2011

Riemann Stieltjes Integration

Colleen McFadden
Virginia Commonwealth University

Riemann Stieltjes Integration

A Thesis submitted in partial fulfillment of the requirements for the degree of Master of Mathematics at Virginia Commonwealth University.

by

Colleen Cook McFadden

Master of Mathematics

Director: Lon Mitchell, Assistant Professor

Department of Mathematics and Applied Mathematics

Virginia Commonwealth University

Richmond, Virginia

May 2011

Acknowledgment

I would like to thank Dr. Lon Mitchell for all of his help and guidance as my thesis advisor and professor. I appreciate his patience and willingness to help and the fact that his guidance, while prompting me in the right direction, allowed me to enjoy the success of forming a conclusion of my own. I would, also, like to thank all of my professors at Virginia Commonwealth University who have given me such a strong education in mathematics and shown the enthusiasm I hope to demonstrate in my classroom. I am grateful for the understanding and support my family has given me while I have worked to complete my thesis and degree.

Contents

Introduction

Provided in the following pages is an expository thesis on Riemann-Stieltjes Integration. The goals of this thesis are to define the Riemann-Stieltjes Integral, and explain the properties of this integral. There are many proofs provided in this paper, some of which are original proofs, some are modifications of proofs of similar properties and others are from the sources used to prepare this paper. All of these proofs will help explain to the reader the calculation, limitations and applications of the Riemann-Stieltjes Integral. The process of Riemann Integration which is taught in calculus classes is a specific case of Riemann-Stieltjes Integration, thus many of the same terms and properties used to describe Riemann Integration will be discussed in this paper. Riemann-Stieltjes integration is useful in the areas of Physics, and Statistics, but of limited use in Stochastic Processes.

Definitions

The following definitions will be used to define and explain Riemann-Stieltjes Integration or in the properties and proofs of Riemann-Stieltjes Integration.

Let $[a,b]$, $a < b$, be a given closed and bounded interval in $\mathbb{R}$. A **partition** $\mathcal{P}$ of $[a,b]$ is a finite set of points $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < \ldots < x_n = b$. There is no requirement that the points x_i be equally spaced.

A partition $\mathcal{P}^*$ of $[a,b]$ is a **refinement** of $\mathcal{P}$ if $\mathcal{P} \subset \mathcal{P}^*$

The **norm of a partition** $\mathcal{P}$ is the length of the largest subinterval of $\mathcal{P}$ and is denoted by $\|\mathcal{P}\|$.

Let E be a nonempty subset of $\mathbb{R}$ that is bounded above. An element $\alpha \in \mathbb{R}$ is called the **least upper bound or supremum** of E if

(i) α is an upper bound of E, and

(ii) if $\beta \in \mathbb{R}$ satisfies $\beta < \alpha$, then β is not an upper bound of E.

Let E be a nonempty subset of $\mathbb{R}$ that is bounded below. An element $\gamma \in \mathbb{R}$ is called the **greatest lower bound or infimum** of E if

(i) γ is a lower bound of E, and

(ii) if $\beta \in \mathbb{R}$ satisfies $\beta > \gamma$, then β is not a lower bound of E.

A real-valued function f defined on a set E is **bounded** on E if there exists a constant M such that $|f(x)| \leq M$ for all $x \in E$.

A subset O of $\mathbb{R}$ is open if every point of O is an interior point of O. A subset F of $\mathbb{R}$ is **closed** if F^c is open.

Let $p \in \mathbb{R}$ and let $\delta > 0$. The set $N_\delta(p) = \{x \in \mathbb{R} : |x-p| < \delta\}$ is called a **δ-neighborhood of the point p**.

Suppose $E \subset \mathbb{R}$ and f is a real-valued function with domain E. The function f has a **local minimum** at a point $q \in E$ if there exists a $\delta > 0$ such that $f(q) \leq f(x)$ for all $x \in E \cap N_\delta(p)$.

Let f be a real-valued function defined on an interval I, f is **monotone increasing** on I if $f(x) \leq f(y)$ for all $x, y \in I$ with $x < y$.

Let E be a subset of $\mathbb{R}$ and f a real-valued function with domain E. The function f is continuous at a point $p \in E$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ for all $x \in E$ with $|x-p| < \delta$. The function f is **continuous on** E if and only if f is continuous at every point $p \in E$.

Let E be a subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$. The function f is **uniformly continuous** on E, if given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in E$ with $|x-y| < \delta$.

Riemann-Stieltjes Integration vs. Riemann Integration

Presented here is the definition and notation of the Riemann-Stieltjes Integral. Let α be a monotone increasing function on $[a,b]$, and let f be a bounded real-valued function on $[a,b]$. For each partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ set

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}), i = 1, \ldots, n.$$

Since α is monotone increasing, $\Delta\alpha_i \geq 0$ for all i. Let

$$m_i = \inf\{f(t) : t \in [x_{i-1}, x_i]\},$$

$$M_i = \sup\{f(t) : t \in [x_{i-1}, x_i]\}.$$

The upper Riemann-Stieltjes sum of f with respect to α and the partition $\mathcal{P}$, is defined by

$$\mathcal{U}(\mathcal{P}, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i.$$

The lower Riemann-Stieltjes sum of f with respect to α and the partition $\mathcal{P}$, is defined by

$$\mathcal{L}(\mathcal{P}, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i.$$

By the given conditions and defintions $m_i \leq M_i$ and $\Delta\alpha_i \geq 0$, we know

$$\mathcal{L}(\mathcal{P}, f, \alpha) \leq \mathcal{U}(\mathcal{P}, f, \alpha).$$

Let $\mathcal{P}$ be any partition of $[a,b]$. Then the following sum can be found

$$\sum_{i=1}^{n} \Delta\alpha_i = (\alpha(x_1) - \alpha(x_0)) + (\alpha(x_2) - \alpha(x_1)) + \ldots + (\alpha(x_n) - \alpha(x_{n-1}))$$

$$= \alpha(x_n) - \alpha(x_0)$$

$$= \alpha(b) - \alpha(a).$$

Since $M_i \leq M$ for all i and $\Delta\alpha_i \geq 0$,

$$\sum_{i=1}^{n} M_i \Delta\alpha_i \leq \sum_{i=1}^{n} M \Delta\alpha_i = M \sum_{i=1}^{n} \Delta\alpha_i = M[\alpha(a) - \alpha(b)],$$

$$\sum_{i=1}^{n} m_i \Delta\alpha_i \geq \sum_{i=1}^{n} m \Delta\alpha_i = m \sum_{i=1}^{n} \Delta\alpha_i = m[\alpha(a) - \alpha(b)].$$

Therefore

$$\mathcal{U}(\mathcal{P}, f, \alpha) \leq M[\alpha(a) - \alpha(b)]$$

and

$$\mathcal{L}(\mathcal{P}, f, \alpha) \geq m[\alpha(a) - \alpha(b)].$$

Thus, if $m \leq f(x) \leq M$ for all $x \in [a,b]$, then

$$m[\alpha(b) - \alpha(a)] \leq \mathcal{L}(\mathcal{P}, f, \alpha) \leq \mathcal{U}(\mathcal{P}, f, \alpha) \leq M[\alpha(b) - \alpha(a)]$$

for all partitions $\mathcal{P}$ of $[a,b]$.

The upper integral of f with respect to α over $[a,b]$, denoted $\overline{\int_a^b} f\,\mathrm{d}\alpha$ and the lower integral of f with respect to α over $[a,b]$, denoted $\underline{\int_a^b} f\,\mathrm{d}\alpha$ are defined

$\overline{\int_a^b} f\,\mathrm{d}\alpha = \inf\{\mathcal{U}(\mathcal{P},f,\alpha) : \mathcal{P} \text{ is a partition of [a,b]}\}$,

$\underline{\int_a^b} f\,\mathrm{d}\alpha = \sup\{\mathcal{L}(\mathcal{P},f,\alpha) : \mathcal{P} \text{ is a partition of [a,b]}\}$. A function f is said to be Riemann-Stieltjes integrable with respect to α on $[a,b]$ if

$$\overline{\int_a^b} f\,\mathrm{d}\alpha = \underline{\int_a^b} f\,\mathrm{d}\alpha$$

when f is a bounded real-valued function on $[a,b]$, and α is a monotone increasing function on $[a,b]$.

The definition of the Riemann Integral is reviewed here to allow the Riemann-Stieltjes and Riemann Integrals to be compared.

