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Parity Domination in Product Graphs

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A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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Contents

Abstract

PARITY DOMINATION IN PRODUCT GRAPHS

By Christopher Alan Whisenant, Master of Science.

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An odd open dominating set of a graph is a subset of the graph's vertices with the property that the open neighborhood of each vertex in the graph contains an odd number of vertices in the subset. An odd closed r-dominating set is a subset of the graph's vertices with the property that the closed r-ball centered at each vertex in the graph contains an odd number of vertices in the subset. We first prove that the n-fold direct product of simple graphs has an odd open dominating set if and only if each factor has an odd open dominating set. Secondly, we prove that the n-fold strong product of simple graphs has an odd closed r-dominating set if and only if each factor has an odd closed r-dominating set.

Preliminaries

A *graph* G is a finite nonempty set $V(G)$ of objects called *vertices*, together with a set $E(G)$ of unordered pairs of distinct vertices of G called *edges*. The set $V(G)$ is called the *vertex set* of G while the set $E(G)$ is called the *edge set* of G. We will conveniently denote an edge $\{u,v\}$ by either uv or vu. A *simple graph* G is a graph with no loops (i.e., no edges of the form vv) and no multiple edges. Throughout this paper, the term *graph* will always mean *simple graph*. Figure 1 illustrates a graph G where $V(G) = \{s,t,u,v,w,x,y,z\}$ and $E(G) = \{st,tu,uv,wx,xy,yz,sw,tx,uy,vz\}$.

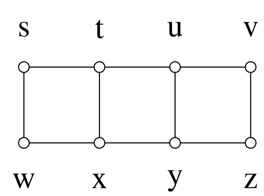
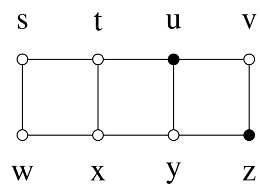


Figure 1

An edge $e = uv$ is said to *join* the vertices u and v. Two vertices u and v are considered to be *adjacent* if they are *joined* by the edge $e = uv$. The edge e and vertex u are *incident*, as are e and v. If e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are *adjacent edges*.

The *open neighborhood* of a vertex v in a graph G, denoted $N(v)$, is the set of all vertices u in $V(G)$ that are adjacent to v. The *closed neighborhood* of a vertex v in a graph G, denoted $N[v]$, is the set of all vertices u in $V(G)$ that are adjacent to v together with the vertex v. Thus $N[v] = N(v) \cup \{v\}$. A comparison of the two types of neighborhoods is given in Figures 2a and 2b, where the dark vertices are elements of the $N(v)$ and $N[v]$ respectively. Notice in Figure 2a that the vertex v is not included in the open neighborhood of v whereas in Figure 2b the vertex v is included in the closed neighborhood of v.



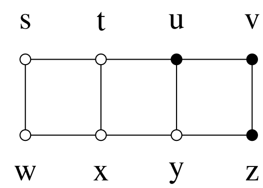


Figure 2a Figure 2b

The *distance* between vertices u and v in G, denoted by $d_G(u,v)$, is the number of edges in a shortest path from u to v. For a vertex $v \in V(G)$, let $B(v,r) = \{u \in V(G) \mid d_G(v,u) \leq r\}$ denote the *r-ball centered at v*. In Figure 3 each vertex is labeled with its distance from u and the dark vertices represent $B(u,2)$, the 2-ball centered at u. Notice that in this graph $d_G(u,v) = 2$, and in general $d_G(u,v) = 0$ if and only if $u = v$.

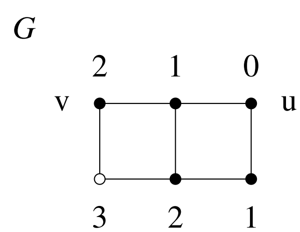


Figure 3

A *dominating set* in a graph G is a subset $D \subseteq V(G)$ with the property that every vertex $v \in V(G)$ is either in D or is adjacent to an element of D. Figure 4 gives an example of a dominating set $D = \{t,u,x,y\}$, as indicated by the dark vertices.

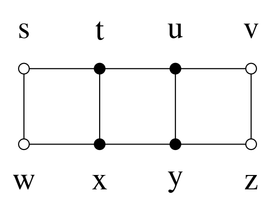


Figure 4

There are many different types of dominating sets that can exist within a graph. The study of these various types of dominating sets is a popular area of graph theory called *domination*. Parity domination is a specific type of domination in which one requires that each vertex in the graph be adjacent to an odd (or even) number of vertices in the dominating set. The two most notable references for domination in graphs are [12] and [13].

Odd Open Dominating Sets in the Direct Product of Graphs

2.1 Odd Open Dominating Sets

An *odd open dominating set* in a simple graph $G = (V(G), E(G))$ is a subset $D \subseteq V(G)$ such that $|\ N(v) \cap D\ |$ is odd for all $v \in V(G)$. That is, the neighborhood $N(v) = \{u \mid uv \in E(G)\}$ contains an odd number of vertices in D. If v is adjacent to $u \in D$, we say v is *dominated* by u. This is illustrated in Figures 5a, 5b and 5c where the dark vertices form an odd open dominating set. A graph may admit several different odd open dominating sets. In the case where $|\ N(v) \cap D\ | = 1$ for all $v \in V(G)$, the set D is called a *total perfect code*. Notice that the odd open dominating sets in Figures 5a and 5b are total perfect codes. Total perfect codes have been studied in [2], [6], [7], [8], [9] and [15] and odd open dominating sets have been studied in [3], [5] and [11].

