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## A Weak Groethendieck Compactness Principle for Infinite Dimensional Banach Spaces

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*Virginia Commonwealth University*

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## ABSTRACT

### **A Weak Grothendieck Compactness Principle for Infinite Dimensional Banach Spaces**

Kaitlin Bjorkman

The goal of this thesis is to give an exposition of the following recent result of Freeman, Lennard, Odell, Turett and Randrianantoanina [3] A Banach space has the Schur property if and only if every weakly compact set is contained in the closed convex hull of a weakly null sequence. This result complements an old result of Grothendieck (now called the Grothendieck Compactness Principle) stating that every norm compact subset of a Banach space is contained in the closed convex hull of a norm null sequence. We include many of the relevant definitions and preliminary results which are required in the proofs of both of these theorems.

Dissertation Director: Dr. Kevin Beanland

# A Weak Grothendieck Compactness Principle for Infinite Dimensional Banach Spaces

by

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Bachelor of Science, Mathematics & Bachelor of Science, Cognitive Science  
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Submitted in Partial Fulfillment of the Requirements

for the Degree of Master of Science in the

Department of Mathematics

Virginia Commonwealth University

2013

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Major Professor

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Dean of The Graduate School

## DEDICATION

This manuscript is dedicated to

Jesus Christ

*who calls me to excellence every day and who made me love mathematics.*

Brad and Nancy Bjorkman

*my wonderful parents who modeled and taught excellence for me.*

Turk

*my faithful dog, for reminding me to take a break every so often.*

## ACKNOWLEDGEMENTS

A big thanks to my advisor Dr. Kevin Beanland for getting me through real analysis and then addicted to analysis in general.

To students in courses I taught and TA'd and who visited me in the tutoring center during my time at VCU, thanks for reminding me of the basics and of how much fun math can be - gold stars for everyone!

## ABSTRACT

The goal of this thesis is to give an exposition of the following recent result of Freeman, Lennard, Odell, Turett and Randrianantoanina [3] A Banach space has the Schur property if and only if every weakly compact set is contained in the closed convex hull of a weakly null sequence. This result complements an old result of Grothendieck (now called the Grothendieck Compactness Principle) stating that every norm compact subset of a Banach space is contained in the closed convex hull of a norm null sequence. We include many of the relevant definitions and preliminary results which are required in the proofs of both of these theorems.

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## INTRODUCTION

The purpose of this work is to introduce a graduate student, a mathematician in another field, or a non-expert in Banach spaces to a useful result characterizing compact sets under the weak topology in certain Banach spaces. Our primary goal is to provide the requisite background information and a clear explanation for the reader to understand the main result presented by Odell et al. in 2012 [3]. The Grothendieck Compactness Principle is a well-known result in Banach space theory that states that every compact subset of a Banach space is contained in the closed convex hull of a norm null sequence. As we will see, when the norm topology is replaced with the weak topology, this result does not hold in general. In this case we have the following characterization: that every weakly compact subset of a Banach space is contained in a weakly null sequence if and only if that Banach space has the Schur property. Now we outline the contents of each chapter.

Chapter 1 we give definitions and results in Banach space necessary to understand the Grothendieck Compactness Principle. These are results not typically covered in an introduction to Banach spaces. A reader entirely unfamiliar with Banach spaces should begin with Appendix A for a more thorough treatment of the basics of Banach space theory.

Chapter 2 contains our first major result. In particular, we present a proof of the Grothendieck Compactness Principle. Then, as a motivation for subsequent material, we conclude with a discussion and example of spaces that lack this property under the weak topology.



Chapter 3 consists of definitions and results essential for the proof of the main theorem. It includes results in the weak topology applied to Banach spaces. A reader unfamiliar with the weak topology applied to Banach spaces should refer to Appendix B for preliminary information.

In Chapter 4 the theorem of Odell et al. is proved. This is the most technically demanding part of the thesis.

## LIST OF SYMBOLS

- $\mathbb{N}$  The natural numbers  $\{1, 2, 3, \dots\}$ .
- $X^*$  The dual space of the Banach space  $X$ .
- $B_X$  The unit ball of the space  $X$ .
- $\mathcal{C}(K)$  The space of bounded linear operators from  $K$  to  $\mathbb{R}$  where  $K$  is a compact topological space.
- $c_0$  The space consisting of sequences of scalars that converge to 0 endowed with the  $\|\cdot\|_\infty$  norm.
- $\ell_1$  The space consisting of sequences  $x = (a_n)_{n=1}^\infty$  whose sums converge under the norm  $\|x\|_1 = \sum_{n=1}^\infty |a_n|$ .

# CHAPTER 1

## IMPORTANT RESULTS FROM BANACH SPACE THEORY

In this work, we assume that Banach spaces are infinite dimensional. A reader unfamiliar with Banach spaces should refer to Appendix A for basic definitions and well-known results in Banach space theory. In this chapter we will look at compactness and a structure called a convex hull. We begin with a definition and then provide results which will be used in the subsequent chapters.

DEFINITION 1.1. *A set  $C$  is convex if for all  $x, y \in C$*

$$\frac{x + y}{2} \in C$$

DEFINITION 1.2. *Let  $X$  be a Banach space and  $A \subseteq X$ . The convex hull of  $A$  is defined as*

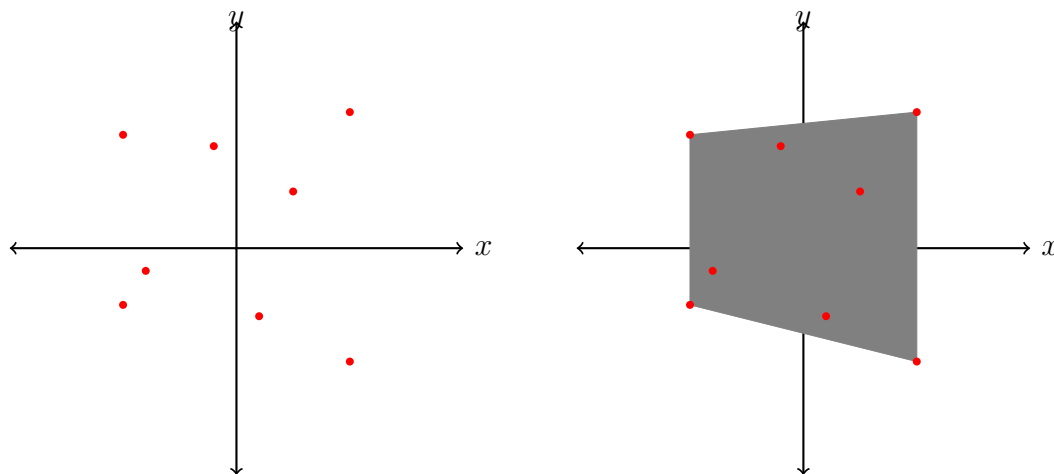
$$\text{conv}(A) = \left\{ \sum_{i \in F} r_i x_i \mid r_i \geq 0, \sum_{i \in F} r_i = 1, x_i \in A, \text{ and } F \text{ a finite subset of } \mathbb{N} \right\}.$$

*The norm closed convex hull is the union of the convex hull and its limit points. In general, the union of set  $A$  and its limit points is called the closure of  $A$  and is denoted  $\overline{A}$ .*

Note that the convex hull of the closure of a convex hull is itself, and that the addition of the limit points is equivalent to allowing infinite sums in the above definition, i.e,

$$\overline{\text{conv}}(A) = \left\{ \sum_{i=1}^{\infty} r_i x_i \mid r_i \geq 0, \sum_{i=1}^{\infty} r_i = 1, \text{ and } x_i \in A \right\}.$$

For the set of red points on the left, the closed convex hull is the gray figure on the right.



LEMMA 1.3. *Let  $X$  be a Banach space and  $(x_n)_{n=1}^\infty$  be a sequence such that  $\|x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $\overline{\text{conv}}(x_n)_{n=1}^\infty$  is compact.*

PROOF. We utilize a standard diagonalization argument. Consider the sequence  $(y_n)_{n=1}^\infty$  in  $\overline{\text{conv}}(x_n)_{n=1}^\infty$ . To show that  $\overline{\text{conv}}(x_n)_{n=1}^\infty$  is compact, it is sufficient to show that  $(y_n)$  has a convergent subsequence. For each  $i \in \mathbb{N}$  let  $(r_{i,j})_{j=1}^\infty$  be non-negative coefficients such that  $\sum_{j=1}^\infty r_{i,j} = 1$  and

$$y_i = \sum_{j=1}^{\infty} r_{i,j} x_j$$

Consider the sequence  $(r_{i,1})_{i=1}^\infty$  which is bounded by 1 and therefore has a convergent subsequence. Let  $N_1 \subseteq \mathbb{N}$  such that  $(r_{i,1})_{i \in N_1}$  is convergent to  $r_1$ . Now consider the sequence  $(r_{i,2})_{i \in N_1}$ , which by the same argument is bounded and therefore has a convergent subsequence. Let  $N_2 \subseteq N_1$  such that  $(r_{i,2})_{i \in N_2}$  is convergent to  $r_2$ . Continue in this fashion, with

$$\mathbb{N} \supseteq N_1 \supseteq N_2 \supseteq \dots$$

such that  $(r_{i,j})_{i \in N_j}$  converges for all  $i \in \mathbb{N}$ . Now define  $n_{i,j}$  to be the  $j^{\text{th}}$  element in the ordered set  $N_i$  and consider the subsequence  $(y_{n_{i,i}})_{i=1}^{\infty}$  of  $(y_n)$ . We show that  $y_{n_{i,i}} \rightarrow y = \sum_{i=1}^{\infty} r_i x_i$  as  $i \rightarrow \infty$ .

