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Uniqueness of Bipartite Factors in Prime Factorizations Over the Direct Product of Graphs

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A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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Abstract

UNIQUENESS OF BIPARTITE FACTORS IN PRIME FACTORIZATIONS OVER THE DIRECT PRODUCT OF GRAPHS

By Owen Puffenberger, Master of Science.

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While it has been known for some time that connected non-bipartite graphs have unique prime factorizations over the direct product, the same cannot be said of bipartite graphs. This is somewhat vexing, as bipartite graphs *do* have unique prime factorizations over other graph products (the Cartesian product, for example). However, it is fairly easy to show that a connected bipartite graph has only one prime bipartite factor, which begs the question: is such a prime bipartite factor unique? In other words, although a connected bipartite graph may have multiple prime factorizations over the direct product, do such factorizations contain the same prime bipartite factor? It has previously been shown that when the prime bipartite factor is K_2, this is in fact true [4]. The goal of this paper is to prove that this is in fact true for any prime bipartite factor, provided the graph being factored is R-thin. The proof of the main result takes the same initial approach as the proof in [4] before moving into new territory in order to prove the final result.

Preliminaries

In this chapter we present some introductory graph theory required for later chapters. Many definitions and results are presented without reference or proof, but may be discovered in one or more of the texts listed in the Bibliography, especially [1] and [2].

DEFINITION 1.1. A *graph* G is a set of vertices, denoted $V(G)$, and a set of edges, $E(G)$, where an element of $E(G)$ is an un-ordered pair of elements of $V(G)$, regarded as a line segment joining the two vertices that belong to it. Vertices are usually denoted by letters, such as u and v, while an edge is denoted by the pair of letters which correspond to the vertices it joins (referred to as endpoints), i.e. we denote the edge running from u to v as uv. The number of vertices in a graph is its *order* and the number of edges in a graph is its *size*. Graphs are depicted visually by representing vertices as nodes and edges as line segments connecting them.

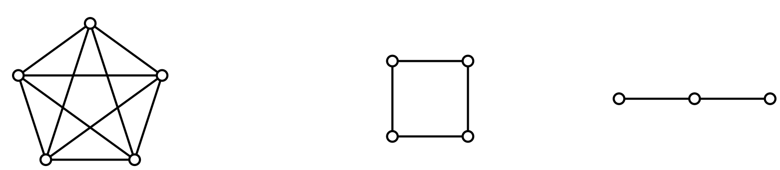

Figure 1.1: Some typical graphs.

We say that two vertices u and v are *adjacent* if they are connected by an edge, denoted $u \sim v$. An edge beginning and ending at the same vertex is called a *loop*. We say that two edges are *incident* if they share an endpoint. The *neighborhood* of a vertex u, denoted $N(u)$, is the set of all vertices adjacent to u. When there is possibility for conufsion, we denote the neighborhood of a vertex u in the graph G as $N_G(u)$. The number of vertices in $N(u)$, or the number of vertices adjacent to u, is called the *degree* of u. A vertex with degree 0 is called *isolated*.

A *subgraph* H of a graph G is a graph H where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* H of G is a subgraph in which pairs of vertices are adjacent whenever they are adjacent in G, the parent graph. A *spanning subgraph* H of G is a subgraph in which $V(H) = V(G)$.

A *path* is a sequence of distinct vertices $v_1 v_2 \dots v_n$ where $v_i \sim v_j$ whenever $i - j = \pm 1$. A graph consisting only of a path with n vertices is called P_n. A *cycle* is a sequence of distinct vertices $v_1 v_2 \dots v_n v_1$ where $v_i \sim v_j$ whenever $i - j = \pm 1$ and $v_1 \sim v_n$. A graph consisting of only a cycle with n vertices is called C_n. A *complete graph* is a graph in which every pair of vertices is adjacent. A complete graph with n vertices is called K_n. The *trivial* graph is the graph consisting of a single isolated vertex. The graph consisting of a single vertex with a loop is called K_1^s. If we now look at Figure 1.1, we can see that the graphs shown there are K_5, C_4, and P_3, respectively.

We say that a graph is *connected* if there exists a path between every pair of vertices. A graph which is not connected is called *disconnected*. Disconnected graphs consist of *connected components*, which are connected subgraphs of maximum order and size.

The *disjoint union* of G and H, written $G + H$, has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$ (Note: we assume $V(G) \cap V(H) = \emptyset$). The disjoint union of G and H is always disconnected.

DEFINITION 1.2. A *bipartite graph* is a graph G for which $V(G)$ can be separated into two distinct sets G_1 and G_2 where vertices in G_1 are only adjacent to vertices in G_2, and vice versa. The sets G_1 and G_2 are called *partite sets* and the pair (G_1, G_2) is called a *bipartition*.

DEFINITION 1.3. Identical graphs are said to be *equal*. We say that two graphs G and H are *isomorphic*, written $G \cong H$, if there is a bijection $\varphi\colon V(G) \to V(H)$ which preserves both adjacence and nonadjacence. This means that $\varphi(u)\varphi(v) \in E(H)$ if and only if $uv \in E(G)$. Such a mapping φ is called an isomorphism. A non-bijective mapping from $V(G)$ to $V(H)$ which preserves adjacence is called a *homomorphism*.

DEFINITION 1.4. A graph with a finite vertex set is called *finite*. The class of all finite graphs allowing loops is denoted Γ_0; the class of all finite graphs without loops (called simple graphs) is denoted Γ.

DEFINITION 1.5. A graph $G \in \Gamma_0$ is called *R-thin* if none of its vertices have identical neighborhoods, that is, $N(u) = N(v)$ implies that $u = v$.

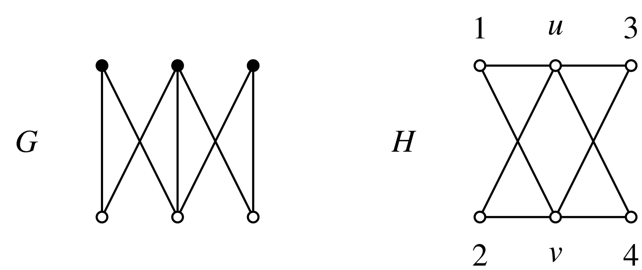

Figure 1.2: The graph G is bipartite–note that black vertices are only adjacent to white vertices, and vice versa. The graph H is *not* R-thin, as $N(u) = \{1,2,3,4\} = N(v)$.