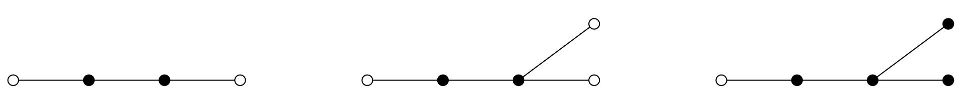


Figure 5a　　Figure 5b　　Figure 5c

2.2 The Direct Product

The *direct product* of G and H is the graph, denoted by $G \times H$, whose vertex set is $V(G) \times V(H)$, and for which vertices (g, h) and (g', h') are adjacent precisely if $gg' \in E(G)$ and

$hh' \in E(H)$. Thus

$$V(G \times H) = \{(g,h) \mid g \in V(G) \text{ and } h \in V(H)\},$$

$$E(G \times H) = \{(g,h)(g',h') \mid gg' \in E(G) \text{ and } hh' \in E(H)\}.$$

Figures 6a and 6b show the direct product $P_3 \times P_4$, where P_n denotes a path on n vertices, and the direct product $P_3 \times C_4$, where C_n denotes a cycle on n vertices. The direct product may also be referred to in literature as *tensor product*, *cross product*, *categorical product*, *Kronecker product*, *cardinal product* and *relational product*.

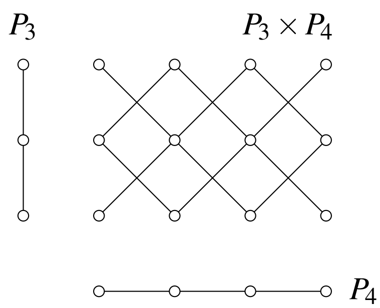


Figure 6a

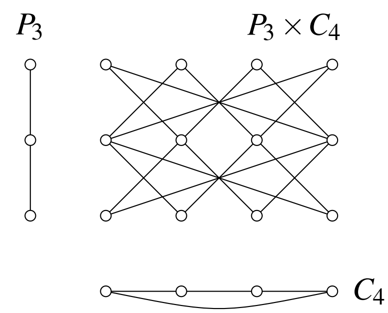


Figure 6b

The definition above of the direct product can easily be extended to finitely many graphs. If $G_1, G_2, \ldots, G_n$ are graphs, the *n-fold direct product* is the graph $G_1 \times G_2 \times \cdots \times G_n$ with vertex set $V(G_1) \times V(G_2) \times \cdots \times V(G_n)$, and for which the vertices $(g_1, g_2, \ldots, g_n)$ and $(g_1', g_2', \ldots, g_n')$ are adjacent precisely if $g_i g_i' \in E(G_i)$ for every $1 \leq i \leq n$. The graphs G_i are called *factors* of the product. For example, in Figure 6a the factors of $P_3 \times P_4$ are P_3 and P_4. The direct product is commutative and associative. A complete treatment of the direct product can be found in [14].

One important observation for the direct product that will be used throughout this chapter is the following: the open neighborhood of a vertex $(g_1, g_2, \ldots, g_n) \in V(G_1 \times G_2 \times \cdots \times G_n)$ is the Cartesian product of the neighborhoods of each $g_i \in V(G_i)$ for $1 \leq i \leq n$. In other words,

$$N(g_1, g_2, \ldots, g_n) = N(g_1) \times N(g_2) \times \cdots \times N(g_n). \tag{2.1}$$

For $g \in V(G_i)$, the *fiber of* g is the set $F_g = \{(g_1, g_2, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_n) \in G_1 \times G_2 \times \cdots \times G_n : \; g_i = g\}$. This is simply the set of vertices in the product that lie directly "above" the vertex g. This is illustrated in Figures 7a and 7b, where the dark vertices form the fiber of g.

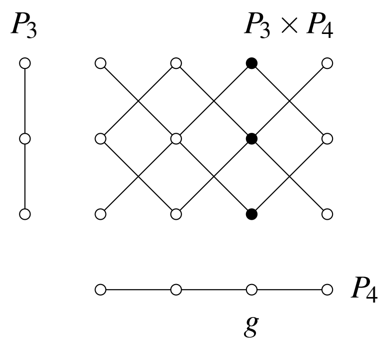


Figure 7a

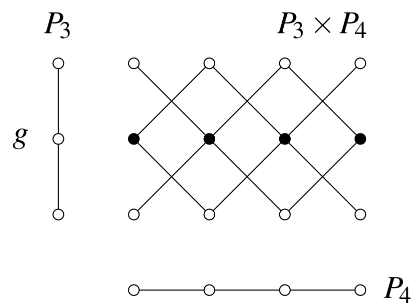


Figure 7b

2.3 Results

The first goal of this paper is to examine the relationship between odd open dominating sets in an n-fold direct product of graphs and odd open dominating sets in each of the factors. We show that an n-fold direct product of graphs has an odd open dominating set if and only if each factor has an odd open dominating set. First we will prove the converse.

THEOREM 2.1. Let $G_1, G_2, \ldots, G_n$ be graphs with odd open dominating sets $D_1, D_2, \ldots, D_n$, respectively. Then $D_1 \times D_2 \times \cdots \times D_n$ is an odd open dominating set in $G_1 \times G_2 \times \cdots \times G_n$.

Proof. Set $G = G_1 \times G_2 \times \cdots \times G_n$. Suppose that $D_i \subseteq V(G_i)$ is an odd open dominating set in G_i for $1 \leq i \leq n$. Form the Cartesian product $D = D_1 \times D_2 \times \cdots \times D_n$. We claim that

D is an odd open dominating set in G.

Let $(g_1, g_2, \ldots, g_n) \in V(G)$. Then for every $g_i \in V(G_i)$ for $1 \le i \le n$, it follows that $|\, N(g_i) \cap D_i \,|$ is odd since the D_i are odd open dominating sets in G_i. By (2.1) we have

$$N(g_1, g_2, \ldots, g_n) = N(g_1) \times N(g_2) \times \cdots \times N(g_n).$$

Thus

$$N(g_1, g_2, \ldots, g_n) \cap (D_1 \times D_2 \times \cdots \times D_n)$$

$$= (N(g_1) \cap D_1) \times (N(g_2) \cap D_2) \times \cdots \times (N(g_n) \cap D_n).$$

Since the cardinalities of each set on the right is odd, it follows that

$$|\, N(g_1, g_2, \ldots, g_n) \cap (D_1 \times D_2 \times \cdots \times D_n) \,|$$

is odd. Hence D is an odd open dominating set in G. □

Figures 8a and 8b illustrate Theorem 2.1.

