2011

Embeddings of Product Graphs Where One Factor is a Hypercube

Bethany Turner
Virginia Commonwealth University

College of Humanities and Sciences
Virginia Commonwealth University

This is to certify that the thesis prepared by Bethany Turner titled "Embeddings of Prodcut Graphs Where One Factor is Q_n" has been approved by his or her committee as satisfactory completion of the thesis requirement for the degree of Master of Science.

Ghidewon Abay-Asmerom, College of Humanities and Sciences

Richard H. Hammack, College of Humanities and Sciences

James E. Ames, School of Engineering

John F. Berglund, Graduate Chair, Mathematics and Applied Mathematics

---

Fred Hawkridge, Dean, College of Humanities and Sciences

F. Douglas Boudinot, Graduate Dean

Date

Embeddings of Prodcut Graphs Where One Factor is Q_n

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

Bethany Turner
Master of Science

Director: Ghidewon Abay-Asmerom, Associate Professor
Department of Mathematics and Applied Mathematics

Virginia Commonwealth University
Richmond, Virginia
May 2011

Acknowledgment

I thank Ghidewon for introducing me to voltage graph theory with enthusiasm and reverence. Your emphasis on joy as the motivation for mathematics, and your willingness to explore the unexpected, influenced the diversity of this thesis. Thank you for your humility as an advisor and your confidence as a teacher.

Contents

List of Figures

Abstract

EMBEDDINGS OF PRODCUT GRAPHS WHERE ONE FACTOR IS Q_N

By Bethany Turner, Master of Science.

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

Virginia Commonwealth University, 2011.

Director: Ghidewon Abay-Asmerom, Associate Professor, Department of Mathematics and Applied Mathematics.

Voltage graph theory can be used to describe embeddings of product graphs if one factor is a Cayley graph. We use voltage graphs to explore embeddings of various products where one factor is a hypercube, describing some minimal and symmetrical embeddings. We then define a graph product, the weak symmetric difference, and illustrate a voltage graph construction useful for obtaining an embedding of the weak symmetric difference of an arbitrary graph with a hypercube.

Introduction

In this chapter we define graphs and product graphs along with some of their basic properties, and ideas from topological graph theory of graph genus and graph embeddings. We will also introduce the theory of voltage graphs which will aid us later in describing some embeddings of product graphs.

1.1 Basic Definitions

A **graph** is a structure consisting of two sets: a **vertex set** and an **edge set**. The vertex set is denoted $V(G)$, and the edge set, denoted $E(G)$, is a set of unordered pairs of distinct vertices. Since a graph can be uniquely defined by its edges and vertices, we may write $G = (V(G), E(G))$. When $\{v_1, v_2\} \in E(G)$ we will denote the edge by v_1v_2 whenever convenient.

We say that two vertices $v_1, v_2 \in V(G)$ are **adjacent** if $v_1v_2 \in E(G)$, and the edge v_1v_2 is **incident with** the vertices v_1 and v_2. Naturally a graph is presented as a collection of dots, the vertices, with edges connecting pairs of adjacent vertices as in Figure 1.1.

The **degree** of a vertex v is denoted $d(v)$ and represents the number of edges incident with v. The cardinality of $V(G)$, $|V(G)|$, is said to be the **order** of G and is denoted by p, while $|E(G)|$ is the **size** of G and is denoted q. If every vertex of G has degree k, G is k**-regular**. An **isolated vertex** is one with degree zero. There is a simple relationship between a graph's order and the sum of the degrees of its vertices.

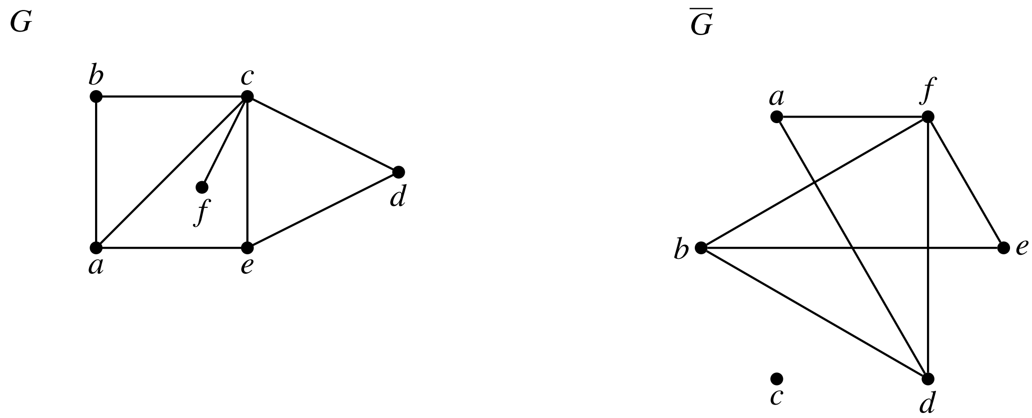


Figure 1.1: A graph G and its complement, $\overline{G}$

THEOREM 1.1. For any graph G,

$$\sum_{v \in V(G)} d(v) = 2q.$$

A **walk** is sequence of vertices, not necessarily distinct, such that any two consecutive vertices are adjacent. A **path** is a walk in which no vertex is repeated. A walk is **closed** if its endpoints are the same and **open** otherwise. A closed path is a **cycle**; a cycle is **even** or **odd** depending on whether it contains an even or odd number of vertices. Two vertices are **connected** if there is a path between them; similary, a graph is **connected** if there is a path between every pair of its vertices. For example in Figure 1.1, $abcf$ is an $a-f$-path, so a is connected to f. In fact, the graph is connected.

We define the **complement** of G by $\overline{G} = (V(G), \overline{E(G)})$, so that vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. Figure 1.1 illustrates a graph and its complement; notice that c is an isolated vertex in $\overline{G}$ but it is adjacent to every vertex in G.

The structure G in Figure 1.2 has some additional features. Every edge in G comes from an ordered pair as indicated by the arrows; this indicates that the edge is a **directed edge**. For example, the edges ab and ba are distinct. Such a structure with directed edges is called

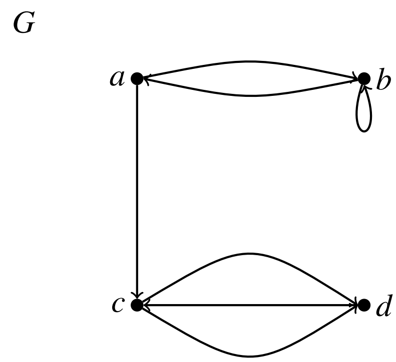


Figure 1.2: A pseudograph with directed edges

a **directed graph**, or **digraph**. An edge from a vertex to itself is called a **loop**. In the figure, there is a bb loop. There are also two distinct edges from c to d. A **pseudograph** is a graph in which both loops and multiple edges are permitted. In the literature, our graphs, those that admit no loops, directed edges or multiple edges are called **simple graphs**, reserving the name graph for what we have called a pseudograph.

Unless otherwise noted, we will assume that all graphs are simple. We will make use of pseudographs in Section 1.4, but only as tools for understanding other simple graphs.

A subset of the vertex set of a graph is **independent** if its elements are mutually nonadjacent. A graph is **bipartite** if its vertices can be partitioned into two independent sets, known in context as **partite sets**. A useful characterization of bipartite graphs is that they are exactly those graphs containing no odd cycles.

A **complete graph** is a graph with an edge between every pair of vertices; the complete graph on n vertices is denoted K_n. A graph with no edges is an **empty graph**.

A graph G' is a **subgraph** of G if $V(G') \subset V(G)$ and $E(G') \subset E(G)$. G' is an **induced subgraph** of G if $V(G') \subset V(G)$ and $E(G') = \{v_1v_2 : v_1, v_2 \in V(G') \text{ and } v_1v_2 \in E(G)\}$. Vertices of an induced subgraph are adjacent if and only if they are adjacent in the parent graph.

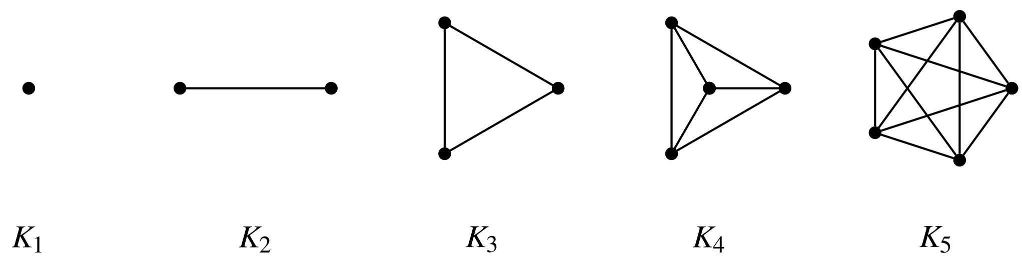


Figure 1.3: Some complete graphs

A graph is **vertex-transitive** if for every $v_1, v_2 \in V(G)$, there is some automorphism $\alpha\colon V(G) \to V(G)$ such that $\alpha(v_1) = v_2$. Similarly a graph is **edge-transitive** if for every $e_1, e_2 \in E(G)$, there is some automorphism $\alpha\colon E(G) \to E(G)$ such that $\alpha(e_1) = e_2$.