Let $[a,b]$, $a<b$, be a given closed and bounded interval in $\mathbb{R}$. Let $\mathcal{P}$ be a partition of $[a,b]$. Given

$$\Delta x_i = x_i - x_{i-1},$$

which is equal to the length of the interval $[x_{i-1},x_i]$. Let

$$m_i = \inf\{f(t) : t \in [x_{i-1},x_i]\},$$

$$M_i = \sup\{f(t) : t \in [x_{i-1},x_i]\}.$$

Since f is bounded, by the least upper bound property, the quantites m_i and M_i exist in $\mathbb{R}$. If f is a continuous function on $[a,b]$, then for each i there exist points $t_i, s_i \in [x_{i-1},x_i]$ such that $M_i = f(t_i)$ and $m_i = f(s_i)$. The upper sum $\mathcal{U}(\mathcal{P},f)$ for the partition $\mathcal{P}$ and function f is defined by

$$\mathcal{U}(\mathcal{P}, f) = \sum_{i=1}^{n} M_i \Delta x_i.$$

The lower sum for the partition $\mathcal{P}$ and function f is defined by

$$\mathcal{L}(\mathcal{P}, f) = \sum_{i=1}^{n} m_i \Delta x_i.$$

Since $m_i \leq M_i$ for all $i = 1, \ldots, n$, we know $\mathcal{L}(\mathcal{P}, f) \leq \mathcal{U}(\mathcal{P}, f)$ for any partition of $\mathcal{P}$ of $[a,b]$. Let $\mathcal{P}$ be any partition of $[a,b]$. Since $M_i \leq M$ and $m_i \geq m$ for all $i = 1, \ldots, n$,

$$\mathcal{U}(\mathcal{P}, f) = \sum_{i=1}^{n} M_i \Delta x_i \leq \sum_{i=1}^{n} M(x_i - x_{i-1}) = M(b-a).$$

$$\mathcal{L}(\mathcal{P}, f) = \sum_{i=1}^{n} m_i \Delta x_i \geq \sum_{i=1}^{n} m(x_i - x_{i-1}) = m(b-a).$$

Thus, if $m \leq f(t) \leq M$ for all $t \in [a,b]$, then

$$m(b-a) \leq \mathcal{L}(\mathcal{P}, f) \leq \mathcal{U}(\mathcal{P}, f) \leq M(b-a)$$

for all partitions $\mathcal{P}$ of [a,b].

The Riemann-Stieltjes Integral is a modification of the Riemann Integral where the function f is integrated with respect to a function α instead of with respect to x, which means in the Riemann-Stieltjes Integral $\Delta\alpha$ is used versus the use of Δx for the Riemann Integral. When evaluating the Riemann-Stieltjes Integral the upper and lower sums are found by multiplying M_i and m_i, respectively, by $\Delta\alpha_i$ where

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

To evaluate the Riemann Integral the upper and lower sums are found by multiplying M_i and m_i by Δx_i. When $\alpha(x)$ is defined to equal x then the Riemann-Stieltjes Integral and the Riemann Integral are equivalent. Let $\alpha(x) = x$ then $\alpha(x_i) = x_i$ and $\alpha(x_{i-1}) = x_{i-1}$. Therefore

$$\Delta\alpha(x_i) = \alpha(x_i) - \alpha(x_{i-1}) = x_i - x_{i-1} = \Delta x_i.$$

Properties and Theorems of Riemann-Stieltjes Integration

Theorem 4.1 ([4])**.** Let f be a bounded real-valued function on $[a,b]$, and α a monotone increasing function on $[a,b]$. Then

$$\underline{\int_a^b} f\,\mathrm{d}\alpha \leq \overline{\int_a^b} f\,\mathrm{d}\alpha.$$

Proof. Given that $\mathcal{P}^*$ is a refinement of the partition $\mathcal{P}$ then

$$\mathcal{L}(\mathcal{P},f,\alpha) \leq \mathcal{L}(\mathcal{P}^*,f,\alpha) \leq \mathcal{U}(\mathcal{P}^*,f,\alpha) \leq \mathcal{U}(\mathcal{P},f,\alpha)$$

Let $\mathcal{P}$ and $\mathcal{Q}$ be any two partitions of $[a,b]$ where

$$\mathcal{P} \neq \mathcal{Q}.$$

Then $\mathcal{P} \cup \mathcal{Q}$ is a refinement of partitions $\mathcal{P}$ and $\mathcal{Q}$. By the above inequality

$$\mathcal{L}(\mathcal{P},f,\alpha) \leq \mathcal{L}(\mathcal{P}\cup\mathcal{Q},f,\alpha) \leq \mathcal{U}(\mathcal{P}\cup\mathcal{Q},f,\alpha) \leq \mathcal{U}(\mathcal{Q},f,\alpha)$$

Thus

$$\mathcal{L}(\mathcal{P},f,\alpha) \leq \mathcal{U}(\mathcal{Q},f,\alpha)$$

for any partitions $\mathcal{P}, \mathcal{Q}$. Hence

$$\underline{\int_a^b} f \,\mathrm{d}\alpha = \sup\{\mathcal{L}(\mathcal{P}, f, \alpha)\} \leq \mathcal{U}(\mathcal{Q}, f, \alpha)$$

for any partition $\mathcal{Q}$. And

$$\overline{\int_a^b} f \,\mathrm{d}\alpha = \inf\{\mathcal{U}(\mathcal{Q}, f, \alpha)\} \geq \mathcal{L}(\mathcal{P}, f, \alpha)$$

Therefore

$$\underline{\int_a^b} f \,\mathrm{d}\alpha \leq \overline{\int_a^b} f \,\mathrm{d}\alpha.$$

$\square$

Example 4.2. Fix $a < c \leq b$. Let $I_c(x) = I(x-c)$ be the indicator function at c defined by

$$I_c(x) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}$$

If f is a bounded real-valued function on $[a,b]$ that is continuous at c, $a < c \leq b$, then f is integrable with respect to I_c and

$$\int_a^b f \,\mathrm{d}I_c = \int_a^b f(x) \,\mathrm{d}I(x-c) = f(c)$$

Proof. Let $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a,b]$. Since $a < c \leq b$, there exists an index k, $1 \leq k \leq n$, such that $x_{k-1} < c \leq x_k$. Thus

$$\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1}) = 1 - 0 = 1$$

and $\Delta\alpha_i = 0$ for all $i \neq k$. Therefore

$$\mathcal{U}(\mathcal{P}, f, \alpha) = M_k \Delta_k = M_k = \sup\{f(t) : x_{k-1} \leq t \leq x_k\}$$

and

$$\mathcal{L}(\mathcal{P}, f, \alpha) = m_k \Delta_k = m_k = \inf\{f(t) : x_{k-1} \leq t \leq x_k\}$$

Since f is continuous at c, given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$f(c) - \varepsilon < f(t) < f(c) + \varepsilon$$

for all $t \in [a,b]$ with $|t - c| < \delta$. If $\mathcal{P}$ is any partition of $[a,b]$ with

$$\|\mathcal{P}\| < \delta$$

then

$$f(c) - \varepsilon \leq m_k \leq M_k \leq f(c) + \varepsilon$$

Therefore

$$f(c) - \varepsilon \leq \mathcal{L}(\mathcal{P}, f, \alpha) \leq \mathcal{U}(\mathcal{P}, f, \alpha) \leq f(c) + \varepsilon$$

As a consequence

$$f(c) - \varepsilon \leq \underline{\int_a^b} f \,\mathrm{d}\alpha \leq \overline{\int_a^b} f \,\mathrm{d}\alpha \leq f(c) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary the upper and lower integrals of f are equal and thus f is integrable with respect to α on $[a,b]$ with

$$\int_a^b f \, \mathrm{d}\alpha = f(c).$$

□

Example 4.3. The function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

is not integrable with respect to any nonconstant monotone increasing function α.

Proof. Suppose α is monotone increasing on $[a,b]$, $a < b$, with $\alpha(a) \neq \alpha(b)$. If $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ is any partition of $[a,b]$ then $m_i = 0$ and $M_i = 1$, since the rational numbers are dense, for all $i = 1, 2, \ldots, n$. Therefore

$$\mathcal{L}(\mathcal{P}, f, \alpha) = 0$$

and

$$\mathcal{U}(\mathcal{P}, f, \alpha) = \sum_{i=1}^{n} \Delta\alpha_i = \alpha(b) - \alpha(a).$$

Thus f is not integrable with respect to alpha. □

Theorem 4.4 ([4])**.** Let α be a monotone increasing function on $[a,b]$. A bounded real-valued function f is Riemann-Stieltjes integrable with respect to α on $[a,b]$ if and only if for all $\varepsilon > 0$ there exists a $\mathcal{P}$ of $[a,b]$ such that

$$\mathcal{U}(\mathcal{P}, f, \alpha) - \mathcal{L}(\mathcal{P}, f, \alpha) < \varepsilon.$$

Furthermore if $\mathcal{P}$ is a partition of $[a,b]$ for the which the aboved stated inequality holds true, then the inequality holds for all refinements of $\mathcal{P}$.

