}.$$

Fix $k \geq K$. Then,

$$\begin{aligned}
\|R_k(y_k) - R(y)\| &\leq \|R_k(y_k) - R_k(y)\| + \|R_k(y) - R(y)\| \\
&< \|R_k\| \|y_k - y\| + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

□

To show $\mathcal{C}_{((a_i), F)}$ is closed, suppose $(R_k, (x_i^k)_i) \rightarrow (R, (x_i)_i)$ in the Polish topology on $\mathcal{L}^1(X, Y) \times X^\mathbb{N}$ and that $(R_k, (x_i^k)_i) \in \mathcal{C}_{((a_i), F)}$. We wish to show that $(R, (x_i)_i) \in \mathcal{C}_{((a_i), F)}$. By definition, $R_k \rightarrow R$ in $\mathfrak{T}_{\text{tot}}$, $\|R_k\| \leq 1$ for all $k \in \mathbb{N}$ and $\lim_{k \rightarrow \infty} \|x_i^k - x_i\| = 0$ for all $i \in \mathbb{N}$. For each $k \in \mathbb{N}$ let $y_k = \sum_{i \in F} a_i x_i^k$ and $y = \sum_{i \in F} a_i x_i$. Since F is finite it follows that $\lim_{k \rightarrow \infty} \|y_k - y\| = 0$. Applying Remark 6.0.37 we have,

$$\frac{1}{m} \|y\| = \lim_{k \rightarrow \infty} \frac{1}{m} \|y_k\| \leq \lim_{k \rightarrow \infty} \|R_k(y_k)\| = \|R(y)\|.$$

This proves the $\mathcal{C}_{((a_i), F)}$ is closed in $\mathcal{L}^1(X, Y) \times X^{\mathbb{N}}$ and thus Borel as a subset of $SS^1(X, Y) \times \mathfrak{B}_X$.

□

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