1.2 Graph Products

Graph products are certain binary operations on a set of graphs. Given two graphs G and H, we can construct a graph $G * H$, whose vertex set is the Cartesian product $V(G) \times V(H)$. Different ways of defining an edge set give rise to the various **graph products**. For this discussion we define $V(G) = \{g_1, g_2, ..., g_p\}$ and $V(H) = \{h_1, h_2, ..., h_{p'}\}$. Some common examples of graph products are described below. With respect to any product graph $G * H$, we refer to G and H as **factors**

The **Cartesian product** of G and H is denoted by $G \Box H$, and has

$$E(G \Box H) = \{(g_1, h_1)(g_2, h_2) | (g_1 = g_2 \text{ and } h_1 h_2 \in E(H)) \text{ or } (h_1 = h_2 \text{ and } g_1 g_2 \in V(G))\}.$$

Consider the graph $K_2 \Box K_2$. Notice in Figure 1.4 (a) that the Cartesian product consists of two identical copies of K_2, with edges between the copies' corresponding vertices. In general, the product $G \Box H$ contains a copy of H for each vertex of G, and corresponding vertices of H placed at adjacent vertices of G are joined. The Cartesian product satisfies

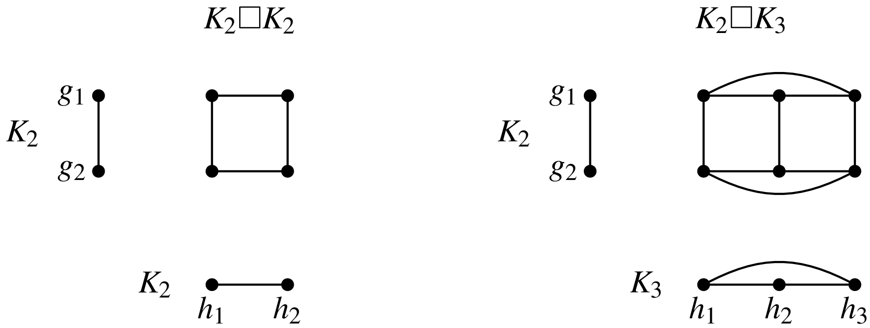


Figure 1.4: The Cartesian products $K_2\Box K_2$ and $K_2\Box K_3$

$|E(G\Box H)| = pq' + p'q$.

The graphs known as the **hypercubes** or ***n*-cubes** are defined recursively as a Cartesian product. Let $Q_1 = K_2$, and $Q_n = Q_{n-1}\Box K_2$ for $n \geq 2$. Figure 1.5 shows the first three n-cubes. Chapter 2 details some properties of these graphs.

The **tensor product** of G and H, also known in the literature as the direct product or Kronecker product, is denoted $G \times H$, with

$$E(G \times H) = \{(g_1,h_1)(g_2,h_2)|g_1g_2 \in E(G) \text{ and } h_1h_2 \in E(H)\}.$$

The tensor product is demonstrated for small graphs in Figure 1.6. Notice that every pair of edges one from each factor gives rise to two distinct edges in the tensor product; therefore, $|E(G \times H)| = 2qq'$.

The **augmented tensor product** $G \boxtimes H$ has edge set

$$E(G\boxtimes H) = \{(g_1,h_1)(g_2,h_2)|(g_1 = g_2 \text{ and } h_1h_2 \in E(H)) \text{ or } (g_1g_2 \in E(G) \text{ and } h_1h_2 \in E(H))\}.$$

The augmented tensor product is not commutative.

There are many possibilities for the edge set of a product graph, and not all types have

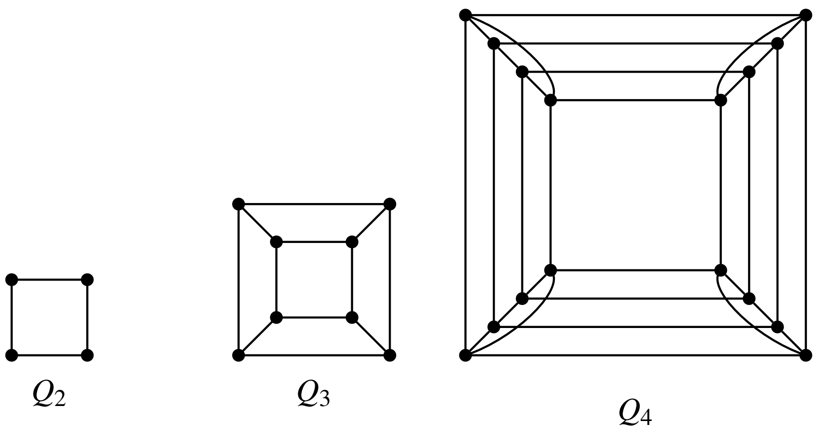


Figure 1.5: Some n-cubes

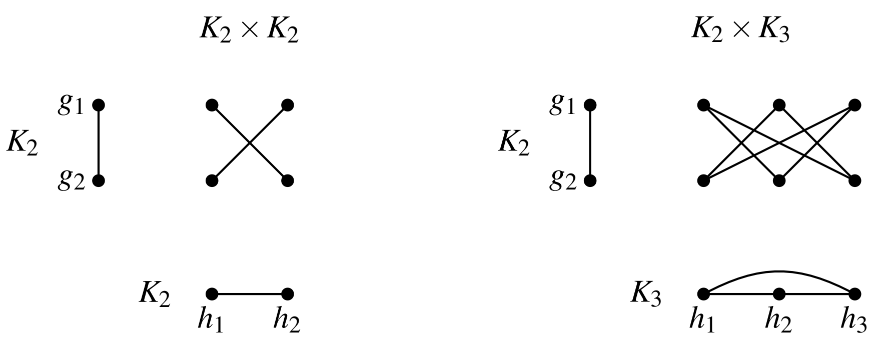


Figure 1.6: The tensor products $K_2 \times K_2$ and $K_2 \times K_3$

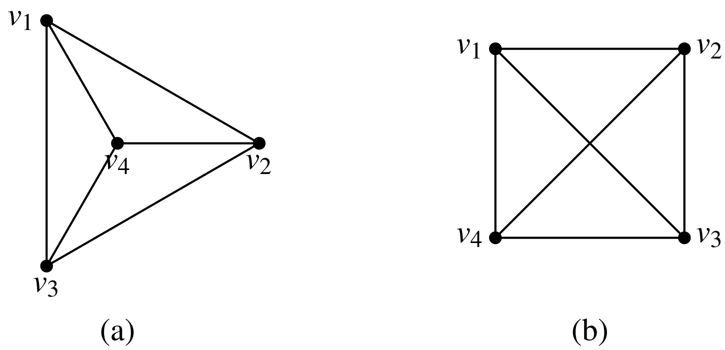


Figure 1.7: Two drawings of K_4 on S_0

been studied at length. In Chapter 4 we discuss the **weak symmetric difference product**, denoted $G \bigtriangledown H$, whose edge set is given by

$$E(G \bigtriangledown H) = \{(g_1h_1, g_2, h_2) | (g_1g_2 \in E(G) \text{ and } h_1h_2 \in E(\overline{H})) \text{ or } (g_1g_2 \in E(\overline{G}) \text{ and } h_1h_2 \in E(H))\}.$$

In several instances properties of product graphs depend on those properties of their factors: if G and H are each connected, for example, then $G \square H$ is connected as well. We also know that $G \times H$ is bipartite if either factor is bipartite. The genus of a product graph, defined in the next section, however, is not known to depend simply on the genera of its factors. Our goal is to attain some of this information with the help of voltage graph theory, defined in the following sections.

1.3 Drawings and Embeddings

Our definition of a graph was combinatorial, with a graph defined uniquely by the connections which exist between its vertices. The diagrams used to represent these connections are helpful, but they are not unique. Consider the two representations of K_4 in Figure 1.7: in (a), the edges of K_4 intersect only at their common vertices, but in (b) the edges v_1v_3 and v_2v_4 cross elsewhere.

One aim of topological graph theory is to study the relationship between a graph's combinatorial structure and its drawings. Some properties of these drawings are given in the following definitions.

We define a **surface** as a closed orientable 2-manifold, and say that a graph is **embedded** on a surface if it is drawn so that edges intersect only at common vertices. By this definition, the drawing of K_4 in Figure 1.7 (a) is an embedding, while the drawing in Figure 1.7 (b) is not. A **region** of an embedding of a graph G on a surface S is a component of $S-G$. We identify regions in an embedding by the edges that bound them; for example in Figure 1.7 (a), the outer region is bounded by the cycle v_1v_2, v_2v_3, v_3v_1, and there are four regions in total. We refer to the number of edges in the cycle bounding a region as its **size**. If a region has size k it is called a k-gon.

An embedding is **2-cell** if every region of the embedding is homeomorphic to an open disk. The **genus** of a graph G, denoted $\gamma(G)$, is the minimum genus among the surfaces on which G has a 2-cell embedding. Similarly, the **maximum genus** of G is denoted $\gamma_M(G)$ and is the maximum genus among the surfaces on which G has a 2-cell embedding. We will now use embedding to mean 2-cell embedding exclusively.

An embedding of G on $S_{\gamma(G)}$ is said to be **minimal**, while an embedding of G on $S_{\gamma_M(G)}$ is **maximal**. A graph is **planar** if it has genus zero; that is, it is embeddable on the sphere. The following theorem of Euler relates the number of vertices, edges and regions in a given embedding.