Proof. Suppose

$$\mathcal{U}(\mathcal{P},f,\alpha)-\mathcal{L}(\mathcal{P},f,\alpha)<\varepsilon$$

holds for a given $\varepsilon>0$. Then

$$0\leq\overline{\int_a^b}f\,\mathrm{d}\alpha-\underline{\int_a^b}f\,\mathrm{d}\alpha\leq\mathcal{U}(\mathcal{P},f,\alpha)-\mathcal{L}(\mathcal{P},f,\alpha)<\varepsilon.$$

Thus f is integrable on $[a,b]$. Suppose f is integrable on $[a,b]$. Let $\varepsilon>0$. There exist partitions $\mathcal{P}_1$, and $\mathcal{P}_2$ of $[a,b]$ such that

$$\mathcal{U}(\mathcal{P}_2,f,\alpha)-\int_a^b f\,\mathrm{d}\alpha<\varepsilon/2$$

and

$$\int_a^b f\,\mathrm{d}\alpha-\mathcal{L}(\mathcal{P}_1,f,\alpha)<\varepsilon/2.$$

Let

$$\mathcal{P}=\mathcal{P}_1\cup\mathcal{P}_2.$$

Then

$$\mathcal{U}(\mathcal{P},f,\alpha)\leq\mathcal{U}(\mathcal{P}_2,f,\alpha)<\int_a^b f\,\mathrm{d}\alpha+\varepsilon/2<\mathcal{L}(\mathcal{P}_1,f,\alpha)+\varepsilon\leq\mathcal{L}(\mathcal{P},f,\alpha)+\varepsilon$$

Therefore

$$\mathcal{U}(\mathcal{P},f,\alpha)-\mathcal{L}(\mathcal{P},f,\alpha)<\varepsilon.$$

If $\mathcal{Q}$ is any refinement of $\mathcal{P}$ then

$$\mathcal{U}(\mathcal{Q}, f, \alpha) \leq \mathcal{U}(\mathcal{P}, f, \alpha)$$

and

$$\mathcal{L}(\mathcal{P}, f, \alpha) \leq \mathcal{L}(\mathcal{Q}, f, \alpha)$$

Therefore

$$0 \leq \mathcal{U}(\mathcal{Q}, f, \alpha) - \mathcal{L}(\mathcal{Q}, f, \alpha) \leq \mathcal{U}(\mathcal{P}, f, \alpha) - \mathcal{L}(\mathcal{P}, f, \alpha) < \varepsilon$$

Thus the inequality holds for any refinement $\mathcal{Q}$ of $\mathcal{P}$. $\square$

Lemma: If both $f(x)$ and $\alpha(x)$ are discontinuous from the same side at a point c then the functions are not Riemann-Stieltjes integrable with respect to α for the interval $[a,b]$ where $c \in [a,b]$

Proof. By contradiction. Suppose that f is Reimann-Stieltjes integrable with respect to α. By Theorem 4.4, given a fixed $\varepsilon > 0$ there exists a partition $\mathcal{P}_1$ of $[a,b]$ such that

$$\mathcal{U}(\mathcal{P}_1, f, \alpha) - \mathcal{L}(\mathcal{P}_1, f, \alpha) < \varepsilon$$

$\mathcal{P}_1$ is a partition of $[a,b]$ and $c \in [a,b]$, let $\mathcal{P}^* = \mathcal{P}_1 \cup c$. Thus $\mathcal{P}^*$ is a refinement of $\mathcal{P}_1$. Choose $\varepsilon_f > 0$ such that for all $\delta_f > 0$ there is an x_f such that

$$|x_f - c| < \delta_f \quad \text{and} \quad |f(x_f) - f(c)| \geq \sqrt{\varepsilon_f}.$$

Choose $\varepsilon_\alpha > 0$ such that for all $\delta_\alpha > 0$ there is an x_α such that

$$|x_\alpha - c| < \delta_\alpha \quad \text{and} \quad |\alpha(x_\alpha) - \alpha(c)| \geq \sqrt{\varepsilon_\alpha}.$$

Since $a < c < b$, there exists an index k, $1 \leq k \leq n$, such that

$$x_{k-1} < c < x_k$$

Let $\varepsilon^* = \inf\{\varepsilon_f, \varepsilon_\alpha\}$ Choose $\varepsilon = \varepsilon^*$ for $\delta^* = \min(x_n - c, c - x_{n-1})$ so there is an x^* such that $|x_f^* - c| < \delta^*$ and $|f(x_f^*) - f(c)| \geq \sqrt{\varepsilon^*}$ and $|x_\alpha^* - c| < \delta^*$ and $|\alpha(x_\alpha^*) - \alpha(c)| \geq \sqrt{\varepsilon^*}$. Therefore

$$\mathcal{U}(\mathcal{P}^*, f, \alpha) - \mathcal{L}(\mathcal{P}^*, f, \alpha) < \varepsilon.$$

By definition

$$\mathcal{U}(\mathcal{P}^*, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$

and

$$\mathcal{L}(\mathcal{P}^*, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

which means

$$\sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i < \varepsilon$$

Thus

$$\sum_{i=1}^{n} M_i \Delta\alpha_i = \sum_{i=1}^{k-1} M_i \Delta\alpha_i + M_k \Delta\alpha_k + \sum_{i=k}^{n} M_i \Delta\alpha_i$$

and

$$\sum_{i=1}^{n} m_i \Delta\alpha_i = \sum_{i=1}^{k-1} m_i \Delta\alpha_i + m_k \Delta\alpha_k + \sum_{i=k}^{n} m_i \Delta\alpha_i$$

Therefore

$$\left(\sum_{i=1}^{k-1} M_i \Delta\alpha_i + M_k \Delta\alpha_k + \sum_{i=k}^{n} M_i \Delta\alpha_i\right) - \left(\sum_{i=1}^{k-1} m_i \Delta\alpha_i + m_k \Delta\alpha_k + \sum_{i=k}^{n} m_i \Delta\alpha_i\right) < \varepsilon$$

Where $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$ since α is monotone increasing $|\alpha(x_k) - \alpha(x_{k-1})| \geq |\alpha(x^*) - \alpha(c)|$. The upper sums minus the lower sums is positive due to the fact that the upper sums are greater than the lower sums, which was shown in the previous proof.

$$\sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i \geq M_k \Delta\alpha_k - m_k \Delta\alpha_k = \Delta\alpha_k (M_k - m_k) \geq \sqrt{\varepsilon^*}\sqrt{\varepsilon^*} \geq \varepsilon$$

Which is a contradiction, thus f is not integrable with respect to α when both f and α are not continuous from the same side at c. □

Theorem 4.5 ([4])**.** Let f be a real-valued function on $[a,b]$ and α a monotone increasing function on $[a,b]$.

a) If f is continuous on $[a,b]$, then f is integrable with respect to α on $[a,b]$.

b) If f is monotone on $[a,b]$ and α is continuous on $[a,b]$, then f is integrable with respect to α on $[a,b]$.

Proof. Proof of part (a) Let $\varepsilon > 0$. Choose $\eta > 0$ such that

$$[\alpha(b) - \alpha(a)]\eta < \varepsilon.$$

The function f is closed and bounded on $[a,b]$ and f is continuous on $[a,b]$ then by the Uniform Continuity Theorem f is uniformly continuous on $[a,b]$. Thus there exists a η such that $|f(x) - f(t)| < \eta$ for all x, t in $[a,b]$ with $|x - t| < \eta$. Choose a partition $\mathcal{P}$ of $[a,b]$ such that $\Delta x_i < \eta$ for all $i = 1, 2, \ldots, n$. Then by the inequality

$$|f(x) - f(t)| < \eta$$

$M_i - m_i \leq \eta$ for all $i = 1, 2, \ldots, n$. Therefore

$$\mathcal{U}(\mathcal{P}, f, \alpha) - \mathcal{L}(\mathcal{P}, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i \leq \eta \sum_{i=1}^{n} \Delta\alpha_i = \eta(\alpha(b) - \alpha(a)) < \varepsilon$$

For this n and corresponding partition $\mathcal{P}$,

$$\mathcal{U}(\mathcal{P}, f, \alpha) - \mathcal{L}(\mathcal{P}, f, \alpha) < \varepsilon.$$

Therefore f is integrable on $[a,b]$. $\square$

Proof. Proof of part (b) For any positive integer n choose a partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = \frac{1}{n}[\alpha(b) - \alpha(a)]$$

Since α is continuous, such a choice is possible by the intermediate value theorem. Assume f is monotone increasing on $[a,b]$. Then

$M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Therefore

$$\begin{aligned}
\mathcal{U}(\mathcal{P}, f, \alpha) - \mathcal{L}(\mathcal{P}, f, \alpha) &= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \Delta\alpha_i \\
&= \frac{[\alpha(b) - \alpha(a)]}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\
&= \frac{[\alpha(b) - \alpha(a)]}{n} [f(b) - f(a)]
\end{aligned}$$

Given $\varepsilon > 0$, choose $n \in \mathcal{N}$ such that

$$\frac{[\alpha(b) - \alpha(a)]}{n} [f(b) - f(a)] < \varepsilon.$$

For this n and corresponding partition $\mathcal{P}$,

$$\mathcal{U}(\mathcal{P}, f, \alpha) - \mathcal{L}(\mathcal{P}, f, \alpha) < \varepsilon.$$

□

Properties of the Riemann-Stieltjes Integral are given below.