Let  $1 > \varepsilon > 0$ . Since  $(x_n)$  is norm null there exists  $M \in \mathbb{N}$  such that  $\|x_n\| < \varepsilon/4$  for all  $n \geq M$ . Then for each  $i \leq M$  there exist  $K_i \in \mathbb{N}$  such that  $|r_{j,k} - r_k| < \varepsilon/2M$  for all  $j \geq K_i$ . Let  $K = \max_{1 \leq i \leq M} K_i$  then for all  $j \geq K$  we have

$$\begin{aligned}
\|y_{n_{i,j}} - y\| &= \left\| \sum_{k=1}^{\infty} r_{n_{i,j},k} x_k - \sum_{k=1}^{\infty} r_k x_k \right\| \\
&\leq \left\| \sum_{k=1}^M r_{n_{i,j},k} x_k - \sum_{k=1}^M r_k x_k \right\| + \left\| \sum_{k=M+1}^{\infty} r_{n_{i,j},k} x_k - \sum_{k=M+1}^{\infty} r_k x_k \right\| \\
&= \left\| \sum_{k=1}^M (r_{n_{i,j},k} - r_k) x_k \right\| + \left\| \sum_{k=M+1}^{\infty} (r_{n_{i,j},k} - r_k) x_k \right\| \\
&\leq \left\| \sum_{k=1}^M (r_{n_{i,j},k} - r_k) x_k \right\| + \varepsilon/4 \left( \sum_{k=M+1}^{\infty} r_{n_{i,j},k} + \sum_{k=M+1}^{\infty} r_k \right) \\
&\leq \left\| \sum_{k=1}^M (r_{n_{i,j},k} - r_k) x_k \right\| + \varepsilon/2 \\
&< M(\varepsilon/2M) + \varepsilon/2 = \varepsilon
\end{aligned}$$

Therefore,  $(y_n)$  has a convergent subsequence and the closed convex hull of a norm null space is compact. □

REMARK 1. The analogous result holds if in Lemma 1.3 we replace the norm null sequence with a weakly null sequence and norm compactness with weak compactness. The proof is almost identical.

## CHAPTER 2

### THE GROTHENDIECK COMPACTNESS PRINCIPLE

Grothendieck [4, p. 112] observed a significant consequence of a paper by Schwartz and Dieudonné [2], as noted in [3], that gives a convenient characterization of compact subsets of a Banach space. This result indicates that compact subsets of Banach spaces are rather small: recall that  $B_X$  is compact if and only if  $X$  is finite dimensional. His result can be used both to prove that a certain subset is compact and also to construct a compact subset when such a set would be of value in a proof.

**THEOREM 2.1** (The Grothendieck Compactness Principle). *A norm closed subset  $K$  of a Banach space  $X$  is norm compact if and only if it is contained in the closed convex hull of a norm null sequence; that is,  $K \subseteq \overline{\text{conv}}(x_n)_{n=0}^\infty$  where  $\|x_n\| \rightarrow 0$ .*

**PROOF.** (this is an expansion of the proof found in [Vol 1, pp. 30-31] [7] ) For the forward direction, by Lemma 1.3,  $\overline{\text{conv}}(x_n)_{n=0}^\infty$  is compact for all  $\|x_n\| \rightarrow 0$ . If  $K \subseteq \overline{\text{conv}}(x_n)_{n=0}^\infty$ , it is a closed subset of a compact set and is therefore compact.

Now for the reverse direction. Let  $K$  be a closed, compact subset of  $X$ . We construct a norm null sequence  $(x_n)_{n=1}^\infty$  such that  $K \subseteq \overline{\text{conv}}(x_n)_{n=0}^\infty$  by constructing a sequence of vectors that are the centers of a progression of finite covers of variations of  $K$  which we define below. The first such variation is

$$2K = \{2x \mid x \in K\} \subseteq \cup_{i=1}^{n_1} B(x_{1,i}, 1/4)$$

For the lefthand side  $2K = \{2x|x \in K\}$  is scalar multiplication of every element in  $K$  by 2 to form  $2K$ . For the right side,  $B(x_{1,j}, 1/4)$  is the ball of diameter  $1/4$  centered at  $x_{1,j}$ ,  $B(x_{1,j}, 1/4) = \{x \in X : \|x - x_{1,j}\| < 1/4\}$ . Since we know that  $2K$  is compact and that these balls around all elements of  $2K$  form an open cover, we can find a finite number of them to be a finite cover  $\cup_{j=1}^{n_1} B(x_{1,j}, 1/4)$ . We fix the elements  $x_{1,1}, x_{1,2}, \dots, x_{1,n_1}$  as the first  $n$  elements of our norm null sequence. Now, we shift these “balls” to the origin by subtracting the center of each from every element of the ball and consider the set

$$K_2 = \cup_{i=1}^{n_1} [(B(x_{1,j}, 1/4) - x_{1,i}) \cap (2K - x_{1,i})] \subseteq B(0, 1/4)$$

So,  $K_2$  is the union of the sets resulting from shifting the center of each element of the finite cover to the origin and removing the portion outside of an identically shifted  $2K$ . This is equivalent to intersecting each open set with  $2K$  and then shifting to the origin, but compactness follows more directly in the former explanation. Each of these balls is compact as a subset of  $B(0, 1/4)$  since they are the intersection of the shifted compact set  $2K$  and  $B(0, 1/4)$  itself. Note that when shifted each ball becomes

$$B(0, 1/4).$$

$K_2$  is compact as a subset of  $B(0, 1/4)$  since it is the finite union of compact sets. Thus  $2K_2$  is compact as a subset of  $B(0, 1/2)$  and we can find a finite cover of the open cover of  $1/4^2$  balls, namely,

$$2K_2 \subseteq \cup_{i=1}^{n_2} B(x_{2,i}, 1/4^2).$$

Take this finite set as the next  $n_2$  elements of our norm null sequence before constructing

$$K_3 = \cup_{i=1}^{n_2} [(B(x_{2,j}, 1/4^2) - x_{2,i}) \cap (2K_2 - x_{2,i})] \subseteq B(0, 1/4^2)$$

We may continue in this fashion, appending the centers of the finite collection of shrinking balls to our norm null sequence each time so that at the  $k^{\text{th}}$  step these balls are derived as the finite subcover of the open cover

$$K_{k+1} = \cup_{i=1}^{n_k} [(B(x_{k,j}, 1/4^k) - x_{k,i}) \cap (2K_k - x_{k,i})] \subseteq B(0, 1/4^k)$$

Now, we need only show that every element  $x \in K$  can be approached arbitrary close by a linear combination of these sequence elements with coefficients that sum to 1. Construct this linear sequence as follows: By construction, we have an element  $x_{1,m_1}$  in the sequence such that

$$\left\| x - \frac{1}{2}x_{1,m_1} \right\| < 1/4$$

Since  $(x - \frac{1}{2}x_{1,m_1}) \in B(0, 1/4)$  it is in one of the  $\frac{1}{4^2}$  balls and there is also an element  $x_{2,m_2}$  such that

$$\left\| x - \frac{1}{2}x_{1,m_1} - \frac{1}{4}x_{2,m_2} \right\| < 1/4^2$$

Continuing in this fashion, we may select one element from each  $K_i$  contributed portion of the sequence, endow it with the coefficient  $1/2^i$  and add it to our approximation to ensure we are within  $1/4^i$  of  $x$ ,

$$\left\| x - \frac{1}{2}x_{1,m_1} - \frac{1}{4}x_{2,m_2} - \dots - \frac{1}{2^i}x_{i,m_i} \right\| \leq 1/4^i$$

This gives

$$\sum_{i=1}^{\infty} \frac{1}{2^i}x_{i,m_i} = x$$

a sum with coefficients that sum to 1 and that converges to  $x$ . Therefore by definition,  $x \in \overline{\text{conv}}(x_n)_{n=0}^{\infty}$ . □

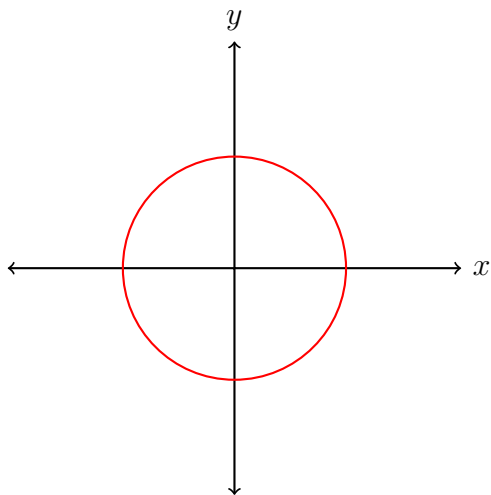
**REMARK 2.** If we consider  $X$  to be a finite-dimensional space, this theorem is equivalent to the Heine-Borel Theorem [5, p.335] as in such spaces a closed subset of the closed convex hull of a norm null sequence is precisely a closed and bounded subset.



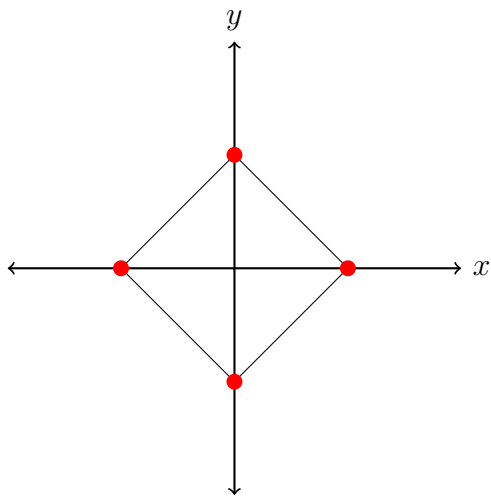
The Grothendieck Compactness Principle concerns Banach spaces considered with their norm topology. When the space is considered under the weak topology, the principal no longer holds in general. Let's unpack why this is the case.