THEOREM 1.2. Any embedding of a connected graph G on the surface S_γ has satisfies $p-q+r=2-2\gamma$, where r is the number of regions in the embedding.

A graph may have distinct embeddings on the same surface. Figure 1.8 shows two embeddings of K_5 on S_1. The embedding in (a) contains two 3-gons and one 8-gon, while the embedding in (b) contains four 3-gons, one 4-gon and two 5-gons. Each embedding has

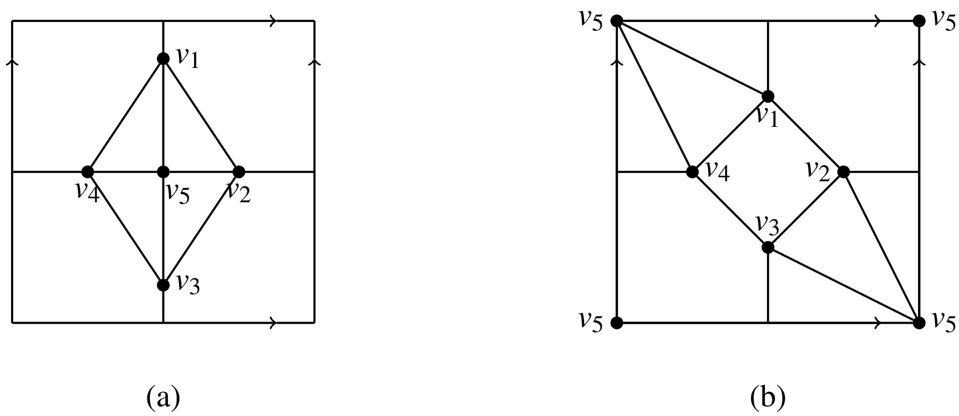


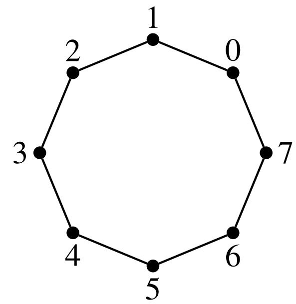
Figure 1.8: Distinct 2-cell embeddings of K_5 on S_1

a total of five regions, ten edges and five vertices, satisfying the equation of Theorem 1.2.

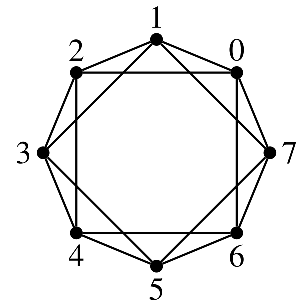
To formalize the distinction between embeddings such as the ones in 1.8, we identify a given embedding by the rotation of edges around each vertex. For each vertex $v_i \in V(G)$, define a cyclic permutation $\rho_i \colon N(v_i) \to N(v_i)$. This is the **rotation** at v_i, and the set $P = \{\rho_1, ..., \rho_p\}$ is a **rotation scheme**. There is a one-to-one correspondence between rotation schemes and embeddings. In Figure 1.8 (b), $\rho_1 = (5324)$.

The relationship from Theorem 1.2 implies that the more regions an embedding has, the lower the genus of the surface on which it is embedded. If a graph has a triangular embedding, then this embedding is necessarily minimal, as every region is as small as possible. Similarly if a graph is bipartite, then any quadrilateral embedding is minimal, as the graph has no odd cycles.

Minimal embeddings are of interest because the construction of a minimal embedding of a graph verifies the graph's genus. In addition to the minimal and maximal embeddings, we may seek **isogonal** embeddings in which every region has the same size. We will refer to **triangular** and **quadrilateral** embeddings, those isogonal embeddings in which every region has size three or four, respectively.



(a) $C_{\{1\}}(\mathbb{Z}_8)$



(b) $C_{\{1,2\}}(\mathbb{Z}_8)$

Figure 1.9: The Cayley graphs $C_{\{1\}}(\mathbb{Z}_8)$ and $C_{\{1,2\}}(\mathbb{Z}_8)$

We have discussed vertex-transitivity and edge-transitivity of graphs. An embedding is **symmetrical** if it is region-transitive as well. That is, an embedding is symmetrical if there is some automorphism which preserves its rotation scheme.

1.4 Voltage Graphs

Given a group Γ and a generating set Δ, the **Cayley graph** $C_\Delta(\Gamma)$ has vertex set $V(C_\Delta(\Gamma)) = \Gamma$. Edges in the Cayley graph are defined as follows:

$$E(C_\Delta(\Gamma)) = \{g_1g_2 | g_1, g_2 \in \Gamma \text{ and } g_2 = g_1\delta \text{ for some } \delta \in \Delta\}.$$

So vertices in $C_\Delta(\Gamma)$ are adjacent whenever they are related by some generator. Cayley graphs encode the structures of groups with respect to certain generating sets, as illustrated in Figure 1.9.

The following definitions and results are detailed in [3].

For a pseudograph G, let G^* be the graph obtained by replacing every edge in G with a pair of oppositely directed edges. A **voltage graph** is a triple (G, Γ, ϕ) where G is a pseudograph, Γ is a group, and $\phi : E(G^*) \to \Gamma$ is a function satisfying $\phi(uv) = \phi(vu)^{-1}$ for every $uv \in E(G^*)$. The values of $\phi[E(G^*)]$ are called **voltages**.

Given a region R in (G, Γ, ϕ), we define the **order of R in** Γ by $|R|_\phi = \sum_{i=1}^{n} \phi(e_i)$, where $e_1, e_2, ..., e_n$ are the edges of (G, Γ, ϕ) bounding R.

The **covering graph** $G \times_\phi \Gamma$ for (G, Γ, ϕ) has vertex set $V(G) \times \Gamma$ and $uv \in E(G)$ implies $(u, g)(v, g\phi(uv)) \in E(G \times_\phi \Gamma)$ for all $g \in \Gamma$.

Whenever (G, Γ, ϕ) is embedded on a surface S, described by the rotation scheme $P = (\rho_1, ..., \rho_p)$, then we may obtain an embedding of $G \times_\phi \Gamma$, described by a new rotation scheme. We define the **lift** $P^{'}$ of P to $G \times_\phi \Gamma$ as follows: if $\rho_v(v, u) = (v, w)$, then

$$\rho^{'}_{(v,g)}((v, g), (u, g\phi(v, u))) = ((v, g), (w, g\phi(v, w))),$$

for each $g \in \Gamma$. Then

$$P^{'} = \{\rho^{'}_{(v,g)} | (v, g) \in V(G \times_\phi \Gamma)\}.$$

The next result, due to Gross and Alpert as it appears in [3], allows us to analyze the embedding of a covering graph obtained from its voltage graph.

THEOREM 1.3. Let (G, Γ, ϕ) be a voltage graph. If R is a k-gonal region of the embedding of G, then the embedding of $G \times_\phi \Gamma$ contains $\frac{|\Gamma|}{|R|_\phi}$ $k|R|_\phi$-gons.

Proof. Let $v_1, ..., v_k$ be a closed walk w bounding R, with $e_i = (v_i, v_{i+1}),$ mod k. Then we can also express w as: $e_1, ..., e_k$. Let $n = |R|_\phi$, the order of $\phi(w)$ in Γ. Then each component of the covering of R will have boundary of the form:

$$
\begin{aligned}
(v_1,g),(v_2,g\phi(e_1)),\dots,(v_k,g\phi(e_1)\phi(e_2)\dots\phi(e_{k-1})),\\
(v_1,g\phi(w)),\dots,(v_k,g\phi(w)\phi(e_1)\phi(e_2)\dots\phi(e_{k-1})),\\
.\\
.\\
.\\
(v_1,g\phi^n(w))=(v_1,g),
\end{aligned}
$$

for some $g \in \Gamma$. So each such component is a kn-gon. The number of such components is $\frac{|\Gamma|}{n}$, as the second coordinates of (v_1,g) range over Γ. $\square$

Figure 1.10 shows a voltage graph which lifts to the Petersen graph. The group used is $\mathbb{Z}_5$. The embedding determined by the voltage graph in the figure is not pictured, but we can describe it. The voltage graph has two 1-gonal regions, bounded by the loops aa and bb, each of which has order 5. Therefore each of these regions lifts to one 5-gonal region in the embedding of the covering graph. The 4-gonal region in the voltage graph, bounded by the walk ab,bb,ba,aa, has order 5 and lifts to one 20-gonal region in the embedding of the covering graph. We can compute using Theorem 1.2 that the embedding is on the surface of genus 2.

For the purpose of embedding product graphs, it is convenient that we obtain an embedding of a graph whose vertex set is $V(G)\times\Gamma$. It will be possible to choose Γ and ϕ so that $G\times_\phi\Gamma$ is in fact isomorphic to a familiar product graph, resulting in an embedding of the product.