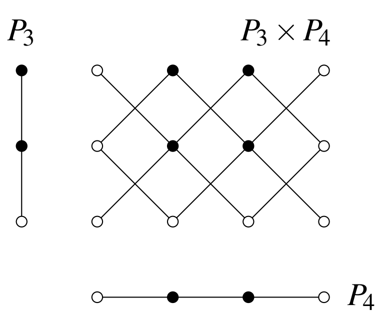


Figure 8a

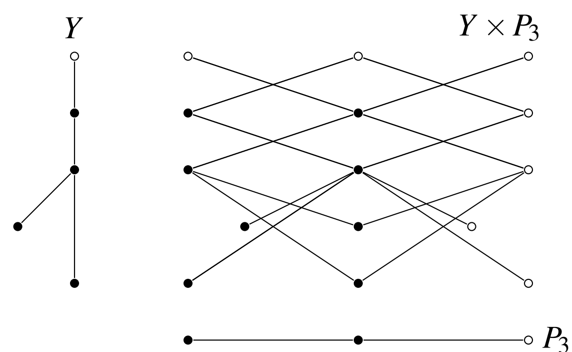


Figure 8b

We now move on to prove the forward direction of our statement. That is, we will show that if an n-fold direct product has an odd open dominating set, then each of its factors has an odd open dominating set. Notice in Figures 9a and 9b that the odd open dominating sets in the product are not Cartesian products of odd open dominating sets in the factors. Thus, it is not possible to simply project the odd open dominating set in the product to an odd open dominating set in each factor of the direct product. A simple projection would lead to the darkened vertices in P_3. Clearly these vertices do not form an odd open dominating set since the central vertex is adjacent to two dark vertices. Further thought indicates that projections of appropriate subsets of an odd open dominating set produce an odd open dominating set in each of its factors.

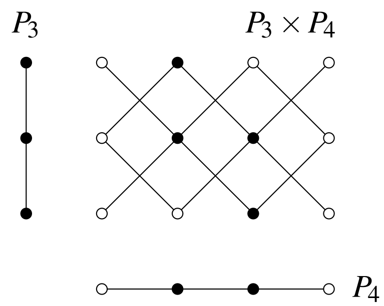


Figure 9a

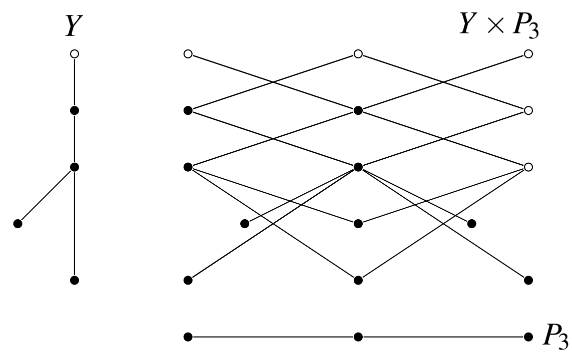


Figure 9b

THEOREM 2.2. Suppose $G_1, G_2, \ldots, G_n$ are graphs and let $G = G_1 \times G_2 \times \cdots \times G_n$. Suppose D is an odd open dominating set in G. Fix a vertex $(g_1', g_2', \ldots, g_n')$ in G. Then $D_i = \{g \in V(G_i) : |D \cap [N(g_1') \times N(g_2') \times \cdots \times N(g_{i-1}') \times V(G_i) \times N(g_{i+1}') \times \cdots \times N(g_n')] \cap F_g| \text{ is odd } \}$ is an odd open dominating set in G_i for each $1 \leq i \leq n$.

Proof. Suppose D is an odd open dominating set in G. Let $x \in V(G_i)$. Then the vertex $(g_1', g_2', \ldots, g_{i-1}, x, g_{i+1}, \ldots, g_n')$ is adjacent to an odd number of vertices in D. Let S be that set of vertices. Necessarily, there is an edge in G_i between x and the i^{th} component of each vertex in S. It is easy to see that each of these vertices lie in $D \cap [N(g_1') \times N(g_2') \times \cdots \times N(g_{i-1}') \times N(x)) \times N(g_{i+1}') \times \cdots \times N(g_n')] \subseteq D \cap [N(g_1') \times N(g_2') \times \cdots \times N(g_{i-1}') \times V(G_i) \times N(g_{i+1}') \times \cdots \times N(g_n')]$. Denote the set $D \cap [N(g_1') \times N(g_2') \times \cdots \times N(g_{i-1}') \times V(G_i) \times N(g_{i+1}') \times \cdots \times N(g_n')]$ by X.

If each of the vertices in S lie in separate fibers then the intersection of each fiber with the set X is a single vertex. Hence the cardinality of each intersection is 1 and x is adjacent to an odd number of vertices in D_i. Thus D_i would be a total perfect code and hence an odd open dominating set in G_i.

Suppose each of the vertices in S do not lie in separate fibers. Since there is an odd number of vertices in S it must be that an odd number of fibers contains an odd number of vertices in S. Thus the cardinality of X intersect each of these fibers is odd. Hence x is adjacent to an odd number of vertices in D_i and D_i is an odd open dominating set in G_i. $\square$

Figures 10a, 10b, 10c and 10d illustrate odd open dominating sets of the type given in Theorem 2.2. As before, each odd open dominating set is indicated by the dark vertices. The dotted lines in Figures 10a and 10b represent the open neighborhoods of g_2' and g_1', respectively. The vertices that form D_1 and D_2 are the vertices of G_1 and G_2 respectively whose fibers contain vertices enclosed by dotted circles. Notice that the vertices g_1'' and g_1''' in Figure 10c are not in the odd open dominating set of their respective factor since the fibers above each of these vertices contains an even number of vertices when intersected with the set $D \cap [V(G_1) \times N(g_2')]$. These vertices are enclosed by the solid circles. Similarly, g_2'' in Figure 10d is not in the odd open dominating set for G_2.

