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ON THE EMBEDDINGS OF THE SEMI STRONG PRODUCT GRAPH

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

Eric Bradley Brooks
Master of Science

Director: Ghidewon Abay Asmerom, Associate Professor
Department of Mathematics and Applied Mathematics



Virginia Commonwealth University
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Abstract

Over the years, a lot has been written about the three more common graph products (Cartesian product, Direct product and the Strong product), as all three of these are commutative products. This thesis investigates a non-commutative product graph, $\mathbf{H} \overline{\times} \mathbf{G}$, we call the Semi-Strong graph product, also referred in the literature as the Augmented Tensor and/or the Strong Tensor. We will start by discussing its basic properties and then focus on embeddings where the second factor, \mathbf{G} , is a regular graph. We will use permutation voltage graphs and their graph coverings to compute the minimum genus for several families of graphs. The results follow work started first by A T White [12], extended by Ghidewon Abay Asmerom [1],[2], and follows the lead of Pisanski [9]. The strategy we use starts with an embedding of a graph \mathbf{H} and then modifying \mathbf{H} creating a pseudograph \mathbf{H}^* . \mathbf{H}^* is a voltage graph whose covering is $\mathbf{H} \overline{\times} \mathbf{G}$. Given the graph product $\mathbf{H} \overline{\times} \mathbf{G}$, where \mathbf{G} is a regular graph and \mathbf{H} meets certain conditions, we will use the embedding of \mathbf{H}^* to study topological properties, particularly the surface on which $\mathbf{H} \overline{\times} \mathbf{G}$ is minimally embedded.

Chapter 1

Basic Graph Theory Definitions

In this chapter, we present some introductory graph theory concepts and definitions that will be used in later chapters. Many results are presented without reference or proof. The reader can find them in the bibliography, especially [4], [8], [5] and [6] .

Definition 1 *A Graph G is a finite nonempty set V of vertices with a possibly empty set E of distinct 2-element subsets of V called edges. The vertex set is denoted $V(G)$ and the edge set by $E(G)$. An edge joining u and v is typically denoted uv or vu . If uv is an edge in G then vertices u and v are said to be **adjacent** vertices.*

Figure 1.1 below shows some very simple graphs.

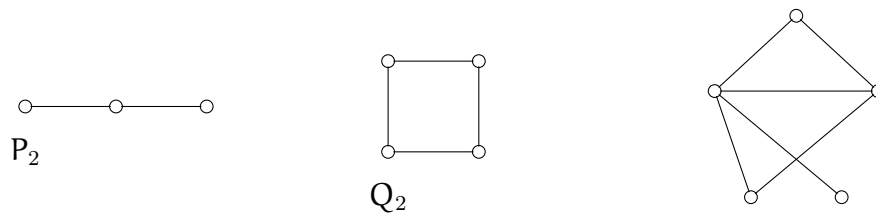


Figure 1.1: Examples of small graphs

Definition 2 The **degree** (valence) of a vertex v in a graph G is the number of vertices in G that are adjacent to v . Equivalently, the degree of a vertex v is the number of edges incident with v . A graph with all of its vertices having degree k is said to be a k -regular graph.

Definition 3 In a simple graph G , every two distinct vertices, u and v , are joined by either one edge or no edge of G . A **multigraph** is a non-empty set of vertices in which every pair are joined by a finite number of edges. Two or more edges that connect the same pair of vertices are called **parallel edges**. An edge joining a vertex to itself is called a **loop**. Any graph which allows loops and parallel edges is called a **pseudograph**.

The figure below shows a pseudograph with four loops called a Bouquet (B_4).

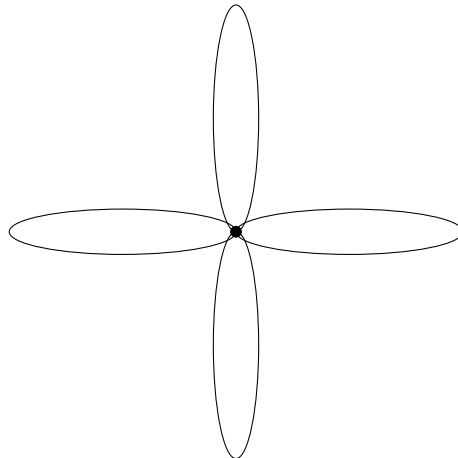


Figure 1.2: Bouquet graph B_4 !