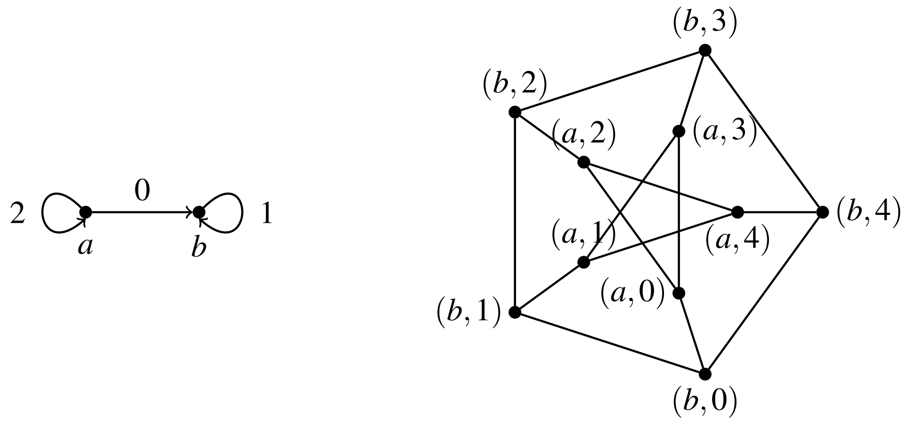


Figure 1.10: The Petersen graph as a covering graph

A **bouquet** B_n is a graph with a single vertex and n loops. Every Cayley graph $C_\Delta(\Gamma)$ is the covering graph of (B_n, Γ, ϕ) where ϕ assigns a unique element of Δ to each loop in B_n. For example letting $\Gamma = \mathbb{Z}_8$ and $\Delta = \{1,2\}$, we may construct a voltage graph with one vertex a and a loop for each element of Γ. The resulting covering graph will have vertex set $\{a\} \times \Gamma \equiv \Gamma$, and edges between exactly those vertices that are related by some generator in Δ. This is exactly the definition of $C_{\{1,2\}}(\mathbb{Z}_8)$, and results in the graph in Figure 1.9(b).

Hypercubes

In this chapter we elaborate on our earlier definition of the hypercubes and derive some of their basic properties. This is in preparation to describe embeddings of various product graphs in which one factor is a hypercube. Specifically, we will define Q_n as a Cayley graph for the group $(Z_2)^n$ and construct some voltage graphs which lift to Q_n.

2.1 Properties of Q_n

In Chapter 1 we defined Q_n as the Cartesian product of n copies of K_2. Equivalently, we may define $V(Q_n)$ as the set of bit strings of length n, and allow that two vertices are adjacent if and only if their corresponding bit strings differ by exactly one bit. These labels may be built through the iteration of the Cartesian product. Beginning with two copies of K_2, name each vertex set $\{0,1\}$. Then name the vertices in the product graph by concatenating the names of their components.

Note that $|V(Q_n)| = 2^n$, because there are 2^n bitstrings of length n. Given a bit string of length n, there are n bit strings which differ from it in exactly one place; this implies that Q_n is n-regular. From Theorem 1.1 we have $|E(Q_n)| = n2^{n-1}$.

Q_n is bipartitesince the Cartesian product of two bipartite graphs is bipartite, and K_2 is bipartite. Alternatively, our association of $V(Q_n)$ with bit strings leads to an explicit partition of the vertices into two independent sets, where every vertex is assigned to one of two sets according to the (mod 2) sum of its digits. Two vertices belonging to the same set

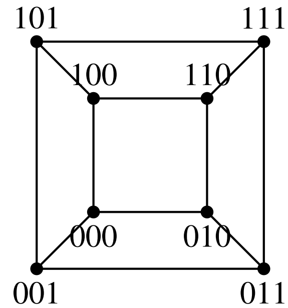


Figure 2.1: Partite sets in $V(Q_3)$

must differ by an even number of digits, so that each set is independent. This is illustrated in Figure 2.1 where the partite sets are $\{000, 011, 110, 101\}$ and $\{111, 100, 001, 010\}$.

For us this vertex labeling has another advantage; we can easily view $V(Q_n)$ as isomorphic to the group $(\mathbb{Z}_2)^n$. This will help us see Q_n as a Cayley graph. The set $\Delta = \{\delta_i \in (\mathbb{Z}_2)^n : \delta_i$ has a 1 in its i^{th} coordinate only$\}$ is a generating set for $(\mathbb{Z}_2)^n$. Our definition of $E(Q_n)$ allows us to view Q_n as the Cayley graph $C_\Delta((\mathbb{Z}_2)^n)$. We will refer to the elements of Δ as the **standard generators** of $(\mathbb{Z}_2)^n$, and denote by δ_i that element of Δ with a 1 in its i^{th} coordinate.

2.2 Q_n as a Covering Graph

We now construct some voltage graphs which lift to Q_n.

Because Q_n is a Cayley graph, it occurs as the covering graph of the n-loop bouquet in which each loop is assigned a distinct voltage from Δ. For example, Figure 2.2 shows a voltage graph of this type which lifts to Q_3. While simple, this construction does not lead to a minimal embedding. Its outer region is a 6-gon of order two, which lifts to four 6-gonal regions by Theorem 1.3. The regions bounded by invidual loops are 1-gons of order two which lift to a total of 12 2-gons in the covering graph; however, we may exclude any 2-gons

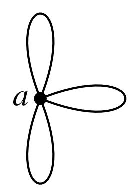


Figure 2.2: B_3 as a voltage graph which lifts to Q_3

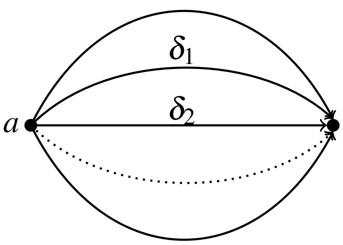


Figure 2.3: A voltage graph which lifts to Q_n

in our count as they do not affect the underlying graph. Using $p = 8, q = 12$ and $r = 4$ in Theorem 1.2 implies that our embedding is on S_1. This is not minimal because Q_3 is planar.

We will consider a different voltage graph which lifts to Q_n; by proving the embedding is minimal and applying 1.2, we will compute $\gamma(Q_n)$. Let $\Gamma = (\mathbb{Z}_2)^{n-1}$. Let $G = K_2$ with $V(G) = \{a,b\}$. We construct the voltage graph (G,Γ,ϕ) by replacing the edge in G with n edges, each assigned a distinct voltage from $\Delta \cup e$.

The covering graph has 2^n vertices and is n-regular. In fact, the edge set of the covering graph is exactly that of $K_2 \boxtimes Q_{n-1}$, which is simply Q_n. This embedding of Q_n allows a proof of the following theorem.

THEOREM 2.1. $\gamma(Q_n) = 1 + 2^{n-3}(n-4)$.

Proof. Construct the voltage graph G as described above and illustrated in Figure 2.3.

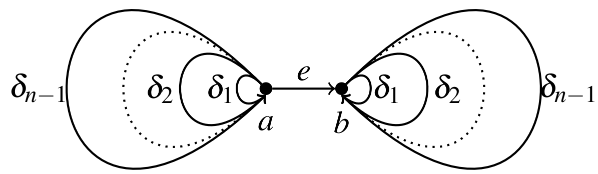


Figure 2.4: A second voltage graph which lifts to Q_n

Then $|\Gamma| = 2^{n-1}$. We have seen that the covering graph is Q_n, so that $p' = 2^n$ and $q' = 2^{n-1}n$. Because the voltage graph embedding contains n 2-gons, each of order 2, by Theorem 1.3 the covering graph embedding contains $2^{n-2}n$ 4-gons. Because Q_n is bipartite, this quadrilateral embedding is minimal. By Theorem 1.2, the embedding is on S_h, where $h = 1 + 2^{n-3}(n-4)$. $\square$

Finally we consider the voltage graph in 2.4. The subgraph induced by $\{a\}$ is a bouquet which lifts to Q_{n-1}, as does the subgraph induced by $\{b\}$. The covering graph contains these two copies of Q_{n-1}, in addition to edges between each pair of corresponding vertices in the subgraphs. So the covering graph is simply $Q_{n-1}\square K_2 = Q_n$. Again the embedding is bipartite and thus minimal.

Product Embeddings

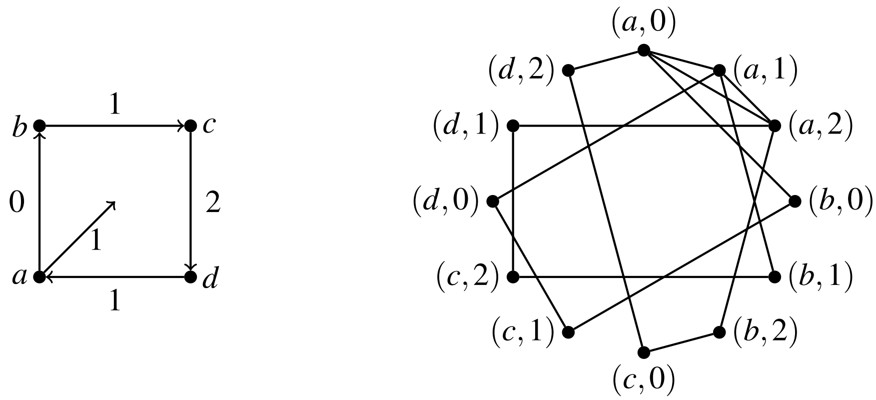
We now investigate embeddings of products of the form $G * Q_n$, concentrating on those products defined in Chapter 1. To illustrate embedding techniques, we consider cases where G is a familiar planar graph. We then look to some products of Q_n with quadrilateral tilings of the torus.