DEFINITION 2.2. Let  $X$  be a Banach space and  $E \subseteq X$ . A vector  $x \in E$  is an extreme point of  $E$  if for all  $x_1 \neq x_2$  in  $E$  and  $r_1, r_2 \in (0, 1)$ ,  $x \neq r_1x_1 + r_2x_2$ . Let  $\text{ext}(E)$  denote the set of all extreme points of  $E$ .

To understand this definition, let's consider two subsets of  $\mathbb{R}^2$ .



This set, the unit circle, has an uncountable number of extreme points since any point on the circumference of the circle does not lie “between” any two other points in the set.



This set, a diamond with diagonal 2, has only four extreme points - one at each corner. The sides of the diamond lie between the corners and so contain no extreme points.

A well-known inequality called the Cauchy-Schwartz inequality implies that

$$a_1b_1 + a_2b_2 \leq (a_1^2 + a_2^2)^{1/2}(b_1^2 + b_2^2)^{1/2}.$$

Where  $a_1, a_2, b_1, b_2 \in \mathbb{R}$ . We shall use this fact in our next proof.

**PROPOSITION 2.3.** *The unit sphere of  $\ell_2$  (see Definition A.16), denoted  $S_{\ell_2}$ , is the set of extreme points of  $B_{\ell_2}$ .*

**PROOF.** Let  $(a_i)_{i=1}^{\infty}, (b_i)_{i=1}^{\infty} \in B_{\ell_2}$  and  $r_1, r_2 \in (0, 1)$  with  $r_1 + r_2 = 1$ , note that  $(r_1^2 + r_2^2)^{1/2} < 1$  since  $r_1, r_2 < 1$ . Using the Cauchy-Schwartz inequality we get

$$\begin{aligned} \|r_1(a_i) + r_2(b_i)\|_2 &= \|(r_1a_i + r_2b_i)_{i=1}^{\infty}\|_2 \\ &= (\sum_{i=1}^{\infty} |r_1a_i + r_2b_i|^2)^{1/2} \\ &\leq (\sum_{i=1}^{\infty} (r_1^2 + r_2^2)(a_i^2 + b_i^2))^{1/2} \\ &= (r_1^2 + r_2^2)^{1/2} (\sum_{i=1}^{\infty} a_i^2 + \sum_{i=1}^{\infty} b_i^2)^{1/2} \\ &< 1 \end{aligned}$$

Therefore,  $(r_1(a_i) + r_2(b_i)) \notin S_{\ell_2}$  and the extreme points are exactly the sphere.  $\square$

Recall that the closed unit ball  $B_{\ell_2}$  is weakly compact and the sphere  $S_{\ell_2}$  is uncountable (Proposition B.8).

**PROPOSITION 2.4.** *Let  $(x_n)_{n=1}^{\infty}$  be a sequence in a Banach space  $X$  then  $\text{ext}(\overline{\text{conv}}(x_n)) \subseteq \{x_n : n \in \mathbb{N}\}$ .*

**PROOF.** We proceed by contradiction. Suppose that  $x \in \overline{\text{conv}}(x_n)$ ,

$$x = \sum_{i=1}^{\infty} r_i x_i$$

and that there exists  $j_1 \neq j_2$  such that  $r_{j_1} \neq 0, r_{j_2} \neq 0$ . Then

$$x = r_{j_1}x_{j_1} + S \sum_{i \neq j_1} \frac{r_i}{S} x_i.$$

where  $S = \sum_{j \neq j_1} r_j \neq 0$ . Since

$$\sum_{i \neq j_1} \frac{r_i}{S} x_i \in \overline{\text{conv}}(x_n)$$

and  $x_{j_1} \in \overline{\text{conv}}(x_n)$  and  $r_{j_1} + S = 1$ ,  $x \notin \text{ext}(\overline{\text{conv}}(x_n))$ , a contradiction. Therefore,  $\text{ext}(\overline{\text{conv}}(x_n)) \subseteq \{x_n : n \in \mathbb{N}\}$ .  $\square$

**PROPOSITION 2.5.**  *$B_{\ell_2}$  is weakly compact but is not contained in  $\text{ext}(\overline{\text{conv}}(x_n))$  for any  $(x_n)_{n=1}^{\infty}$  that is weakly null.*

**PROOF.** For the sake of contradiction, assume that  $B_{\ell_2} \subseteq \text{ext}(\overline{\text{conv}}(x_n))$  where  $(x_n)_{n=1}^{\infty}$  is weakly null. Then by Proposition 2.3

$$\text{ext}(B_{\ell_2}) \subseteq \text{ext}(\overline{\text{conv}}(x_n)) \subseteq \{x_n : n \in \mathbb{N}\}$$

but  $\text{ext}(B_{\ell_2})$  is uncountable, so it cannot be the subset of a countable set.  $\square$

So for  $\ell_2$  at least, the Grothendieck compactness principle does not hold under the weak topology. A result of Lindenstrauss [6] implies that the unit ball of every reflexive space is has uncountably many extreme points. Since for a reflexive Banach space  $X$ ,  $B_X$  is weakly compact (see Proposition B.8), Proposition 2.5 holds if  $B_{\ell_2}$  is replaced by  $B_X$  for any reflexive Banach space  $X$ .

## CHAPTER 3

### BUILDING UP TO A WEAK GROTHENDIECK COMPACTNESS PRINCIPLE

Here we will recall some definitions and results needed for the statement of our major result and its proof in the next chapter.

DEFINITION 3.1. *Let  $K$  be a compact topological space. The space  $C(K)$  is the space of all continuous functions from  $K$  into  $\mathbb{R}$ . Equip  $C(K)$  with norm*

$$\|f\|_\infty = \sup_{x \in K} |f(x)|$$

and  $(C(K), \|\cdot\|_\infty)$  is a Banach space.

THEOREM 3.2. *[The Banach-Mazur Theorem] Every separable Banach space  $X$  is isomorphically isometric to a subspace of  $C[2^\mathbb{N}]$  where  $2^\mathbb{N}$  is the Cantor set.*

PROOF. We need only to find an isometric isomorphism from  $X$  to  $C[2^\mathbb{N}]$ . Let  $B_{X^*}$  be the unit ball of the dual of  $X$  considered under the weak\* topology. Since  $X$  is separable, we have that  $B_{X^*}$  is both separable and metrizable as well as compact under the weak\* topology [9, p.204]. By [5, p. 17 ff] this means that there exists a function  $\phi$  such that

$$\phi : 2^\mathbb{N} \rightarrow B_{X^*}$$

is continuous and onto. Now define

$$T : X \rightarrow C[2^\mathbb{N}]$$

by  $T(x)(\sigma) = \phi(\sigma)(x)$ . This is isometric since

$$\|T(x)\|_\infty = \sup_{\sigma \in 2^\mathbb{N}} |\phi(\sigma)(x)| = \sup_{x^* \in B_{X^*}} |x^*(x)| = \|x\|$$

Note that  $\|x\| = \sup_{x^* \in B_{X^*}} |x^*(x)|$  by Corollary B.17. This completes the proof.  $\square$

**PROPOSITION 3.3.** *Every weakly null sequence in  $\ell_1$  is also norm null*

**PROOF.** Let  $(x_n)_{n=1}^\infty$  be a weakly null sequence in  $\ell_1$  and for contradiction assume that it does not converge in norm to 0. Now this sequence contains a subsequence that can be normalized and is equivalent to the unit vector basis of  $\ell_1$  (this fact is proven in [5]). But the unit vector basis of  $\ell_1$  is not weakly null since it is not weakly Cauchy (Proposition B.13), a contradiction. Therefore, every weakly null sequence in  $\ell_1$  is also norm null.  $\square$

Proposition 3.3 was first observed by much different methods by Schur in 1921 [10]. Proposition 3.3 is enough to imply that every weakly convergent subsequence of  $\ell_1$  is also norm convergent. Additional spaces were later found to share this property and it was named after Schur.

**DEFINITION 3.4.** *A Banach space  $X$  has the Schur Property if every weakly convergent sequence in  $X$  is norm convergent. That is, if there is a sequence  $(x_n)_{n=1}^\infty$  that weakly converges to  $x \in X$ , then under the norm  $(x_n)$  also converges to  $x$ .*

The space  $\ell_1$  is the canonical example of a space which has the Schur property,  $c_0$  is a classical space that does not have the Schur property. For example, the sequence  $(e_n)_{n=1}^\infty$  (the unit vector basis) converges to 0 in the weak topology on  $c_0$  but does not converge at all under the norm topology. To see this, let  $f = (a_i) \in c_0^* = \ell_1$  then we have  $\lim_{n \rightarrow \infty} f(e_n) = \lim_{n \rightarrow \infty} a_n = 0$ , but  $(e_n)_{n \in \mathbb{N}}$  is not norm Cauchy since  $\|e_n - e_m\|_\infty = 1$  for all  $n \neq m$ .

**PROPOSITION 3.5.** *Let  $X$  be a space with the Schur property and let  $K$  be a weakly compact subset of  $X$ . Then  $K$  is norm compact.*

PROOF. By the Eberlain-Smulian Theorem (Theorem B.7),  $K$  is weakly compact if and only if every sequence in  $K$  has a weakly convergent subsequence in  $K$ . Since  $X$  has the Schur property, every weakly convergent sequence in  $K$  is also norm convergent in  $K$ . Therefore  $K$  is norm compact.  $\square$

It is easy to see that separable, reflexive, infinite-dimensional spaces do not have the Schur property. Let  $X$  be a separable reflexive infinite dimensional space. In reflexive spaces the weak and weak\* topologies coincide. Since  $X$  is separable  $B_X$  is weak\* and therefore weakly compact. By the above proposition  $B_X$  would also be norm compact but  $B_X$  is only norm compact when  $X$  is finite dimensional [5, Appendix C], a contradiction.