Definition 4 A **Walk** W in a graph G is a sequence of vertices in G , beginning at vertex u and ending at vertex v , such that consecutive vertices in W are adjacent in G . A "closed" walk is a walk that begins and ends at the same vertex. A **Path** P is a walk in G where no vertex is repeated. The path on n vertices is denoted by P_n . A **cycle** is a closed walk in G where no edges or vertex other than the first is repeated. A cycle on n vertices is denoted by C_n . Figure 1.3 below depicts a walk, a path and a cycle.

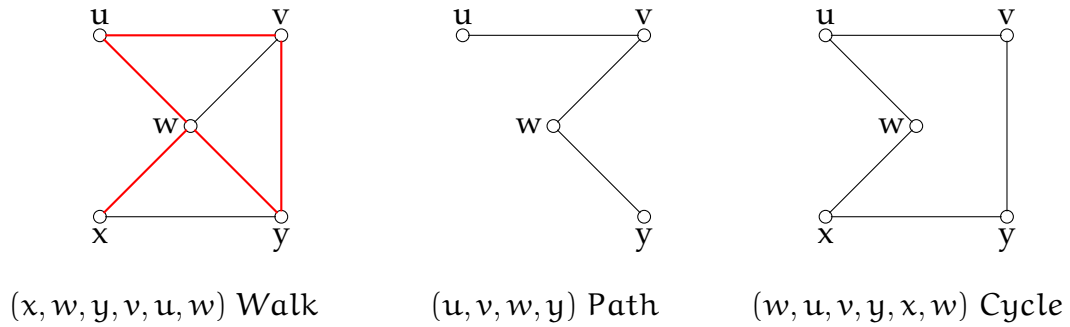


Figure 1.3: A walk, path and cycle

We will appreciate these terms as they become important later on in Chapters 4 and 5, when we discuss voltage graphs and the results.

Definition 5 Two vertices u and v are **connected** in a graph G if G contains a $u - v$ path. The graph G is **connected** if G contains a $u - v$ path for every pair of vertices u and v in $V(G)$. A graph that is not connected is **disconnected**.

See figure 1.4.

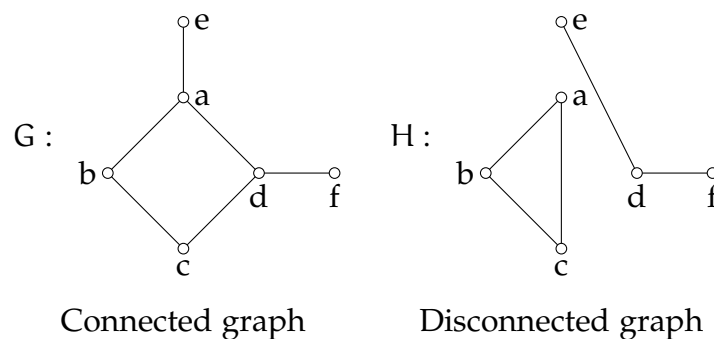


Figure 1.4: G is connected, and H is disconnected.

Definition 6 A **complete** graph on n vertices, denoted K_n , is a graph where every two of its vertices are adjacent. It has size $\binom{n}{2} = n(n-1)/2$. The figure below shows the first five complete graphs.