Graph Products

In this chapter we introduce two different graph products: the direct product and the Cartesian product. Although there are several other types of graph products, only these two are relevant to this paper. We will discuss many definitions and theorems that will be later used in the main result. Unless otherwise noted, all results are taken from [2].

Generally speaking, a product of two graphs G and H always has the same vertex set: the set-Cartesian product of $V(G)$ and $V(H)$. Where graph products differ is in their edge sets; different products have different rules for how edge sets are defined. So, for *any* graph product $*$, it is the case that $|V(G*H)| = |V(G)||V(H)|$.

2.1 The Direct Product

DEFINITION 2.1. For graphs G and H in Γ_0, the direct product of G and H is written as $G \times H$. It is defined as follows:

$$\begin{aligned} V(G\times H) &= \; V(G)\times V(H) \\ E(G\times H) &= \; \{(g,h)(g'h') \mid gg' \in E(G) \text{ and } hh' \in E(H)\}. \end{aligned}$$

Note: on the left-hand side, the symbol $\times$ refers to the direct product of the graphs G and H. On the right-hand side, it refers to the set-Cartesian product of $V(G)$ and $V(H)$.

Figures 2.1 and 2.2 illustrate some typical examples. The direct product can be extended to include an arbitrary number of factors, and it is both associative and commutative. In other words, for graphs X, Y, and Z in Γ_0, it is the case that $X \times Y \cong Y \times X$ and $X \times$

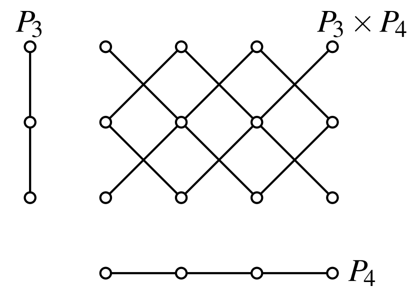

Figure 2.1: The direct product of the graphs P_3 and P_4.

$(Y \times Z) \cong (X \times Y) \times Z$. Also, the direct product distributes over a disjoint union, i.e., $X \times (Y + Z) = (X \times Y) + (X \times Z)$.

In later chapters, we will make frequent use of the following lemma. It allows us to swap the left-hand components of the endpoints of an edge in the direct product of two graphs:

LEMMA 2.2. If $(g,h)(g',h') \in E(G \times H)$, then $(g',h)(g,h') \in E(G \times H)$ as well.

Proof. This follows directly from the definition of the direct product. □

Notice in Figure 2.1 that the product $P_3 \times P_4$ is disconnected, and that both factors P_3 and P_4 are bipartite. This observation leads us to the following theorem:

THEOREM 2.3. Let G and H be nontrivial connected graphs in Γ_0. If at most one of G or H is bipartite, then $G \times H$ is connected. If G and H are both bipartite, $G \times H$ is disconnected and has exactly two components. Moreover, $G \times H$ is bipartite if and only if at least one of G and H is bipartite. Further, if H is bipartite with bipartition (X,Y), then $G \times H$ has partite sets $V(G) \times X$ and $V(G) \times Y$.

A proof of this theorem is given in Chapter 5 of [2]. Extending this to the general case allows us to see that a direct product of connected nontrivial graphs is connected if and only if at most one factor is bipartite, and that a direct product of connected nontrivial graphs is bipartite if and only if at least one factor is bipartite.

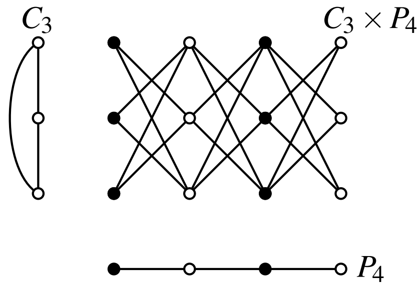

Figure 2.2: The graph $C_3 \times P_4$.

Note that in Figure 2.2, the factor P_4 is bipartite, while the factor C_3 is not. The resulting graph $C_3 \times P_4$ is both connected and bipartite (the bipartitions are marked with black and white vertices).

It turns out that the graph K_1^s (the graph consisting of a single vertex with a loop) acts as a "unit" for the direct product; that is, for a graph $G \in \Gamma_0$, it is the case that $G \times K_1^s \cong G \cong K_1^s \times G$. This fact prompts the following definition.

DEFINITION 2.4. A nontrivial graph $G \in \Gamma_0$ is said to be *prime over the direct product* if any factoring $G \cong G_1 \times G_2$ where $G_1, G_2 \in \Gamma_0$ implies that either G_1 or G_2 is K_1^s. If a graph H is not prime, we say that H has *prime factorization* $H \cong H_1 \times H_2 \times \cdots \times H_k$ if each of the H_i is prime for $1 \leq i \leq k$. Every nontrivial graph has a prime factorization over the direct product.

Figure 2.3: Two different prime factorizations of C_6.

It is a known fact that prime factorizations over the direct product are *not* unique in general. This fact is stated and proved as Theorem 8.1 of [2]. However, as a consequence of Theorem 2.3, it turns out that if a connected bipartite graph G has prime factorization $G \cong G_1 \times G_2 \times \cdots \times G_k$, then exactly one prime factor is bipartite. Figure 2.3 illustrates two different prime factorizations of the connected bipartite graph C_6. Notice that although the prime factorizations are different, the prime bipartite factor–K_2–is the same!

This realization prompts the following conjecture. Proving it is the main goal of this paper.

CONJECTURE 2.5. Suppose $\mathbf{G}$ is a connected bipartite graph in Γ_0. Suppose that $\mathbf{G}$ factors as $\mathbf{G} \cong A \times B$ and $\mathbf{G} \cong A' \times B'$, where B and B' are prime bipartite graphs. Then $B \cong B'$.

In order to prove this, we require some other background information related to graph products, specifically, the Cartesian product.

2.2 The Cartesian Product

Along with the direct product, the Cartesian product is one of the most frequently studied graph products. We discuss it here because unlike the direct product, connected bipartite graphs *do* have unique prime factorizations over the Cartesian product (subject to a few restrictions–but more on that later). We will be able to use this fact to our advantage in later chapters, but first we must define the Cartesian product and explain why it possesses this useful property.