For a given monotone increasing function α on $[a,b]$, $\mathcal{R}(\alpha)$ denotes the set of bounded real-valued functions f on $[a,b]$ that are Riemann-Stieltjes integrable with respect to α.

Theorem 4.6 ([4])**.** 1. If $f,g \in \mathcal{R}(\alpha)$ the sum $f+g$ and cf are in $\mathcal{R}(\alpha)$ for every $c \in \mathbb{R}$ and

   (a) $\int_a^b (f+g)\,\mathrm{d}\alpha = \int_a^b f\,\mathrm{d}\alpha + \int_a^b g\,\mathrm{d}\alpha$

   (b) $\int_a^b cf\,\mathrm{d}\alpha = c\int_a^b f\,\mathrm{d}\alpha$

2. If $f \in \mathcal{R}(\alpha_i), i = 1,2$ then $f \in \mathcal{R}(\alpha_1 + \alpha_2)$ and

$$\int_a^b f\,\mathrm{d}(\alpha_1+\alpha_2) = \int_a^b f\,\mathrm{d}\alpha_1 + \int_a^b f\,\mathrm{d}\alpha_2$$

3. If $f \in \mathcal{R}(\alpha)$ and $a < c < b$ then f is integrable with respect to α on $[a,c]$ and $[c,b]$ with

$$\int_a^b f\,\mathrm{d}\alpha = \int_a^c f\,\mathrm{d}\alpha + \int_c^b f\,\mathrm{d}\alpha$$

4. If $f,g \in \mathcal{R}(\alpha)$ with $f(x) \le g(x)$ for all $x \in [a,b]$, the

$$\int_a^b f\,\mathrm{d}\alpha \le \int_a^b g\,\mathrm{d}\alpha$$

5. If $|f(x)| \le M$ on $[a,b]$ and $f \in \mathcal{R}(\alpha)$ then $|f| \in \mathcal{R}(\alpha)$ and

$$\left|\int_a^b f\,\mathrm{d}\alpha\right| \le \int_a^b |f|\,\mathrm{d}\alpha \le M[\alpha(b)-\alpha(a)]$$

Proof. Proof of part (1a) Let $\mathcal{P} = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$. For each $i = 1, \ldots, n$, let

$$M_i(f) = \sup\{f(t) : t \in [x_{i-1}, x_i]\},$$

$$M_i(g) = \sup\{g(t) : t \in [x_{i-1}, x_i]\}.$$

Then

$$f(t) + g(t) \leq M_i(f) + M_i(g)$$

for all $t \in [x_{i-1}, x_i]$ and thus

$$\sup\{f(t) + g(t) : t \in [x_{i-1}, x_i]\} \leq M_i(f) + M_i(g)$$

Therefore, for all partitions $\mathcal{P}$ of $[a,b]$

$$\mathcal{U}(\mathcal{P}, f+g, \alpha) \leq \mathcal{U}(\mathcal{P}, f, \alpha) + \mathcal{U}(\mathcal{P}, g, \alpha)$$

Given $\varepsilon > 0$. Since $f, g \in \mathcal{R}(\alpha)$, there exist partitions $\mathcal{P}_f$ and $\mathcal{P}_g$ of $[a,b]$ such that

$$\mathcal{U}(\mathcal{P}_f, f, \alpha) < \int_a^b f \, \mathrm{d}\alpha + \frac{\varepsilon}{2}$$

and

$$\mathcal{U}(\mathcal{P}_g, g, \alpha) < \int_a^b g \, \mathrm{d}\alpha + \frac{\varepsilon}{2}.$$

Let $\mathcal{Q} = \mathcal{P}_f \cup \mathcal{P}_g$. $\mathcal{Q}$ is a refinement of both $\mathcal{P}_f$ and $\mathcal{P}_g$, Thus

$$\mathcal{U}(\mathcal{Q}, f, \alpha) < \int_a^b f \, \mathrm{d}\alpha + \frac{\varepsilon}{2}$$

and

$$\mathcal{U}(\mathcal{Q}, g, \alpha) < \int_a^b g \, \mathrm{d}\alpha + \frac{\varepsilon}{2}$$

$$\mathcal{U}(\mathcal{Q}, f+g, \alpha) \leq \mathcal{U}(\mathcal{Q}, f, \alpha) + \mathcal{U}(\mathcal{Q}, g, \alpha) < \int_a^b f \, \mathrm{d}\alpha + \int_a^b g \, \mathrm{d}\alpha + \varepsilon.$$

Therefore,

$$\overline{\int_a^b} (f+g) \, \mathrm{d}\alpha < \int_a^b f \, \mathrm{d}\alpha + \int_a^b g \, \mathrm{d}\alpha + \varepsilon.$$

Since the above holds for all $\varepsilon > 0$,

$$\overline{\int_a^b} (f+g) \, \mathrm{d}\alpha \leq \int_a^b f \, \mathrm{d}\alpha + \int_a^b g \, \mathrm{d}\alpha.$$

Now the same logic will be used with the lower sum. For each $i = 1, \ldots, n$, let

$$m_i(f) = \inf\{f(t) : t \in [x_{i-1}, x_i]\},$$

$$m_i(g) = \inf\{g(t) : t \in [x_{i-1}, x_i]\}.$$

Then $m_i(f) + m_i(g) \leq f(t) + g(t)$ for all $t \in [x_{i-1}, x_i]$ and thus

$$m_i(f) + m_i(g) \leq inf\{f(t) + g(t) : t \in [x_{i-1}, x_i]\}$$

Therefore, for all partitions $\mathcal{P}$ of $[a, b]$

$$\mathcal{L}(\mathcal{P}, f, \alpha) + \mathcal{L}(\mathcal{P}, g, \alpha) \leq \mathcal{L}(\mathcal{P}, f+g, \alpha)$$

Given $\varepsilon > 0$. Since $f, g \in \mathcal{R}(\alpha)$, there exist partitions $\mathcal{P}_f$ and $\mathcal{P}_g$ of $[a, b]$ such that

$$\mathcal{L}(\mathcal{P}_f, f, \alpha) > \int_a^b f \, \mathrm{d}\alpha + \frac{\varepsilon}{2}$$

and

$$\mathcal{L}(\mathcal{P}_g, g, \alpha) > \int_a^b g \, \mathrm{d}\alpha + \frac{\varepsilon}{2}.$$

Let

$$\mathcal{Q} = \mathcal{P}_f \cup \mathcal{P}_g.$$

$\mathcal{Q}$ is a refinement of both $\mathcal{P}_f$ and $\mathcal{P}_g$,

$$\mathcal{L}(\mathcal{Q}, f, \alpha) > \int_a^b f \, \mathrm{d}\alpha + \frac{\varepsilon}{2}$$

$$\mathcal{L}(\mathcal{Q}, g, \alpha) > \int_a^b g \, \mathrm{d}\alpha + \frac{\varepsilon}{2}$$

$$\mathcal{L}(\mathcal{Q}, f+g, \alpha) \geq \mathcal{L}(\mathcal{Q}, f, \alpha) + \mathcal{L}(\mathcal{Q}, g, \alpha) > \int_a^b f \, \mathrm{d}\alpha + \int_a^b g \, \mathrm{d}\alpha + \varepsilon.$$

Therefore,

$$\underline{\int_a^b} (f+g) \, \mathrm{d}\alpha > \int_a^b f \, \mathrm{d}\alpha + \int_a^b g \, \mathrm{d}\alpha + \varepsilon.$$