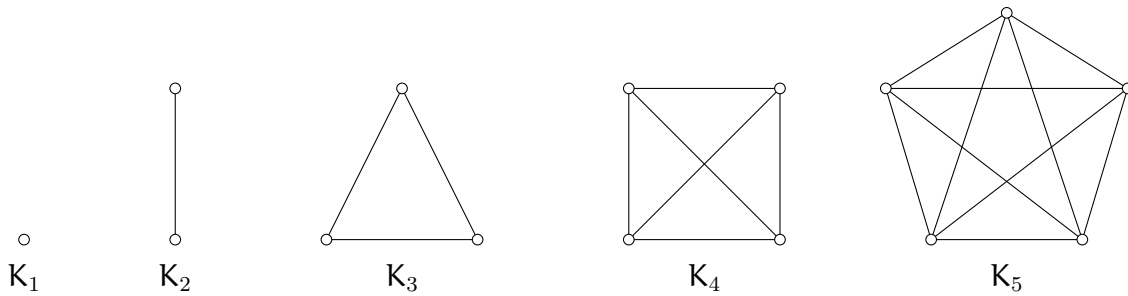


Figure 1.5: Complete graphs

Definition 7 A graph G is said to be **bipartite**, if the vertex set $V(G)$ can be partitioned into two sets, U and W (called partite sets), so that every edge of G connects a vertex of U with a vertex of W . A complete bipartite graph, $V(G)$ is a bipartite graph where every vertex of U is adjacent to all vertices of W . A complete bipartite graph where the cardinality of vertex set $|U| = n$ and vertex set $|W| = m$ is denoted $K_{m,n}$.

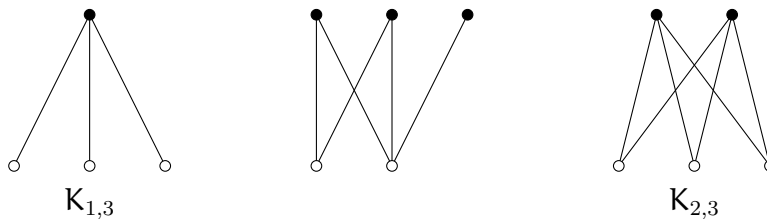


Figure 1.6: Examples of bipartite graphs

The **Cartesian Product**, denoted $G \square H$, has a vertex set $V(G) \times V(H)$, the cartesian product of the vertex sets. Two vertices (g, h) and (g', h') are adjacent if $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$. The individual graphs G and H are called factors of the product $G \square H$. From the definition of the product every pair of edges, one in G and another in H , gives rise to four edges in the product. Two edges are vertical and two edges are horizontal creating a \square and hence the use of the symbol \square in $G \square H$. In summary:

$$V(G \square H) = \{(g, h) \mid g \in V(G) \text{ and } h \in V(H)\}$$

$$E(G \square H) = \{(g, h)(g', h') \mid g = g', hh' \in E(H) \text{ or } h = h', gg' \in E(G)\}.$$

The **Direct graph** product of G and H is denoted $G \times H$ and has a vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent when $gg' \in E(G)$ and $hh' \in E(H)$. This definition clearly describes the X pattern visible in the product and why $G \times H$ uses the symbol X to identify the product. Unfortunately we are using "X" in two senses here. The first is for the the cartesian product of the vertex sets and the second is the direct product of graphs. The direct product is also referred to as the *Tensor* or *Kronecker* product. The product is commutative.

$$V(G \times H) = \{(g, h) \mid g \in V(G) \text{ and } h \in V(H)\}$$

$$E(G \times H) = \{(g, h)(g', h') \mid gg' \in E(G) \text{ and } hh' \in E(H)\}.$$

The **Strong Product** of G and H is denoted $G \boxtimes H$ and has the following vertex set $\{(g, h) \mid g \in V(G) \text{ and } h \in V(H)\}$. The edge set of the Strong Direct product is the union of the edge sets of the Cartesian product and the Direct product graphs. The edge set for the Strong product is simply a union of the direct and cartesian product edge sets and why a combination of their symbols is used for $G \boxtimes H$.

