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KE Theory & the Number of Vertices Belonging to All Maximum Independent Sets in a
Graph

A thesis submitted in partial fulfillment of the requirements for the degree of Master of
Science at Virginia Commonwealth University.

by

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Abstract

KE THEORY & THE NUMBER OF VERTICES BELONGING TO ALL MAXIMUM INDEPENDENT SETS IN A GRAPH

By Taylor Mitchell Short, Master of Science.

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

Virginia Commonwealth University, 2011.

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For a graph G , let $\alpha(G)$ be the cardinality of a maximum independent set, let $\mu(G)$ be the cardinality of a maximum matching and let $\xi(G)$ be the number of vertices belonging to all maximum independent sets. Boros, Golumbic and Levit showed that in connected graphs where the independence number $\alpha(G)$ is greater than the matching number $\mu(G)$, $\xi(G) \geq 1 + \alpha(G) - \mu(G)$. For any graph G , we will show there is a distinguished induced subgraph $G[X]$ such that, under weaker assumptions, $\xi(G) \geq 1 + \alpha(G[X]) - \mu(G[X])$. Furthermore $1 + \alpha(G[X]) - \mu(G[X]) \geq 1 + \alpha(G) - \mu(G)$ and the difference between these bounds can be arbitrarily large. Lastly some results toward a characterization of graphs with equal independence and matching numbers is given.

Graph Theory and the Independence Number, α

1.1 Basic Definitions and Terminology

A *graph* G is a finite, nonempty set of elements called *vertices* (single element is a vertex) together with a set of unordered pairs of distinct vertices of G called *edges*. The *vertex set* of G is denoted by $V(G)$ and the *edge set* is denoted $E(G)$. The *order* of a graph G is the cardinality of the vertex set, denoted $n = |V(G)|$. The *size* of a graph is the cardinality of the edge set and is commonly denoted $m = |E(G)|$ although this notation will not be used in this thesis.

The edge $e = \{u, v\}$, or conveniently denoted uv or vu , is said to be *incident* to the vertices u and v . If $e = uv$ is an edge of a graph G , then u and v are *adjacent vertices*. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are *adjacent edges*. An *isolated vertex* is a vertex with no adjacent vertices. A vertex v that is adjacent to itself forms a *loop*, vv , and two or more edges between the same pair of vertices are called *multiple edges*.

The *degree* of a vertex v is the number of edges adjacent to v and is denoted $d(v)$ or sometimes $deg(v)$. The maximum degree of all the vertices of G is $\Delta(G)$ and the minimum degree of all the vertices of G is $\delta(G)$. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N(v) = \{u : u \in V(G) \text{ and } uv \in E(G)\}$. We can also talk about the neighborhood of a subset $S \subseteq V(G)$, $N(S) = \bigcup\{N(v) : v \in S\}$.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $S \subseteq V(G)$, the *induced subgraph* on S is the graph $G[S]$ with vertex set $V(G[S]) = S$ and edge set

$$E(G[S]) = \{uv \in E(G) : u, v \in S\}.$$

Graphs may be defined or described by a diagram, as seen in Figure 1.1, in which each vertex of G is represented by a point and each edge $e = uv$ is represented by a line segment or curve joining the points u and v . Graphs where the edges are assigned a direction are called *directed graphs*.

A *walk* in a graph is a sequence of vertices such that pairs of successive vertices are adjacent and a *path* is a walk with no repeated vertices. Graphs that are paths of order n are denoted P_n . A *trail* is a walk with no repeated edges and a *circuit* is a non-trivial closed trail. A circuit with distinct vertices is called a *cycle* and graphs that are cycles of order n are denoted C_n .

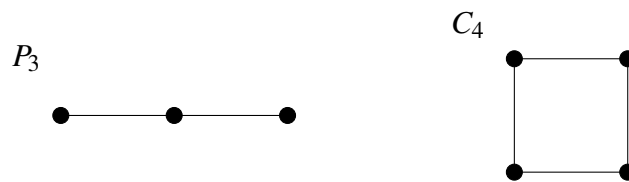


Figure 1.1: On the left is a path of order 3 and on the right is a cycle with order 4.

If there is a path between every pair of vertices, then that graph is said to be *connected*. If a graph is not connected, then the graph is *disconnected*. A *connected component* is a connected subgraph not properly contained in any other connected subgraph.

A *complete graph* is a graph with an edge between every pair of vertices and complete graphs of order n are denoted K_n . A *k-partite graph*, $k \geq 1$, is a graph G whose vertex set can be partitioned into k subsets V_1, V_2, \dots, V_k such that every edge of G joins a vertex of V_i to a vertex in V_j where $i \neq j$. A *complete k-partite graph* is a k-partite graph with partite sets V_1, V_2, \dots, V_k with the extra property that if $u \in V_i$ and $v \in V_j$ where $i \neq j$ then $uv \in E(G)$. Complete k-partite graphs are denoted K_{n_1, n_2, \dots, n_k} where n_i is the cardinality of the set V_i . The complete bipartite graph $K_{1, n-1}$ is called a *star* and is denoted S_n .

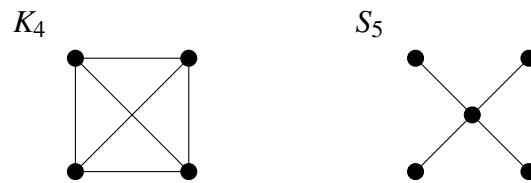


Figure 1.2: On the left is a complete graph of order 4 and on the right is a star with order 5.

1.2 Definition of the Independence Number

All graphs considered from here on are *simple graphs*, which are finite, undirected, loopless and without multiple edges. Before we begin looking at previous results we must introduce several key definitions, the first of which is an independent set of vertices and the corresponding invariant, the independence number of a graph.

DEFINITION 1.1. An *independent set of vertices* in a graph is a set of vertices, no two of which are adjacent. A *maximum independent set (MIS)* is an independent set of largest cardinality. The *independence number* of a graph $\alpha(G)$ is the cardinality of a maximum independent set.

An example of an independent set as well as a maximum independent set is given in Figure 1.3 below. As for the independence number of some general classes of graphs we have $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$, $\alpha(P_n) = \lceil \frac{n}{2} \rceil$, $\alpha(K_n) = 1$, and $\alpha(S_n) = (n - 1)$.

1.3 Results on the Independence Number

Finding the independence number of a graph is a widely-studied, NP-hard problem and it's of interest for theoretical and practical purposes. It's related to many other major invariants which we present here, the first of which involves sets of independent edges.

DEFINITION 1.2. A *matching* in a graph is a set of non-adjacent edges. A *maximum matching* is a matching of largest cardinality. The *matching number* of a graph $\mu(G)$ is the

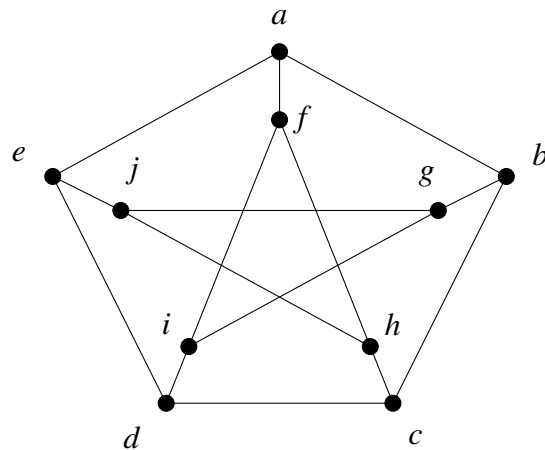


Figure 1.3: An example of independent set of vertices is $I_1 = \{a, c, g\}$. A maximum set of vertices is $I_2 = \{a, c, i, j\}$, so $\alpha(G) = |I_2| = 4$.

cardinality of a maximum matching.