Since the above holds for all $\varepsilon > 0$,

$$\underline{\int_a^b}(f+g)\,\mathrm{d}\alpha \geq \int_a^b f\,\mathrm{d}\alpha + \int_a^b g\,\mathrm{d}\alpha.$$

Thus

$$\int_a^b f\,\mathrm{d}\alpha + \int_a^b g\,\mathrm{d}\alpha + \varepsilon \leq \underline{\int_a^b}(f+g)\,\mathrm{d}\alpha \leq \overline{\int_a^b}(f+g)\,\mathrm{d}\alpha \leq \int_a^b f\,\mathrm{d}\alpha + \int_a^b g\,\mathrm{d}\alpha$$

Thus the upper and lower integrals of $\int_a^b (f+g)\,\mathrm{d}\alpha$ are equal to each other and the quantity $\int_a^b f\,\mathrm{d}\alpha + \int_a^b g\,\mathrm{d}\alpha$. □

Proof. Proof of part (1b) By the given cf are in $\mathcal{R}(\alpha)$ for every $c \in \mathcal{R}$ which means that

$$\sup\{\mathcal{L}(\mathcal{P},cf,\alpha)\} = \underline{\int_a^b} cf\,\mathrm{d}\alpha = \overline{\int_a^b} cf\,\mathrm{d}\alpha = \inf\{\mathcal{U}(\mathcal{P},cf,\alpha)\}$$

$$\mathcal{L}(\mathcal{P},cf,\alpha) = \sum_{i=1}^{n} cm_i \Delta\alpha_i = c\sum_{i=1}^{n} m_i \Delta\alpha_i = c(\mathcal{L}(\mathcal{P},f,\alpha))$$

Thus

$$c\underline{\int_a^b} f\,\mathrm{d}\alpha = c(\sup\{\mathcal{L}(\mathcal{P},f,\alpha)) = \sup\{\mathcal{L}(\mathcal{P},cf,\alpha) = \underline{\int_a^b} cf\,\mathrm{d}\alpha$$

Therefore

$$c\underline{\int_a^b} f\,\mathrm{d}\alpha = \underline{\int_a^b} cf\,\mathrm{d}\alpha$$

The same logic is applied to upper integral.

$$\mathcal{U}(\mathcal{P},cf,\alpha) = \sum_{i=1}^{n} cM_i \Delta\alpha_i = c\sum_{i=1}^{n} M_i \Delta\alpha_i = c(\mathcal{U}(\mathcal{P},f,\alpha))$$

Thus

$$c\overline{\int_a^b} f\,\mathrm{d}\alpha = c(\inf\{\mathcal{U}(\mathcal{P}, f, \alpha)) = \inf\{\mathcal{U}(\mathcal{P}, cf, \alpha) = \overline{\int_a^b} cf\,\mathrm{d}\alpha$$

Therefore

$$c\overline{\int_a^b} f\,\mathrm{d}\alpha = \overline{\int_a^b} cf\,\mathrm{d}\alpha$$

So

$$c\underline{\int_a^b} f\,\mathrm{d}\alpha = \underline{\int_a^b} cf\,\mathrm{d}\alpha = \overline{\int_a^b} cf\,\mathrm{d}\alpha = c\overline{\int_a^b} f\,\mathrm{d}\alpha$$

Thus

$$\int_a^b cf\,\mathrm{d}\alpha = c\int_a^b f\,\mathrm{d}\alpha$$

□

Proof. Proof of part (2) Since $f \in \mathcal{R}(\alpha_i)$, given $\varepsilon > 0$, there exists a partition $\mathcal{P}_i, i = 1,2$ such that

$$\mathcal{U}(\mathcal{P}_i, f, \alpha_i) - \mathcal{L}(\mathcal{P}, f, \alpha_i) < \frac{\varepsilon}{2}$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Since $\mathcal{P}$ is a refinement of both $\mathcal{P}_1$ and $\mathcal{P}_2$ the inequality

$$\mathcal{U}(\mathcal{P}, f, \alpha) - \mathcal{L}(\mathcal{P}, f, \alpha) < \frac{\varepsilon}{2}$$

holds true for $\mathcal{P}$. Thus since

$$\Delta(\alpha_1 + \alpha_2)_i = \Delta(\alpha_1)_i + \Delta(\alpha_2)_i$$

for all $i = 1, \ldots, n$,

$$\begin{aligned}\mathcal{U}(\mathcal{P}, f, \alpha_1 + \alpha_2) - \mathcal{L}(\mathcal{P}, f, \alpha_1 + \alpha_2) &= \mathcal{U}(\mathcal{P}, f, \alpha_1) - \mathcal{L}(\mathcal{P}, f, \alpha_1) + \mathcal{U}(\mathcal{P}, f, \alpha_2) - \mathcal{L}(\mathcal{P}, f, \alpha_2) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Therefore $f \in \mathcal{R}(\alpha_1 + \alpha_2)$.

And for any partition $\mathcal{P}$ of $[a, b]$

$$\begin{aligned}\mathcal{L}(\mathcal{P}, f, \alpha_1 + \alpha_2) &= \mathcal{L}(\mathcal{P}, f, \alpha_1) + \mathcal{L}(\mathcal{P}, f, \alpha_2) \\ &\leq \int_a^b f \, \mathrm{d}\alpha_1 + \int_a^b f \, \mathrm{d}\alpha 2 \\ &\leq \mathcal{U}(\mathcal{P}, f, \alpha_1) + \mathcal{U}(\mathcal{P}, f, \alpha_2) \\ &= \mathcal{L}(\mathcal{P}, f, \alpha_1 + \alpha_2)\end{aligned}$$

Thus since $f \in \mathcal{R}(\alpha_1 + \alpha_2)$,

$$\int_a^b f \, \mathrm{d}(\alpha_1 + \alpha_2) = \int_a^b f \, \mathrm{d}\alpha_1 + \int_a^b f \, \mathrm{d}\alpha_2.$$

$\square$

Proof. Proof of part (3) By the given $f \in \mathcal{R}$ there exists a partition $\mathcal{P}_1$ of $[a, b]$ such that

$$\mathcal{U}(\mathcal{P}_1, f, \alpha) - \mathcal{L}(\mathcal{P}_1, f, \alpha) < \varepsilon$$

Let the partition $\mathcal{P}_2 = \mathcal{P}_1 \cup \{c\}$ By the fact that $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$

$$\mathcal{U}(\mathcal{P}_2, f, \alpha) - \mathcal{L}(\mathcal{P}_2, f, \alpha) < \varepsilon$$

Let $\mathcal{P}_2^a$ be the section of partition of $\mathcal{P}_2$ from $[a,c]$ and $\mathcal{P}_2^b$ be the section of partition of $\mathcal{P}_2$ from $[c,b]$.

$$\mathcal{U}(\mathcal{P}_2, f, \alpha) = \mathcal{U}(\mathcal{P}_2^a, f, \alpha) + \mathcal{U}(\mathcal{P}_2^b, f, \alpha)$$

$$\mathcal{L}(\mathcal{P}_2, f, \alpha) = \mathcal{L}(\mathcal{P}_2^a, f, \alpha) + \mathcal{L}(\mathcal{P}_2^b, f, \alpha)$$

$$\varepsilon > \mathcal{U}(\mathcal{P}_2, f, \alpha) - \mathcal{L}(\mathcal{P}_2, f, \alpha) = \mathcal{U}(\mathcal{P}_2^a, f, \alpha) + \mathcal{U}(\mathcal{P}_2^b, f, \alpha) - (\mathcal{L}(\mathcal{P}_2^a, f, \alpha) + \mathcal{L}(\mathcal{P}_2^b, f, \alpha))$$

$$\varepsilon > (\mathcal{U}(\mathcal{P}_2^a, f, \alpha) - \mathcal{L}(\mathcal{P}_2^a, f, \alpha)) + (\mathcal{U}(\mathcal{P}_2^b, f, \alpha) + \mathcal{L}(\mathcal{P}_2^b, f, \alpha))$$

By the fact that the upper sum $>$ lower sum

$$(\mathcal{U}(\mathcal{P}_2^a, f, \alpha) - \mathcal{L}(\mathcal{P}_2^a, f, \alpha)) \geq 0$$

and

$$(\mathcal{U}(\mathcal{P}_2^b, f, \alpha) + \mathcal{L}(\mathcal{P}_2^b, f, \alpha)) \geq 0$$

which means

$$(\mathcal{U}(\mathcal{P}_2^a, f, \alpha) - \mathcal{L}(\mathcal{P}_2^a, f, \alpha) < \varepsilon$$

and

$$(\mathcal{U}(\mathcal{P}_2^b, f, \alpha) - \mathcal{L}(\mathcal{P}_2^b, f, \alpha) < \varepsilon$$