DEFINITION 2.6. For graphs G and H in Γ, the Cartesian product of G and H is written as $G \,\square\, H$. It is defined as follows:

$$\begin{aligned} V(G \,\square\, H) = &\quad V(G) \times V(H) \\ E(G \,\square\, H) = &\quad \{(g,h)(g'h') \mid g = g' \text{ and } hh' \in E(H), \text{ or } gg' \in E(G) \text{ and } h = h'\}. \end{aligned}$$

Figure 2.4 shows an example. Notice that unlike $P_3 \times P_4$, the product $P_3 \,\square\, P_4$ is connected. In general, the Cartesian product of connected graphs is connected, whether or not any of

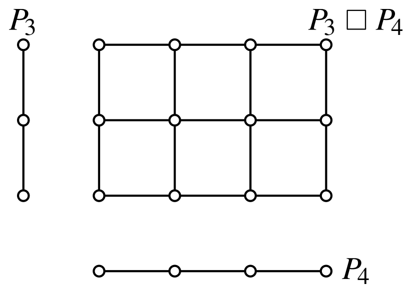

Figure 2.4: The graph $P_3 \square P_4$.

the factors are bipartite. Like the direct product, the Cartesian product is associative and commutative, and it also distributes over a disjoint union.

The graph K_1, also known as the trivial graph, is a unit for the Cartesian product in the sense that $K_1 \square G \cong G \cong G \square K_1$ for all graphs G. This leads to the following definition, which is very similar to Definition 2.4.

DEFINITION 2.7. A nontrivial graph $G \in \Gamma$ is *prime over the Cartesian product* if $G \cong G_1 \square G_2$ implies that either G_1 or G_2 is K_1. If a graph H is not prime, we say that H has *prime factorization* $H \cong H_1 \square H_2 \square \cdots \square H_k$ if each of the H_i is prime for $1 \leq i \leq k$. Every nontrivial graph has a prime factorization over the Cartesian product.

Unlike the direct product, every connected graph has a unique prime factorization in Γ up to order and isomorphism of factors, as proved by Sabidussi and Vizing ([5], [6]). This is stated in much greater detail in Theorem 6.8 of [2], and we restate it here for the sake of completeness.

THEOREM 2.8. Let $G, H \in \Gamma$ be isomorphic connected graphs with prime factorings $G \cong G_1 \square \cdots \square G_k$ and $H \cong H_1 \square \cdots \square H_l$. Then $k = l$, and for any isomorphism $\varphi\colon G \to H$, there is a permutation π of $\{1, 2, \ldots, k\}$ and isomorphisms $\varphi_i\colon G_{\pi(i)} \to H_i$ for which

$$\varphi(x_1, x_2, \ldots, x_k) = (\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), \ldots, \varphi_k(x_{\pi(k)})).$$

It is essential to note that this theorem only applies to graphs in Γ (in fact, it can easily be shown to be false in Γ_0).

As a result of Theorem 2.8, it is natural to associate each H_i with $G_{\pi^{-1}(i)}$, which yields the following corollary (Corollary 6.9 of [2]).

COROLLARY 2.9. If $\varphi\colon G_1 \,\square \cdots \square\, G_k \to H_1 \,\square \cdots \square\, H_k$ is an isomorphism, and each G_i and H_i is prime, then the vertices of each H_i can be relabeled so that

$$\varphi(x_1, x_2, \ldots, x_k) = (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)})$$

for some permutation π of $\{1, \ldots, k\}$.

So, we can sum up Theorem 2.8 and Corollary 2.9 as follows: if a graph G has two different prime factorizations over the Cartesian product, then each prime factorization must contain the same number of factors, and the factors can be ordered so that corresponding (i.e. similarly indexed) factors are isomorphic.

The Cartesian Skeleton

In this chapter we introduce the notion of the Cartesian skeleton $S(G)$ of an arbitrary graph G in Γ_0. The Cartesian skeleton $S(G)$ shares G's vertex set and has the important property $S(A \times B) = S(A) \,\square\, S(B)$ as long as A and B are R-thin and contain no isolated vertices. Note that this is *equality*, rather than isomorphism. This will allow us to relate prime factorings of graphs over the direct product (which are not generally unique) to prime factorings of graphs over the Cartesian product (which have properties of uniquness as outlined in the previous chapter). First, a definition.

DEFINITION 3.1. The *Boolean square* of a graph $G \in \Gamma_0$ is denoted G^s and is defined as follows:

$$\begin{aligned} V(G^s) = &\ \ V(G) \\ E(G^s) = &\ \ \{uv \mid N_G(u) \cap N_G(v) \neq \emptyset\}. \end{aligned}$$

Figure 3.1: The graph P_4 (left) and its Boolean square (right).

In other words, G^s contains a loop at every non-isolated vertex, and contains edges between any two vertices with at least one common neighbor. Figure 3.1 shows a typical example. Moreover, it illustrates the property that the Boolean square of a connected bipartite graph G is disconnected and contains exactly two components–the vertex sets of which are the two partite sets of G. This happens because vertices in a bipartite graph can only share neighbors with vertices in the same bipartition, and because vertices in different bipartitions have no neighbors in common. Thus, the Boolean square of a bipartite graph G will only have edges between vertices that are in the same bipartition in G.

The Cartesian skeleton $S(G)$ is constructed as a specific spanning subgraph of the Boolean square of G.

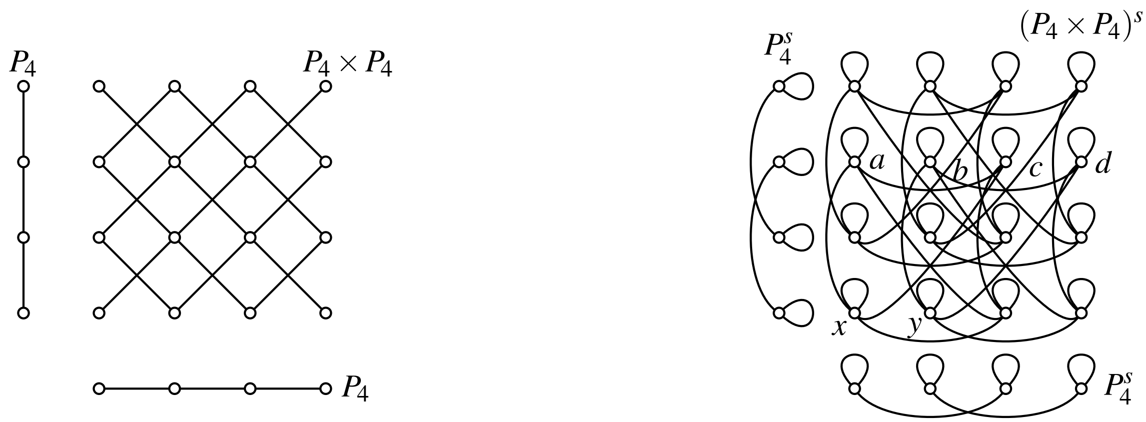

Figure 3.2: The graph $P_4 \times P_4$ (left) and its Boolean square (right).