A vertex v is said to be *matched* under a matching M if $v \in V(M)$. Note that for any graph G , $\mu(G) \leq \frac{n}{2}$ since each edge in a matching is incident to two vertices in the graph. If $\mu(G) = \frac{n}{2}$, then all the vertices of G are matched by a maximum matching and G is said to have a *perfect matching*.

Now that we have defined a matching we can introduce a well-known relation between the independence number and the matching number in the following theorem.

THEOREM 1.3. If G is a connected graph, then

$$n - 2\mu(G) \leq \alpha(G) \leq n - \mu(G). \quad (1.1)$$

Proof. Suppose G is a connected graph and let M be a maximum matching. Consider the set of vertices, $V' = V(G) - V(M)$. Then V' is an independent set since if any two vertices of V' were adjacent then we could add the edge between them to M . So $\alpha(G) \geq |V'|$ and

$n = |V'| + |V(M)|$ so $|V'| = n - |V(M)| = n - 2\mu(G)$. Then

$$\alpha(G) \geq |V'| = n - 2\mu(G)$$

and hence, $\alpha(G) \geq n - 2\mu(G)$.

Also, given a set of independent edges M separating vertices $V(M)$, the largest possible independent set of vertices in $V(M)$ has at most $|M|$ vertices. Then any independent set has at most $|V'| + |M| = |V'| + \mu(G)$ vertices. So $\alpha(G) \leq |V'| + \mu(G)$. Then

$$\alpha(G) \leq |V'| + \mu(G) = n - 2\mu(G) + \mu(G) = n - \mu(G).$$

Thus $\alpha(G) \leq n - \mu(G)$ and therefore $n - 2\mu(G) \leq \alpha(G) \leq n - \mu(G)$. \square

Another well-known bound on the independence number of the graph is due to Turán and involves the maximum degree of a graph, $\Delta(G)$.

THEOREM 1.4. For any graph G ,

$$\alpha(G) \geq \frac{n}{1 + \Delta(G)}.$$

Proof. Suppose I is a maximum independent set of vertices of G . Then

$$V(G) = I \cup \left(\bigcup_{v \in I} N(v) \right)$$

since if some vertex $v \in V(G)$ did not belong to $I \cup (\bigcup_{v \in I} N(v))$ then we could add v to our independent set I , which is already maximum. This implies, since $|\bigcup_{v \in I} N(v)| \leq \sum_{v \in I} |N(v)|$

and $|N(v)| \leq \Delta(G)$ for any vertex $v \in V(G)$, that

$$n = |I| + \left| \bigcup_{v \in I} N(v) \right| \leq |I| + |I| \cdot \Delta(G) = \alpha(G) + \alpha(G) \cdot \Delta(G).$$

Then

$$n \leq \alpha(G) + \alpha(G) \cdot \Delta(G) = \alpha(G)(1 + \Delta(G)) \quad (1.2)$$

Therefore from (1.2) we get $\frac{n}{1+\Delta(G)} \leq \alpha(G)$. □

A vertex and an edge are said to *cover* each other if they are incident.

DEFINITION 1.5. A *vertex cover* in a graph is a set of vertices that covers all the edges of a graph. The *vertex cover number* $\tau(G)$ is the cardinality of a minimum vertex cover.

A set of vertices C is a vertex cover of a graph G if, and only if, $V(G) - C$ is an independent set. This property of vertex covers leads us to the following relation between the vertex cover number and the independence number due to Gallai.

THEOREM 1.6. If G has no isolated vertices, then

$$\alpha(G) + \tau(G) = n.$$

Proof. Suppose G has no isolated vertices and let I be a maximum independent set and let C be a minimum vertex cover. Then $V(G) - I$ is a vertex cover of G since if any vertex was not covered then we could add it to our independent set I . So $\tau(G) \leq n - \alpha(G)$ or equivalently, $\alpha(G) + \tau(G) \leq n$.

Also, $V(G) - C$ is an independent set of vertices since if two vertices were adjacent then C would not cover the edge between them. So $\alpha(G) \geq n - \tau(G)$ or equivalently, $\alpha(G) + \tau(G) \geq n$. Thus $\alpha(G) + \tau(G) \leq n \leq \alpha(G) + \tau(G)$ and therefore, $\alpha(G) + \tau(G) = n$. □

The next invariant related to the independence number comes from an area of graph theory that has received the most attention over the years. This attention is due to the famous Four Color Problem and the invariant is the chromatic number of a graph.

DEFINITION 1.7. A *proper coloring* of a graph G is an assignment of colors to the vertices of G , one color to each vertex, so that adjacent vertices are assigned different colors. The *chromatic number* $\chi(G)$ is the minimum number of colors needed for a proper coloring.

A *proper k -coloring* of a graph is a proper coloring using k colors.

THEOREM 1.8. For any graph G ,

$$\alpha(G)\chi(G) \geq n.$$

Proof. Suppose $\chi(G) = k$ and let V_1, V_2, \dots, V_k be subsets of the vertex set such that V_i is assigned color i under a proper k -coloring of G . Then each V_i is an independent set of vertices, so

$$\alpha(G) \geq M = \max\{|V_1|, |V_2|, \dots, |V_k|\}.$$

Also, since V_1, V_2, \dots, V_k partition $V(G)$ we have $M \cdot k \geq n$. Then

$$\alpha(G)\chi(G) \geq M \cdot k \geq n.$$

Therefore, $\alpha(G)\chi(G) \geq n$. □

The next invariant deals with types of sets that appear to have their origins in games like chess, where the goal is to dominate squares of the chess board using different pieces. A vertex v in a graph is said to *dominate* itself and each of its neighbors.

DEFINITION 1.9. A set of vertices of a graph is a *dominating set* if every vertex in the graph is dominated by at least one vertex from the set. The *domination number* $\gamma(G)$ is the

minimum cardinality among the dominating sets of a graph G .

THEOREM 1.10. For any graph G ,

$$\alpha(G) \geq \gamma(G).$$

Proof. Suppose I is a maximum independent set of vertices G . Then I is a dominating set since if any vertex was not dominated by I then that vertex does not belong to $I \cup N(I)$, so we could add it to our independent set I . Therefore $\gamma(G) \leq \alpha(G)$. \square

The final relation on the independence number given in this chapter will be using the minimum degree of a graph. For many types of graphs the bound given below is not a good indicator of the size of a maximum independent set, but for complete graphs this bound is sharp.