Therefore f is integrable with respect to α on $[a,c]$ and $[c,b]$. Now show that

$$\int_a^b f\,\mathrm{d}\alpha = \int_a^c f\,\mathrm{d}\alpha + \int_c^b f\,\mathrm{d}\alpha.$$

As stated above we know

$$\mathcal{U}(\mathcal{P}_2, f, \alpha) = \mathcal{U}(\mathcal{P}_2^a, f, \alpha) + \mathcal{U}(\mathcal{P}_2^b, f, \alpha)$$

$$\mathcal{L}(\mathcal{P}_2, f, \alpha) = \mathcal{L}(\mathcal{P}_2^a, f, \alpha) + \mathcal{L}(\mathcal{P}_2^b, f, \alpha)$$

By definition

$$\overline{\int_a^b} f\,\mathrm{d}\alpha = \inf\{\mathcal{U}(\mathcal{P}, f, \alpha) : \mathcal{P}\}$$

$$\inf\{\mathcal{U}(\mathcal{P}_2, f, \alpha) : \mathcal{P}\} = \inf\{\mathcal{U}(\mathcal{P}_2^a, f, \alpha) : \mathcal{P}\} + \inf\{\mathcal{U}(\mathcal{P}_2^b, f, \alpha) : \mathcal{P}\} = \overline{\int_a^c} f\,\mathrm{d}\alpha + \overline{\int_c^b} f\,\mathrm{d}\alpha$$

and

$$\underline{\int_a^b} f\,\mathrm{d}\alpha = \sup\{\mathcal{L}(\mathcal{P}, f, \alpha) : \mathcal{P}\}$$

$$\sup\{\mathcal{L}(\mathcal{P}_2, f, \alpha) : \mathcal{P}\} = \sup\{\mathcal{L}(\mathcal{P}_2^a, f, \alpha) : \mathcal{P}\} + \sup\{\mathcal{L}(\mathcal{P}_2^b, f, \alpha) : \mathcal{P}\} = \underline{\int_a^c} f\,\mathrm{d}\alpha + \underline{\int_c^b} f\,\mathrm{d}\alpha$$

Therefore

$$\int_a^b f\,\mathrm{d}\alpha = \int_a^c f\,\mathrm{d}\alpha + \int_c^b f\,\mathrm{d}\alpha.$$

$\square$

Proof. Proof of part (4) By the given $f(x) \leq g(x)$. Let $\mathcal{P} = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$. For each $i = 1, \ldots, n$, let

$$m_i(f) = \inf\{f(t) : t \in [x_{i-1}, x_i]\},$$

and

$$m_i(g) = \inf\{g(t) : t \in [x_{i-1}, x_i]\}.$$

By the given $m_i(f) \leq m_i(g)$ Therefore, for all partitions $\mathcal{P}$ of $[a,b]$

$$\mathcal{L}(\mathcal{P}, f, \alpha) \leq \mathcal{L}(\mathcal{P}, g, \alpha)$$

So as the

$$\lim_{||\mathcal{P}|| \to 0} \mathcal{L}(\mathcal{P}, f, \alpha) \leq \lim_{||\mathcal{P}|| \to 0} \mathcal{L}(\mathcal{P}, g, \alpha)$$

Since $f, g \in \mathcal{R}(\alpha)$ then

$$\int_a^b f \, \mathrm{d}\alpha = \lim_{||\mathcal{P}|| \to 0} \mathcal{L}(\mathcal{P}, f, \alpha) \leq \lim_{||\mathcal{P}|| \to 0} \mathcal{L}(\mathcal{P}, g, \alpha) = \int_a^b fg \, \mathrm{d}\alpha$$

□

Proof. Proof of part (5) Suppose $f \in \mathcal{R}(\alpha)$ and $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$. For each $i = 1, 2, \ldots, n$ let

$$M_i(f) = \sup\{f(t) : t \in [x_{i-1}, x_i]\},$$

$$M_i^*(f) = \sup\{|f(t)| : t \in [x_{i-1}, x_i]\}.$$

$$m_i(f) = \inf\{f(t) : t \in [x_{i-1}, x_i]\},$$

$$m_i^*(f) = \inf\{|f(t)| : t \in [x_{i-1}, x_i]\}.$$

If $t, x \in [x_{i-1}, x_i]$ then

$$||f(t)| - |f(x)|| \leq |f(t) - f(x)| \leq M_i - m_i$$

Thus $M_i^* - m_i^* \leq M_i - m_i$ for all $i = 1, 2, \ldots, n$ and as a consequence

$$\mathcal{U}(\mathcal{P}, |f|, \alpha) + \mathcal{L}(\mathcal{P}, |f|, \alpha) \leq \mathcal{U}(\mathcal{P}, f, \alpha) + \mathcal{L}(\mathcal{P}, f, \alpha)$$

Therefore $|f| \in \mathcal{R}(\alpha)$. Choose $c = \pm 1$ such that

$$c \int_a^b f \, \mathrm{d}\alpha \geq 0.$$

Then

$$|\int_a^b f \, d\alpha| = c \int_a^b f \, \mathrm{d}\alpha = \int_a^b cf \, \mathrm{d}\alpha \leq \int_a^b |f| \, \mathrm{d}\alpha \leq M \int_a^b \mathrm{d}\alpha = M[\alpha(b) - \alpha(a)].$$

$\square$

Theorem 4.7. Mean Value Theorem([4]). Let f be a continuous real-valued function on $[a, b]$ and α a monotone increasing function on $[a, b]$. Then there exists a $c \in [a, b]$ such that

$$\int_a^b f \, \mathrm{d}\alpha = f(c)[\alpha(b) - \alpha(a)]$$

Proof. Let m and M denote the minimum and maximum of f on $[a,b]$ respectively. Then

$$m[\alpha(b)-\alpha(a)] \leq \int_a^b f\,\mathrm{d}\alpha \leq M[\alpha(b)-\alpha(a)]$$

If $\alpha(b)-\alpha(a)=0$ then any $c\in[a,b]$ will work. If $\alpha(b)-\alpha(a)\neq 0$ then by the Intermediate Value Theorem there exists a $c\in[a,b]$ such that

$$f(c)=\frac{1}{\alpha(b)-\alpha(a)}\int_a^b f\,\mathrm{d}\alpha$$

□

Theorem 4.8. Integration by Parts Formula([4]). Suppose α and β are monotone increasing functions on $[a,b]$.

a) Then $\alpha\in\mathcal{R}(\beta)$ if and only if $\beta\in\mathcal{R}(\alpha)$.

b) If this is the case,

$$\int_a^b \alpha\,\mathrm{d}\beta = \alpha(b)\beta(b)-\alpha(a)\beta(a)-\int_a^b \beta\,\mathrm{d}\alpha.$$

Proof. Proof of part (a) For any partition $\mathcal{P}$ of $[a,b]$

$$\mathcal{U}(\mathcal{P},\alpha,\beta)=\alpha(b)\beta(b)-\alpha(a)\beta(a)-\mathcal{L}(\mathcal{P},\beta,\alpha)$$

and

$$\mathcal{L}(\mathcal{P},\alpha,\beta)=\alpha(b)\beta(b)-\alpha(a)\beta(a)-\mathcal{U}(\mathcal{P},\beta,\alpha).$$

By the subtraction of these 2 equations.

$$\mathcal{U}(\mathcal{P},\alpha,\beta)-\mathcal{L}(\mathcal{P},\alpha,\beta)=\mathcal{U}(\mathcal{P},\beta,\alpha)-\mathcal{L}(\mathcal{P},\beta,\alpha)$$

If $\alpha \in \mathcal{R}(\beta)$ then

$$\mathcal{U}(\mathcal{P},\alpha,\beta) - \mathcal{L}(\mathcal{P},\alpha,\beta) < \varepsilon$$

and due to the fact that these 2 quantities are equal if

$$\mathcal{U}(\mathcal{P},\alpha,\beta) - \mathcal{L}(\mathcal{P},\alpha,\beta) < \varepsilon$$

then

$$\mathcal{U}(\mathcal{P},\beta,\alpha) - \mathcal{L}(\mathcal{P},\beta,\alpha) < \varepsilon$$

as well. This means both are Riemann Stieltjes Integrable. □

Proof. Proof of part (b) Furthermore if $\beta \in \mathcal{R}(\alpha)$ then given $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that

$$\mathcal{L}(\mathcal{P},\beta,\alpha) > \int_a^b \beta \,\mathrm{d}\alpha - \varepsilon.$$

Hence

$$\int_a^b \alpha \,\mathrm{d}\beta \leq \mathcal{U}(\mathcal{P},\alpha,\beta) < \alpha(b)\beta(b) - \alpha(a)\beta(a) - \int_a^b \beta \,\mathrm{d}\alpha + \varepsilon.$$

Since the above holds for any $\varepsilon > 0$

$$\int_a^b \alpha \,\mathrm{d}\beta \leq \alpha(b)\beta(b) - \alpha(a)\beta(a) - \int_a^b \beta \,\mathrm{d}\alpha.$$

A similar argument using the lower sum proves the reverse inequality. □

The following theorem allows for the evaluation of Reimann-Stieltjes Integrals using the method of Riemann Integration given that α has a continuous derivative α'.