3.1 Embedding Techniques

Section 2.2 described a way to modify a single edge or vertex into a voltage graph lifting to Q_n. In this chapter we develop the method further, modifying an arbitrary graph G into a voltage graph which lifts to a product $G * Q_n$ of our choosing. Throughout this discussion we will use $\Gamma = (\mathbb{Z}_2)^n$. When considering a product $G * Q_n$, we will denote $|V(G * Q_n)|$ by p' and $|E(G * Q_n)|$ by q'.

Every element of Γ is self-inverse, so to simplify voltage graph constructions we need not consider the edges of G as directed edges. Another convenience is that the regions in these voltage graphs will have order one or two.

Some of the voltage graphs that follow contain **half-edges**. A half-edge has exactly one endpoint and contributes one to the size of a region while contributing nothing to the net voltage; intuitively, we traverse the half-edge in both directions while computing the net voltage of a region. The inner region of the voltage graph in Figure 3.1 is bounded by the walk a, b, c, d, a, a, so its net voltage is 0+1+2+1+1+1^{-1}=1, which has order three in $\mathbb{Z}_3$. We consider the region a 5-gon, so it will lift to a 15-gon in the covering graph embedding.



(a) $(G, \mathbb{Z}_3, \phi)$ (b) $G \times_\phi \mathbb{Z}_3$

Figure 3.1: A half-edge in a voltage graph

Table 3.1: Platonic Solids

Name	p	q	r
Tetrahedron (PS_4)	4	6	4
Octahedron (PS_8)	6	12	8
Icosahedron (PS_{20})	12	30	20
Dodecahedron (PS_{12})	20	30	12

Half-edges are useful for minimizing the size of regions in the voltage graph while forcing certain edges into the covering graph. In Figure 3.1, the effect of the half-edge at a is to ensure that the edge $(a,g)(a,g+1)$ belongs to the edge set of the covering graph for every $g \in \mathbb{Z}_3$. Only the underlying graph of the covering graph is shown in the figure, and it is not represented as an embedding.

The following techniques are due to Abay-Asmerom and appear in [3] along with similar techniques for embedding additional product graphs.

3.1.1 Cartesian Product Embeddings

We have seen already that Q_n arises as the covering graph of a bouquet. By similar logic, the single-vertex graph with n half-edges assigned distinct elements of Δ also lifts to Q_n. We exploit this property to obtain an embedding of $G\square Q_n$.

Given an embedding of G, construct a voltage graph (G,Γ,ϕ) by assigning voltage $e \in \Gamma$ to each edge of G and adding n half-edges at each vertex, one labeled by each of the n standard generators of Γ. The covering graph may be described similarly to that of Figure 2.4. For each vertex in G, the subgraph induced by that vertex lifts to a copy of Q_n in the covering graph, while the edges between the vertices in G lift to those edges connecting adjacent copies of Q_n.

We can define the edge set of the covering graph formally. The vertices (v,g_1) and (u,g_2) are adjacent in $G\times_\phi \Gamma$ if and only if one of the following conditions hold. Either $v=u$ and $g_2=g_1\delta_i$ for some $\delta_i\in\Delta$, in which case the edge results from some half-edge in the voltage graph, or $g_1=g_2$ and $uv\in E(G)$, in which case the edge results from some original edge from G. This definition is equivalent to that of $E(G\square Q_n)$, using the definition of Q_n as a Cayley graph given in Chapter 2.

Figure 3.2 shows a voltage graph which will lift to $PS_8\square Q_8$. Each full edge has voltage $e\in(\mathbb{Z}_2)^8$ while the n half-edges around each vertex have distinct voltages from the standard generators of $(\mathbb{Z}_2)^8$; beyond this, the specific assignment of voltages for the half-edges is irrelevant, as each half-edge contributes nothing to the order of the region it bounds.

We may use Theorem 1.3 to describe the embedding of $PS_8\square Q_8$ associated with the voltage graph in Figure 3.2. The product is embedded on S_h where $h=1+2^6(44)=2817$, but we can say more. Because $n\equiv 0(\mathrm{mod}\ 4)$, we can obtain an isogonal embedding of $PS_8\square Q_8$ on S_h by distributing the half-edges evenly between the regions of PS_8, as shown in the figure. The covering graph is isogonal because each region of the voltage graph is an

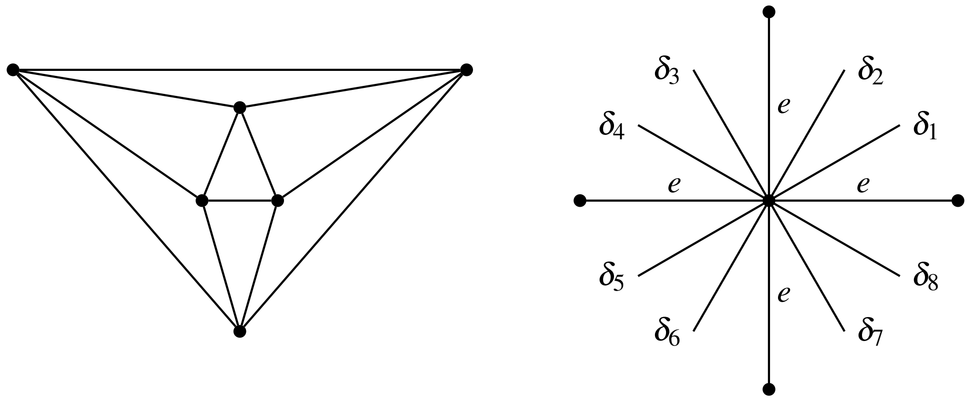


Figure 3.2: A voltage graph which lifts to $PS_8 \square Q_8$

order one 9-gon.

The following theorem generalizes the above argument to embeddings of $G \square Q_n$.

THEOREM 3.1. Given an embedding of G on S_k, we may define an embedding of $G \square Q_n$ on the surface of genus $h = 1 + 2^{n-2}(pn + 4k - 4)$. Given an isogonal embedding of an r-regular graph, we may ensure that this embedding of $G \square Q_n$ is isogonal if $n \equiv 0 (\mathrm{mod}\ r)$.

Proof. We have $p' = 2^n p$ and $q' = 2^n q + 2^{n-1} pn$. Because each of the $2 - 2k + q - p$ regions in the embedding of G has order one in the voltage graph, $G \square Q_n$ has $r' = 2^n(2 - 2k + p - q)$ regions. Applying Theorem 1.2 gives $h = 1 + 2^{n-2}(pn + 4k - 4)$. If $n \equiv 0 (\mathrm{mod}\ r)$ and the embedding of G is isgonal, we may place at each vertex exactly n/r half-edges in each region bounded by that vertex. Then each s-gon in G becomes an $(s + s(n/r))$-gon in the voltage graph, lifting to 2^n $(s + s(n/r))$-gons in the embedding of $G \square Q_n$. $\square$

3.1.2 Tensor Product Embeddings

To produce the tensor product $G \times Q_n$, we construct a voltage graph (G, Γ, ϕ) by first replacing each edge of G with n edges, then assigning voltages to these edges corresponding to the n standard generators of Γ. Then, by definition of $E(G \times_\phi \Gamma)$, (v, g_1) and (u, g_2)

are adjacent in $G \times_\phi \Gamma$ if and only if $vu \in E(G)$ and $g_2 = g_1\phi(vu)$. Because $\phi(vu)$ is some standard generator of Γ, this is equivalent to the statement that $g_1 g_2 \in E(Q_n)$. So $E(G \times_\phi \Gamma) = E(G \times Q_n)$.

One consequence of this construction is that the resulting embedding of $G \times Q_n$ will necessarily contain at least $2^{n-1}q(n-1)$ 4-gons, corresponding to the $q(n-1)$ 2-gonal regions in the voltage graph. The ratio of numbers of regions of different sizes will remain unchanged, unless we adjust the ordering of the voltages to ensure that regions of even size have order one.

THEOREM 3.2. Given an embedding of G on S_k, we may obtain an embedding of $G \times Q_n$ on the surface of genus $h \leq 1 + 2^{n-2}(qn - (p+2) + 2k)$, with equality holding whenever every region in G has order 2. If t is the number of regions in the voltage graph having order one, then $h = 1 + 2^{n-2}(qn - (p+2+t) + 2k)$.

Proof. We have $p' = 2^n p$ and $q' = 2^n qn$. We compute the number of regions in the voltage graph by adding the $2 - 2k + q - p$ regions in the original embedding of G to the $q(n-1)$ 2-gons created by adding edges in the voltage graph, so that the voltage graph has $2 - 2k + qn - p$ regions. Each region in the voltage graph has order at most two, so that $r' \geq 2^{n-1}(2 - 2k + qn - p)$. Applying Theorem 1.2 gives $h \leq 1 + 2^{n-2}(qn - (p+2) + 2k)$.

We may of course adjust the voltages so that certain regions have order one. If t is the number of regions in the voltage graph having order one, then there are $2 - 2k + qn - p - t$ regions having order two, so that $r' = 2^n t + 2^{n-1}(2 - 2k + qn - p - t)$. As above, we use the values of p', q' and r' along with Theorem 1.2 to conclude that $h = 1 + 2^{n-2}(qn - (p+2+t) + 2k)$. □

For example, by Theorem 3.2 we can embed $PS_{20} \times Q_n$ on S_h where $h = 1 + 2^{n-2}(30n - 14)$. Because every region in PS_{20} has odd size, each region in our voltage graph must have order two. In the voltage graph there are 20 3-gons and $30(n-1)$ 2-gons. Therefore the associated embedding of $PS_{20} \times Q_n$ contains $20 \cdot 2^{n-1}$ 6-gons and $30(n-1) \cdot 2^{n-1}$ 4-gons. So this embedding is not isogonal, despite the fact that the embedding of the voltage graph was isogonal.