THEOREM 1.11. For any graph G ,

$$\alpha(G) \leq n - \delta(G).$$

Proof. Let I be a maximum independent set and $v \in I$. Then $I \cap N(v) = \emptyset$, so $I \subseteq V(G) \setminus N(v)$ which implies

$$|I| \leq |V(G) \setminus N(v)| = n - |N(v)| \leq n - \delta(G).$$

Thus $\alpha(G) = |I| \leq n - \delta(G)$ and therefore, $\alpha(G) \leq n - \delta(G)$. \square

Core of a graph, ξ

When studying the independence number of a graph, it is natural to question how many vertices belong to every possible maximum independent set. This was first studied by Hammer, Hansen and Simeone in [5], where they showed $\xi(G) \geq 1$ whenever $\alpha(G) > \frac{n}{2}$. These results were later improved in [7] by Levit and Mandrescu who showed for graphs G with $\alpha(G) > (n+k-1)/2$ the stronger inequality $\xi(G) \geq k+1$ must hold. We will discuss the most recent results due to Boros, Golumbic and Levit from [3] here.

2.1 Definition of the Core

Before we look at several results we must introduce the formal definition for the set of vertices belonging to all maximum independent sets. Let $\Omega(G)$ be the collection of all maximum independent sets of a graph G .

DEFINITION 2.1. The *core* of a graph is the set of vertices that belong to all maximum independent sets, $core(G) = \bigcap \{I : I \in \Omega(G)\}$. The cardinality of $core(G)$ is denoted $\xi(G)$.

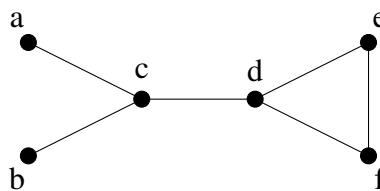


Figure 2.1: Maximum independent sets are $I_1 = \{a, b, d\}$, $I_2 = \{a, b, e\}$ and $I_3 = \{a, b, f\}$, so $core(G) = I_1 \cap I_2 \cap I_3 = \{a, b\}$.

The cardinality of the core of a graph gives a lower bound for the independence number and also gives structural information about the graph. Related to the core of a graph are the vertices that belong to some, but not all, maximum independent sets.

DEFINITION 2.2. The *corona* of a graph is the set of vertices that belong to some maximum independent set, $corona(G) = \bigcup\{I : I \in \Omega(G)\}$. The cardinality of $corona(G)$ is denoted $\zeta(G)$.

In Figure 2.1, $corona(G) = \{d, e, f\}$ since these vertices belong to some maximum independent set but not all.

2.2 Some Previous Results on the Core

The following is a result due to Levit and Mandrescu in [7] that we will use later. Note that this theorem implies that the core is nonempty, that is, $\xi(G) > 0$.

THEOREM 2.3. (Levit, Mandrescu [7]) If $\alpha(G) > \frac{n}{2}$, then $|core(G)| > |N(core(G))|$.

Before proving their main result, Boros, Golubic and Levit in [3] prove several small lemmas that are both interesting and useful when studying the core of a graph. We state the lemmas here and the proofs can be found in [3].

LEMMA 2.4. (Boros, Golubic, Levit [3]) For any graph G , there are no edges between the set $core(G)$ and $corona(G)$, that is,

$$N(core(G)) \subseteq V(G) \setminus corona(G). \quad (2.1)$$

LEMMA 2.5. (Boros, Golubic, Levit [3]) For any graph G and for any maximum independent set $I \in \Omega(G)$, there is a matching from $(I \setminus core(G))$ into $(corona(G) \setminus I)$, that

is,

$$|corona(G) \setminus I| \geq |I \setminus core(G)|. \quad (2.2)$$

Note that Lemma 2.5 implies for any graph, there exists a matching M of size $|M| \geq |I \setminus core(G)|$ where I is a MIS. This is key for the proof of Theorem 2.6 below, since $\alpha(G) > \mu(G)$ by assumption, so $\mu(G) \geq |M|$ implies $\alpha(G) > |I \setminus core(G)|$ which shows the core is nonempty.

Now onto the main result in [3] which we will improve upon in the results section of this thesis.

THEOREM 2.6. (Boros, Golumbic, Levit [3]) If G is a connected graph with $\alpha(G) > \mu(G)$, then

$$\xi(G) \geq 1 + \alpha(G) - \mu(G). \quad (2.3)$$

Proof. By Lemma 2.5, for any maximum independent set I of G there exists a matching M in G of size $|M| \geq |I \setminus core(G)|$. Since $\mu(G) \geq |M|$ and $\alpha(G) > \mu(G)$ is assumed, then $\alpha(G) > \mu(G) \geq |I \setminus core(G)|$ and hence, $core(G) \neq \emptyset$.

Since G is assumed to be connected, $N(core(G)) \neq \emptyset$, and so there exists an edge $uv \in E(G)$ with $u \in core(G)$ and $v \in N(core(G))$. By Lemma 2.4 $N(core(G)) \cap corona(G) = \emptyset$, so uv does not have a common endpoint with the edges in M . Thus $M \cup \{uv\}$ is also a matching, so

$$\mu(G) \geq |M| + 1 \geq |I \setminus core(G)| + 1 = \alpha(G) - \xi(G) + 1$$

and hence, $\xi(G) \geq 1 + \alpha(G) - \mu(G)$. □

The following result was first proved in [7] by Levit and Mandrescu but was later presented as a corollary to Theorem 2.6 in [3].

COROLLARY 2.7. (Levit, Mandrescu [7]) If G is a connected graph, and for some $k \geq 1$ the inequality

$$2\alpha(G) \geq n + k$$

holds, then

$$\xi(G) \geq k + 1$$

Proof. Suppose G is a connected graph and the inequality $2\alpha(G) \geq n + k$ holds for some $k \geq 1$. Then

$$\alpha(G) = \frac{n+k}{2} > \frac{n}{2} \geq \mu(G),$$

so by Theorem 2.6, $\xi(G) \geq 1 + \alpha(G) - \mu(G)$. Then by Theorem 1.3, $\mu(G) \leq n - \alpha(G)$, so

$$\xi(G) \geq 1 + \alpha(G) - \mu(G) \geq 1 + \alpha(G) - (n - \alpha(G)) = 1 + 2\alpha(G) - n.$$

and thus, by our assumption

$$\xi(G) \geq 1 + 2\alpha(G) - n \geq 1 + (n + k) - n = 1 + k.$$

Therefore, $\xi(G) \geq k + 1$. □

KE Theory

In this chapter we will develop a decomposition of any graph that is due to Larson in [1]. This decomposition will be our tool for improving Theorem 2.6 and will preserve the relevant structure of graphs by looking at induced subgraphs.

3.1 Definition of Critical Independent Sets

To form one of the induced subgraphs in our decomposition we will choose a subset of the vertices that is based on an independent set subjected to an extra condition.

DEFINITION 3.1. An independent set of vertices I_c is a *critical independent set* if $|I_c| - |N(I_c)| \geq |J| - |N(J)|$ for any independent set J . A maximum critical independent set is a critical independent set of maximum cardinality. The critical independence number of a graph $\alpha'(G)$ is the cardinality of a maximum critical independent set.

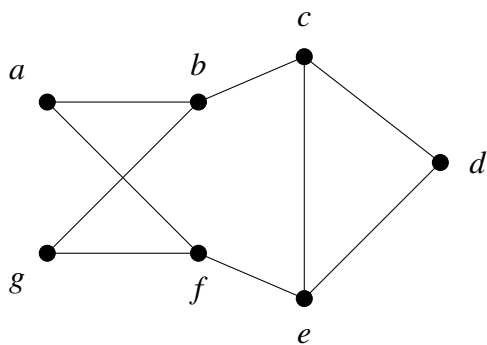


Figure 3.1: The vertices $I_c = \{a, g\}$ form a maximum critical independent set.