Theorem 4.9 ([1])**.** Assume $f \in \mathcal{R}$ on $[a,b]$ and assume that α has a continuous derivative α' on $[a,b]$. Then the Riemann integral $\int_a^b f(x)\alpha'(x)\mathrm{d}x$ exists and we have

$$\int_a^b f(x)\,\mathrm{d}\alpha(x) = \int_a^b f(x)\alpha'(x)\,\mathrm{d}x.$$

Proof. Let $g(x) = f(x)\alpha'(x)$ and consider a Riemann sum

$$S(\mathcal{P},g) = \sum_{k=1}^{n} g(t_k)\Delta x_k = \sum_{k=1}^{n} f(t_k)\alpha'(t_k)\Delta x_k.$$

The same partition $\mathcal{P}$ and the same choice of t_k can be used to form the Riemann-Stieltjes sum

$$S(\mathcal{P},f,\alpha) = \sum_{k=1}^{n} f(t_k)\Delta\alpha_k.$$

Applying the Mean-Value Theorem, we can write

$$\Delta\alpha_k = \alpha'(v_k)\Delta x_k, \quad \text{where} \quad v_k \in (x_{k-1},x_k),$$

and hence

$$S(\mathcal{P},f,\alpha) - S(\mathcal{P},g) = \sum_{k-1}^{n} f(t_k)[\alpha'(v_k) - \alpha'(t_k)]\Delta x_k.$$

Since f is bounded, we have $|f(x)| \le M$ for all x in $[a,b]$, where $M > 0$. Continuity of α' on $[a,b]$ implies uniform continuity on $[a,b]$. Hence, if $\varepsilon > 0$ is given, there exists a $\delta > 0$ (depending on ε) such that

$0 \le |x-y| < \delta$ implies $|\alpha'(x) - \alpha'(y)| < \varepsilon/(2M(b-a))$. If we take a partition $\mathcal{P}'_\varepsilon$ with norm $||\mathcal{P}'_\varepsilon|| < \delta$, then for any finer partition $\mathcal{P}$ we will have $|\alpha'(v_k) - \alpha'(t_k)| < \varepsilon/[2M(b-a)]$ in the preceding equation. For such $\mathcal{P}$ we therefore have

$$|S(\mathcal{P}, f, \alpha) - S(\mathcal{P}, g)| < \varepsilon/2.$$

On the other hand, since $f \in \mathcal{R}(\alpha)$ on $[a,b]$, there exists a partition $\mathcal{P}''_\varepsilon$ such that $\mathcal{P}$ finer than $\mathcal{P}''_\varepsilon$ implies

$$|S(\mathcal{P}, f\alpha) - \int_a^b f \, \mathrm{d}\alpha| < \varepsilon/2.$$

Combining the last two inequalities, we see that when $\mathcal{P}$ is finer than $\mathcal{P}_\varepsilon = \mathcal{P}'_\varepsilon \cup \mathcal{P}''_\varepsilon$, we will have $|S(\mathcal{P}, g) - \int_a^b f \, \mathrm{d}\alpha| < \varepsilon$, and this proves the theorem. $\square$

Functions of Bounded Variation and Riemann-Stieltjes Integration

The variation of a function is defined as: Let $\phi : [a,b] \to \mathcal{R}$. To measure how much ϕ wiggles on an interval $[a,b]$ set

$$V(\phi,\mathcal{P}) = \sum_{j=1}^{n} |\phi(x_j) - \phi(x_{j-1})|$$

for each partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$. The variation of ϕ is defined by

$\operatorname{Var}(\phi) = \sup\{V(\phi,\mathcal{P}) : \mathcal{P}$ is a partition of $[a,b]\}$. Let $[a,b]$ be a closed, bounded, non-degenerate ($a \neq b$) interval and $\phi : [a,b] \to \mathcal{R}$. Then ϕ is said to be of bounded variation on $[a,b]$ if $\operatorname{Var}(\phi) < \infty$.

Remarks on Bounded Variation:([5])

a) If ϕ is continuously differentiable on $[a,b]$, then ϕ is of bounded variation on $[a,b]$.

Proof. Proof of (a) Let $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. By the Extreme Value Theorem, there is an $M > 0$ such that

$|\phi'(x)| \leq M$ for all $x \in [a,b]$. Therefore it follows from the Mean Value Theorem that for each k between 1 and n there is a point $c_k \in [x_{k-1}, x_k]$ such that

$$|\phi(x_k) - \phi(x_{k-1})| = |\phi'(c_k)|(x_k - x_{k-1}) \leq M(x_k - x_{k-1}).$$

By telescoping, we obtain $V(\phi,\mathcal{P}) \leq M(b-a)$ for any partition $\mathcal{P}$ of $[a,b]$. Therefore,

$$\operatorname{Var}(\phi) \leq M(b-a).$$

□

However, there exist functions of bounded variation which are not continuously differentiable. An example of this is the indicator function which is of bounded variation but not continously differentiable. It is not differentiable at the point of discontinuity.

b) If ϕ is monotone on $[a,b]$, then ϕ is of bounded variation on $[a,b]$.

Proof. Proof of (b) Let ϕ be increasing on $[a,b]$ and $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Then by telescoping,

$$\sum_{j=1}^{n} |\phi(x_j) - \phi(x_{j-1})| = \sum_{j=1}^{n} (\phi(x_j) - \phi(x_{j-1})) = \phi(x_n) - \phi(x_0) = \phi(b) - \phi(a) =: M < \infty.$$

Thus, $\mathrm{Var}(f) \leq M$.

But, there exist functions of bounded variation which are not monotone. The function $\phi(x) = \sin(x)$ is of bounded variation on $[0, 2\pi]$ but is not monotone. □

c) If ϕ is of bounded variation on $[a,b]$, then ϕ is bounded on $[a,b]$.

Proof. Proof of (c) Let $x \in [a,b]$ and by defintion

$$|\phi(x) - \phi(a)| \leq |\phi(x) - \phi(a)| + |\phi(b) - \phi(x)| \leq \mathrm{Var}(\phi).$$

Hence, by the triangle inequality,

$$|\phi(x)| \leq |\phi(a)| + \mathrm{Var}(\phi).$$

□

However, there exist bounded functions which are not of bounded variation. An example of this is the function

$$y = \begin{cases} \sin(\frac{1}{x}) & \text{x} \neq 0 \\ 0 & \text{x} = 0 \end{cases}$$

which is bounded but not of bounded variation on the interval $[0, b]$. Given below is a graph of the function.

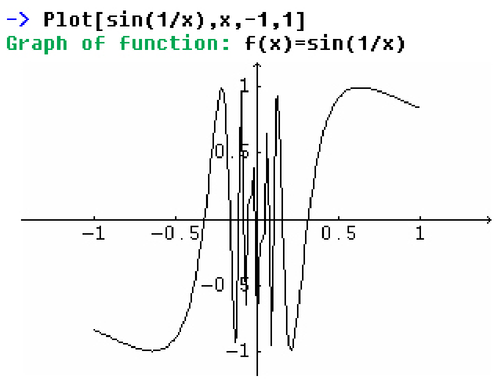

Figure 5.1: Image

Bounded variation is important to the study of Riemann-Stieltjes Intgration due to the following Theorem.

Theorem 5.1 ([6])**.** Suppose that f is continuous on $[a, b]$ and that α is of bounded variation on $[a, b]$. Then the Riemann-Stietjes Integral $\int_a^b f \, d\alpha$ exists.

To prove this theorem, Jordan's Theorem which is stated below will be used.

Theorem 5.2. Jordan's Theorem.([6]) A function f is of bounded variation on $[a, b]$ if and only if it can be written as the difference of two bounded increasing functions on $[a, b]$.