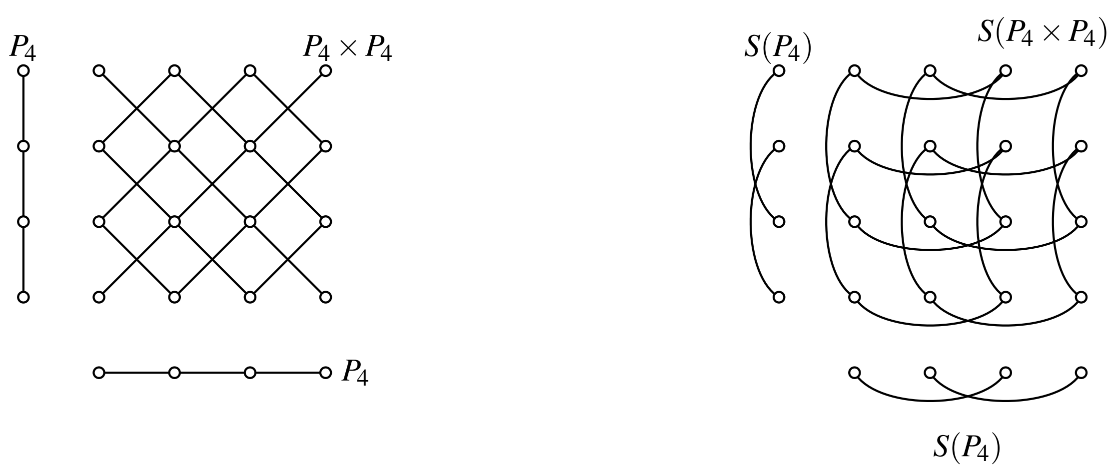

Figure 3.3: The graph $P_4 \times P_4$ (left) and its Cartesian skeleton (right).

Given any factorization $G \cong A \times B$, an edge $(a,b)(a'b')$ of G^s is called *Cartesian* relative to the factorization $A \times B$ if either $a = a'$ and $b \neq b'$ or $a \neq a'$ and $b = b'$. The goal is to construct $S(G)$ from G^s by removing the edges of G^s which are not Cartesian. We begin by examining Figures 3.2 and 3.3, in particular noting which edges of $(P_4 \times P_4)^s$ which do not appear in $S(P_4 \times P_4)$. This will help us determine three criteria for identifying which edges to eliminate to form the Cartesian skeleton from the Boolean square. First, let $G = P_4 \times P_4$.

(1) Clearly, loops are not Cartesian, as the Cartesian skeleton does not contain any. So, we can say that if an edge uv is a loop (i.e. if $u = v$) then uv is not Cartesian.

(2) The edge yd of G^s is not Cartesian, and there is a vertex $b \in V(G)$ such that $N_G(y) \cap N_G(d) \subset N_G(y) \cap N_G(b)$ and $N_G(y) \cap N_G(d) \subset N_G(d) \cap N_G(b)$. Note that $\subset$ denotes strict inclusion.

(3) The edge xc of G^s is not Cartesian, and there is a vertex $a \in V(G)$ such that $N_G(x) \subset N_G(a) \subset N_G(c)$.

So, we want to remove all edges from G^s that meet one of the above criteria. We can write this as a definition.

DEFINITION 3.2. An edge uv of G^s is *dispensable* if $u = v$ or there is a $w \in V(G)$ for which both of the following statements hold:

(i) $N_G(u) \cap N_G(v) \subset N_G(u) \cap N_G(w)$ or $N_G(u) \subset N_G(w) \subset N_G(v)$

(ii) $N_G(v) \cap N_G(u) \subset N_G(v) \cap N_G(w)$ or $N_G(v) \subset N_G(w) \subset N_G(u)$

Note that both of these conditions hold if and only if at least one of conditions (2) and (3) above hold.

This allows us to now properly define the Cartesian skeleton of a graph G.

DEFINITION 3.3. The *Cartesian skeleton* of a graph G is the spanning subgraph $S(G)$ of G^s obtained by removing all dispensable edges from G^s.

We conclude this chapter by stating several theorems regarding certain properties of the Cartesian skeleton. Proofs may be found in [2].

THEOREM 3.4. If A and B are R-thin graphs with no isolated vertices, then $S(A \times B) = S(A) \Box S(B)$.

This has been mentioned previously. It is extremely useful, as it allows us to associate the prime factorization of a graph over the direct product (which may not be unique) with the prime factorization of the Cartesian skeletons of its factors over the Cartesian product (which is unique).

THEOREM 3.5. If $\varphi\colon G \to H$ is an isomorphism defined as a map from $V(G)$ to $V(H)$, then $\varphi\colon S(G) \to S(H)$ is an isomorphism as well.

This is also very useful, since it removes any doubt as to whether taking skeletons of graphs will affect their isomorphism.

THEOREM 3.6. Suppose G is a connected graph.

(1) If G is not bipartite, then $S(G)$ is connected.

(2) If G is bipartite, then $S(G)$ has two connected components, the vertex sets of which are the two partite sets of G.

Finally, this theorem will come in handy during the proof of our main result, where we will be specifically concerning ourselves with the Cartesian skeletons of bipartite graphs.

Main Result

In this chapter we set out to prove the main result of this paper: that a (connected, R-thin) bipartite graph, while it may have several prime factorizations over the direct product, contains a unique prime bipartite factor. So, let's restate Conjecture 2.5 as a threorem, and prove it.

THEOREM 4.1. Suppose $\mathbf{G}$ is a connected, bipartite, R-thin graph in Γ_0. Suppose that $\mathbf{G}$ factors as $\mathbf{G} \cong A \times B$ and $\mathbf{G} \cong A' \times B'$, where B and B' are prime bipartite graphs. Then $B \cong B'$.