One characterization of the Schur property comes from Rosenthal's  $\ell_1$  Theorem. This is appropriate since the Schur property is closely connected to  $\ell_1$ .

**THEOREM 3.6** (Rosenthal's  $\ell_1$  Theorem). [8] *Every bounded sequence in a Banach space either has a weakly Cauchy subsequence or has a subsequence equivalent to the unit vector basis of  $\ell_1$ .*

Rosenthal's  $\ell_1$  theorem is a dichotomy, a bounded sequence has either but not both a weakly Cauchy subsequence or a subsequence equivalent to the unit vector basis of  $\ell_1$ . This comes from the fact that the unit vector basis of  $\ell_1$  is not weakly Cauchy (Proposition B.13), and if a sequence has a subsequence which is not weakly Cauchy, by definition the sequence cannot be weakly Cauchy.

**DEFINITION 3.7.** *A Banach space  $X$  is said to be  $\ell_1$  saturated if every subspace  $Y$  of  $X$  contains a further subspace isomorphic to  $\ell_1$ .*

**COROLLARY 3.8.** *If a Banach Space has the Schur Property then it is  $\ell_1$  saturated.*

PROOF. We proceed by contradiction. Assume that  $X$  has the Schur Property. Let  $Y$  be a subspace of  $X$ , by Theorem A.12 there is a basic sequence  $(y_n)_{n=1}^{\infty}$  in  $Y$ .

Assume it is not equivalent to the unit vector basis of  $\ell_1$ . Then by Rosenthals'  $\ell_1$  Theorem we can assume it has a weakly Cauchy subsequence. Let  $(x_n)_{n=1}^\infty$  be the weakly Cauchy subsequence of  $(y_n)$ . Define a new sequence  $z_n = x_{2n+1} - x_{2n}$  for each  $n \in \mathbb{N}$ . Since  $(x_n)$  is weakly Cauchy,  $z_n \xrightarrow{w} 0$ , but since  $X$  has the Schur property, this means that  $\|z_n\| \rightarrow 0$  which is a contradiction since our original sequence was a basic sequence (see Remark 5 in Appendix A) and thus  $\inf \|z_n\| > 0$ .

Therefore, any subspace of a space with the Schur Property must contain a sequence equivalent to the unit vector basis of  $\ell_1$ . □

## CHAPTER 4

### A WEAK GROTHENDIECK COMPACTNESS PRINCIPLE

This chapter contains the the main theorem of this thesis [3]. As seen in the previous chapter, the Grothendieck Compactness Principle does not in general apply under the weak topology. However, characterizing compact sets is useful under that topology, and in some spaces, the principle applies to all weakly compact subsets just as it applies to norm compact subsets.

Before we begin we will need one definition and several lemmas.

DEFINITION 4.1. *A basis  $(x_i)_{i=1}^{\infty}$  of a Banach space  $X$  is bimonotone if the projections  $P_{[n,m]}$  defined by*

$$P_{[n,m]} \left( \sum_{i=1}^{\infty} a_i x_i \right) = \sum_{i=n}^{m-1} a_i x_i$$

*have norm 1 for all  $n < m$ .*

Suppose  $(X, \|\cdot\|)$  is a Banach space with a basis  $(x_n)$ . Then there is an equivalent norm  $\|\!\|\!\cdot\!\|\!$  on  $X$  such that  $(x_n)$  is a bimonotone basis with respect to this norm.

This can be shown by defining a new norm and considering the basis under that norm. Suppose  $(x_i)_{i=1}^{\infty}$  is a basis for  $X$ . Define the following new norm on  $X$ .

$$\left\| \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \right\| = \sup_{\substack{I \subseteq \mathbb{N} \\ I \text{ interval}}} \left\| \sum_{i \in I} a_i x_i \right\|$$

Now, compare to the original norm we want to show that there exists  $K \in \mathbb{R}$  such that

$$\frac{1}{K} \left\| \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \right\| \leq \left\| \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \right\| \leq K \left\| \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \right\|$$



For all  $x = \sum_{i=1}^{\infty} a_i x_i \in X$ .

LEMMA 4.2. *Let  $X$  be a Banach space with a normalized bimonotone basis  $(e_n)_{n=1}^{\infty}$  and associated projections  $P_{[n,m]}$ . If a sequence  $(y_k)_{k=1}^{\infty} \subseteq X$  is weakly null then for all  $\varepsilon > 0$  and  $n \in \mathbb{N}$ , there exists  $K \in \mathbb{N}$  such that for all  $k \geq K$ ,*

$$\|P_{[1,n]}y_k\| < \varepsilon$$

PROOF. Let  $(e_n^*)$  denote the biorthogonals associated with the basis  $(e_n)$  (see Definition B.3) so that

$$y_k = \sum_{i=1}^{\infty} e_i^*(y_k)e_i.$$

Since  $(y_k)$  is weakly null for each  $i \in \mathbb{N}$  the sequence  $(e_i^*(y_k))_{k=1}^{\infty}$  converges to 0. For each  $i$  find  $K_i \in \mathbb{N}$  such that for all  $k > K_i$  the projection

$$\|P_{[i,i+1]}y_k\| = |e_i^*(y_k)| < \varepsilon/n.$$

Let  $K = \max_{1 \leq i \leq n} K_i$ . Then we have

$$\|P_{[1,n]}y_k\| \leq \sum_{i=1}^n \|P_{[i,i+1]}y_k\| < n(\varepsilon/n) = \varepsilon.$$

□

LEMMA 4.3. *Let  $X$  be a Banach space with a normalized bimonotone basis  $(e_n)_{n=1}^{\infty}$  and associated projections  $P_{[n,m]}$ . For each  $x \in X$  and  $\varepsilon > 0$  there exists  $n \in \mathbb{N}$  such that*

$$\|x - P_{[1,n]}(x)\| < \varepsilon$$

PROOF. Let  $x = \sum_{i=1}^{\infty} a_i e_i$ . There exists  $N \in \mathbb{N}$  such that for all  $n > N$

$$\|x - \sum_{i=1}^n a_i e_i\| < \varepsilon$$

But this is precisely

$$\|x - P_{[1,n]}(x)\| < \varepsilon$$

□

LEMMA 4.4. *Let  $X$  be a Banach space with a normalized bimonotone basis and let  $x_n \xrightarrow{w} 0$  in  $X$ . Then there exist two monotonically increasing sequences of natural numbers  $(m_n)_{n=1}^\infty$  and  $(q_n)_{n=1}^\infty$  such that*

- (i)  $\|x_{m_j} - P_{[q_j, q_{j+1})}(x_{m_j})\| < \left(\frac{1}{2^{j+1}}\right)^2$ .
- (ii)  $\|P_{[q_j, q_{j+1})}(x_{m_i})\| < \frac{1}{2^{i+1}} \cdot \frac{1}{2^{j+1}}$  for all  $i \neq j$ .

PROOF. We will be using Lemmas 4.2 and 4.3 repeatedly. Our preliminary work is shown below:

Let  $m_1 = q_1 = 1$ . Find  $q_2 > q_1$  such that

$$\|P_{[q_2, \infty)}x_m\| < \left(\frac{1}{2^2}\right)^2 / 2$$

Now find  $m_2 > m_1$  such that for all  $k \geq m_2$  we have

$$\|P_{[1, q_2)}x_k\| < \left(\frac{1}{2^3}\right)^2 / 2$$

Assume for  $j \geq 2$  that  $m_j$  and  $q_j$  have been defined. Find  $q_{j+1}$  such that for all  $1 \leq i \leq j$

$$(*) \quad \|P_{[q_{j+1}, \infty)}x_{m_i}\| < \left(\frac{1}{2^{j+2}}\right)^2 / 2$$

Now find  $m_{j+1} > m_j$  such that for all  $k \geq m_{j+1}$

$$(**) \quad \|P_{[1, q_{j+1})}x_k\| < \left(\frac{1}{2^{j+2}}\right)^2 / 2$$

Proof of (i). Let  $j \in \mathbb{N}$

$$\begin{aligned} \|(P_{[1, q_j)} + P_{[q_{j+1}, \infty)})x_{m_j}\| &\leq \|P_{[1, q_j)}x_{m_j}\| + \|P_{[q_{j+1}, \infty)}x_{m_j}\| \\ &\leq \left(\frac{1}{2^{j+1}}\right)^2 / 2 + \left(\frac{1}{2^{(j+1)+1}}\right)^2 / 2 \\ &< \left(\frac{1}{2^{j+1}}\right)^2 \end{aligned}$$

With the second inequality due to (\*\*) and (\*), respectively.