$$V(G \boxtimes H) = \{(g, h) \mid g \in V(G) \text{ and } h \in V(H)\}$$

$$E(G \boxtimes H) = \{E(G \square H) \cup E(G \times H)\}.$$

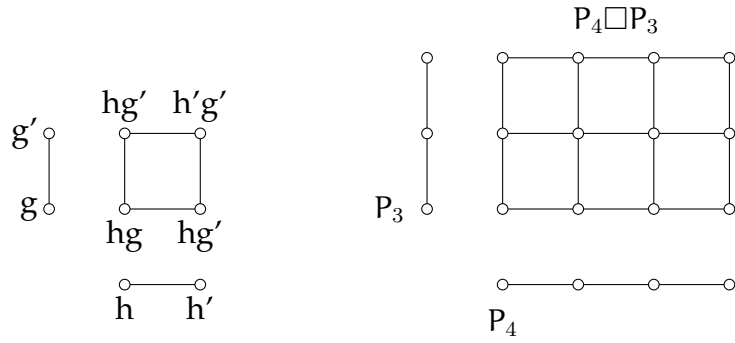


Figure 1.7: Cartesian product of $P_4 \square P_3$

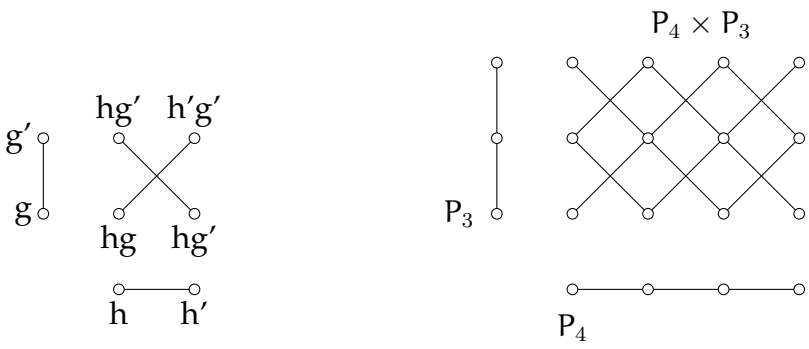


Figure 1.8: Example of Direct product $P_4 \times P_3$

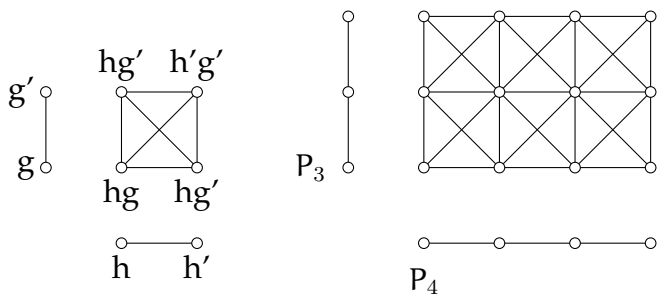


Figure 1.9: Example of Strong Direct product

Chapter 2

The Semi-Strong Product and its Properties

The three common product graphs, the Cartesian, the Direct and the Strong products have been investigated extensively [8].

All of these products are both commutative and associative, i.e. $H * G = G * H$. As we shall see, the Semi-Strong product graph, however, does not exhibit these properties [12].

Definition 8 *The Semi Strong Product of H and G is denoted $H \overline{\times} G$ and has the vertex set $V(H) \times V(G)$. Two edges (h_1, g_1) and (h_2, g_2) are adjacent if $(h_1 h_2) \in E(H)$ and $(g_1 g_2) \in E(G)$ or $g_1 = g_2$ and $(h_1 h_2) \in E(H)$. That is:*

$$V(H \overline{\times} G) = \{(h, g) \mid h \in V(H) \text{ and } g \in V(G)\}$$

$$E(H \overline{\times} G) = \{(h_1, g_1)(h_2, g_2) \mid g_1 g_2 \in E(G) \text{ and } h_1 h_2 \in E(H) \text{ or } g_1 = g_2 \text{ and } h_1 h_2 \in E(H)\}.$$

From the definition of the edge set we can observe that our product is not commutative. Consider the two products $P_4 \overline{\times} P_3$ and $P_3 \overline{\times} P_4$, the first product has 21 edges and the second has 20. This makes it necessary that we develop and introduce a "convention" for the order of the factors in the Semi Strong product. In the Semi-Strong product,

$\mathbf{H} \overline{\times} \mathbf{G}$, the \mathbf{H} factor will always run along the x – axis, while the \mathbf{G} factor runs along the y – axis. This will provide a consistent orientation of the factors and help in the discussions of later chapters. Furthermore, the \mathbf{G} factor will always be a regular graph which will allow us to see it as a covering graph of a permutation voltage graph. We will more formally discuss the definitions of coverings and voltage graphs in Chapter 4.