Proof. Let $\varphi : A \times B \to A' \times B'$ be an isomorphism. Then φ is also an isomorphism from $S(A \times B)$ to $S(A' \times B')$ via Theorem 3.5. Using this, along with Theorems 3.4 and 3.6, we can generate the following diagram: (The double vertical lines indicate equality, and the horizontal arrows are isomorphisms.)

$$\begin{tikzcd}
A \times B \arrow[r, "\varphi"] \arrow[d, "S"'] & A' \times B' \arrow[d, "S"'] \\
S(A \times B) \arrow[r, "\varphi"] \arrow[d, equal] & S(A' \times B') \arrow[d, equal] \\
S(A) \Box S(B) \arrow[r, "\varphi"] \arrow[d, equal] & S(A') \Box S(B') \arrow[d, equal] \\
S(A) \Box (B_0 + B_1) \arrow[r, "\varphi"] \arrow[d, equal] & S(A') \Box (B'_0 + B'_1) \arrow[d, equal] \\
S(A) \Box B_0 + S(A) \Box B_1 \arrow[r, "\varphi"] & S(A') \Box B'_0 + S(A') \Box B'_1
\end{tikzcd} \tag{4.1}$$

Note that B_0 and B_1 are the connected components of $S(B)$, while B_0' and B_1' are the connected components of $S(B')$ (Theorem 3.6). Note also that since B is bipartite, the connected components of $S(A)\square S(B)$, namely, $S(A)\square B_0$ and $S(A)\square B_1$, are the partite sets of $A\times B$. The same is true for the connected components of $S(A')\square S(B')$.

Therefore, since $S(A)\square S(B)$ and $S(A')\square S(B')$ are both disconnected graphs with two components, the fact that they are isomorphic requires that their connected components are isomorphic. In other words, φ induces two isomorphisms, φ_0 and φ_1, on the connected components of $S(A)\square S(B)$ and $S(A')\square S(B')$:

$$\begin{array}{ccc} S(A)\square B_0 & + & S(A)\square B_1 \\ \varphi_0 \Big\downarrow & & \varphi_1 \Big\downarrow \\ S(A')\square B_0' & + & S(A')\square B_1' \end{array} \tag{4.2}$$

Note: We are assuming without loss of generality that φ_0 and φ_1 run between similarly-labeled components. If they do not, we can simply relable the components as necessary.

Now, let's break down $S(A)$, B_0, B_1, $S(A')$, B_0', and B_1' into their prime factorizations with respect to the *Cartesian* product, since we know that such prime factorizations are unique (Theorem 2.8). This will give us the following:

$$S(A) = A_1\square A_2\square\cdots\square A_i$$

$$S(A') = A_1'\square A_2'\square\cdots\square A_l'$$

$$B_0 = B_{01}\square B_{02}\square\cdots\square B_{0j}$$

$$B_1 = B_{11}\square B_{12}\square\cdots\square B_{1k}$$

$$B_0' = B_{01}'\square B_{02}'\square\cdots\square B_{0m}'$$

$$B_1' = B_{11}'\square B_{12}'\square\cdots\square B_{1n}'$$

With these prime factorizations in hand, we can re-draw Figure 4.2 in greater detail:

$$\begin{array}{ccc} \overbrace{(A_1\Box A_2\Box\cdots\Box A_i)}^{S(A)}\Box\overbrace{(B_{01}\Box B_{02}\Box\cdots\Box B_{0j})}^{B_0} & + & \overbrace{(A_1\Box A_2\Box\cdots\Box A_i)}^{S(A)}\Box\overbrace{(B_{11}\Box B_{12}\Box\cdots\Box B_{1k})}^{B_1} \\ \Big\downarrow \varphi_0 & & \Big\downarrow \varphi_1 \\ \underbrace{(A'_1\Box A'_2\Box\cdots\Box A'_l)}_{S(A')}\Box\underbrace{(B'_{01}\Box B'_{02}\Box\cdots\Box B'_{0m})}_{B'_0} & + & \underbrace{(A'_1\Box A'_2\Box\cdots\Box A'_l)}_{S(A')}\Box\underbrace{(B'_{11}\Box B'_{12}\Box\cdots\Box B'_{1n})}_{B'_1} \end{array} \tag{4.3}$$

Now, let's define the following products of factors of $S(A)\Box S(B)$ and $S(A')\Box S(B')$. We will be using Theorem 2.8 liberally, as it guarantees that $S(A)\Box B_0 + S(A)\Box B_1$ and $S(A')\Box B'_0 + S(A')\Box B'_1$ have the same number of prime factors, and that their prime factorizations can be rearranged so that similarly-indexed factors are isomorphic.

Define K to be the product of A_i's that φ_0 and φ_1 both send to $S(A')$. Define α_1 and β_1 to be the component functions of φ_0 and φ_1, respectively, which act on K. Also, define $\alpha_1(K) = \beta_1(K) = K'$.

Define L to be the product of A_i's that φ_0 sends to factors of $S(A')$ and φ_1 sends to factors of B'_1. Define α_2 and β_2 to be the component functions of φ_0 and φ_1, respectively, which act on L. Also, define $\alpha_2(L) = L'$ and $\beta_2(L) = L''$.

Define M to be the product of A_i's that φ_0 sends to factors of B'_0 and φ_1 sends to factors of $S(A')$. Define α_3 and β_3 to be the component functions of φ_0 and φ_1, respectively, which act on M. Also, define $\alpha_3(M) = M'$ and $\beta_3(M) = M''$.

Define X to be the product of A_i's that φ_0 sends to factors of B'_0 and φ_1 sends to factors of B'_1. Define α_4 and β_4 to be the component functions of φ_0 and φ_1, respectively, which act on X. Also, define $\alpha_4(X) = X'$ and $\beta_4(X) = X''$.

Define P to be the product of B_{0j}'s that φ_0 sends to factors of B'_0. Define α_6 to be the component function of φ_0 which acts on P, and define $\alpha_6(P) = P'$.

Define Q to be the product of B_{1k}'s that φ_1 sends to factors of B'_1. Define β_6 to be the component function of φ_1 which acts on Q, and define $\beta_6(Q) = Q'$.

Define Y to be the product of A'_l's that φ_0^{-1} sends to factors of B_0 and φ_1^{-1} sends to factors of B_1. Define α_7^{-1} and β_7^{-1} to be the component functions of φ_0^{-1} and φ_1^{-1}, respectively, which act on Y. Also, define $\alpha_7^{-1}(Y) = Y'$ and $\beta_7^{-1}(Y) = Y''$.