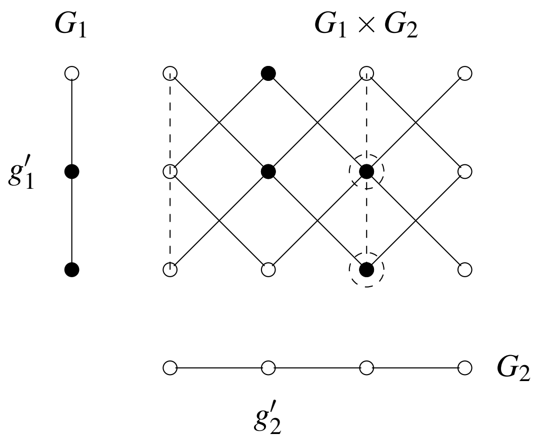


Figure 10a

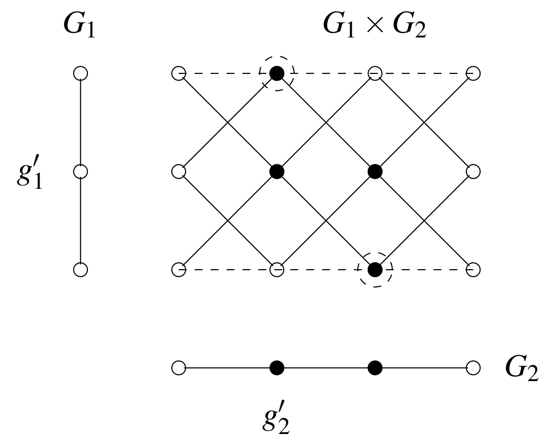


Figure 10b

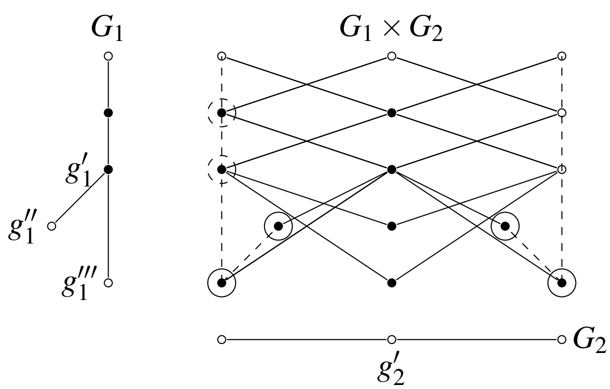


Figure 10c

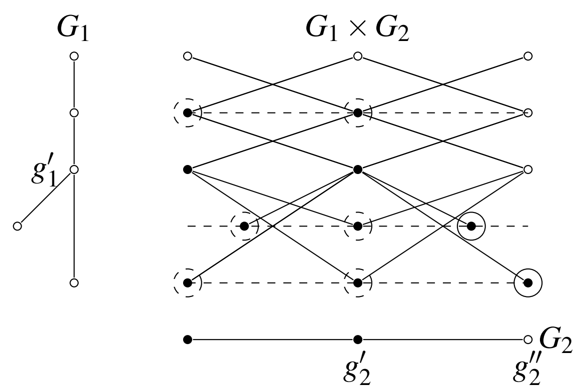


Figure 10d

Odd Closed r-Dominating Sets in Strong Products of Graphs

3.1 Odd Closed r-Dominating Sets

Recall that in the direct product we use the open neighborhood, $N(v)$, of each vertex v in G to find odd open dominating sets within G. For the strong product, we will turn our attention to the closed neighborhood, $N[v]$, of each vertex v in G.

For a positive integer r, an *odd closed r-dominating set* in a simple graph $G = (V(G), E(G))$ is a subset $D \subseteq V(G)$ such that for each $v \in V(G)$, $B(v, r)$ contains an odd number of vertices in D. That is, the balls of radius r centered at each vertex $v \in V(G)$ each contain an odd number of vertices in D. If $u \in B(v, r)$, where $u \in D$, we say v is *r-dominated by u*. Hence an odd closed r-dominating set in a simple graph G is a subset D of $V(G)$ such that each vertex of G is r-dominated by an odd number of vertices in D. This is illustrated in Figures 11a, 11b, and 11c where the dark vertices form odd closed r-dominating sets with $r = 1, 1$ and 3, respectively. Similar to odd open dominating sets, a graph may admit several different odd closed r-dominating sets. In the case for which the balls of radius r centered at the vertices of D form a partition of $V(G)$, the set D is called a *perfect r-code*. That is, each vertex in the graph is r-dominated by exactly one vertex in D. Notice that the odd closed 1-dominating set in Figure 11a is a perfect 1-code. Perfect 1-codes are simply referred to as perfect codes. Perfect r-codes have been studied in [3], [4], [10] and [12]. A complete characterization of perfect r-codes in the strong product appears in [1]. Our goal is to extend these results in the strong product to the more generalized odd closed r-dominating sets.

Figure 11a

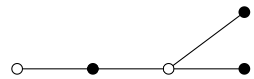


Figure 11b

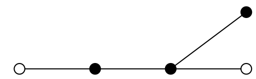


Figure 11c

3.2 The Strong Product

The *strong product* of G and H is the graph, denoted by $G \boxtimes H$, whose vertex set is $V(G) \times V(H)$, and for which distinct vertices (g,h) and (g',h') are adjacent precisely if one of the following holds:

1. $g = g'$ *and* $hh' \in E(H)$
2. $gg' \in E(G)$ *and* $h = h'$
3. $gg' \in E(G)$ *and* $hh' \in E(H)$.

Figures 12a and 12b show the strong products $P_3 \boxtimes P_4$ and $P_3 \boxtimes C_4$. Notice that the strong product is a combination of the direct product and the Cartesian product of graphs, it contains both types of edges. The strong product may also be referred to in literature as *strong direct product* or *symmetric composition*.