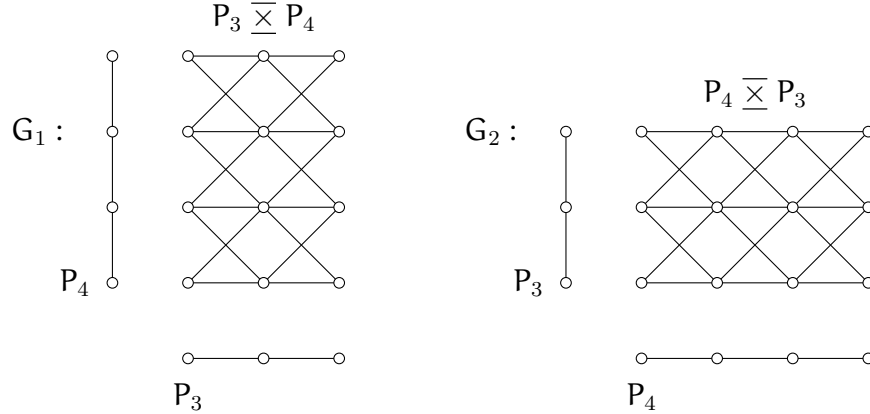


Figure 2.1: $G_1 \neq G_2$

Properties of Semi-Strong product graphs

Proposition 1 *A Semi-Strong product graph, $\mathbf{H} \overline{\times} \mathbf{G}$ is connected if and only if \mathbf{H} and \mathbf{G} are connected, and $\mathbf{H} \neq K_1$.*

Proof: Assume $\mathbf{H} \overline{\times} \mathbf{G}$ is a connected graph, then there is a path $(h_i, g_j) (h_r, g_t)$ for any pair of vertices (h_i, g_j) and (h_r, g_t) in the product. Let $(h_i, g_j) = (u_0, v_0) - (u_1, v_1) - \dots, (u_k, v_l) = (h_r, g_t)$ be a path in $\mathbf{H} \overline{\times} \mathbf{G}$, then $h_i = u_0, u_1, \dots, u_k = h_r$ is a walk in \mathbf{H} and $g_j = v_0, v_1, \dots, v_l = g_t$ is a walk in \mathbf{G} . Hence, $h_i - h_r$ and $g_j - g_t$ are paths in \mathbf{H} and \mathbf{G} in that order, and thus \mathbf{G} and \mathbf{H} are connected.

Conversely, let the factors \mathbf{G} and \mathbf{H} be connected graphs. Let (h_i, g_j) and (h_r, g_t) be two arbitrary vertices of the product. Then there is a $g_i g_t$ path in \mathbf{G} and $h_i h_r$ path in \mathbf{H} . By the definition of the Semi-Strong product and the connectedness of factors \mathbf{H} and \mathbf{G} , there is a $(h_i, g_j) (h_r, g_t)$ walk in the product of $\mathbf{H} \overline{\times} \mathbf{G}$ showing it is connected.

Proposition 2 A Semi-Strong product graph $H \overline{\times} G$ is bipartite if and only if the factor H is bipartite.

Proof: Let the graph $H \overline{\times} G$ be bipartite, then by definition the product graph contains a copy of H for every vertex in G . Since the resulting product graph contains no odd cycles, then H must be bipartite.

Conversely, let H be a bipartite with partition sets H_1 and H_2 . Then $H_1 \times V(G)$ and $H_2 \times V(G)$ form a partition of the vertex set of $H \overline{\times} G$. In figure 2.2 we see Semi-Strong product graphs where H is not bipartite and H is bipartite.