Now, consider M', the image of M under β_3. We know that M' is a factor of $S(A)$, but we haven't yet discussed where φ_0^{-1} sends M'. But since we have already exhausted all the possibilities as to where φ_0 sends the factors of $S(A)$, it is clear that M' is not the image under φ_0 of any factor of $S(A)$. Thus, it must be the case that φ_0^{-1} in fact sends M' to a factor of B_0. So, we define α_5^{-1} to be the component function of φ_0^{-1} which acts on M', and we define $\alpha_5^{-1}(M') = M'''$.

By the same logic, it is apparent that φ_1^{-1} must send L' to some factor of B_1. So,we define β_5^{-1} to be the component function of φ_1^{-1} which acts on L', and we define $\beta_5^{-1}(L') = L'''$.

Now, we can modify Diagram 4.3, consolidating factors using the above definitions, and illustrating exactly which factors of $S(A)\square B_0 + S(A)\square B_1$ and $S(A')\square B'_0 + S(A')\square B'_1$ are isomorphic.

$$
\begin{tikzcd}[column sep=tiny]
(K \square \arrow[d, "\alpha_1"'] & L \square \arrow[d, "\alpha_2"'] & M \square \arrow[drr, "\alpha_3"'] & X)\square \arrow[drrr, "\alpha_4"] & (M'''\square \arrow[dll, "\alpha_5"'] & P \square \arrow[d, "\alpha_6"] & Y')+ \arrow[dlll, "\alpha_7"'] & (K \square \arrow[d, "\beta_1"'] & L \square \arrow[drrr, "\beta_2"'] & M \square \arrow[d, "\beta_3"'] & X) \square \arrow[drrr, "\beta_4"] & (L'''\square \arrow[dlll, "\beta_5"'] & Q \square \arrow[d, "\beta_6"] & Y'') \arrow[dlll, "\beta_7"'] \\
(K' \square & L' \square & M' \square & Y)\square & (M''\square & P' \square & X')+ & (K' \square & L' \square & M' \square & Y) \square & (L''\square & Q' \square & X'')
\end{tikzcd}
\tag{4.4}
$$

Now, given an edge in $A \times B$ (i.e. an edge with one endpoint in $S(A)\Box B_0$ and one endpoint in $S(A)\Box B_1$), we can see exactly where φ sends each component of each endpoint. For example, if $((a,b,c,d)(e,f,g))((h,i,j,k)(l,m,n)) \in E(A \times B)$, then

$$\varphi(((a,b,c,d)(e,f,g))((h,i,j,k)(l,m,n))) =$$

$$((\alpha_1(a),\alpha_2(b),\alpha_5(e),\alpha_7(g))(\alpha_3(c),\alpha_6(f),\alpha_4(d)))((\beta_1(h),\beta_5(l),\beta_3(j),\beta_7(n))(\beta_2(i),\beta_6(m),\beta_4(k)))$$

$$\in E(A' \times B').$$

The remainder of the proof that $B \cong B'$ consists of two sections. First, we will show that $X \cong X' \cong X'' \cong Y \cong Y' \cong Y'' \cong K_1$, which will allow us to greatly reduce our factorings of $S(A)\Box S(B)$ and $S(A')\Box S(B')$. This in turn will allow us to more easily see why it must be the case that $B \cong B'$.

Now, the first part of our plan:

LEMMA 4.2. Given these factorings of $S(A)\Box B_0 + S(A)\Box B_1$ and $S(A')\Box B'_0 + S(A')\Box B'_1$, it is the case that $X \cong X' \cong X'' \cong Y \cong Y' \cong Y'' \cong K_1$.

Proof. Given our factoring of $S(A)\Box S(B)$, let's define two new graphs G and H, constructed from the factors of $S(A)\Box S(B)$ as follows:

$$V(G) = V(M'''\Box P) \cup V(L'''\Box Q)$$
$$E(G) = \{(m,p)(l,q) \mid ((*,*,*,*)(m,p,*))((*,*,*,*)(l,q,*)) \in E(A \times B)\},$$

$$V(H) = V(Y)$$
$$E(H) = \{\alpha_7(y_1)\beta_7(y_2) \mid ((*,*,*,*)(*,*,y_1))((*,*,*,*)(*,*,y_2)) \in E(A \times B)\}.$$

It's easy to see that G is bipartite, since each edge $(m,p)(l,q)$ has one endpoint in $V(M'''\Box P)$ and one endpoint in $V(L'''\Box Q)$. Our goal is to show that $B \cong G \times H$, which will force $H \cong K_1^s$ (since B is prime), which will then imply that $Y \cong Y' \cong Y'' \cong K_1$. So, consider the direct product $G \times H$:

$$\begin{aligned} V(G\times H) &= V(G)\times V(H) = [V(M'''\Box P)\cup V(L'''\Box Q)]\times V(Y) \\ &= [V(M''')\times V(P)\times V(Y)]\cup[V(L''')\times V(Q)\times V(Y)], \end{aligned}$$

$$\begin{aligned} E(G\times H) &= \{((m,p),\alpha_7(y_1))((l,q),\beta_7(y_2)) \mid (m,p)(l,q)\in E(G) \\ &\quad \text{and } \alpha_7(y_1)\beta_7(y_2)\in E(H)\}. \end{aligned}$$

Now, let's define the map $\psi\colon V(B)\to V(G\times H)$ as follows:

$$\psi(a,b,c) = \begin{cases} ((a,b),\alpha_7(c)) & \text{if } (a,b,c)\in B_0 \\ ((a,b),\beta_7(c)) & \text{if } (a,b,c)\in B_1. \end{cases}$$

We can see right away that ψ is a bijection which preserves the bipartitions of B and $G\times H$. This means that B and $G\times H$ have the same number of vertices. So in order to show that $B\cong G\times H$, we must show that $\psi\colon E(B)\to E(G\times H)$ where $\psi((m,p,y_1)(l,q,y_2)) = ((m,p),\alpha_7(y_1))((l,q),\beta_7(y_2))$ is an isomorphism.

So, suppose that $(a,b,c)(d,e,f)\in E(B)$. Then:

$$\begin{aligned} (a,b,c)(d,e,f)\in E(B) &\Rightarrow ((*,*,*,*)(a,b,c))((*,*,*,*)(d,e,f))\in E(A\times B) \\ &\Rightarrow (a,b)(d,e)\in E(G) \text{ and } \alpha_7(c)\beta_7(f)\in E(H) \\ &\Rightarrow ((a,b),\alpha_7(c))((d,e),\beta_7(f))\in E(G\times H). \end{aligned}$$

So $\psi\colon E(B)\to E(G\times H)$ is at the very least a homomorphism.