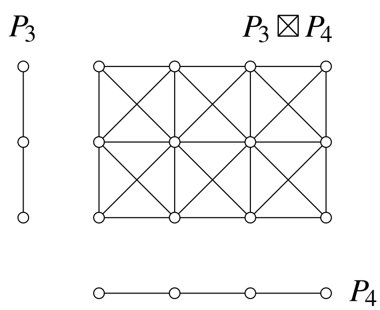


Figure 12a

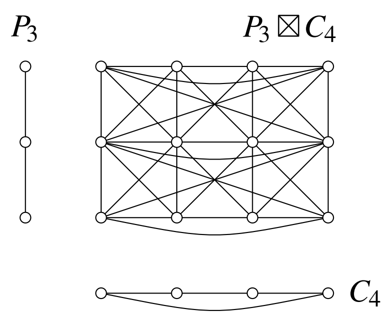


Figure 12b

We can also extend the above definition of strong product to finitely many graphs. If $G_1, G_2, \ldots, G_n$ are graphs, the *n-fold strong product* is the graph $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$ with the vertex set $V(G_1) \times V(G_2) \times \cdots \times V(G_n)$, and for which distinct vertices $(g_1, g_2, \ldots, g_n)$

and $(g'_1, g'_2, \ldots, g'_n)$ are adjacent precisely if $g_i = g'_i$ or $g_i g'_i \in E(G_i)$ for each $1 \leq i \leq n$. Similar to the direct product, the graphs G_i are called *factors* of the product. For example, in Figure 11a the factors of $P_3 \boxtimes P_4$ are P_3 and P_4. The strong product also has the two properties of commutativity and associativity. In addition to these properties, the strong product holds an interesting distance property. By [14, Lemma 5.1], the distance between two vertices $g = (g_1, g_2, \ldots, g_n)$ and $g' = (g'_1, g'_2, \ldots, g'_n)$ in the graph $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$ is

$$d_G(g, g') = \max_{1 \leq i \leq n} d_{G_i}(g_i, g'_i).$$

See [14] for more details on the strong product.

The strong product has an analogous property regarding closed neighborhoods to that of the direct product and open neighborhoods from (2.1). That is, the closed neighborhood of a vertex in the strong product is the Cartesian product of the respective closed neighborhoods in each of the factors, i.e. for $(g_1, g_2, \ldots, g_n) \in V(G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n)$,

$$N[g_1, g_2, \ldots, g_n] = N[g_1] \times N[g_2] \times \cdots \times N[g_n]. \tag{3.1}$$

More generally,

$$B((g_1, g_2, \ldots, g_n), r) = B(g_1, r) \times B(g_2, r) \times \cdots \times B(g_n, r). \tag{3.2}$$

For $g \in V(G_i)$, we use the same notation and definition for F_g as we did in the previous chapter with the exception of using the strong product in place of the direct product. This is illustrated for the strong product in Figures 13a and 13b where the dark vertices form the fiber of g.

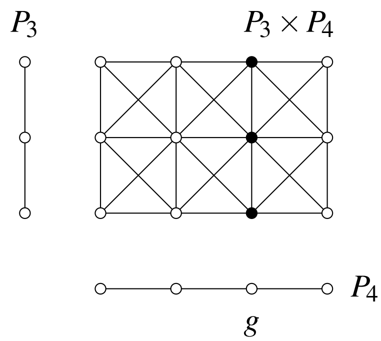


Figure 13a

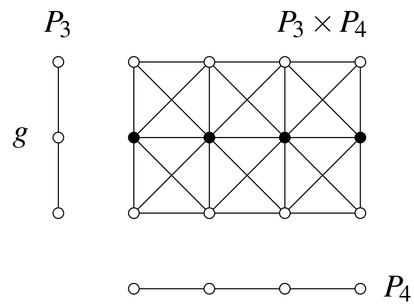


Figure 13b

3.3 Results

In this section we examine the relationship between odd closed r-dominating sets in n-fold strong products and odd closed r-dominating sets of their factors. We show that an n-fold strong product of graphs has an odd closed r-dominating set if and only if each factor has an odd closed r-dominating set. First we will prove the converse.

THEOREM 3.1. Suppose $G_1, G_2, \ldots, G_n$ are graphs and G_i has an odd closed r-dominating set $D_i \subseteq V(G_i)$ for $1 \leq i \leq n$. Then $D_1 \times D_2 \times \cdots \times D_n$ is an odd closed r-dominating set for $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$.

Proof. Set $G = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$. Suppose that $D_i \subseteq V(G_i)$ is an odd closed r-dominating set in G_i for $1 \leq i \leq n$. Form the Cartesian product $D = D_1 \times D_2 \times \cdots \times D_n$. We claim that D is an odd closed r-dominating set in G.

Let $g = (g_1, g_2, \ldots, g_n) \in V(G)$. Then every $g_i \in V(G_i)$ for $1 \leq i \leq n$ is within a distance r of an odd number of $d_i \in D_i$ since the D_i are odd closed r-dominating sets in G_i. In the strong product, by (3.2) we have

$$B((g_1, g_2, \ldots, g_n), r) = B(g_1, r) \times B(g_2, r) \times \cdots \times B(g_n, r).$$

Thus

$$B((g_1,g_2,\ldots,g_n),r)\cap(D_1\times D_2\times\cdots\times D_n)$$

$$=(B(g_1,r)\cap D_1)\times(B(g_2,r)\cap D_2)\times\cdots\times(B(g_n,r)\cap D_n).$$

Since the cardinalities of each set on the right is odd, it follows that

$$\mid B((g_1,g_2,\ldots,g_n),r)\cap(D_1\times D_2\times\cdots\times D_n)\mid$$

is odd. Hence D is an odd closed r-dominating set in G. □

Figures 14a and 14b illustrate Theorem 4.1 where $r=1$ and $r=3$, respectively.