Proof of (ii). For  $i < j$  by (\*) and bimonotonicity we have

$$\|P_{[q_j, q_{j+1})}x_{m_i}\| \leq \|P_{[q_j, \infty)}x_{m_i}\| < \left(\frac{1}{2^{j+1}}\right)^2 / 2$$

For  $i > j$  by (\*\*) and bimonotonicity we have

$$\|P_{[q_j, q_{j+1})}x_{m_i}\| \leq \|P_{[1, q_i)}x_{m_i}\| < \left(\frac{1}{2^{i+1}}\right)^2 / 2 < \left(\frac{1}{2^{i+1}}\right)^2$$

□

LEMMA 4.5. *Let  $X$  be a Banach space with a normalized bimonotone basis and  $y_n \xrightarrow{w} 0$  in  $X$  and let  $(q_i)_{i=1}^\infty$  be an increasing sequence of natural numbers. Then there are strictly increasing sequences  $(p_n)_{n=1}^\infty$  and  $(k_n)_{n=1}^\infty$  such that*

$$(iii) \|P_{[q_{k_i}, q_{k_i+1})}(y_i)\| < \left(\frac{1}{2^{j+2}}\right)^2 \text{ for all } i \notin [p_j, p_{j+1})$$

PROOF. First some preliminaries: Let  $p_1 = 1$ . Find  $p_2 > p_1$  such that for all  $i > p_2$

$$\|P_{[q_1, q_2)}y_i\| < \frac{1}{2^2}.$$

Let  $k_1 = 1$  and find  $k_2 > k_1$  such that for all  $i \in [p_1, p_2)$

$$\|P_{[q_{k_2}, \infty)}y_i\| < \frac{1}{2^3}.$$

Find  $p_3 > p_2$  such that for all  $i \geq p_3$

$$\|P_{[1, q_{k_2+1})}y_i\| < \frac{1}{2^3}.$$

Find  $k_3$  such that for all  $i \in [p_1, p_3)$

$$\|P_{[q_{k_3}, \infty)}y_i\| < \frac{1}{2^4}.$$

Continuing in this fashion we have that for each  $j \geq 2$ ,  $j \in \mathbb{N}$  for all  $i \geq p_j$

$$(b) \|P_{[q_{k_{j-1}}, q_{k_{j-1}+1})}y_i\| < \frac{1}{2^j}$$

and for all  $i \in [1, p_j)$

$$(bb) \|P_{[q_{k_j}, \infty)}y_i\| < \frac{1}{2^{j+1}}.$$

Proof of (iii). Let  $i \notin [p_j, p_{j+1})$ .

If  $i \geq p_{j+1}$  then by (b)

$$\|P_{[q_{k_j}, q_{k_{j+1}})} y_i\| \leq \|P_{[1, q_{k_{j+1}})} y_i\| < \frac{1}{2^{j+1}}.$$

If  $i < p_j$  then (bb) gives us

$$\|P_{[q_{k_j}, q_{k_{j+1}})} y_i\| \leq \|P_{[q_{k_j}, \infty)} y_i\| < \frac{1}{2^{j+1}}.$$

□

Now we are ready to state and prove the theorem.

**THEOREM 4.6** (A Weak Grothendieck Compactness Principle). [**3**, Odell et al.]  
*Every weakly compact subset  $K$  of a Banach space  $X$  is contained in the closed convex hull of some weakly null sequence if and only if  $X$  has the Schur property.*

**PROOF.** The reverse direction follows directly from the definition of the Schur property. Let  $X$  be a space with the Schur property and let  $K$  be a weakly compact subset of  $X$ . Since  $K$  is weakly compact, by Theorem B.7 every sequence has a convergent subsequence. Since  $X$  has the Schur property, these subsequences are also norm convergent. Therefore,  $K$  is norm compact and the Grothendieck Compactness Principle gives that it is contained in the closed convex hull of some norm null sequence. Since norm convergence implies weak convergence (Proposition B.5) the norm null sequence is also weakly null. Therefore, if a Banach space has the Schur property, every weakly compact set  $K$  is contained in the closed convex hull of a weakly null sequence.

The forward direction is more involved. We proceed by contradiction. Let  $X$  be a Banach space without the Schur property such that every weakly compact subset  $K$  of  $X$  is contained in the closed convex hull of a weakly null sequence.

**CLAIM 4.7.** *There exists a sequence  $(x_n)_{n=1}^\infty$  in  $X$  such that  $(x_n)$  is normalized but weakly converges to 0.*

Since  $X$  by assumption does not have the Schur property there exists a weakly convergent sequence  $(x'_n)_{n=1}^\infty$  such that  $x'_n \xrightarrow{w} x$  but  $x'_n \not\rightarrow x$ . This sequence can be perturbed by a factor of  $x$  to become a weakly null sequence,  $x''_n = x'_n - x$ . Finally by passing to a subsequence  $(x''_{n_k})$  with  $\inf_{k \in \mathbb{N}} \|x''_{n_k}\| > 0$  that can be normalized so  $x_n = \frac{x''_{n_k}}{\|x''_{n_k}\|}$  and  $x_n \xrightarrow{w} 0$ .

For each  $j \in \mathbb{N}$  define the set

$$K_j = (j \overline{\text{conv}}(x_n)_{n=1}^\infty) \cap \frac{1}{j} B_X$$

where  $j \overline{\text{conv}}(x_n)_{n=1}^\infty$  denotes the set consisting of all elements of the closed convex hull multiplied by the scalar  $j$  and  $B_X$  is the closed unit ball of the space  $X$ . Let

$$K = \bigcup_{j=1}^{\infty} K_j$$

CLAIM 4.8. *Each  $K_j$  is weakly compact.*

The closed convex hull of a weakly null sequence is weakly compact (see Remark 1). The closed unit ball  $B_X$  is weakly closed by Proposition B.8. Since scalar multiplication (by  $j$  or  $\frac{1}{j}$ ) does not alter compactness and since the intersection of a weakly compact set with a weakly closed set is compact,  $K_j$  is compact for all  $j$ .

CLAIM 4.9. *The set  $K$  is weakly compact.*

Let  $\mathcal{A}$  be a weakly open cover of  $K$  and let  $A$  be an open set in  $\mathcal{A}$  containing 0. Since  $A$  is weakly open it contains a neighborhood  $N_{\varepsilon, x_1^*, \dots, x_k^*}(0)$  (see Definition B.1 and Proposition B.14) for all  $j$  such that  $\frac{1}{j} < \varepsilon$  since  $K_j \subseteq \varepsilon B_X \subseteq A$ . Thus,  $K \subseteq \left( \bigcup_{j=1}^{k-1} K_j \right) \cup A$ . But since each  $K_j$  is compact, the open cover  $\mathcal{A}$  has a finite subcover  $\mathcal{A}_j$  for each  $K_j$ , specifically for  $j < k$ . Therefore,  $K \subseteq \left( \bigcup_{j=1}^{k-1} \mathcal{A}_j \right) \cup A$ , a finite cover, and  $K$  is weakly compact.

By our assumption, since  $K$  is weakly compact it is contained in the closed convex hull of a weakly null sequence  $(y_n)_{n=1}^\infty$ .

CLAIM 4.10.  $Y = \overline{\text{span}}(y_n)_{n=1}^\infty$  is a separable Banach space.

The following set is clearly countably and dense in  $Y$ :

$$D = \left\{ \sum_{i=1}^n r_i y_i : n \in \mathbb{N}, r_i \in \mathbb{Q} \right\}.$$

This is countable since it is the union of countable sets. It is dense since every vector in  $\overline{\text{span}}(y_n)_{n=1}^\infty$  can be approximated by a vector in  $\text{span}(y_n)_{n=1}^\infty$ , which can be approximated by a rationally supported finite sum of the vectors in  $(y_n)$ .

Now, every separable Banach space is isomorphic to a subspace of  $C[2^{\mathbb{N}}]$  (Theorem 3.2) and  $C[2^{\mathbb{N}}]$  has a normalized bimonotone basis. It follows that  $Y$  is a subspace of some Banach space  $Z$  with a normalized bimonotone basis  $(e_n)_{n=1}^\infty$ . In this space,  $(y_n)$  is still weakly null by Proposition B.10.

The goal is to show that  $\sup \|y_i\| = \infty$  and thereby contradict the fact that it is a weakly null sequence, since by Proposition B.11 a weakly null sequence must be bounded.

CLAIM 4.11. *There exist three monotonically increasing sequences of natural numbers  $(p_n)_{n=1}^\infty$ ,  $(m_n)_{n=1}^\infty$ , and  $(q_n)_{n=1}^\infty$  such that*

- (1)  $\|x_{m_j} - P_{[q_{2j-1}, q_{2j}]}(x_{m_j})\| < \left(\frac{1}{2^{j+1}}\right)^2$
- (2)  $\|P_{[q_{2j-1}, q_{2j}]}(x_{m_i})\| < \frac{1}{2^{i+1}} \frac{1}{2^{j+1}}$  for all  $i \neq j$
- (3)  $\|P_{[q_{2j-1}, q_{2j}]}(y_i)\| < \left(\frac{1}{2^{j+1}}\right)^2$  for all  $i \notin [p_j, p_{j+1})$

From Lemmas 4.4 and 4.5 we have three sequences  $(q_{k_i})$ ,  $(p_i)$ , and  $(m_i)$  such that

(1) Follows directly from Lemma 4.4

$$\|(I - P_{[q_{k_j}, q_{k_{j+1}}]})y_i\| = \|(P_{[1, q_{k_j}]} + P_{[q_{k_{j+1}}, \infty)})y_i\| < \left(\frac{1}{2^{k_j+1}}\right)^2$$

(2) Also from Lemma 4.4, for  $i \neq k_j$

$$\|P_{[q_{k_j}, q_{k_j}]}x_{m_i}\| < \left(\frac{1}{2^{\max\{k_j, i\}+1}}\right)^2$$

(3) And from Lemma 4.5 For all  $i \notin [p_j, p_{j+1})$

$$\|P_{\{q_{k_j}, q_{k_j}\}} y_i\| < \frac{1}{2^{i+j}}$$

Relabelling  $q_{2i-1} = q_{k_i}$  and  $q_{2i} = q_{k_{i+1}}$  completes the claim as stated and we have increasing sequences of natural numbers  $(m_n)_{n=1}^\infty$ ,  $(p_n)_{n=1}^\infty$  and  $(q_n)_{n=1}^\infty$  such that (1), (2), and (3) hold simultaneously.

Let  $N \in \mathbb{N}$ .