Now, suppose that $((m,p),\alpha_7(y_1))((l,q),\beta_7(y_2))\in E(G\times H)$. Then $(m,p)(l,q)\in E(G)$

and $\alpha_7(y_1)\beta_7(y_2) \in E(H)$. Then:

$$((*,*,*,*)(m,p,*))((*,*,*,*)(l,q,*)) \in E(A\times B) \tag{4.5}$$

$$((*,*,*,*)(*,*,y_1))((*,*,*,*)(*,*,y_2)) \in E(A\times B). \tag{4.6}$$

Applying φ to (4.5) gives

$$((*,*,\alpha_5(m),*)(*,\alpha_6(p),*))((*,\beta_5(l),*,*)(*,\beta_6(q),*)) \in E(A'\times B').$$

Then by Lemma 2.2,

$$((*,\beta_5(l),*,*)(*,\alpha_6(p),*))((*,*,\alpha_5(m),*)(*,\beta_6(q),*)) \in E(A'\times B').$$

Applying φ^{-1} to this gives

$$((*,\alpha_2^{-1}\beta_5(l),*,*)(*,p,*))((*,*,\beta_3^{-1}\alpha_5(m),*)(*,q,*)) \in E(A\times B),$$

which means that

$$(*,\alpha_2^{-1}\beta_5(l),*,*)(*,*,\beta_3^{-1}\alpha_5(m),*) \in E(A), \tag{4.7}$$

$$(*,p,*)(*,q,*) \in E(B). \tag{4.8}$$

Now, applying φ to (4.6) implies that

$$((*,*,*,\alpha_7(y_1))(*,*,*))((*,*,*,\beta_7(y_2))(*,*,*)) \in E(A'\times B'),$$

and thus implies that

$$(*,*,*,\alpha_7(y_1))(*,*,*,\beta_7(y_2)) \in E(A'). \tag{4.9}$$

By definition of the direct product, (4.8) imples that

$$((*,*,*,*)(*,p,*))((*,*,*,*)(*,q,*) \in E(A \times B).$$

Applying φ to this implies that

$$((*,*,*,*)(*,\alpha_6(p),*))(*,*,*,*)(*,\beta_6(q),*)) \in E(A' \times B'),$$

thus by definition of the direct product,

$$(*,\alpha_6(p),*)(*,\beta_6(q),*) \in E(B').$$

Combing this with (4.9) implies that

$$((*,*,*,\alpha_7(y_1))(*,\alpha_6(p),*))((*,*,*,\beta_7(y_2))(*,\beta_6(q),*)) \in E(A' \times B'),$$

and by Lemma 2.2,

$$((*,*,*,\beta_7(y_2))(*,\alpha_6(p),*))((*,*,*,\alpha_7(y_1))(*,\beta_6(q),*)) \in E(A' \times B').$$

Applying φ^{-1} to this gives

$$((*,*,*,*)(*,p,\alpha_7^{-1}\beta_7(y_2)))((*,*,*,*)(*,q,\beta_7^{-1}\alpha_7(y_1))) \in E(A \times B).$$

In particular, this means that

$$(*, p, \alpha_7^{-1}\beta_7(y_2))(*, q, \beta_7^{-1}\alpha_7(y_1)) \in E(B). \tag{4.10}$$

Now, combining (4.7) and (4.10) gives us

$$((*, \alpha_2^{-1}\beta_5(l), *, *)(*, p, \alpha_7^{-1}\beta_7(y_2)))((*, *, \beta_3^{-1}\alpha_5(m), *)(*, q, \beta_7^{-1}\alpha_7(y_1))) \in E(A \times B).$$

Applying φ to this edge takes us to

$$((*, \beta_5(l), *, \beta_7(y_2))(*, \alpha_6(p), *))((*, *, \alpha_5(m), \alpha_7(y_1))(*, \beta_6(q), *)) \in E(A' \times B').$$

Then by Lemma 2.2,

$$((*, *, \alpha_5(m), \alpha_7(y_1))(*, \alpha_6(p), *))((*, \beta_5(l), *, \beta_7(y_2))(*, \beta_6(q), *)) \in E(A' \times B').$$

Finally, applying φ^{-1} takes us to

$$((*, *, *, *)(m, p, y_1))((*, *, *, *)(l, q, y_2)) \in E(A \times B).$$

In particular, this implies that $(m, p, y_1)(l, q, y_2) \in E(B)$.

It turns out that $B \cong G \times H$. But B is both prime and bipartite, so since G is also bipartite (and hence nontrivial), it must be the case that $H \cong K_1^s$. Then it follows that $Y \cong Y' \cong Y'' \cong K_1$. A symmetric argument involving the factors of $A' \times B'$ can be used to show that $X \cong X' \cong X'' \cong K_1$. □

Now that we have shown that all of the X- and Y-factors of $A \times B$ and $A' \times B'$ are trivial, we can drop them from our diagram, and appropriately relabel the component isomorphisms which make up φ:

$$
\begin{tikzcd}[column sep=small]
 & S(A) & & & B_0 & & & & S(A) & & & B_1 & \\
(K \arrow[d, "\alpha_1"'] \;\Box & L \arrow[d, "\alpha_2"'] \;\Box & M) \arrow[drr, "\alpha_3"] & \Box & (M''' \arrow[dll, "\alpha_4"'] \;\Box & P) \arrow[d, "\alpha_5"] & + & (K \arrow[d, "\beta_1"'] \;\Box & L \arrow[drrr, "\beta_2"] \;\Box & M) \arrow[d, "\beta_3"] & \Box & (L''' \arrow[dll, "\beta_4"'] \;\Box & Q) \arrow[d, "\beta_5"] \\
(K' \;\Box & L' \;\Box & M') & \Box & (M'' \;\Box & P') & + & (K' \;\Box & L' \;\Box & M') & \Box & (L'' \;\Box & Q') \\
 & S(A') & & & B_0' & & & & S(A') & & & B_1' &
\end{tikzcd}
\tag{4.11}
$$