3.2 Some Types of Graphs

Note that critical independent sets can have different cardinalities. The graph K_2 has critical independent sets of cardinalities 0 and 1. We now introduce three classes of graphs that are based on the size of their maximum critical independent set.

DEFINITION 3.2. A graph G is *independence irreducible* (II or II-graph) if $\alpha'(G) = 0$.

So if a graph is independence irreducible then the empty set is the only critical independent set. These graphs include odd cycles and complete graphs with at least 3 vertices and will be important to our decomposition below. See examples of II graphs in Figure 3.2.

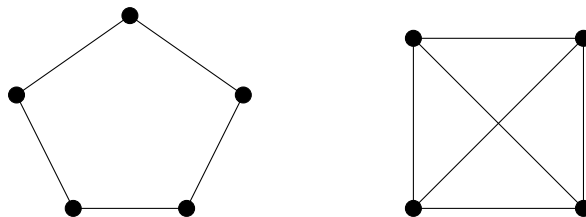


Figure 3.2: Independence irreducible graphs. For any non-empty independent set I , $|I| < |N(I)|$.

DEFINITION 3.3. A graph G is *independence reducible* if $\alpha'(G) > 0$, that is, the graph has a nonempty critical independent set.

Figure 3.1 is an example of an independence reducible graph, where $I_c = \{a, g\}$ is a critical independent set. Note that for this graph, $\alpha(G) = 3$, so the cardinality of a maximum critical independent set is less than the cardinality of a maximum independent set. This brings us to our next definition.

DEFINITION 3.4. A graph G is *totally independence reducible* if $\alpha'(G) = \alpha(G)$.

Totally independence reducible graphs will also be important to our decomposition. These graphs include all bipartite graphs and an examples of totally independence reducible

graphs can be seen in Figure 3.3.

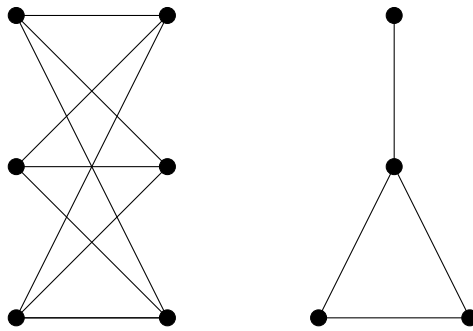


Figure 3.3: Totally independence reducible graphs. The graph on the left is bipartite with $\alpha = \alpha' = 3$. The graph on the right is not bipartite with $\alpha = \alpha' = 2$.

3.3 Results on Critical Independent Sets

As far as results dealing with critical independent sets, this next theorem is responsible for much of the interest in the topic.

THEOREM 3.5. (Butenko, Trukhanov [6]) If I_c is a critical independent set in a graph G then there is a maximum independent set I in G such that $I_c \subseteq I$.

Critical independent sets can be computed in polynomial time, first shown by Zhang in [2]. As mentioned before, finding a maximum independent set is a NP-hard problem and the above theorem relates finding a critical independent set to a maximum independent set.

These next three lemmas are required for the proof of the decomposition we will be using and proofs can be found in [1].

LEMMA 3.6. (Larson [1]) For any graph G with maximum critical independent set I_c , $\alpha = \alpha'$ if, and only if, $I_c \cup N(I_c) = V(G)$.

LEMMA 3.7. (Larson [1]) If I_c is a critical independent set of G , then there is a matching of the vertices $N(I_c)$ into (a subset of) the vertices of I_c .

LEMMA 3.8. (Larson [1]) If G is a graph with critical independent sets I_c and J_c , where $J = J_c \setminus (I_c \cup N(I_c))$, and $I = I_c \cup J$ then,

1. $|I_c \cap N(J_c)| = |J_c \cap N(I_c)|$,
2. $|J| \geq |N(J_c) \setminus (I_c \cup N(I_c))|$,
3. I is a critical independent set.

We are now ready to present the main result in [1] and this will give us the decomposition we need to improve the previous results on the core.

THEOREM 3.9. (Larson [1]) For any graph G , there is a unique set $X \subseteq V(G)$ such that

1. $\alpha(G) = \alpha(G[X]) + \alpha(G[X^c])$,
2. $G[X]$ is totally independence reducible,
3. $G[X^c]$ is independence irreducible, and
4. for every maximum critical independent set J_c of G , $X = J_c \cup N(J_c)$.

3.4 Introduction to KE Theory

Theorem 3.9 can be stated in a more useful way in the corollary below, but first we must introduce a type of graph whose name originates from the König-Egerváry Theorem.

THEOREM 3.10. (Theorem 9.13 in [4]) If G is a bipartite graph, then $\tau(G) = \mu(G)$.

Note in the above theorem, $\tau(G) = \mu(G)$ is equivalent to $n - \alpha(G) = \mu(G)$. So by Theorem 3.10 all bipartite graphs enjoy the identity, $\alpha(G) + \mu(G) = n$, which coincides with our definition for König-Egerváry graphs.

DEFINITION 3.11. A graph G is a *König-Egerváry graph* (KE or KE-graph) if $\alpha(G) + \mu(G) = n$.

So all bipartite graphs are König-Egerváry but there are also non-bipartite König-Egerváry graphs, see Figure 3.3 for examples of KE graphs. There are many results on König-Egerváry graphs that make up the growing study of KE theory but in this thesis we will present only those of interest to us.

Also required for the restatement of Theorem 3.9 is the following result which was first conjectured by Ermelinda DeLaVina's program Graffiti.pc and proved by Larson in [1].

THEOREM 3.12. (Larson [1]) For any graph G , $\alpha(G) = \alpha'(G)$ if, and only if, $\tau(G) = \mu(G)$.

The above theorem is equivalent to: a graph G is totally independence reducible if, and only if, G is a KE graph. Using this we can now restate Theorem 3.9 in the following way.

COROLLARY 3.13. (Larson [1]) For any graph G , there is a unique set $X \subseteq V(G)$ such that

1. $\alpha(G) = \alpha(G[X]) + \alpha(G[X^c])$,
2. $G[X]$ is a König-Egerváry graph,
3. for every non-empty independent set I in $G[X^c]$, $|N(I)| > |I|$, and
4. for every maximum critical independent set J_c of G , $X = J_c \cup N(J_c)$.

Results on the Core

In this chapter, we will develop results that give new bounds on the core of a graph and show that our bounds can be arbitrarily better.

4.1 Motivation

Recall that Hammer, Hansen and Simeone in [5] first showed $\xi(G) \geq 1$ whenever $\alpha(G) > \frac{n}{2}$. By Theorem 2.6 due to Boros, Golubic and Levit, $\xi(G) \geq 1 + \alpha(G) - \mu(G)$ whenever G is connected with $\alpha(G) > \mu(G)$, so $\xi(G) = 1$ cannot happen. Comparing these results leads to the question for what graphs is $\xi(G) = 1$ with $\alpha(G) > \frac{n}{2}$? One class of examples are graphs with an isolated vertex whose other connected components have $\alpha = \frac{n}{2}$ with an empty core. Connected graphs with the desired properties are characterized in the proposition below.

PROPOSITION 4.1. For any connected graph G , if $\xi(G) = 1$ and $\alpha(G) > \frac{n}{2}$, then $G = K_1$.