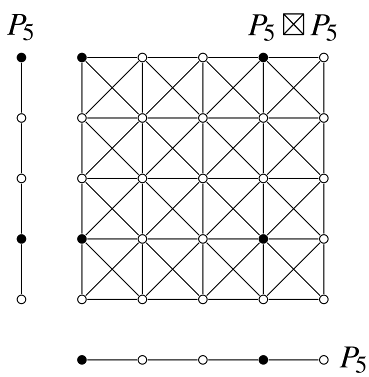


Figure 14a

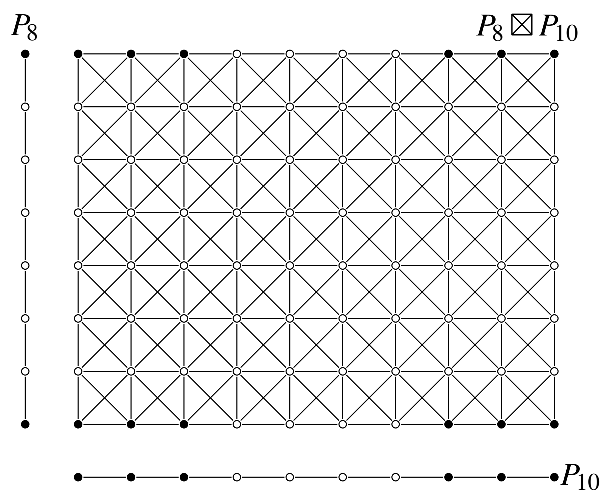


Figure 14b

Now we will prove the forward direction of our statement. We will show that if an n-fold strong product has an odd closed r-dominating set then each of its factors has an odd closed r-dominating set. Since the strong product contains the edges of the direct product, it is not surprising that we encounter a similar problem when attempting to simply project an odd closed r-dominating set onto each factor. Just as with the direct product, a strong product may admit odd closed r-dominating sets that do not arise from Cartesian products of odd closed r-dominating sets in the factors. Figures 15a and 15b where $r=1$ and $r=3$, respectively, illustrate this. In Figure 15a, the central vertex in both of the factors are both within distance 1 of two dark vertices. In Figure 15b there are two dark vertices within distance 3 of each end vertex in P_8 and there are four dark vertices within distance 3 of each end vertex in P_{10}. However, just as in the direct product, projections of appropriate subsets of an odd closed r-dominating set produce odd closed r-dominating sets in each of its factors.

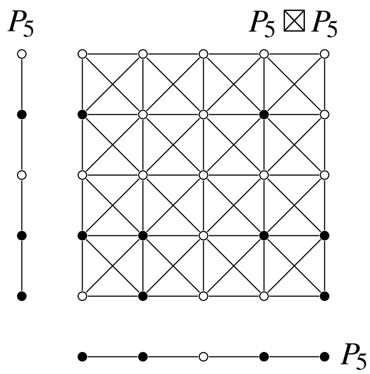


Figure 15a

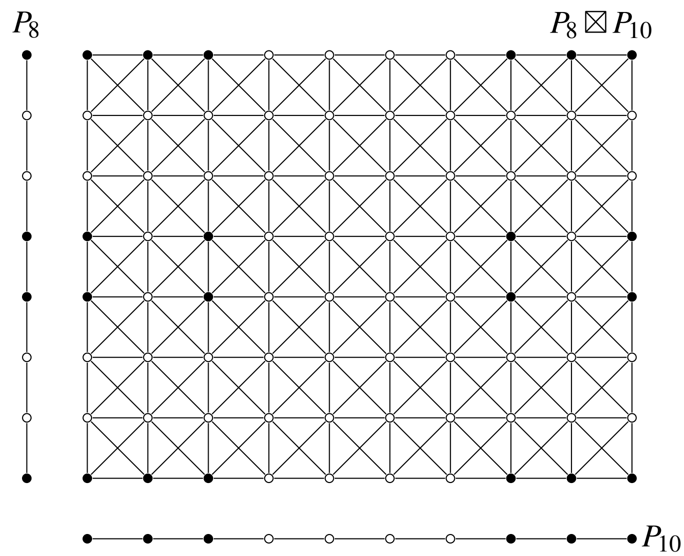


Figure 15b

THEOREM 3.2. Suppose $G_1, G_2, \ldots, G_n$ are graphs and let $G = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$. Suppose D is an odd closed r-dominating set in G. Fix a vertex $(g_1', g_2', \ldots, g_n')$ in G. Then $D_i = \{g \in V(G_i) : |\, D \cap [B(g_1', r) \times B(g_2', r) \times \cdots \times B(g_{i-1}', r) \times V(G_i) \times B(g_{i+1}', r) \times \cdots \times B(g_n', r)] \cap F_g \,| \textit{ is odd}\}$ is an odd closed r-dominating set in G_i for each $1 \le i \le n$.

Proof. Suppose D is an odd closed r-dominating set in G. Let $x \in V(G_i)$. Then the vertex $(g_1', g_2', \ldots, , g_{i-1}, x, g_{i+1}, \ldots, g_n')$ is within distance r of an odd number of vertices in D. Let S be that set of vertices. It is easy to see that each of these vertices lie in $D \cap [B(g_1', r) \times B(g_2', r) \times \cdots \times B(g_{i-1}', r) \times B(x, r) \times B(g_{i+1}', r) \times \cdots \times B(g_n', r)] \subseteq D \cap [B(g_1', r) \times B(g_2', r) \times \cdots \times B(g_{i-1}', r) \times V(G_i) \times B(g_{i+1}', r) \times \cdots \times B(g_n', r)]$. Denote the set

$D \cap [B(g'_1, r) \times B(g'_2, r) \times \cdots \times B(g'_{i-1}, r) \times V(G_i) \times B(g'_{i+1}, r) \times \cdots \times B(g'_n, r)]$ by X.

If each of the vertices in S lie in separate fibers then the intersection of each fiber with the set X is a single vertex. Hence the cardinality of each intersection is 1 and x is within distance r to an odd number of vertices in D_i. Thus D_i would be a perfect r-code and hence an odd closed r-dominating set in G_i.