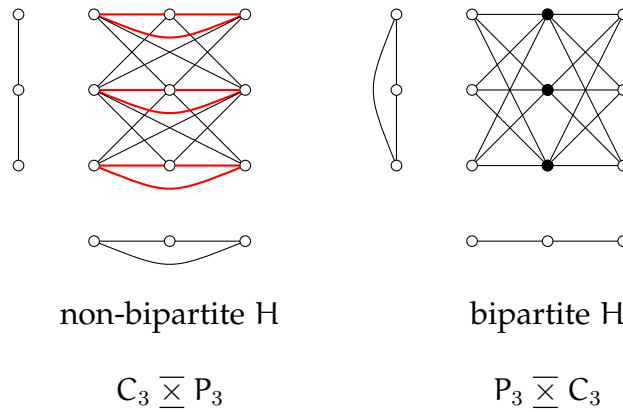


Figure 2.2:

We state the following proposition without proof. The reader can refer to [12].

Proposition 3 If G is bipartite then $K_2 \overline{\times} G = K_2 \square G$.

Corollary 1 Let $G_1 = K_2$ and for $n \geq 2$, $G_n = K_2 \overline{\times} G_{n-1}$, then $G_n = Q_n$.

The proof of this follows the same argument of a cartesian product of K_2 and K_2 and subsequent cartesian products with K_2 .

One can see this by the figure 2.3 and the proof would follow on induction.

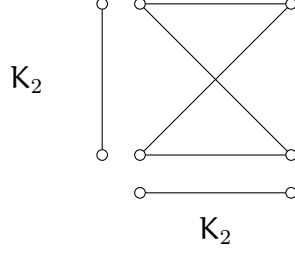


Figure 2.3:

Proposition 4 $K_n \overline{\times} K_2 = K_{2,2,2,\dots,2}$ (n times).

From the definition, we have two copies of K_n , one for each vertex of K_2 . If (h_i, g_1) is a vertex in the product it is adjacent to all (h_j, g_2) , where $j \neq 1$. Thus the graph is $K_{2,2,2,\dots,2}$.

Proposition 5 $K_2 \overline{\times} K_n = K_{n,n}$.

The proof by induction on k . Once again, by figure 2.3 we see how it is true for the base case K_2 . Now assume the result is true for the n th case, and we will show it is true for the $n+1$ case. For each vertex of the K_{n+1} we place a copy of the graph K_2 . By the definition of the Semi-Strong product, which does not allow an edge between vertices of H when $g = g$ in G , we end up with two partite sets containing $n+1$ vertices each.

Proposition 6 $K_{p_1,p_2,p_3,\dots,p_m} \overline{\times} K_n = K_{np_1,np_2,np_3,\dots,np_m}$

Proof. By the definition of the Semi-Strong product we have a copy of $K_{p_1,p_2,p_3,\dots,p_n}$ for every vertex of K_n . Each partite set will be connected to all $n-1$ copies of the partite sets with the exception of its own, because there is no h_1, h_2 edge when $g_1 = g_2$. This creates $K_{p_1n,p_2n,p_3n,\dots,p_nn}$ as desired.

Proposition 7 $H_1 = K_2$ and for $n \geq 2$ $H_n = H_{n-1} \overline{\times} K_2$. $H_n = K_{2(n-1),2(n-1)}$ First note that H_n is bipartite by (prop 2). It is also easy to observe that $H_2 = K_2 \overline{\times} K_2$ is $K_{2,2}$. Let us assume $H_n = K_{2(n-1),2(n-1)}$; we want to show $H_{n+1} = H_n \overline{\times} K_2 = K_{2(n-1),2(n-1)} \overline{\times} K_2 = K_{2n,2n}$. In this case we have two copies of H_n (one for each vertex of K_2). H_n is bipartite, and following the definition of Semi-Strong product, we connect the partite sets of n vertices, but we do not connect

the vertices within each set. This gives us two partite sets of double the number of vertices or $K_{2n,2n}$.

Chapter 3

Surfaces, Embeddings and Genus

In this chapter we define terms associated with surfaces, graph embeddings and theorems important for determining on which surface a particular graph may be embedded. We will begin with older results that will become important later on Chapter 5. There are many applications for embedding graphs on a sphere or plane. Consider placing the mapping of an electronic circuit on a circuit board. It is imperative to eliminate the crossing of edges to avoid an electric short. What is done if this becomes impossible on the plane or sphere? This is why graph embeddings are studied and explored. For more background on the definitions and theorems the reader can refer to [4], [6], [11], [13].