CLAIM 4.12. *There exists  $M \in \mathbb{N}$  and sequence of scalars  $(\lambda_n)_{n=1}^\infty$  with  $0 \leq \lambda_n \leq 1$  for all  $n \in \mathbb{N}$  such that  $\|\sum_{i=1}^M \lambda_i x_{m_i}\| \leq \frac{1}{N}$  and  $\sum_{i=1}^M \lambda_i = N$*

This follows directly from Theorem B.15 applied  $N$  times to disjoint blocks of  $(x_n)$  with  $\varepsilon = \frac{1}{N^2}$ , then taking the sum of  $N$  of them gives  $\|\sum_{i=1}^M \lambda_i x_{m_i}\| \leq N \cdot \frac{1}{N^2} = \frac{1}{N}$

CLAIM 4.13.  $\sum_{i=1}^M \lambda_i x_{m_i} \in K_N$

This is because the coefficients sum to  $N$  with vectors from  $(x_n)$ , so this vector is in  $N \cdot \overline{\text{conv}}(x_n)$ . It is also in  $\frac{1}{N} B_X$  since  $\|\sum_{i=1}^M \lambda_i x_{m_i}\| \leq \frac{1}{N}$ . Therefore, by definition it is in  $K_N$ .

CLAIM 4.14. *Then there exists  $y = \sum_{i=1}^\infty \alpha_i y_i \in \overline{\text{conv}}(y_n)$  such that*

$$\left\| \sum_{i=1}^M \lambda_i x_{m_i} - \sum_{i=1}^\infty \alpha_i y_i \right\| < \frac{1}{M}$$

This follows immediately from the fact that  $\sum_{i=1}^M \lambda_i x_{m_i} \in K_N \subseteq K \subseteq \overline{\text{conv}}(y_n)$ , so we may get as close as we like with some element of the convex hull. Note that since  $\sum_{i=1}^\infty \alpha_i y_i \in \overline{\text{conv}}(y_n)$ ,  $\sum_{i=1}^\infty \alpha_i = 1$ .

We now look at projections of the inequality just established. For each  $1 \leq j \leq M$  we have the following inequalities, first since the basis is bimonotone and then by

repeated applications of the triangle inequality,

$$\begin{aligned}
\frac{1}{M} &> \left\| \sum_{i=1}^M \lambda_i x_{m_i} - \sum_{i=1}^{\infty} \alpha_i y_i \right\| \\
&\geq \left\| P_{[q_{2j-1}, q_{2j}]} \left( \sum_{i=1}^M \lambda_i x_{m_i} - \sum_{i=1}^{\infty} \alpha_i y_i \right) \right\| \\
&\geq \left\| P_{[q_{2j-1}, q_{2j}]} \sum_{i=1}^M \lambda_i x_{m_i} \right\| - \left\| P_{[q_{2j-1}, q_{2j}]} \sum_{i=1}^{\infty} \alpha_i y_i \right\| \\
&\geq \lambda_j \|x_{m_j}\| - \lambda_j \|x_{m_j} - P_{[q_{2j-1}, q_{2j}]} x_{m_j}\| - \left\| P_{[q_{2j-1}, q_{2j}]} \sum_{\substack{i \leq M \\ i \neq j}} \lambda_i x_{m_i} \right\| \\
&\quad - \sum_{i \in [p_j, p_{j+1}]} \alpha_i \|y_i\| - \sum_{i \notin [p_j, p_{j+1}]} \alpha_i \|P_{[q_{2j-1}, q_{2j}]}(y_i)\|
\end{aligned}$$

Note that the last inequality involves adding and subtracting  $x_{m_j}$  before using the triangle inequality, and that positive scalars can pass through the norm. Now apply properties (1), (2), and (3) as defined in Claim 4.11 and note that  $(x_i)$  is a normalized sequence so we have,

$$\begin{aligned}
\dots &> \lambda_j - \frac{1}{2^{j+1}} - \sum_{\substack{i \leq M \\ i \neq j}} \frac{1}{2^{i+1}} \frac{1}{2^{j+1}} - \sum_{i \in [p_j, p_{j+1}]} \alpha_i \|y_i\| - \frac{1}{2^{j+1}} \\
&> \lambda_j - \sum_{i \in [p_j, p_{j+1}]} \alpha_i \|y_i\| - 2 \frac{1}{2^{j+1}}
\end{aligned}$$

Considering the sum of these over  $1 \leq j \leq M$ , and we have

$$\begin{aligned}
M \frac{1}{M} = 1 &> \sum_{j=1}^M \left\| P_{[q_{2j-1}, q_{2j}]} \left( \sum_{i=1}^M \lambda_i x_{m_i} - \sum_{i=1}^{\infty} \alpha_i y_i \right) \right\| \\
&> \sum_{j=1}^M \left( \lambda_j - \sum_{i=1}^{\infty} \alpha_i \|y_i\| - 2 \sum_{j=1}^M \frac{1}{2^{j+1}} \right) \\
&\geq N - \sup_{i \in \mathbb{N}} \|y_i\| - 2
\end{aligned}$$

Rearranging gives

$$\sup_{i \in \mathbb{N}} \|y_i\| > N - 1 - 2 \geq N - 3$$

Allowing  $N$  to go to infinity we have  $\sup_{i \in \mathbb{N}} \|y_i\| = \infty$ . Since  $(y_n)$  weakly converges to 0 this is a contradiction. Therefore, our original assumption was false and  $X$  has the Schur property.  $\square$

**REMARK 3.** We have only addressed one direction of Grothendieck's Compactness Principle applied directly under the weak topology. The other direction follows almost immediately, if  $K$  is weakly closed and is a subset of  $\overline{\text{conv}}(x_n)$  where  $x_n \xrightarrow{w} 0$ , then it is the closed subset of a weakly compact set and is weakly compact.



# Appendices

## APPENDIX A

### A REVIEW OF BANACH SPACE THEORY

The purpose of this appendix is to provide a reader unfamiliar with Banach spaces with sufficient background to understand the remainder of the work. It is assumed that the reader has a fundamental understanding of real vector spaces, including the definitions of dimension and basis. In this work, all Banach spaces will be infinite dimensional.

Vector spaces are not necessarily endowed with a topology. There may not be a concept of size or distance, nor of open, closed, or compact sets on the space. Banach spaces are vector spaces with topologies induced by a function called a *norm* that is specified on the given space.

**DEFINITION A.1.** *A norm  $\|\cdot\|$  on a real linear space  $X$  is an assignment of a positive real number to every vector of the space such that*

- (1)  $\|x\| = 0$  if and only if  $x = 0$ .
- (2)  $\|ax\| = |a|\|x\|$  for all  $x \in X$  and  $a \in \mathbb{R}$ .
- (3)  $\|x + y\| \leq \|x\| + \|y\|$  for all  $x, y \in X$ .

*A linear space together with a norm  $\|\cdot\|$  is called a normed linear space.*

**DEFINITION A.2.** *Let  $X$  be a normed linear space. A Cauchy sequence is a sequence of vectors  $(x_n)_{n=1}^{\infty} \in X$  such that for all  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that  $\|x_n - x_m\| < \epsilon$  whenever  $n, m > N$ .*

DEFINITION A.3. A normed linear space  $X$  is said to be complete if every Cauchy sequence converges in the space. That is, if  $(x_n)_{n=1}^{\infty}$  is Cauchy in  $X$  there exists  $x \in X$  such that for all  $\epsilon > 0$  there is an  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $\|x_n - x\| < \epsilon$ .

DEFINITION A.4. A Banach space is a complete normed linear space.

One class of Banach spaces are the separable Banach spaces,

DEFINITION A.5. A Banach space  $X$  is separable if there exists a countable dense subset of  $X$ . That is, if there exists a sequence of vectors  $(x_n)_{n=1}^{\infty}$  such that for all  $x \in X$  and  $\epsilon > 0$ , there is an  $m \in \mathbb{N}$  such that  $\|x - x_m\| < \epsilon$ .

For example,  $\mathbb{R}$  is separable since  $\mathbb{Q}$  is countable and dense in  $\mathbb{R}$ .

DEFINITION A.6. A basis of an infinite dimensional Banach space  $X$  is a sequence of elements  $(e_n)_{n=1}^{\infty}$  such that for each  $x \in X$ ,

$$x = \sum_{n=1}^{\infty} a_n e_n$$

for a unique sequence of real numbers  $(a_n)_{n=1}^{\infty}$  and where the partial sums

$$S_N = \sum_{n=1}^N a_n e_n$$

form a Cauchy sequence.

REMARK 4. Note that this definition presumes convergence of the sum under the norm topology on  $X$  and that the uniqueness of the sum provides linear independence of basis elements.

DEFINITION A.7. The linear span of a set of vectors  $(e_n)_{n=1}^{\infty}$  is the collection of finite linear combinations

$$\text{span}(x_n) = \left\{ \sum_{n=1}^N a_n e_n \mid (a_n)_{n=1}^N \subseteq \mathbb{R}, N \in \mathbb{N} \right\}$$

The closed linear span of a set of vectors is the linear span and its limits points, which are the infinite sums,

$$\overline{\text{span}}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n e_n \mid (a_n)_{n=1}^{\infty} \in \mathbb{R} \right\}$$

DEFINITION A.8. A basic sequence  $(e_n)_{n=1}^{\infty}$  of a Banach space  $X$  is a sequence that forms a basis for the closed linear span of  $(e_n)_{n=1}^{\infty}$ .

PROPOSITION A.9. [5, Proposition 1.1.9] A sequence  $(e_k)_{k=1}^{\infty}$  of nonzero elements of a Banach space  $X$  is basic if and only if there is a positive constant  $K$  such that

$$\left\| \sum_{k=1}^m a_k e_k \right\| \leq K \left\| \sum_{k=1}^n a_k e_k \right\|$$

for any sequence of scalars  $(a_k)$  and any  $m \leq n$ .