Proof. From Theorem 4.6 2 we can rewrite the integral with respect to α as the difference of the integrals respect to α_1 and α_2 if $\alpha_1 - \alpha_2 = \alpha$. So by Jordan's Theorem if α is of bounded variation then $\alpha_1 - \alpha_2 = \alpha$. Thus α is the difference of two monotone increasing functions which can be evaluated separately and then subtracted. Let $\mathcal{P}$ be a partition of $[a,b]$. Then

$$\sup\{\mathcal{L}(\mathcal{P},f,\alpha)\} \leq \int_a^b f\,\mathrm{d}\alpha \leq \inf\{\mathcal{U}(\mathcal{P},f,\alpha)\}.$$

It is enough to show

$$\sup\{\mathcal{L}(\mathcal{P},f,\alpha)\} = \inf\{\mathcal{U}(\mathcal{P},f,\alpha)\}$$

to prove the Riemann-Stieltjes Integral exists. If α is constant on $[a,b]$ then $\Delta\alpha = 0$ thus $\int_a^b f d\alpha = 0$ as do the upper and lower sums. If α is not constant though let $\varepsilon > 0$, the uniform continuity of f implies there exists $\delta > 0$ such that if

$|x_i - x_{i-1}| < \delta$, then $M_i - m_i < \frac{\varepsilon}{\alpha(b)-\alpha(a)}$. Therefore, if

$$|x_i - x_{i-1}| < \delta,$$

$$0 \leq \inf\{\mathcal{U}(\mathcal{P},f,\alpha)\} - \sup\{\mathcal{L}(\mathcal{P},f,\alpha)\} = \sum_{i=1}^{n}(M_i - m_i)[\phi(x_i) - \phi(x_{i-1}) < \varepsilon.$$

Thus

$$\inf\{\mathcal{U}(\mathcal{P},f,\alpha)\} - \sup\{\mathcal{L}(\mathcal{P},f,\alpha)\} < \varepsilon$$

so the Riemann-Stieltjes Integral of f with respect to α exists. $\square$

Stochastic Processes

A stochastic process is one in which there are many different ways the process could evolve or different paths the process can take. An example of a stochastic process is the stock market fluctuations.

Definition 6.1. A random variable is a variable that has a single numerical value, determined by chance, for each outcome of a procedure.

A stochastic process is a collection of random variables

$$(X_t, t \in T) = (X_t(\omega), t \in T, \omega \in \Omega),$$

where T is time and ω is a function of time defined on some space Ω.

Brownian Motion, an idea which first came about when the random movements of particles suspended in a liquid, is a stochastic process. A real-valued stochastic process $B = B(t) : t \in \mathcal{R}_+$ is a Brownian Motion if it satisfies the following properties.

(i) B(0) = 0.

(ii) B has a independent increments and for $s < t$, the increment B(t) - B(s) has a normal distribution with mean 0 and variance t - s.

(iii) The paths of B are continuous.

Researchers would like to be able to evaluate an integral of the form

$$\int_{\infty}^{\infty} f(t)\, \mathrm{d}B_t(\omega),$$

where f is a function or a stochastic process on [0,1] and $B_t(\omega)$ is a Brownian sample path. This integral is in the form of a Riemann-Stieltjes Integral. It was previously stated in this thesis the Riemann-Stieltjes Integral exists when $f(t)$ and $B_t(\omega)$ are not disconinuous at the same point from the same side and $B_t(\omega)$ has bounded variation. The latter requirement though is not possible with Brownian sample paths. Brownian sample paths do not have bounded variation. It has been found that a weaker requirement can be used to allow for the evaluation of Riemann-Stieltjes Integrals with Brownian sample paths. The real-valued function h on $[0,1]$ is said to have bounded p-variation for some $p > 0$ if

$$\sup_{\tau} \sum_{i=1}^{n} |h(t_i) - h(t_{i-1})|^p < \infty,$$

where the supremum is taken over all partition τ of $[0,1]$. (If $p = 1$, then h has bounded variation) The Riemann-Stieltjes Integral exists if f has bounded p-variation and the function $B_t(\omega)$ has bounded q-variation for some $p > 0$ and $q > 0$ such that $p^{-1} + q^{-1} > 1$. Not all Riemann-Stieltjes integrals can be evaluated with respect to a Brownian sample path. An example of this is the integral

$$I(B)(\omega) = \int_0^1 B_t(\omega)\, \mathrm{d}B_t(\omega).$$

Brownian motion has bounded p-variation for $p > 2$, thus the condition of $p^{-1} + q^{-1} > 1$ fails. Thus the Riemann-Stieltjes Integral is limited in use with respect to stochastic processes. [3]

Applications of Riemann-Stieltjes Integration

In Probability Theory, the ability to evaluate the following integrals is often desired

$$\int_b^a \mathrm{d}F(\alpha)$$

$$\int_B \mathrm{d}F(\alpha)$$

and

$$\int_b^a g(\alpha)\,\mathrm{d}F(\alpha)$$

where F is a cummulative distribution function, a and b are real numbers, B is a Borel Set and $g : \mathcal{R}^* \to \mathcal{R}^*$. The evaluation of such integrals is desired since the expected value of a random variable X is given by

$$EX = \int_{-\infty}^{\infty} f\,\mathrm{d}F_x(T)$$

where F_x is the distribution function of X.

Definition 7.1. The expected value of a discrete random variable is denoted by E and it represents the average value of the outcomes. It is obtained by finding the value of $\sum(xP(x))$. For a continuous probability function this becomes $E(x) = \int_{-\infty}^{\infty} xf(x)\,\mathrm{d}x$

Definition 7.2. A Borel set is any set that can be formed from open sets.

Definition 7.3. A random variable is a variable that has a single numerical value, determined by chance, for each outcome of a procedure.A continuous random variable has infinitely many values, and those values can be associated with measurements on a continuous scale without gaps or interuptions.

A discrete random variable will have jumps on a continuous scale.

A mixed random variable will have intervals where the variable is continuous and intervals where the variable is discrete.

Definition 7.4. A probability distribution is a description that gives the probability for each value of the random variable.

Properties of a Probability Distribution

a) $\sum P(x) = 1$ where x assumes all possible values

b) $0 \leq P(x) \leq 1$ for every individual value of x

A cummulative probability distribution is a monotone increasing function.

As was previously stated in this paper if $g(\alpha) = 1$ then the Riemann Stieltjes Integral is also a Riemann Integral and evaluated as such. If this is the case and $B = (a, b]$ then all three of these integrals are the same and a Riemann Integral can be used to evaluate all three. The Riemann-Stieltjes Integral, though, allows there to be a standard process for evaluation of such integrals regardless of the random variable being continuous, discrete or mixed. [2]

In Physics the Riemann-Stieltjes Integral is used in the following situation, consider n-masses, each of mass $m_i, i = 1, 2, \ldots, n$, located along the x-axis at distances r_i from the origin with $0 < r_1 < \ldots < r_n$. The moment of inertia I, about an axis through the origin at right angles to the system of masses is given by $I = \sum_{i=1}^{n} r_i^2 m_i$. This is the discrete case. If instead we have a length of wire l along the x-axis with one end at the origin the moment of Inertia is

$$I = \int_0^l x^2 \rho(x)\, \mathrm{d}x,$$

where for each $x \in [0, l], \rho(x)$ equals the cross-sectional density at x. [4]

Conclusion

The definition and properties of the Riemann-Stieltjes Integral have been given, with an explanation of how to calculate the integral. The Riemann-Stieltjes Integral is a useful mathematical tool when working with discrete and random variables simultaneously. It has applications in physics and statistics, but is limited in its use with Stochastic processes.

Bibliography

Bibliography

[1] T. M. Apostol *Mathematical Analysis*, Addison Wesley, Reading, MA, 1975.

[2] J.D. Enderle *Basis Probability Theory for Biomedical Engineers*, Morgan and Claypool Publishers, San Rafael, CA, 2006.

[3] T. Mikosch *Elementary Stochastic Calculus with Finance in View*, World Scientific, Singapore, 1998.

[4] M. Stoll *Introduction to Real Analysis*, Addison-Wesley, Boston, MA, 2001.

[5] W. R. Wade *An Introduction to Analysis*, Prentice Hall, Upper Saddle River, NJ, 2000.

[6] R. L. Wheeden, A. Zygmund *Measure and Integral An Introduction to Real Analysis*, Marcel Dekker, Inc., New York, New York, 1977.

Vita

Colleen Cook McFadden was born in Falls Church, Virginia on June 25,1977. She lived in Falls Church until attending college at Virginia Polytechnic Institute and State University. At Virginia Tech she earned her Bachelor of Science degree in Science of Human Nutrition, Foods and Exercise in 1999. Upon graduation she began teaching high school mathematics. In May 2011 she will graduate from Virginia Commonwealth University, with a Master of Science degree in Mathematics.