Definition 9 *A Surface is a closed, orientable 2-manifold, where a 2-manifold is a connected topological space where every point has a neighborhood homeomorphic to an unit disk.*

We will skip the details of each of the terms used in the above definition, in order to avoid straying too far into discussions of algebraic topology.

Definition 10 *An Embedding of a graph G on an orientable surface S is a representation of G on S where the vertices are points on the S and the edges are arcs such that the endpoints of an arc (edge e) are the vertices or (endpoints) of e . No arcs include points associated with other vertices and arcs only intersect at their points or vertices.(i.e., there is no crossing of edges).*

Definition 11 A graph G is said to be plane if G can be drawn on a plane without any two of its edges crossing. Embedding a graph on a plane is equivalent to embedding it on the sphere. One can perform a stereographic projection to see this.

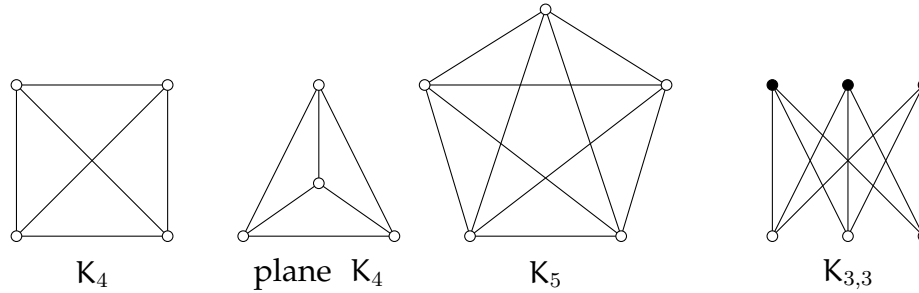


Figure 3.1: Planar and non-planar graphs.

The sphere, S_0 is the first topological surface and each subsequent orientable surface S_k is obtained by adding k handles to the sphere. Below are S_0 , S_1 and S_2 ; the sphere, the torus, and double torus in that order.

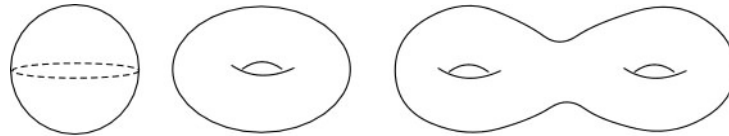


Figure 3.2: The sphere, torus and double torus

Definition 12 Suppose graph G is embedded on a surface S_k , $k \geq 0$. A region of this embedding is **2-cell** if every closed curve in that region can be continuously deformed in that region to a single point. From a topological standpoint, a region is 2-cell if it is homeomorphic to a disk.

Below is Euler's Polyhedra Formula, a true mathematical gem. It was listed as one of the 50 *Most Beautiful* mathematical formulas of all time [10].

Definition 13 A **Tree** is a connected acyclic graph. An edge $e = uv$ in a connected graph G whose removal results in a disconnected graph is a **Bridge**. Every edge in a tree is a bridge.

Theorem 1 (The Euler Polyhedra Formula) *For every connected plane graph of order n , size m and having r regions,*

$$n - m + r = 2.$$

Proof: We proceed by induction on the size m of a connected plane graph. There is only one connected graph of size 0, K_1 . Here, $n = 1$, $m = 0$ and $r = 1$. Since $n - m + r = 2$, $(1 - 0 + 1) = 2$, the base case of the induction holds. Assume for a positive integer m that if H is a connected plane graph of order n' and size m' , where $m' \leq m$ such that there are r' regions, then $n' - m' + r' = 2$. Let G be a connected plane graph of order n and size m with r regions. We consider two cases:

Case 1. G is a tree. In this case, $m = n - 1$ and $r = 1$. Thus $n - m + r = n - (n - 1) + 1 = 2$, the desired result.