REMARK 5. The above proposition in particular gives that basis elements are “far” from each other. Suppose  $(e_k)$  is a basis for  $X$  such that  $\|e_n\| = 1$  for all  $n \in \mathbb{N}$ . Then there exists  $\varepsilon > 0$  for all  $n \neq m$ ,  $\|e_n - e_m\| > \varepsilon$ .

DEFINITION A.10. Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be normed linear spaces and let  $(f_n)_{n=1}^{\infty} \in X$  and  $(g_n)_{n=1}^{\infty} \in Y$  be basic sequences.  $(f_n)$  is equivalent  $(g_n)$  if there exists a constant  $C$  such that

$$\frac{1}{C} \left\| \sum_{n=1}^{\infty} a_n g_n \right\|_Y \leq \left\| \sum_{n=1}^{\infty} a_n f_n \right\|_X \leq C \left\| \sum_{n=1}^{\infty} a_n g_n \right\|_Y$$

for every sequence  $(a_n)_{n=1}^{\infty}$  of scalars.

DEFINITION A.11. A basis  $(e_n)_{n=1}^{\infty}$  of a Banach space  $X$  is called normalized if  $\|e_n\| = 1$  for all  $n \in \mathbb{N}$ .

REMARK 6. If  $(x_n)_{n=1}^{\infty}$  is a basic sequence in a Banach space  $X$ . Then  $\left( \frac{x_n}{\|x_n\|} \right)_{n=1}^{\infty}$  is also a basic sequence in  $X$ .

THEOREM A.12. [5, p. 19] *Every separable Banach space contains a basic sequence.*

We will use two spaces several times in examples throughout this work. There are defined here.

DEFINITION A.13. *Let  $c_0$  denote the vector space of real sequences converging to 0. Let  $x = (a_n)_{n=1}^{\infty}$ , then we define the norm on  $c_0$  by  $\|x\|_{\infty} = \sup_{n \in \mathbb{N}} |a_n|$ .*

DEFINITION A.14. *Let  $\ell_1$  denote the space of real sequences whose sums are convergent. Let  $x = (a_n)_{n=1}^{\infty}$  with  $a_n \in \mathbb{R}$  then we define the norm on  $\ell_1$  by  $\|x\|_1 = \sum_{n=1}^{\infty} |a_n|$ .*

DEFINITION A.15. *The unit vector basis of  $\ell_1$  is the basis formed by the sequence  $(e_n)_{n=1}^{\infty}$  where*

$$e_n = (\underbrace{0, \dots, 0}_{n-1 \text{ times}}, 1, 0, \dots)$$

*That is, it is an infinite sequence of elements consisting entirely of 0's except for a 1 in the  $n^{\text{th}}$  position.*

DEFINITION A.16. *Let*

$$\ell_2 = \left\{ (a_i)_{i=1}^{\infty} \mid \sum_{i=1}^{\infty} |a_i|^2 < \infty \right\}$$

*we define the norm on  $\ell_2$  by  $\|x\|_2 = (\sum_{n=1}^{\infty} |a_n|^2)^{1/2}$ .*

DEFINITION A.17. *Given two linear spaces  $X$  and  $Y$ , a linear operator is a function  $T : X \rightarrow Y$  such that  $T(\lambda x + y) = \lambda Tx + Ty$  for all  $\lambda \in \mathbb{R}$  and  $x, y \in X$ . The space of such continuous operators from  $X$  to  $Y$  is denoted  $\mathcal{L}(X, Y)$  and has the norm*

$$\|T\| = \sup\{\|T(x)\|_Y : \|x\|_X \leq 1\}$$

*If  $\|T\| < \infty$  we say that  $T$  is bounded.*

THEOREM A.18. [5, The Uniform Boundedness Principle p. 339] *Suppose  $(T_\lambda)_{\lambda \in \Lambda}$  is a family of bounded linear operators from a Banach space  $X$  to a Banach space  $Y$ . If*

$$\sup\{\|T_\lambda x\| : \lambda \in \Lambda\} < \infty$$

*for all  $x \in X$ , then*

$$\sup\{\|T_\lambda\| : \lambda \in \Lambda\} < \infty$$

DEFINITION A.19. *The dual space of a Banach space  $X$ ,  $X^* := \mathcal{L}(X, \mathbb{R})$  is the set of continuous linear functionals (operators)  $x^* : X \rightarrow \mathbb{R}$ . Under the norm*

$$\|x^*\| = \sup_{\|x\| \leq 1} |x^*(x)|$$

Note that  $c_0^* = \ell_1$  and  $\ell_1^* = \ell_\infty$  where “=” in both cases is understood to mean isometric.

In this work *operator* will be used to refer to linear mappings between Banach spaces and *functional* will be used to refer to linear mappings from a Banach space into  $\mathbb{R}$ .

We can consider the double-dual as the linear functionals on  $X^*$  ( $X^*$  is defined in Appendix B), we denote the double-dual by  $X^{**}$ .

DEFINITION A.20. *The natural embedding of a Banach space  $X$  is denoted  $J : X \rightarrow X^{**}$  defined by*

$$J(x)[\phi] = \phi(x) \quad \text{for all } x \in X, \phi \in X^*$$

DEFINITION A.21. *A Banach space  $X$  is reflexive if it is isometric to its double dual via the natural embedding, i.e.,*

$$J(X) = X^{**}$$

## APPENDIX B

### AN OVERVIEW OF THE WEAK TOPOLOGY

This appendix will give an overview of the weak topology as applied to Banach spaces.

DEFINITION B.1. *The weak topology is the topology induced by the pre-image of all open sets under all linear functionals contained in the dual space. A neighborhood in the weak topology is defined as follows. Let  $\epsilon > 0$  and  $x_1^*, x_2^*, \dots, x_n^* \in X^*$  then*

$$N_{\epsilon, x_1^*, \dots, x_k^*}(x) = \{y \in X : x_i^*(x) - x_i^*(y) < \epsilon, i = 1, 2, \dots, n\}$$

*A sequence  $(x_n)_{n=1}^\infty$  converges weakly to a point  $x \in X$  if for all  $\epsilon > 0$  and  $x_1^*, x_2^*, \dots, x_n^* \in X^*$ , there exists  $i \in \mathbb{N}$  such that  $x_i \in N_{\epsilon, x_1^*, \dots, x_k^*}(x)$  and we say that  $(x_n)$  is weakly convergent to  $x$ .*

DEFINITION B.2. *The weak\* topology is the topology induced on  $X^*$  by the functionals  $X \subseteq X^{**}$ .*

REMARK 7. Under the weak\* topology  $B_{X^*}$  is both separable and metrizable for any separable Banach space  $X$ .

DEFINITION B.3. *The biorthogonals of a normalized basis  $(e_n)_{n=1}^\infty$  of a Banach space  $X$  is a sequence  $(e_n^*)_{n=1}^\infty \subseteq X^*$  such that*

$$e_i^*(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Since each of these is a linear functional, we have that for all  $x$  in the closed linear span of  $(e_n)$ ,

$$x = \sum_{i=1}^{\infty} e_i^*(x)e_i.$$

Of particular importance will be the notion of convergence and the following theorem gives a useful characterization.

**THEOREM B.4.** *Let  $X$  be a Banach space. A series  $(x_n)_{n=1}^{\infty} \subseteq X$  converges weakly to a point  $x \in X$  if and only if  $|x^*(x_n) - x^*(x)| \rightarrow 0$  for all  $x^* \in X^*$ .*

**PROOF.** Let  $(x_n)_{n=1}^{\infty}$  converge weakly to  $x \in X$ . Let  $\epsilon > 0$  and  $x^* \in X^*$ . Consider the neighborhood  $N_{\epsilon, x^*}(x)$  then by definition, there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|x^*(x_n) - x^*(x)| < \epsilon$ .

Now, let  $(x_n)_{n=1}^{\infty}$  such that  $|x^*(x_n) - x^*(x)| \rightarrow 0$  for all  $x^* \in X^*$  and let  $N_{\epsilon, x_1^*, \dots, x_k^*}(x)$  be a weak neighborhood of  $x$ . Since  $|x^*(x_n) - x^*(x)| \rightarrow 0$  for all  $x^* \in X^*$ , there exist  $N_i \in \mathbb{N}$  such that for all  $n \geq N_i$ ,  $|x_i^*(x_n) - x_i^*(x)| < \epsilon$ . Let  $N = \max_{0 < i \leq k} N_i$ , then for all  $n \geq N$ ,  $|x_i^*(x_n) - x_i^*(x)| < \epsilon$  for all  $0 < i \leq k$  and  $x_n \in N_{\epsilon, x_1^*, \dots, x_k^*}(x)$  so  $(x_n)_{n=1}^{\infty}$  converges weakly to  $x$ .  $\square$

**COROLLARY B.5.** *Let  $X$  be a Banach space. If a sequence  $(x_n)_{n=1}^{\infty} \subseteq X$  converges in norm to a point  $x \in X$ , then the sequence also converges weakly to  $x$ .*

**PROOF.** The dual space is a space of continuous linear maps  $x^* : X \rightarrow \mathbb{R}$  so by definition if  $x_n \xrightarrow{\|\cdot\|} x$  then  $x^*(x_n) \rightarrow x^*(x)$  for all  $x^* \in X^*$ . Therefore by the definition of weak convergence, norm convergence implies weak convergence.  $\square$

**DEFINITION B.6.** *A subset  $A$  of a topological space  $X$  is compact if every open cover  $\cup_{\alpha \in I} A_{\alpha} \supseteq A$ ,  $A_{\alpha}$  open for all  $\alpha \in I$ , there is a finite subset  $\alpha_1, \alpha_2, \dots, \alpha_n$  such that  $A \subseteq \cup_{i=1}^n A_{\alpha_i}$ .*

This definition can be applied to both the norm and weak topologies on a Banach space, but can also be cumbersome in practice. Therefore, it is of great value to



be able to characterize compactness with properties that more straightforward to prove or disprove in particular cases. The Eberlein-Šmulian Theorem gives us such characterizations in the weak topology. The proof is omitted by may be found in [5, Chapter 1.6].