In general, because our voltage graph construction forces 4-gons into the corresponding covering graph embedding, we may not obtain an isogonal embedding of $G \times Q_n$ by this method unless we begin with a quadrilateral embedding of G.

3.1.3 Augmented Tensor Product Embeddings

Finally, we obtain an embedding of the augmented tensor product $G \boxtimes Q_n$ by a slight adjustment of the voltage graph used for the tensor product above; we replace each edge of G with $n+1$ edges instead of n, with the extra edge in each case receiving the voltage $e \in \Gamma$. We may sometimes assign these identity voltages to edges bounding the regions of odd length in G so as to ensure that each such region has order one. The following theorem describes the embeddings obtained from these constructions.

THEOREM 3.3. Given an embedding of G on S_k, we may obtain an embedding of $G \boxtimes Q_n$ on the surface of genus $h \leq 1 + 2^{n-2}(qn - (p+2) + 2k + q)$, with equality holding whenever each region has order two. If t is the number of regions in the voltage graph having order one, then $h = 1 + 2^{n-2}(qn - (p+2+t) + 2k + q)$.

Proof. We have $p' = 2^n p$ and $q' = 2^n q + 2^n qn = 2^n q(n+1)$. We compute the number of regions in the voltage graph by adding the $2 - 2k + q - p$ regions in the original embedding of G to the qn 2-gons created by adding edges in the voltage graph, so that the voltage graph has $2 - 2k + qn - p + q$ regions. Each region in the voltage graph has order at most two, so that $r' \geq 2^{n-1}(2 - 2k + qn - p + q)$. Applying Theorem 1.2 gives $h \leq 1 + 2^{n-2}(qn - (p + 2) + 2k + q)$.

Now if t is the number of regions in the voltage graph having order one, then there are $2-2k+qn-p+q-t$ regions having order two, so that $r' = 2^n t + 2^{n-1}(2-2k+qn-p-+qt)$. As above, we use the values of p', q' and r' along with Theorem 1.2 to conclude that $h = 1+2^{n-2}(qn-(p+2+t)+2k+q)$. □

For example when constructing $PS_{20} \boxtimes Q_n$, we may adjust the ordering of the voltages along the $n+1$ new edges between each pair of adjacent vertices in PS_{20} so that each of the 20 3-gons in the voltage graph has order one. The remaining $30n$ 2-gons each have order 4, so that the covering graph embedding has $20 \cdot 2^n$ 3-gons and $30n \cdot 2^{n-1}$ 4-gons. Again, this embedding is not isogonal.

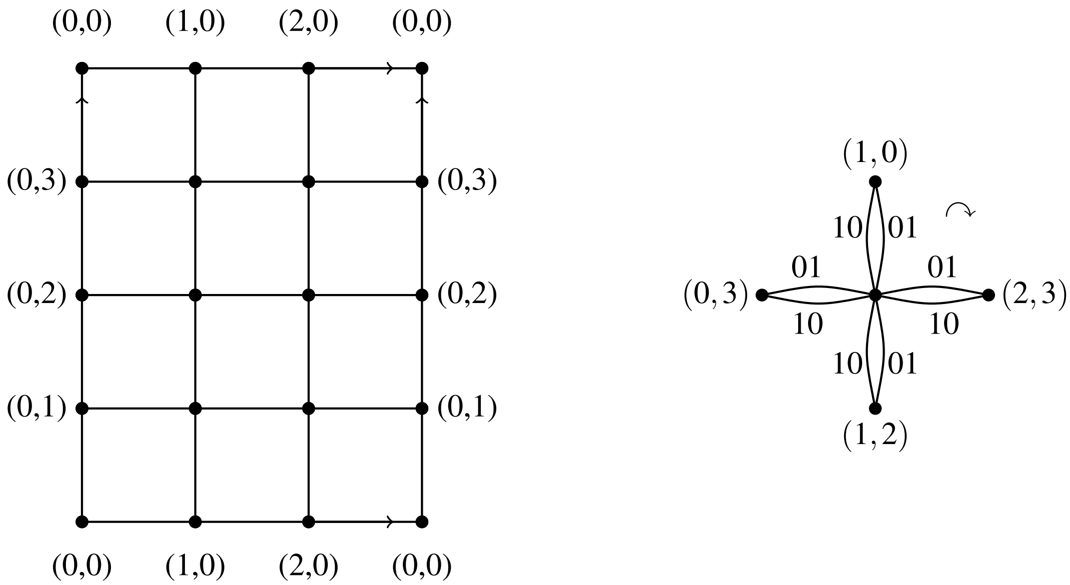
To obtain an isgonal embedding of $G \boxtimes Q_n$, we again need a voltage graph in which each region is an order one 4-gon or an order two 2-gon. In the following section we consider some graphs with quadrilateral embeddings, from which we may obtain isgonal embeddings of their products with Q_n.

3.2 Symmetrical Embeddings

Symmetrical embeddings are of interest because of their great symmetry; in addition to being regular, isogonal, vertex-transitive and edge-transitive, they can be thought of as region-transitive as well.

We will define an infinite class of minimal, symmetrical embeddings of the form $G \times Q_n$, where G is a tiling of the torus by quadrilaterals. To prove that a given embedding is symmetrical, we use the following condition.

A **Cayley map** is a Cayley graph $C_\Delta(\Gamma)$ together with a rotation scheme ρ and a cyclic permutation $r : \Delta^* \to \Delta^*$ such that $\rho_g(h) = gr(g^{-1}h)$ for every $g \in \Gamma$ and $h \in N(g)$. That is, a Cayley map is an embedding of a Cayley graph such that the rotation of generators around



(a) $C_3 \square C_4$ on S_1

(b) Rotation around (1,3) in $(C_3 \square C_4, (\mathbb{Z}_2)^2, \phi$

Figure 3.3: Modifying $C_3 \square C_4$ into a voltage graph

each vertex is fixed. Given a Cayley map, if there is some $\alpha \in \operatorname{Aut}(\Gamma)$ such that $\alpha|_{\Delta^*} = r$, then the embedding is symmetrical.

To illustrate the general approach we consider the Cartesian product $C_3 \square C_4$ embeddeded on S_1 as in Figure 3.3 (a). Note that $C_3 \square C_4$ is the Cayley graph $C_{\Delta_1}(\mathbb{Z}_3 \times \mathbb{Z}_4)$ where $\Delta_1 = \{(0,1),(0,-1),(1,0),(-1,0)\}$. This is a tiling of the torus by quadrilaterals where $p = 12, q = 24$ and, because G is 4-regular, $r = 12$.

We obtain a quadrilateral embedding of $C_3 \square C_4 \times Q_2$ by the methods of Section 3.1.2. This embedding is minimal because $C_3 \square C_4 \times Q_2$ is bipartite. The associated voltage graph $(C_3 \square C_4, (\mathbb{Z}_2)^2, \phi)$ is represented in Figure 3.3. By Theorem 3.2 and the fact that the voltage graph has 12 regions of order one, this embedding of $C_3 \square C_4 \times Q_2$ is on the surface of genus $1 + 24(2) - (12 + 2 + 12) + 2(1) = 25$.

Because G and Q_2 are both Cayley graphs, the tensor product $C_3 \square C_4 \times Q_2$ is the Cayley graph $C_{\Delta_1 \times \Delta_2}((\mathbb{Z}_3 \times \mathbb{Z}_4) \times (\mathbb{Z}_2)^2)$, where $\Delta_2 = \{10, 01\}$. To verify that the given embedding is a Cayley map, we must formalize its rotation scheme, which is defined as the lift of the rotation scheme for $C_3 \Box C_4$. For a vertex $(a,b) \in V(C_3 \Box C_4)$, denote the edges in $(C_3 \Box C_4, (\mathbb{Z}_2)^2, \phi)$ which are incident to (a,b) by $(a',b')_i$, where $\phi((a',b')(a,b)) = i$. Now the rotation of edges at a vertex (a,b) is given by

$$\begin{aligned}\rho_{(a,b)} =&((a,b+1)_{10}(a,b+1)_{01}(a+1,b)_{01}(a+1,b)_{10}\\ &(a,b-1)_{01}(a,b-1)_{10}(a-1,b)_{10}(a-1,b)_{01}).\end{aligned}$$

For example, $\rho_{(1,3)} = ((1,0)_{10},(1,0)_{01},(2,3)_{01},(2,3)_{10},(1,2)_{01},(1,2)_{10},(0,3)_{10},(0,3)_{01})$, which is represented clockwise in Figure 3.3 (b).

Now let $\rho^{'}$ be the lift of ρ to $C_3 \Box C_4 \times_\phi (\mathbb{Z}_2)^2 \simeq C_3 \Box C_4 \times Q_2$, as defined in Section 1.4. For an edge $(a',b')_i$, denote $\rho_{(a,b)}(a'b')_i$ by $(a^*,b^*)_k$, where k is the voltage assigned to the next edge in the rotation about (a,b). Fix $g \in (\mathbb{Z}_2)^2$.