Proof. Let G be a connected graph with $\xi(G) = 1$ and $\alpha(G) > \frac{n}{2}$. We will show by contradiction that G must be K_1 . Suppose $n > 1$. Since $\alpha(G) > \frac{n}{2}$, by Theorem 2.3 $|core(G)| > |N(core(G))|$. But $core(G) = 1 > |N(core(G))|$, so we must have $|N(core(G))| = 0$. Thus $core(G)$ is an isolated vertex, so G is disconnected which is a contradiction. Hence $n = 1$ and therefore, G must be K_1 . \square

For the remainder of this chapter, for any graph G let X be a subset of the vertex set as defined in Corollary 3.13, $X = I_c \cup N(I_c)$, where I_c is a maximum critical independent set and let $X^c = V(G) \setminus X$.

Our main result is a corollary to Theorem 2.6 and will show if G is a connected graph with $\alpha(G[X]) > \mu(G[X])$, then

$$\xi(G) \geq 1 + \alpha(G[X]) - \mu(G[X]). \quad (4.1)$$

To motivate why we would want to pursue the bound in (4.1) we present two theorems on the relations between the independence and matching numbers for both KE and II graphs. Both these results are well-known but presented here for context.

THEOREM 4.2. If G is KE, then $\alpha(G) \geq \mu(G)$.

Proof. Suppose G is a KE graph. Then $\alpha(G) + \mu(G) = n$. Since for any graph $\mu(G) \leq \frac{n}{2}$, we have

$$n = \alpha(G) + \mu(G) \leq \alpha(G) + \frac{n}{2}$$

which implies $\frac{n}{2} \leq \alpha(G)$. So we have

$$\mu(G) \leq \frac{n}{2} \leq \alpha(G)$$

and hence, $\alpha(G) \geq \mu(G)$. □

The proof of the relation for II graphs relies on a well-known result called Hall's Marriage Theorem presented as Theorem 4.3 below.

THEOREM 4.3. (Theorem 9.4 in [4]) For a collection S_1, S_2, \dots, S_k , $k \geq 1$, of finite sets there exists a set $\{s_1, s_2, \dots, s_k\}$ of distinct elements such that $s_i \in S_i$ for $1 \leq i \leq k$ if, and only if, the union of any j of these sets contains at least j elements, for each j such that $1 \leq j \leq k$.

Now we are ready to state and prove the relation between the independence and matching numbers for II graphs.

THEOREM 4.4. If G is II, then $\mu(G) \geq \alpha(G)$.

Proof. Suppose G is an II graph. Let I be a maximum independent set of G . Since G is II, $|I| < |N(I)|$ and for every subset $I' \subseteq I$ we have $|I'| < |N(I')|$ both of which imply by Theorem 4.3 there is a matching M of size $|I|$ from the vertices of I into $N(I)$. Thus $\mu(G) \geq |M| = \alpha(G)$ and therefore $\mu(G) \geq \alpha(G)$. \square

By the above theorems, the bound on $\xi(G)$ in (4.1) will always be greater than or equal to the bound in Theorem 2.6, that is $1 + \alpha(G[X]) - \mu(G[X]) \geq 1 + \alpha(G) - \mu(G)$, since in the KE part, $G[X]$, we know that $\alpha(G[X]) \geq \mu(G[X])$ always holds and in the II part, $G[X^c]$, $\alpha(G[X^c]) \leq \mu(G[X^c])$ holds. So for any graph, the difference $\alpha - \mu$ will be maximized within the KE part and this difference is what gives us our bound on the core, $\xi \geq 1 + \alpha - \mu$.

4.2 Main Results

Since we are focusing on the KE part of any graph, we are finding a bound on the core of $G[X]$ and not G . It is then necessary to check whether vertices in the $core(G[X])$ are also in $core(G)$ which this next lemma shows. The following result was also proved independently by DeLaVina and Larson.

LEMMA 4.5. For any graph G , $core(G[X]) \subseteq core(G)$.

Proof. Let v be in $core(G[X])$. We will show that v must also belong to $core(G)$. Suppose not, that v is not in $core(G)$. Then there exists some maximum independent set I such that $v \notin I$. Now consider the induced subgraph $G[X]$ of G . Then $I \cap X$ is an independent set of $G[X]$ but since $v \in core(G[X])$ and $v \notin I \cap X$ we get $|I \cap X| < \alpha(G[X])$. But then $|I| = |I \cap X| + |I \cap X^c| = \alpha(G)$ and also $\alpha(G) = \alpha(G[X]) + \alpha(G[X^c])$ by Corollary 3.13 (1),

so we must have $|I \cap X^c| > \alpha(X^c)$ where $I \cap X^c$ is an independent set of $G[X^c]$. Thus we found a larger maximum independent set of $G[X^c]$, which is a contradiction. Hence v must also belong to $core(G)$ and since v was arbitrary we get $core(G[X]) \subseteq core(G)$. \square

With the above lemma we are now ready to prove our main result, a corollary to Theorem 2.6.

COROLLARY 4.6. If G is a connected graph with $\alpha(G[X]) > \mu(G[X])$, then

$$\xi(G) \geq 1 + \alpha(G[X]) - \mu(G[X]). \quad (4.2)$$

Proof. Suppose G is a connected graph with $\alpha(G[X]) > \mu(G[X])$. Since $G[X]$ is an induced subgraph and $\alpha(G[X]) > \mu(G[X])$, by Theorem 2.6

$$\xi(G[X]) \geq 1 + \alpha(G[X]) - \mu(G[X]). \quad (4.3)$$

By Lemma 4.5, $core(G[X]) \subseteq core(G)$ which implies the inequality $\xi(G) \geq \xi(G[X])$ and along with (4.3) yields

$$\xi(G) \geq 1 + \alpha(G[X]) - \mu(G[X]).$$

\square

With the improved bound on the core at our disposal, we now present the following improvements to Corollary 2.7.

COROLLARY 4.7. If G is a connected graph with $2\alpha(G) \geq n + k$ where $k \geq 1$ and $m = |X^c| - 2\alpha(G[X^c]) \geq 1$, then

$$\xi(G) \geq k + m + 1.$$

Proof. Suppose G is a connected graph with $2\alpha(G) \geq n+k$ and $m = |X^c| - 2\alpha(G[X^c]) \geq 1$. By Corollary 3.13 (1), we have $2\alpha(G) = 2\alpha(G[X]) + 2\alpha(G[X^c])$ and $n+k = |X| + |X^c| + k$. So by our assumption $2\alpha(G[X]) + 2\alpha(G[X^c]) \geq |X| + |X^c| + k$, or $2\alpha(G[X]) \geq |X| + |X^c| + k - 2\alpha(G[X^c])$. Since $m = |X^c| - 2\alpha(G[X^c])$, then

$$2\alpha(G[X]) \geq |X| + k + |X^c| - 2\alpha(G[X^c]) = |X| + k + m. \quad (4.4)$$