Suppose now that the vertices in S do not lie in separate fibers. Since the cardinality of S is odd, it must be the case that there is an odd number of fibers that contain an odd number of vertices in D. Thus the intersection of X with each of the fibers containing vertices of S is odd. Hence x is within distance r to an odd number of vertices in D_i and D_i is an odd closed r-dominating set in D_i. $\square$

Figures 16a and 16b where $r = 1$ and Figures 16c and 16d where $r = 3$ illustrate odd closed r-dominating sets of the type given in Theorem 4.2. Again, each odd closed r-dominating set is indicated by the dark vertices. The dotted rectangles enclose the sets $V(G_1) \times B(g'_2, r)$ and $B(g'_1, r) \times V(G_2)$, respectively. The vertices in D_1 and D_2 are again the vertices in G_1 and G_2 respectively whose fibers contain vertices enclosed by dotted circles. Notice that the vertex g''_1 in Figure 16a is not in the odd closed r-dominating set of its respective factor since the fiber of this vertex contains an even amount of vertices when intersected with the set $D \cap [V(G_1) \times B(g'_2, r)]$. These vertices are again enclosed by the solid circles. Similarly, g''_2 and g'''_2 in Figure 16b and g''_1 and g'''_1 in Figure 16c are not in the odd closed r-dominating sets for their respective graphs.