Now, B and B' look even more similar than before (recall that $B_0 + B_1 = S(B)$, and likewise for B'). It's clear right away that $P \cong P'$ via α_5, and similarly that $Q \cong Q'$ via β_5. However, upon closer inspection we can see that $M''' \cong M''$ via the isomorphism $\alpha_3\beta_3^{-1}\alpha_4$, and also that $L''' \cong L''$ via the isomorphism $\beta_2\alpha_2^{-1}\beta_4$. This means that there is a bijection between $V(S(B))$ and $V(S(B'))$, i.e. there is a bijection between $V(B)$ and $V(B')$. At the very least, this guarantees that $|V(B)| = |V(B')|$. But it also begs the question: does the bijection from $V(B)$ to $V(B')$ give an isomorphism from B to B'? If so, our proof would be complete. In order to see this, let's define the map $\Phi\colon V(B) \to V(B')$ as follows:

$$
\Phi(a,b) = \begin{cases} (\alpha_3\beta_3^{-1}\alpha_4(a), \alpha_5(b)) & \text{if } (a,b) \in B_0 \\ (\beta_2\alpha_2^{-1}\beta_4(a), \beta_5(b)) & \text{if } (a,b) \in B_1. \end{cases}
$$

If we can show that Φ is an isomorphism from $E(B)$ to $E(B')$, then our proof will be complete.

So, suppose that $(m''',p)(l''',q) \in E(B)$. Then there is an edge

$$((*,*,*)(m''',p))((*,*,*)(l''',q)) \in E(A \times B)$$

since A must be nontrivial (otherwise B would not be prime).

Applying φ to this edge takes us to the edge

$$((*,*,\alpha_4(m'''))(*,\alpha_5(p)))((*,\beta_4(l'''),*)(*,\beta_5(q))) \in E(A' \times B').$$

Then by Lemma 2.2, we know that

$$((*,\beta_4(l'''),*)(*,\alpha_5(p)))((*,*,\alpha_4(m'''))(*,\beta_5(q))) \in E(A' \times B')$$

as well. If we apply φ^{-1} to this edge, we'll move back into $A \times B$ and get the edge

$$((*,\alpha_2^{-1}\beta_4(l'''),*)(*,p))((*,*,\beta_3^{-1}\alpha_4(m'''))(*,q)) \in E(A \times B).$$

Applying Lemma 2.2 again yields the edge

$$((*,*,\beta_3^{-1}\alpha_4(m'''))(*,p))((*,\alpha_2^{-1}\beta_4(l'''),*)(*,q)) \in E(A \times B).$$

Finally, applying φ one last time takes us to

$$((*,*,*)(\alpha_3\beta_3^{-1}\alpha_4(m'''),\alpha_5(p)))((*,*,*)(\beta_2\alpha_2^{-1}\beta_4(l'''),\beta_5(q))) \in E(A' \times B').$$

Then by the definition of the direct product, we know that

$$(\alpha_3\beta_3^{-1}\alpha_4(m'''),\alpha_5(p))(\beta_2\alpha_2^{-1}\beta_4(l'''),\beta_5(q)) \in E(B').$$

In other words, Φ is at least a homomorphism from $E(B)$ to $E(B')$.

To see that Φ^{-1} is a homomorphism as well, suppose that $(m'', p')(l'', q') \in E(B')$. Then there is an edge

$$((*,*,*)(m'', p'))(*,*,*)((l'', q')) \in E(A' \times B')$$

since A' must be nontrivial (otherwise B' would not be prime).

Applying φ^{-1} to this edge takes us to the edge

$$((*,*,\alpha_3^{-1}(m''))(*,\alpha_5^{-1}(p')))((*,\beta_2^{-1}(l''),*)(*,\beta_5^{-1}(q'))) \in E(A \times B).$$

Then by Lemma 2.2, we know that

$$((*,\beta_2^{-1}(l''),*)(*,\alpha_5^{-1}(p')))((*,*,\alpha_3^{-1}(m''))(*,\beta_5^{-1}(q'))) \in E(A \times B)$$

as well. If we apply φ to this edge, we'll move back into $A' \times B'$ and get the edge

$$((*,\alpha_2\beta_2^{-1}(l''),*)(*,p'))((*,*,\beta_3\alpha_3^{-1}(m''))(*,q')) \in E(A' \times B').$$

Applying Lemma 2.2 again yields the edge

$$((*,*,\beta_3\alpha_3^{-1}(m''))(*,p'))((*,\alpha_2\beta_2^{-1}(l''),*)(*,q')) \in E(A' \times B').$$

Finally, applying φ^{-1} one last time takes us to

$$((*,*,*)(\alpha_4^{-1}\beta_3\alpha_3^{-1}(m''),\alpha_5^{-}1(p')))((*,*,*)(\beta_4^{-1}\alpha_2\beta_2^{-1}(l''),\beta_5^{-}1(q'))) \in E(A \times B).$$

Then by the definition of the direct product, we know that

$$(\alpha_4^{-1}\beta_3\alpha_3^{-1}(m''), \alpha_5^{-}1(p'))(\beta_4^{-1}\alpha_2\beta_2^{-1}(l''), \beta_5^{-}1(q')) \in E(B).$$

Therefore, Φ^{-1} is a homomorphism as well. Thus, Φ is an isomorphism, and $B \cong B'$, i.e. the prime bipartite factor in the prime factorization of **G** is unique. $\square$

While this result may not be as powerful as Theorem 2.8, it still allows us to characterize prime factorizations over the direct product in greater detail, and helps us to further nail down this very abstract concept. In the future, we hope to remove the restriction that **G** must be *R*-thin in order to produce a more general result (this has been done previously when the prime bipartite factor is K_2–see [4]).

Bibliography

Bibliography

[1] *Graphs & Digraphs*, fifth edition, by G. Chartrand, L. Lesniak and P. Zhang, CRC Press, Boca Raton, FLA, 2011.

[2] *Handbook of Product Graphs,* second edition, by R. Hammack, W. Imrich and S. Klavžar, CRC Press, Boca Raton, FLA, 2011.

[3] R. Hammack and W. Imrich, On Cartesian skeletons of graphs, *Ars Math. Contemp.,* **2** (2009) 191-205.

[4] R. Hammack, On uniqueness of prime bipartite factors of graphs, *Discrete Mathematics* **313** (9) (2013) 1018-1027.

[5] G. Sabidussi, Graph Multiplication, *Math. Z.*, **72** (1960) 446-457.

[6] G.V. Vizing, The Cartesian product of graphs (Russian), *Vyčisl Sistemy,* **9** (1963) 30-43.

Vita

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