By Corollary 3.13 (2), $G[X]$ is KE so $\mu(G[X]) = |X| - \alpha(G[X])$. Since $\alpha(G) > \mu(G)$ by Corollary 4.6 $\xi(G) \geq 1 + \alpha(G[X]) - \mu(G[X])$, so by the above

$$\xi(G) \geq 1 + \alpha(G[X]) - (|X| - \alpha(G[X])) = 1 + 2\alpha(G[X]).$$

So by (4.4)

$$\xi(G) \geq 1 + 2\alpha(G[X]) \geq 1 + k + m.$$

Therefore, $\xi(G) \geq 1 + k + m$. □

COROLLARY 4.8. If G is connected with $2\alpha(G) \geq n+k$ and $G[X^c] \neq \emptyset$, then

$$\xi(G) \geq k+2 \quad (4.5)$$

Proof. Suppose G is a connected graph with $2\alpha(G) \geq n+k$ and $G[X^c] \neq \emptyset$. Since $G[X^c] \neq \emptyset$,

$$m = |X^c| - 2\alpha(G[X^c]) \geq |X^c| - 2 \frac{|X^c| - 1}{2} = 1.$$

Since $2\alpha(G) \geq n+k$ and $m \geq 1$, by Corollary 4.7,

$$\xi(G) \geq k + m + 1 \geq k + 1 + 1 = k + 2.$$

Therefore $\xi(G) \geq k + 2$. □

Another result that can be generalized using the decomposition is Theorem 2.3 which states if $\alpha(G) > \frac{n}{2}$, then $|core(G)| > |N(core(G))|$ and we give this generalization here.

COROLLARY 4.9. If $\alpha(G[X]) > \frac{|X|}{2}$, then $|core(G[X])| > |N(core(G[X]))|$.

Since $G[X]$ is an induced subgraph, the proof follows as a direct result of Theorem 2.3. Notice that the condition $\alpha(G[X]) > \frac{|X|}{2}$ is much weaker than the previous condition $\alpha(G) > \frac{n}{2}$. Also since $core(G[X]) \subseteq core(G)$ the corollary still gives us information about the size of $core(G)$ in relation to its neighborhood.

4.3 Showing Our Bound can be Better

Before proving other useful results on our improved bounds we must show a result on the matching numbers of $G[X]$ and $G[X^c]$ which has not been formally proven until now. The matching number of G is equal to the sum of the matching numbers of the subgraphs, $G[X]$ and $G[X^c]$.

THEOREM 4.10. For any graph G , $\mu(G[X]) + \mu(G[X^c]) = \mu(G)$.

Proof. Let M' be a maximum matching of $G[X]$ and M'' of $G[X^c]$. Then $M' \cup M''$ is a matching of G , since we are not using any edges between vertices in X to vertices in X^c . So we have $\mu(G[X]) + \mu(G[X^c]) \leq \mu(G)$.

Now we will show that $\mu(G[X]) + \mu(G[X^c]) \geq \mu(G)$ by showing for any maximum matching that contains edges not in $G[X]$ or $G[X^c]$, that we can find a matching of the same cardinality using edges from only $G[X]$ and $G[X^c]$. Suppose that there exists some maximum matching M of G such that

$$\mu(G[X]) + \mu(G[X^c]) < \mu(G) = |M|$$

and there are edges $u_1v_1, \dots, u_kv_k \in M$ such that $u_i \in X$ and $v_i \in X^c$ for $1 \leq i \leq k$. Recall that $X = I_c \cup N(I_c)$ where I_c is a maximum critical independent set of G . Then since $u_i \in X$ we must have $u_i \in N(I_c)$ since if $u_i \in I_c$, then $v_i \in N(I_c)$ so we would have $v_i \in X$. Let M_X be the edges in M with both endpoints in X and M_{X^c} be the edges in M with both endpoints in X^c . Since I_c is an independent set and by Lemma 3.7 there is a matching from $N(I_c)$ into I_c , then M can contain at most $|N(I_c)|$ edges from $G[X]$. Since $u_i \in N(I_c)$, $v_i \in X^c$ and $u_iv_i \in M$ then $|M_X| = |N(I_c)| - k$. Let M' be a maximum matching of $G[X]$. Then $|M'| = |N(I_c)| = |M_X \cup \{u_1v_1, \dots, u_kv_k\}|$ and since edges in $G[X]$ are not adjacent to edges in $G[X^c]$, $M' \cup M_{X^c}$ is a matching of G . Furthermore, $|M' \cup M_{X^c}| = |M|$ and M', M_{X^c} only contain edges from $G[X], G[X^c]$ respectively. This contradicts the fact $\mu(G[X]) + \mu(G[X^c]) < \mu(G) = |M|$. Thus $\mu(G[X]) + \mu(G[X^c]) \geq \mu(G)$ and hence,

$$\mu(G[X]) + \mu(G[X^c]) \leq \mu(G) \leq \mu(G[X]) + \mu(G[X^c]).$$

Therefore, $\mu(G[X]) + \mu(G[X^c]) = \mu(G)$. □

Recall that the previous result on the core Theorem 2.6 has the condition $\alpha(G) > \mu(G)$. The following proposition shows that for any graph where we could apply Theorem 2.6, we can also apply our main result, Corollary 4.6, which requires $\alpha(G[X]) > \mu(G[X])$.

PROPOSITION 4.11. If G is a graph with $\alpha(G) > \mu(G)$, then $\alpha(G[X]) > \mu(G[X])$.

Proof. Suppose $\alpha(G) > \mu(G)$. By Corollary 3.13 (1) and Theorem 4.10, our assumption is equivalent to

$$\alpha(G[X]) + \alpha(G[X^c]) > \mu(G[X]) + \mu(G[X^c])$$

Subtracting $\mu(G[X^c])$ from each side of the inequality gives us

$$\alpha(G[X]) + \alpha(G[X^c]) - \mu(G[X^c]) > \mu(G[X]) \tag{4.6}$$

Since $G[X^c]$ is independence irreducible, $\mu(G[X^c]) \geq \alpha(G[X^c])$ so we have $0 \geq \alpha(G[X^c]) - \mu(G[X^c])$. Then

$$\alpha(G[X]) \geq \alpha(G[X]) + \alpha(G[X^c]) - \mu(G[X^c]) \quad (4.7)$$

Combining (4.6) and (4.7) we obtain $\alpha(G[X]) > \mu(G[X])$ as desired. \square

In addition to graphs where $\alpha(G) > \mu(G)$ holds, there is also a new class of graphs for which our main result holds and Theorem 2.6 does not. These graphs have $\alpha(G[X]) > \mu(G[X])$ but $\alpha(G) \leq \mu(G)$. See Figure 4.1 below for an example.

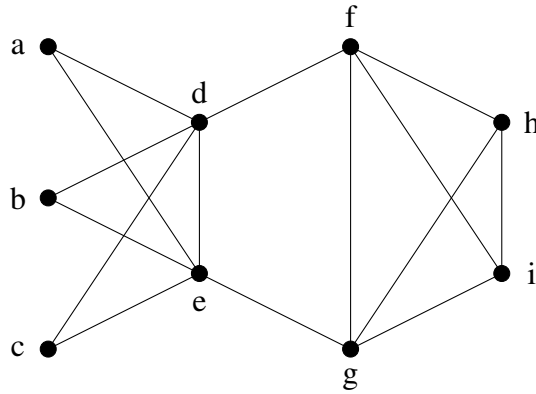


Figure 4.1: This graph has $\alpha(G) = 4 = \mu(G)$ so Theorem 2.6 does not apply, but $\alpha(G[X]) = 3 > 2 = \mu(G[X])$ so Corollary 4.6 applies and yields $\xi(G) \geq 2$.