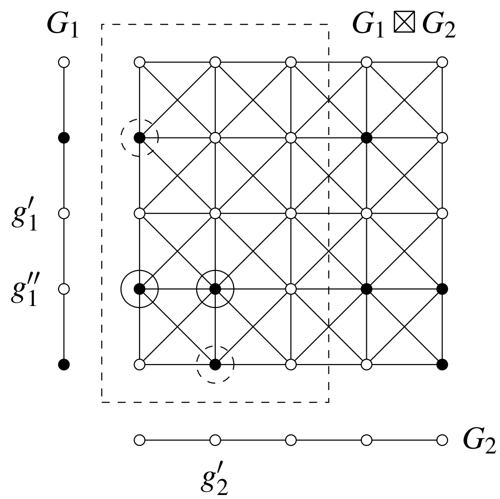


Figure 16a

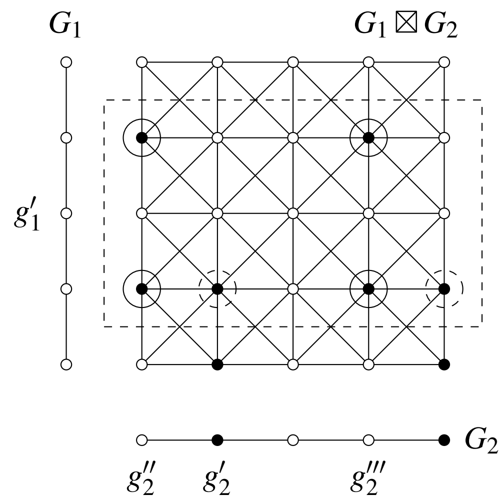


Figure 16b

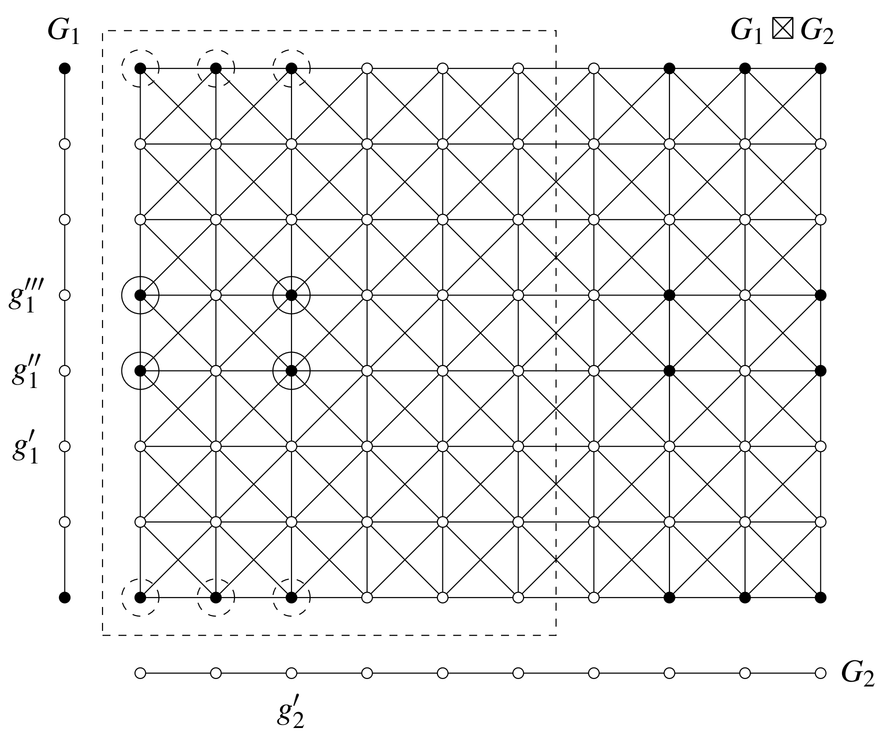


Figure 16c

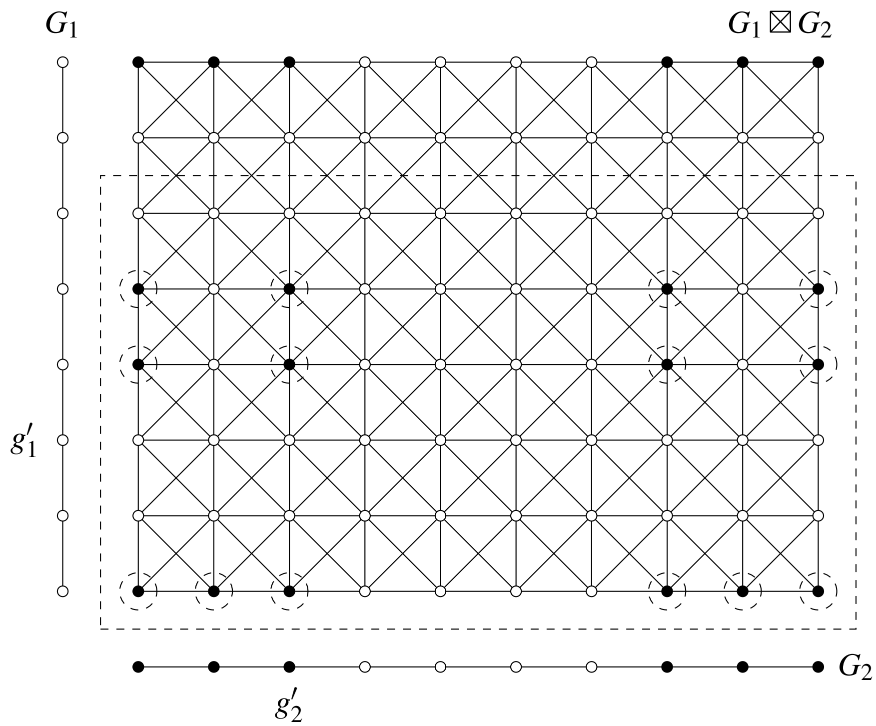


Figure 16d

The Problem of Enumeration

Naturally, we now examine the relationship between the number of odd open dominating sets in the factors and the number of odd open dominating sets in the direct product as well as the relationship between the number of odd closed r-dominating sets in the factors and the number of odd closed r-dominating sets in the strong product. It is important to note that these four theorems are generalizations of [1] and [2]. Thus, the same problems arise when one attempts to examine these two relationships.

Since total perfect codes are special types of odd open dominating sets, Figures 17a and 17b provide an example in which it is not possible in general to determine the number of odd open dominating sets in a direct product from the number of odd open dominating sets in its factors. Notice that each product contains two components. For clarity, one component in each product is drawn in bold.

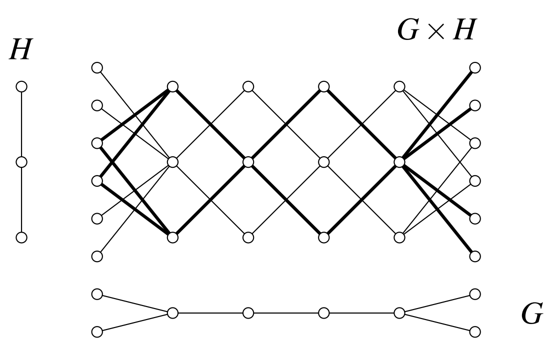


Figure 17a

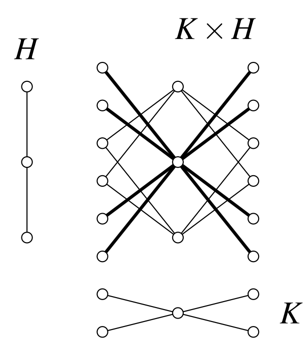


Figure 17b

In each case, the factor H admits exactly two total perfect codes. Factors G and K each admit four total perfect codes, as follows. Any code in G consists of two adjacent vertices incident with one of the two edges on the far left, together with two adjacent vertices incident with one of the two edges on the far right, for a total of four distinct codes. Any code in K consists of any two adjacent vertices. We see that the bold component of $G \times H$ admits 16 codes formed by the choice of two vertices incident with any one of the four edges on the far right, together with two vertices incident with any one of the four edges on the far right. Similarly, the other component of $G \times H$ also has 16 codes, giving $G \times H$ a total of 256 distinct codes. However, we see that the bold component of $K \times H$ has a total of 8 codes formed by any two adjacent vertices and the other component also admits 8 codes formed by any two adjacent vertices, giving $K \times H$ a total of only 64 distinct codes. Thus, we can not determine any correlation between the number of odd open dominating sets in the factors and the number of odd open dominating sets in the direct product.

Similarly, since perfect r-codes are special types of odd closed r-dominating sets, Figures 18a and 18b provide an example in which it is not possible in general to determine the number of odd closed r-dominating sets in a strong product from the number of odd closed r-dominating sets in its factors.

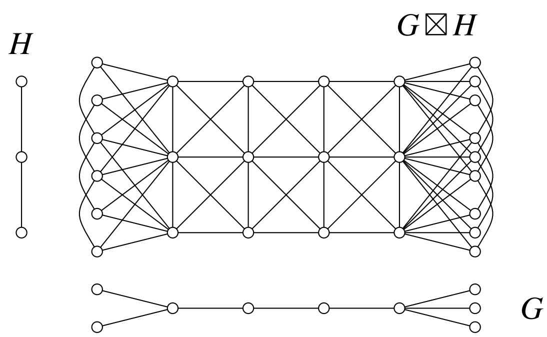


Figure 18a

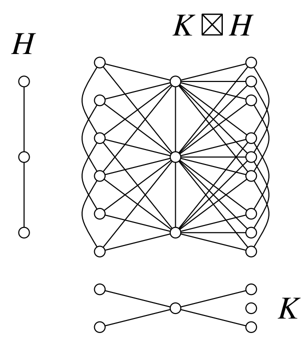


Figure 18b

In each case, the graph H admits three perfect 2-codes and the graphs G and K both admit six perfect 2-codes, as follows. Any code in G consists of any one vertex on the far left, together with any one vertex on the far right and any code in K simply consists of any one vertex. However, $G \boxtimes H$ has 54 distinct perfect 2-codes consisting of any one vertex on the far left together with any one vertex on the far right whereas $K \boxtimes H$ has only

18 distinct perfect 2-codes consisting of any one vertex. Thus, we also can not determine any correlation between the number of odd closed r-dominating sets in the factors and the number of odd closed r-dominating sets in the direct product.

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Vita

Christopher Alan Whisenant was born on April 6, 1987 in Franklin, Virginia. He graduated with honors from Windsor High School in June 2005 and went on to attend Lynchburg College in Lynchburg, Virginia in the Fall of 2005. Throughout his tenure at Lynchburg College, Chris studied pure mathematics and secondary education. In addition to being a math tutor at Lynchburg College, he held many positions, the highest being president, in Sigma Nu Fraternity, Inc., was a member of Kappa Delta Pi Honor Society, Order of Omega Greek Honor Society, Student Judicial Board and was very involved performing services throughout the city of Lynchburg. During Spring 2009, he completed his student teaching at Appomattox County High School and graduated Lynchburg College cum laude with a Bachelor of Science degree in May 2009. In the Fall 2009, he began his tenure at Virginia Commonwealth University studying applied mathematics with the anticipation of graduating with a Master of Science degree in August 2011.