Then $\rho^{'}_{(a,b,g)}(a',b',gi) = (a^*,b^*,gk)$. We know, for example, that $\rho_{(1,3)}(1,0)_{01} = (2,3)_{01}$. Letting $g = 11$, we can compute $\rho^{'}_{(1,3,11)}(1,0,10) = (2,3,10)$. To compute $\rho^{'}_{(1,3,11)}(2,3,10)$, note that $(2,3)_{10} = \rho_{(1,3)}(2,3)_{01}$ and use the same formula as before to obtain $\rho^{'}_{(1,3,11)}(2,3,10) = (2,3,01)$. Figure 3.4 illustrates the complete rotation of vertices around $(1,3,11)$ in $C_3 \Box C_4 \times Q_2$. Note that because $(\mathbb{Z}_2)^2$ contains only 4 elements, the rotation in the figure looks misleadinginly similar to the rotation of edges around (1,3) in the voltage graph.

Define a cyclic permutation of $\Delta_1 \times \Delta_2$ by

$$r = ((0,1,10)(0,1,01)(1,0,01)(1,0,10)(0,-1,01)(0,-1,10)(-1,0,10)(-1,0,01)).$$

We can now verify that

$$\begin{aligned}\rho_{(a,b,g)}(a',b',gi) &= (a^*,b^*,gk)\\ &= (a,b,g)+r((a,b,g)^{-1}+(a',b',gi))\\ &= (a,b,g)+r((a'-a,b'-b,i)), \end{aligned} \tag{3.1}$$

which implies that our embedding of $C_3\square C_4\times Q_2$ is a Cayley map. To demonstrate the above equality, consider again the vertex $(1,3,11)$ and its neighbor $(1,0,10)$. Then $(1,3,11)+r((1,3,11)^{-1}+(1,0,10)) = (1,3,11)+r(0,1,01) = (2,3,10)$. This was exactly our result for $\rho'_{(1,3,11)}(1,0,10)$.

Finally, we define

$$\alpha(a,b,g) = \begin{cases} (a,b,10) & \text{if } (a,b,g) = (0,1,01) \text{ or } (-1,0,01)\\ (a,b,01) & \text{if } (a,b,g) = (0,-1,10) \text{ or } (1,0,10)\\ (-b,a,10) & \text{if } (a,b,g) = (0,1,10) \text{ or } (1,0,01)\\ (-b,a,01) & \text{if } (a,b,g) = (-1,0,10) \text{ or } (0,-1,01)\\ (a,b,g) & \text{otherwise} \end{cases}$$

for $(a,b,g)\in V(C_3\square C_4\times Q_2)$. Because $\alpha\in \mathrm{Aut}(C_3\square C_4\times Q_2)$ and satisfies $\alpha|_{\Delta_1\times\Delta_2} = r$, we may conclude that this embedding of $C_3\square C_4\times Q_n$ is symmetrical.

Generalizing, we obtain a minimal and symmetrical embedding of $G\times Q_n$ for each $G\simeq C_s\square C_t$. In the general case, we may view G as the Cayley graph $C_{\Delta_1}(\mathbb{Z}_s\times\mathbb{Z}_t)$ where again $\Delta_1 = \{(0,1),(0,-1),(1,0),(-1,0)\}$. In this case we have $p = st$, $q = 2st$ and $r = st$. The voltage graph for this construction is illustrated in Figure 3.5.

To verify that $G\times Q_n$ is a Cayley map, we now use the rotation of edges around a vertex (a,b) given by

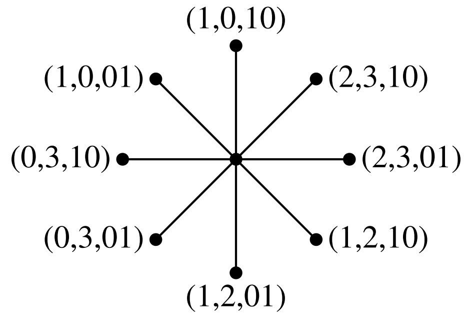


Figure 3.4: Rotation of vertices around (1,3,11) in $C_3\Box C_4 \times Q_2$

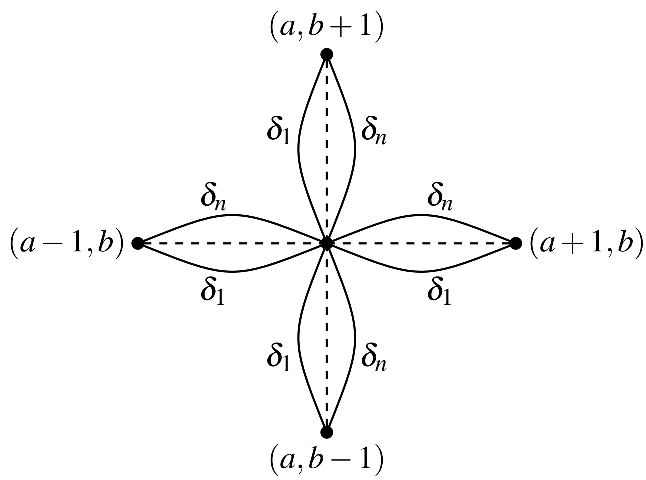


Figure 3.5: Rotation of edges around (a,b) in (G,Γ,ϕ)

$$\begin{aligned}\rho_{(a,b)} =&((a,b+1)_{\delta_1}...(a,b+1)_{\delta_n}(a+1,b)_{\delta_n}...(a+1,b)_{\delta_1}\\ &(a,b-1)_{\delta_n}(a,b-1)_{\delta_1}(a-1,b)_{\delta_1}...(a-1,b)_{\delta_n}).\end{aligned}$$

We define $\rho^{'}$, the lift of ρ to $G\times_\phi \Gamma$, just as before so that $\rho^{'}_{(a,b,g)}(a',b',g\delta_i) = (a^*,b^*,g\delta_k)$. Figure 3.6 illustrates this rotation.

We use the cyclic permutation r of $\Delta_1^* \times \Delta_2^* = \Delta_1 \times \Delta_2$ defined as

$$\begin{aligned} r =& ((0,1,\delta_1)...(0,1,\delta_n)(1,0,\delta_n)...(1,0,\delta_1) \\ & (0,-1,\delta_n)(0,-1,\delta_1)(-1,0,\delta_1)...(-1,0,\delta_n)), \end{aligned}$$

which again satisfies the condition from Equation 3.1. So the embedding of $G \times Q_n$ generated by the voltage graph in Figure 3.5 is in fact a Cayley map. To prove that the embedding is symmetrical, it remains to demonstrate the neccesary automorphism $\alpha \in \mathrm{Aut}(G)$, which is defined in a similar way as in the example.

Define

$$\alpha(a,b,g) = \begin{cases} (a,b,\delta_{i+1}) & \text{if } (a=0, b=1 \text{ and } i<n) \text{ or } (a=-1, b=0 \text{ and } i<n) \\ (a,b,\delta_{i-1}) & \text{if } (a=0, b=-1 \text{ and } i>1) \text{ or } (a=1, b=0 \text{ and } i>1) \\ (-b,a,\delta_n) & \text{if } (a=0, b=1 \text{ and } i=n) \text{ or } (a=1, b=0 \text{ and } i=1) \\ (-b,a,\delta_1) & \text{if } (a=-1, b=0 \text{ and } i=n) \text{ or } (a=0, b=-1 \text{ and } i=1) \\ (a,b,g) & \text{otherwise} \end{cases}$$

where the first four conditons suppose that $g \in \Delta_2$. Then $\alpha \in Aut(\mathbb{Z}_s \times \mathbb{Z}_t \times (\mathbb{Z}_2)^n)$ and $\alpha|_{\Delta_1 \times \Delta_2} = r$.

This embedding of $G \times Q_n$ is again minimal, as it is a quadrilateral embedding and $G \times Q_n$ is bipartite. Using Theorem 3.2 and the fact that there were st regions in the voltage graph of order one, we may conclude that our embedding of $G \times Q_n$ is on the surface of genus $1 + 2^{n-1}(st(n-1))$.

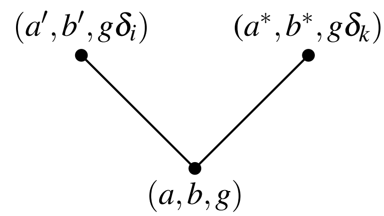


Figure 3.6: Two consecutive edges in the rotation around (a,b,g) in $G \times Q_n$

As G was arbitrary, we have shown that $\gamma(C_s \Box C_t \times Q_n) = (1 + 2^{n-1}(st(n-1)))$.

A similar argument yields a quadrilateral embedding of $G \times Q_n$ where G is now the infinite tiling of the plane by quadrilaterals, corresponding to the Cayley graph $C_\Delta(\mathbb{Z} \times \mathbb{Z})$.