**THEOREM B.7** (Eberlein-Šmulian Theorem). *Let  $X$  be a Banach space and  $A \subseteq X$ . Then  $A$  is weakly compact if and only if every sequence in  $A$  has a weakly convergent subsequence.*

**PROPOSITION B.8.** *Let  $X$  be a reflexive Banach space, then  $B_X$  is weakly compact.*

A proof of Proposotion B.8 which is due to Kakutani and may be found in [9, p. 301].

**REMARK 8.** This is contrast to the fact that the closed unit ball of a Banach space  $X$  is norm compact if and only if  $X$  is finite dimensional. This can be easily seen by considering the unit vector basis  $(e_n) \subseteq B_X$ . Each  $e_n$  can be isolated in a unique, non-overlapping open set. Having an infinite number of them therefore precludes a finite subcover.

**DEFINITION B.9.** *The adjoint of a functional  $T : X \rightarrow Y$  is a map  $T^* : Y^* \rightarrow X^*$  defined by*

$$T^*y^*(x) = y^*(Tx)$$

$T^* \in \mathcal{L}(Y^*, X^*)$  and  $\|T\| = \|T^*\|$ .

**PROPOSITION B.10.** *Let  $X, Y$  be Banach spaces. A linear operator  $T : X \rightarrow Y$  is norm-to-norm continuous if and only if it is weak-to-weak continuous.*

**PROOF.** For the forward direction let  $T$  be norm-to-norm continuous and let  $(x_\alpha)_\alpha$  be a weakly convergent net in  $X$  such that  $x_\alpha \xrightarrow{w} x$  in  $X$ . Then consider the net  $(Tx_\alpha)_\alpha$

in  $Y$  and let  $y^* \in Y^*$ .  $T$  has an adjoint,  $T^* : Y^* \rightarrow X^*$  such that  $(T^*y^*)(x) = y^*(Tx)$  and  $(T^*y^*) \in X^*$ . By definition since  $(x_\alpha)_\alpha$  weakly converges to  $x$  in  $X$  we have

$$((T^*y^*)(x_\alpha))_\alpha \rightarrow (T^*y^*)(x)$$

which by the definition of adjoint means exactly

$$y^*(Tx_\alpha) \rightarrow y^*(Tx)$$

for all  $y^* \in Y^*$  therefore  $T$  is weak-to-weak continuous.  $\square$

**PROPOSITION B.11.** *Let  $X$  be a Banach space and  $(x_n)$  be a weakly convergent sequence in  $X$ . Then there is an  $M > 0$  such that  $\|x_n\| \leq M$  for all  $n \in \mathbb{N}$ .*

**PROOF.** Let  $(x_n)_{n=1}^\infty$  be a weakly convergent sequence with  $x_n \xrightarrow{w} x$ . By Theorem A.18 it suffices to show that for all  $x^* \in X^*$ ,  $\sup_{n \in \mathbb{N}} |x^*(x_n)| < \infty$ . Since  $(x_n)$  is weakly convergent, there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|x^*(x_n) - x^*(x)| < 2$ . Then let  $M = \max\{|x^*(x_1)|, |x^*(x_2)|, \dots, |x^*(x_N)|, |x^*(x)|\}$  and  $\sup_{n \in \mathbb{N}} |x^*(x_n)| \leq M + 2 < \infty$ .  $\square$

**DEFINITION B.12.** *A sequence is weakly Cauchy if for each  $x^* \in X^*$   $\{x^*(x_n)\}_{n=1}^\infty$ , a sequence of real numbers, is convergent.*

This does not imply that there is an  $x \in X$  such that  $x^*(x_n) \rightarrow x^*(x)$  for every  $x^* \in X^*$ , that is, weakly Cauchy does not imply weak convergence, but a weakly convergent sequence will naturally also be weakly Cauchy.

**PROPOSITION B.13.** *The unit vector basis of  $\ell_1$  is not weakly Cauchy.*

**PROOF.** The dual of  $\ell_1$  is isometric to  $\ell_\infty$ , and so elements  $f \in \ell_1^*$  can be represented as  $(a_n) \in \ell_\infty$ . Consider one such element defined coordinatewise by

$$f = (a_n)_{n=1}^\infty$$

where

$$a_i = \begin{cases} 1 & i = 2k \\ 0 & \text{otherwise} \end{cases}$$

then  $f(e_i) = a_i$ . So the sequence  $(f(e_n))_{n=1}^{\infty} = (a_n)_{n=1}^{\infty}$ , which alternates between 0 and 1, is not weakly Cauchy.  $\square$

**PROPOSITION B.14.** *Let  $X$  be a Banach space and let  $A$  be a weakly open subset of  $X$  such that  $0 \in A$ . Then there exists  $\varepsilon > 0$  such that  $\varepsilon B_X \subseteq A$ .*

**PROOF.** Since  $A$  is weakly open there exists  $\delta > 0$  and  $x_1^*, \dots, x_n^*$  such that

$$N_{\delta, x_1^*, \dots, x_n^*}(0) = \{y : |x_i^*(y)| < \delta \text{ for all } i \in \{1, \dots, n\}\} \subseteq A$$

Let  $\varepsilon < \frac{\delta}{\max_{1 \leq i \leq n} \|x_i^*\|}$ . Then since for all  $x \in \varepsilon B_X$  we have  $\|x\| < \frac{\delta}{\max_{1 \leq i \leq n} \|x_i^*\|}$  we can let  $i \in \{1, \dots, n\}$  and have

$$\frac{|x_i^*(x)|}{\|x_i^*\|} < \frac{\delta}{\max_{1 \leq i \leq n} \|x_i^*\|}$$

so  $|x_i^*(x)| \frac{\delta \|x_i^*\|}{\max_{1 \leq i \leq n} \|x_i^*\|} < \delta$  and

$$\varepsilon B_X \subseteq N_{\delta, x_1^*, \dots, x_n^*}(0) \subseteq A$$

$\square$

**THEOREM B.15.** *If  $x_n \xrightarrow{w} 0$  then for all  $\epsilon > 0$  there exists  $M$  and a sequence of scalars  $(\lambda_i)_{i=1}^{\infty}$  with  $\lambda_i \geq 0$  such that  $\sum_{i=1}^M \lambda_i = 1$  and*

$$\left\| \sum_{i=1}^M \lambda_i x_i \right\| < \epsilon$$

**THEOREM B.16 (Hahn-Banach Corollary).** *Let  $X$  be a Banach space and  $Y$  be a closed subspace of  $X$ . Suppose  $f \in Y^*$ . Then there exists  $\tilde{f} \in X^*$  such that*

$$\|f\|_{Y^*} = \|\tilde{f}\|_{X^*}$$

and  $\tilde{f}|_Y = f$ .

COROLLARY B.17. *Let  $X$  be a Banach space then*

$$\|x\|_X = \sup_{f \in B_{X^*}} |f(x)|.$$

PROOF. Let  $x \in X$ . Define  $f \in \{\lambda x : \lambda \in \mathbb{R}\}^*$  by  $f(x) = \|x\|$ .  $f$  is continuous and linear on the 1-dimensional subspace  $\{\lambda x : \lambda \in \mathbb{R}\}$ . By the Hahn-Banach Corollary B.16 find  $\tilde{f} \in X^*$  such that  $\tilde{f}(x) = \|x\|$ . Since  $\|f\| = 1$

$$\|f\| = \sup_{\|\lambda x\| \leq 1} f(\lambda x) = \sup \|\lambda x\| = 1.$$

Thus we have  $\|\tilde{f}\| = 1$  which implies

$$\|x\| = f(x) = \tilde{f}(x) \leq \sup_{g \in B_{X^*}} |g(x)| \leq \|x\|.$$

This completes the proof. □

## BIBLIOGRAPHY

- [1] S. Banach, *Theorie des opérations linéaires*. Chelsea Publishing Co. (1955).
- [2] J. Dieudonné and L. Schwartz *La dualité dans les espaces  $\mathcal{F}$  et  $(\mathcal{LF})$* . Ann. Inst. Fourier Grenoble, v. 1, (1950), 61-101.
- [3] P. N. Dowling, D. Freeman, C. J. Lennard, E. Odell, B. Randrianantoanina, and B. Turett *A weak Grothendieck compactness principle*. Journal of Functional Analysis, v. 263, i. 5, (2012), 1378 - 1381.
- [4] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires.*, Mem. Amer. Math. Soc., no. 16 (1955).
- [5] N. J. Kalton and F. Albiac *Topics in Banach Space Theory* Springer (2000).
- [6] J. Lindenstrauss *On nonlinear projections in Banach spaces*. Michigan Math. J., v. 11 (1964), 263-287.
- [7] J. Lindenstrauss and L. Tzafriri *Classical Banach Spaces I and II* Springer (1996).
- [8] H. P. Rosenthal *A characterization of Banach spaces containing  $\ell_1$* . Proc. Nat. Acad. Sci. U.S.A., v. 71 (1974), 2411-2413.
- [9] H. L. Royden and P.M. Fitzpatrick *Real Analysis 4th Ed* Prentice Hall (2010).
- [10] J. Schur *Über lineare Transformationen in der Theorie der unendlichen Reihen*, Journal für die reine und angewandte Mathematik, v. 151 (1921), 79-111.