By the above example we can see that our main result is an improvement to Theorem 2.6 in that it applies to a wider class of graphs. Here we will characterize the graphs for which our bound is strictly greater than the bound in Theorem 2.6, so graphs where $1 + \alpha(G[X]) - \mu(G[X]) > 1 + \alpha(G) - \mu(G)$. We can see graphs with $1 + \alpha(G[X]) - \mu(G[X]) > 1 + \alpha(G) - \mu(G)$ are exactly the graphs where $\alpha(G[X]) - \mu(G[X]) > \alpha(G) - \mu(G)$, for which the next proposition gives the following characterization.

PROPOSITION 4.12. For any graph G , $\alpha(G[X]) - \mu(G[X]) > \alpha(G) - \mu(G)$ if, and only if, $\mu(G[X^c]) > \alpha(G[X^c])$.

Proof. Suppose $\alpha(G[X]) - \mu(G[X]) > \alpha(G) - \mu(G)$. From Corollary 3.13 (1) and Theorem 4.10, we have

$$\alpha(G) - \mu(G) = \alpha(G[X]) + \alpha(G[X^c]) - \mu(G[X]) - \mu(G[X^c]). \quad (4.8)$$

Substituting (4.8) into our assumption we get

$$\alpha(G[X]) - \mu(G[X]) > \alpha(G[X]) - \mu(G[X]) + \alpha(G[X^c]) - \mu(G[X^c])$$

and simplifying this inequality yields $\mu(G[X^c]) > \alpha(G[X^c])$.

Conversely, suppose $\mu(G[X^c]) > \alpha(G[X^c])$. Then $\alpha(G[X^c]) - \mu(G[X^c]) < 0$ and since for any graph $\alpha(G[X]) - \mu(G[X]) \geq 0$ we get

$$\alpha(G[X]) - \mu(G[X]) + \alpha(G[X^c]) - \mu(G[X^c]) < \alpha(G[X]) - \mu(G[X]).$$

The left side of this inequality is equivalent to $\alpha(G) - \mu(G)$ by (4.8), hence we have

$$\alpha(G) - \mu(G) < \alpha(G[X]) - \mu(G[X]).$$

□

By the above proposition, our bound will be better for all graphs with $\alpha(G[X]) > \mu(G[X])$ and $\alpha(G[X^c]) < \mu(G[X^c])$. Such graphs exists as Figure 4.1 is an example and now we turn to examine how large the difference can be between our bound, $1 + \alpha(G[X]) - \mu(G[X])$ and the previous bound $1 + \alpha(G) - \mu(G)$.

For the remainder of this paper, let

$$T(G) = 1 + \alpha(G[X]) - \mu(G[X])$$

$$B(G) = 1 + \alpha(G) - \mu(G).$$

We also define the graph $H(r, s)$ to be the graph with a star of order r connected by a single edge from the center to any vertex of a complete graph of order s . See Figure 4.2 for an example as well as the general construction of the graph $H(r, s)$. These graphs were first recognized by Levit and Mandrescu in [8] as graphs where every relation between the sizes of $core(G)$ and $N(core(G))$ could be realized. Here we will use them to show that our bound $T(G)$ can be arbitrarily better than the previous bound, $B(G)$.

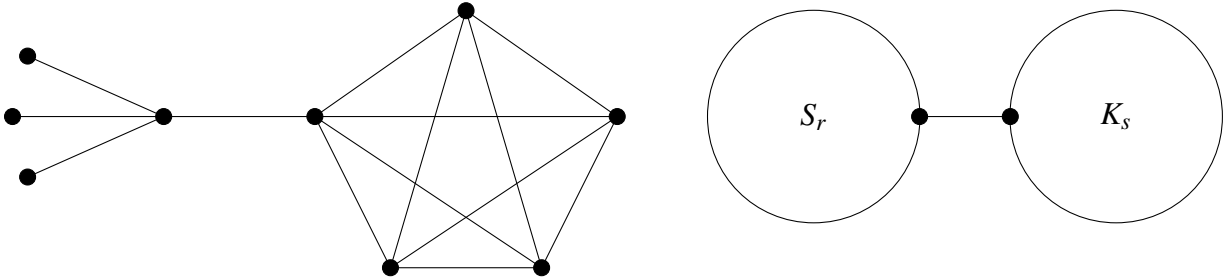


Figure 4.2: On the left is the graph $H(4,5)$ and on the right represents the general construction of the graph $H(r,s)$.

THEOREM 4.13. For any $n \in \mathbb{N}$, there exists a graph G such that $T(G) - B(G) \geq n$.

Proof. Let $n \in \mathbb{N}$ be given. Consider the graph $G = H(r, s)$. Then

$$B(G) = 1 + \alpha(G) - \mu(G) = 1 + r - (1 + \lfloor \frac{s}{2} \rfloor) = r - \lfloor \frac{s}{2} \rfloor$$

and,

$$T(G) = 1 + \alpha(G[X]) - \mu(G[X]) = 1 + (r-1) - 1 = r-1.$$

So $T(G) - B(G) = \lfloor \frac{s}{2} \rfloor - 1$. Since a necessary condition for $B(G)$ is that $\alpha(G) > \mu(G)$, then we must have $r > 1 + \lfloor \frac{s}{2} \rfloor$. Choose $s = 2n + 2$ and any $r > n + 2$. Then

$$T(G) - B(G) = \lfloor \frac{s}{2} \rfloor - 1 = \lfloor \frac{2n+2}{2} \rfloor - 1 = n.$$

Since n was arbitrary we have $T(G) - B(G) \geq n$ for any $n \in \mathbb{N}$. Therefore, for any $n \in \mathbb{N}$ there exists a graph G such that $T(G) - B(G) \geq n$. \square

Other Results

In addition to results on the core an attempt was made to characterize graphs where the independence number equals the matching number. Also during our investigation we stumbled upon some unrelated results which are presented at the end of the chapter.

5.1 Graphs with Independence Number Equal to Matching Number

Recall Corollary 3.13 allows us to decompose any graph into a KE induced subgraph and an II induced subgraph. For KE graphs, the relation $\alpha \geq \mu$ always holds and for II graphs, the relation $\alpha \leq \mu$ holds. If we can characterize for what KE graphs does $\alpha = \mu$ and for what II graphs $\alpha = \mu$, then using Corollary 3.13 we could possibly give a class of graphs where $\alpha = \mu$ for the entire graph.

For a KE graph G , the independence number equals the matching number whenever G has a perfect matching. To verify this, if G has a perfect matching, then $\mu(G) = \frac{n}{2}$ and we know $n = \alpha(G) + \mu(G) = \alpha(G) + \frac{n}{2}$ so $\alpha(G) = \frac{n}{2} = \mu(G)$. Conversely, if $\alpha(G) = \mu(G)$, then $n = \alpha(G) + \mu(G) = 2\alpha(G)$ so $\frac{n}{2} = \alpha(G) = \mu(G)$.

Characterizing the II graphs for which $\alpha = \mu$ is not as easy since all we know for II graphs is $|I| < |N(I)|$ holds for any non-empty independent set I , but the following result may prove to be useful in our attempts.

To prove this result we will make use of Berge's Theorem stated as Theorem 5.1 below. For a matching M in a graph G , an M -alternating path is a path whose edges are alternately in

M and not in M . An M -augmenting path is an M -alternating path both of whose end-vertices are not in $V(M)$.