Weak Symmetric Difference Product

In Chapter 1 we defined the weak symmetric difference of simple graphs G and H, $G \bigtriangledown H$, as the product graph whose edge set is

$$E(G \bigtriangledown H) = \{(g_1,h_1)(g_2,h_2) : (g_1g_2 \in E(G) \text{ and } h_1h_2 \in E(\overline{H})) \text{ or } (g_1g_2 \in E(\overline{G}) \text{ and } h_1h_2 \in E(H))\}.$$

Figure 4.1 illustrates the weak symmetric difference of two small graphs. In each example, edges in $\overline{G}$ and $\overline{H}$ are indicated by dashed lines.

4.1 Properties

We now derive some basic properties of the weak symmetric difference.

Notice that $E(G \bigtriangledown H) = E(G \times \overline{H}) \cup E(\overline{G} \times H)$. Recall that $|E(G \times H)| = 2qq'$. To compute $|E(G \bigtriangledown H)|$, we'll use this along with the fact that $|E(\overline{G})| = \binom{p}{2} - q$ for all graphs G. This gives

$$\begin{aligned}
|E(G \bigtriangledown H)| &= |E(G \times \overline{H})| + |E(\overline{G} \times H)| \\
&= 2q\left(\binom{p'}{2} - q'\right) + 2q'\left(\binom{p}{2} - q\right) \\
&= qp'(p'-1) + q'p(p-1) - 4qq'. \qquad (4.1)
\end{aligned}$$

If $g \in V(G)$ has degree d_g and $h \in V(H)$ has degree d_h, then we may compute the degree

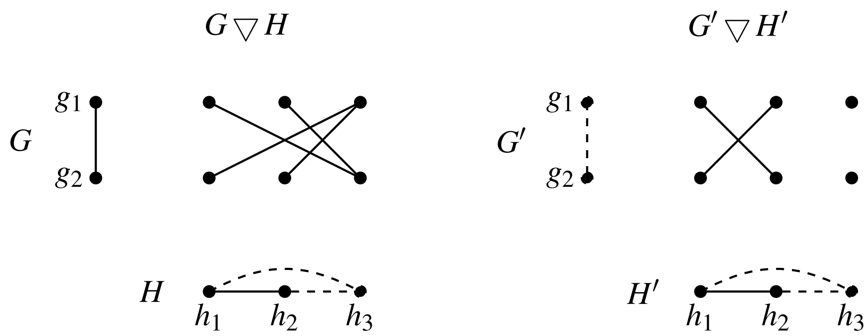


Figure 4.1: Two examples of the weak symmetric difference

of $(g,h) \in V(G \bigtriangledown H)$. The computation relies on the observation that g has degree $p-1-d_g$ when viewed as a vertex in $\overline{G}$, and that h has degree $p'-1-d_h$ when viewed as a vertex in $\overline{H}$. Then

$$\deg((g,h)) = d_g(p'-1-d_h) + d_h(p-1-d_g). \tag{4.2}$$

Equations 4.1 and 4.2 confirm that the weak symmetric difference of two empty graphs is empty, as is the weak symmetric difference of two complete graphs. No weak symmetric difference will be a complete graph, as there will be no edges between, for example, the vertices (g,h_1) and (g,h_2); this is because there are no loops gg in either G or $\overline{G}$. Because there are $p\binom{p'}{2} + p'\binom{p}{2}$ such pairs of vertices in $V(G \bigtriangledown H)$, and $\binom{pp'}{2}$ pairs of vertices total, the maximum number of edges in $G \bigtriangledown H$ is given by $\binom{pp'}{2} - p\binom{p'}{2} - p'\binom{p}{2} = 2\binom{p}{2}\binom{p'}{2}$. If one of G and H is complete while the other is empty, or if $H = \overline{G}$, then this maximum number of edges is obtained in $G \bigtriangledown H$. Whenever the maximum number of edges is obtained, $G \bigtriangledown H$ is connected.

4.2 Embedding techniques

Given a graph G, we obtain an embedding of $G \bigtriangledown Q_n$ via some embedding of K_p. As usual, let $\Gamma = (\mathbb{Z}_2)^n$ and denote by Δ the standard generators of Γ. Consider G as an induced subgraph of K_p.

Notice that $\overline{Q_n}$ is the Cayley graph for Γ with respect to $\Delta^{'} = \Gamma - (\Delta \cup \{e\})$, and that $|\Delta^{'}| = 2^n - n - 1$. Define $\Delta^{'} = \{d_1, ..., d_m\}$. We construct a voltage graph (K_p, Γ, ϕ) from our embedding of K_p by replacing each edge of $K_p[V(G)]$ with $2^n - n - 1$ edges and assign voltages corresponding to the elements of $\Delta^{'}$. We then replace each remaining edge of K_p with n edges, and assign voltages to these edges corresponding to the elements of Δ. Figure 4.2 illustrates such a voltage graph, with the edges from $\overline{G}$ indicated by dashed lines in the original embedding of K_p.

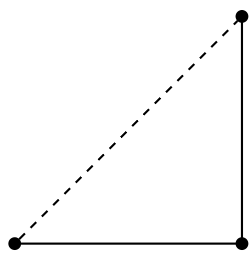
Now an edge $(g_1, h_1)(g_2, h_2)$ occurs in the covering graph $K_p \times_\phi \Gamma$ if and only if $g_1 g_2 \in E(G)$ and $h_2 = h_1 \delta_i$ for some $\delta_i \in \Delta^{'}$, or $g_1 g_2 \in E(\overline{G})$ and $h_2 = h_1 \delta_i$ for some $\delta_i \in \Delta$. That is, the edge set of the covering graph is given by $\{(g_1, h_1)(g_2, h_2) : (g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(\overline{Q_n}))$ or $(g_1 g_2 \in E(\overline{G})$ and $h_1 h_2 \in E(Q_n))\}$, which is exactly the edge set of $G \bigtriangledown Q_n$.

For example, consider C_5. From the quadrilateral embedding of K_5 on S_1 given in (Figure), we construct a voltage graph which lifts to $C_5 \bigtriangledown Q_3$. The voltage graph has 5(2) order two 2-gons corresponding to the edges in C_5, 5(3) order two 2-gons corresponding to the edges in $\overline{C_5}$, and 5 order two 4-gons corresponding to the original regions in the embedding of C_5. This gives a total of $4(10+15+5) = 120$ regions in the covering graph.

We have from Equation 4.1 that $|E(C_5 \bigtriangledown Q_3| = (5)(8)(7) + (12)(5)(4) - (12)(5)(4) = 280$. So by Theorem 1.2, this embedding of $C_5 \bigtriangledown Q_3$ is on S_{61}. Note that it is not minimal, as it does contain 20 8-gons.

The general case is described in the following theorem.

THEOREM 4.1. Given a graph G and an embedding of K_p on S_k, we may construct an



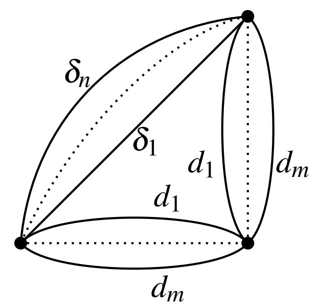


Figure 4.2: Voltage assignment in a graph lifting to $G \bigtriangledown Q_n$

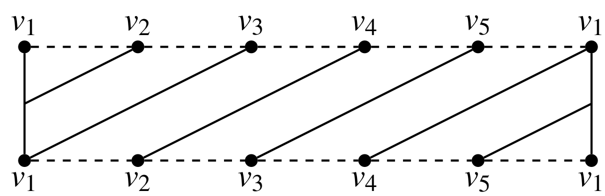


Figure 4.3: K_5 on S_1, with edges not from C_5 dashed

embedding of $G \bigtriangledown Q_n$ on S_h where $h = 1 + 2^{n-2}(-2p + 2q(2^n - 1) + np(p-1) - 4qn - q(2^n - n - 2) - (\binom{p}{2} - 2) - \binom{p}{2} + p - 2 + 2k)$.

Proof. Let the voltage graph be as illustrated in Figure 4.2, and assume that the net voltage around each region has order 2 in Γ. The original embedding of K_p has $\binom{p}{2} - p + 2 - 2k$ regions. Each of the q edges of $K_p[V(G)]$ corresponds to $2^n - n - 2$ regions in the voltage graph, and each of the remaining $\binom{p}{2} - q$ edges corresponds to $n - 1$ regions in the voltage graph, so that the covering graph contains $2^{n-1}(\binom{p}{2} - p + 2 - 2k + q(2^n - n - 2) + (\binom{p}{2} - q)(n-1)) = 2^{n-1}(n(\binom{p}{2} + q - 1) - q - p - 2k + 2^n)$ regions.

By Equation 4.1, there are $q2^n(2^n - 1) + n2^{n-1}p(p-1) - 4qn2^{n-1}$ edges in the covering graph. Applying Theorem 1.2 gives the result. $\square$

When G is a self-complementary graph, as in the example, $G \bigtriangledown Q_n \equiv G \times K_{2^n}$, since

$E(G \bigtriangledown Q_n) = E(G \times \overline{Q_n}) \cup E(G \times Q_n)$. In this way, a minimal embedding of the weak symmetric difference of a self-complementary graph with Q_n can give information about the genus of a different tensor product.

Bibliography

Bibliography

[1] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Chapman and Hall, Florida, 2005.

[2] Jonathan L. Gross and Thomas W. Tucker, *Topological Graph Theory*, Wiley Interscience, 1987.

[3] Arthur T. White, *Graphs of Groups on Surfaces*, Elsevier, 2001.