THEOREM 5.1. (Theorem 9.2 in [4]) A matching M in a graph G is a maximum matching if, and only if, there exists no M -augmenting path in G .

Now onto our potentially useful result on II graphs. With Theorem 5.2 below, we can begin to characterize the II graphs for which every edge of a maximum matching is incident to a vertex in a maximum independent set, and these are precisely the graphs where $\alpha = \mu$.

THEOREM 5.2. If G is II and I is a maximum independent set, then there is a maximum matching M such that I is matched under M .

Proof. Suppose G is II, I is a maximum independent set and every maximum matching does not match all the vertices in I . Let M be a maximum matching that matches the largest number of vertices of I . Let $U \subseteq I$ be the vertices in I not matched by M . Let N be the vertices matched to the vertices in I under M . Let $E = V(G) - (I \cup N)$. Note that no vertex in U is adjacent to a vertex in E , for if so, then M could either be extended or adjusted to match more vertices in I . Let J be the vertices that can be reached by an M -alternating path beginning with a vertex in U . Let $J_N = J \cap N$ and $J_I = J \cap (I - U)$. Then $J = J_N \cup J_I$. By construction $|J_I| = |J_N|$ and no vertex in J_I is adjacent to a vertex in E , since if so, either M could be extended if there is an M -alternating path or M can be adjusted to match a vertex in U which contradicts our assumption that M matches the largest number of vertices in I . So $N(J_I) \subseteq N$ and since G is II and J_I is an independent set, $|N(J_I)| > |J_I|$. Now $N(J_I)$ in N must be in J_N by construction, so $N(J_I) = J_N$. Then $|N(J_I)| = |J_N|$ so $|N(J_I)| = |J_I|$, which is a contradiction. Therefore I is matched under M . \square

After looking at many examples of II graphs where $\alpha = \mu$, we noticed if a certain vertex was removed, then the graph became KE. We define the following types of graphs in an

attempt to characterize $\alpha = \mu$ in Π graphs.

DEFINITION 5.3. A graph G is almost KE if G is not KE and there exists $v \in V(G)$ such that $G - v$ is KE.

DEFINITION 5.4. A graph G is strongly almost KE if G is not KE and for all $v \in V(G)$ such that $G - v$ is KE.

Note that if a graph is strongly almost KE, then the graph is almost KE. The following theorem characterizes an property of KE graphs that will be helpful to us in our efforts.

THEOREM 5.5. If G is almost KE with vertex $v \in V(G)$ such that $G - v$ is KE, then $\alpha(G - v) = \alpha(G)$ and $\mu(G - v) = \mu(G)$.

Proof. Suppose G is almost KE with vertex $v \in V(G)$ such that $G - v$ is KE. Then $\alpha(G - v) + \mu(G - v) = n - 1$. Since $G - v$ is an induced subgraph of G , $\alpha(G - v) \leq \alpha(G)$ and $\mu(G - v) \leq \mu(G)$. Then

$$\alpha(G - v) + \mu(G - v) = n - 1 \leq \alpha(G) + \mu(G) < n.$$

Thus

$$\alpha(G - v) + \mu(G - v) = \alpha(G) + \mu(G) = n - 1$$

and since $\alpha(G - v) \leq \alpha(G)$, $\mu(G - v) \leq \mu(G)$ we must have $\alpha(G - v) = \alpha(G)$ and $\mu(G - v) = \mu(G)$. \square

Recall that we already know for KE graphs the independence number equals the matching number whenever the graph has a perfect matching. This idea along with Proposition 5.5 leads us to the following characterization of almost KE graphs with independence equal to the matching number.

THEOREM 5.6. Let G be almost KE with vertex $v \in V(G)$ such that $G - v$ is KE. Then $\alpha(G) = \mu(G)$ if, and only if $G - v$ has a perfect matching.

Proof. Suppose $\alpha(G) = \mu(G)$. Since G is almost KE, by Proposition 5.5 $\alpha(G - v) = \alpha(G) = \mu(G) = \mu(G - v)$. Since G is almost KE, $\alpha(G - v) + \mu(G - v) = n - 1$ and since $\alpha(G - v) = \mu(G - v)$,

$$\alpha(G - v) + \mu(G - v) = 2\mu(G - v) = n - 1.$$

Thus $2\mu(G - v) = n - 1$ or equivalently, $\mu(G - v) = \frac{n-1}{2}$ and hence, $G - v$ has a perfect matching.

Conversely, suppose $G - v$ has a perfect matching. So $\mu(G - v) = \frac{n-1}{2}$ and since $G - v$ is KE $\alpha(G - v) + \mu(G - v) = n - 1$. Then

$$\alpha(G - v) + \mu(G - v) = \alpha(G - v) + \frac{n-1}{2} = n - 1.$$

So $\alpha(G - v) + \frac{n-1}{2} = n - 1$ or equivalently, $\alpha(G - v) = \frac{n-1}{2}$. Thus $\alpha(G - v) = \frac{n-1}{2} = \mu(G - v)$ so by Proposition 5.5 $\alpha(G) = \alpha(G - v) = \mu(G - v) = \mu(G)$. Therefore $\alpha(G) = \mu(G)$. \square

Now that we've characterized when the independence number equals the matching number for almost KE graphs, what's missing is the relation between almost KE graphs and II graphs. It turns out that some graphs are almost KE and not II, for example the graph in Figure 3.1.

Although we were unable to prove this, we conjectured that if a graph G is strongly almost KE, then G is II. Note, the converse is not true since a complete graph with order $n > 3$ is a counterexample. If this conjecture is true then by Theorem 5.6 we will have

characterized when the independence number equals the matching number for a proper subset of Π graphs.

5.2 More Results

Although unrelated to our earlier efforts in this chapter, an interesting property follows about the *core* of strongly almost KE graphs.

PROPOSITION 5.7. If G is strongly almost KE, then $\xi(G) = 0$.

Proof. Let G be strongly almost KE. Then $G - v$ is KE for all $v \in V(G)$. So by Proposition 5.5, $\alpha(G - v) = \alpha(G)$ for all $v \in V(G)$. We will show by contradiction that $\xi(G) = 0$. Suppose $\xi(G) > 0$. Then there exists some vertex $u \in \text{core}(G)$. Since $u \in \text{core}(G)$ and $G - u$ is an induced subgraph of G , $\alpha(G - u) < \alpha(G)$. This contradicts the fact that $\alpha(G - v) = \alpha(G)$ for all $v \in V(G)$. Therefore $\xi(G) = 0$. \square

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Vita

Taylor Short was born 1985 in Richmond, Virginia. Entering high school he was accepted to the region's Math and Science Specialty Center. He went on to attend the College of William and Mary where he began preparing for medical school. Eventually his interests led him to major in mathematics and he graduated in 2008 with the new intention of pursuing mathematics professionally. In 2009, Taylor began work on his masters degree in pure mathematics at Virginia Commonwealth University as further preparation for a Ph.D. program. He hopes to begin working toward his doctorate in mathematics beginning in the fall of 2011.

When not doing mathematics Taylor enjoys playing basketball, working out and cooking. On weekends Taylor spends time with old friends still in Richmond and his family. He also enjoys making trips up and down the east coast visiting college friends.