Case 2. G is not a tree. Because G is connected and is not a tree, it follows that G contains an edge e that is not a bridge. In G , the edge is on the boundaries of two regions. So, in $G - e$ these two regions merge into a single region. Because $G - e$ has order n , size $m - 1$ and $r - 1$ regions and $m - 1 < m$, it follows that by the induction hypothesis that $n - (m - 1) + (r - 1) = 2$ and $n - m + r = 2$. \square

Theorem 2 *If G is a planar graph of order $n \geq 3$ and size m , then*

$$m \leq 3n - 6.$$

It is easy to see that the sum of the edges of the regions of a graph equals two times the total number of edges in G or $2q$. Because each region must be at least 3 edges, it follows that $2q \geq 3r$. Combining this fact with Euler's Polyhedra Formula and using some algebraic arguments, we are able to obtain $m \leq 3n - 6$. Figure 3.1 above displayed a drawing of K_5 with five edge crossings. Were these crossings avoidable? No, we show

that as a consequence of the result above. The graph K_5 has order $n = 5$ (vertices) and size $m = 10$ (edges). Using the formula we get the result: $10 \leq 9$. This clearly demonstrates that K_5 is non-planar (cannot be drawn in the plane without edge crossing). Below we can see where (on what surface) K_5 can be drawn without edge crossings, the torus.

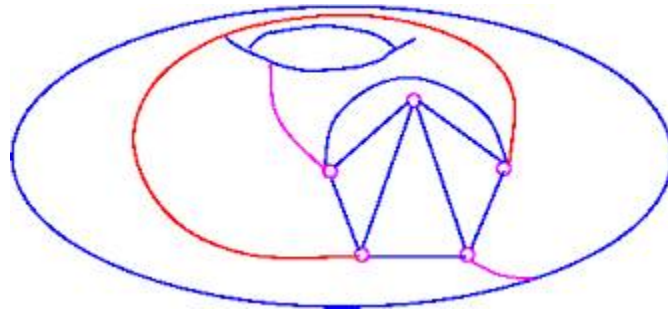


Figure 3.3: K_5 drawn on the torus.

Corollary 2 *If G is a bipartite graph of order n and size m , then*

$$m \leq 2n - 4.$$

This corollary uses the same logic as the previous theorem, with the exception that regions of a bipartite graph have no odd cycles and thus have at least 4 sides. Using $2q \geq 4$ in place of r yields the desired result.

Figure 3.1 above showed $K_{3,3}$ with several edge crossings. Using the previous theorem we see the graph $K_{3,3}$ has order $n = 6$ and size $m = 9$. Plugging these values into the formula ($m < 2n - 4$) produces a contradiction and $K_{3,3}$ is non-planar.

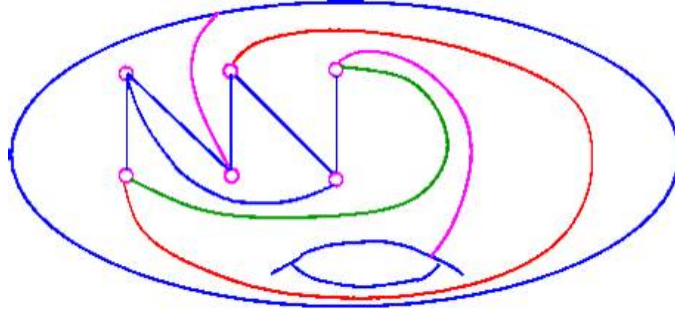


Figure 3.4: $K_{3,3}$ drawn on the torus.

Euler's Polyhedra Formula has a generalization to other orientable surfaces S_k . These are obtained from the sphere by attaching k handles and these surfaces are said to be of genus k .

Theorem 3 (The Generalized Euler Polyhedra Formula) *If G is a connected graph of order n and size m that is 2-cell embedded on a surface of genus $k \geq 0$, resulting in r regions, then*

$$n - m + r = 2 - 2k.$$

Proof is by induction on k . The case $k = 0$ is simply Euler's Polyhedra Formula. Assume the theorem is true for less than k handles, where $k \geq 1$, and let G be a connected pseudograph, with a 2-cell embedding in S_k . Without loss of generality, assume all vertices of G to be on the "sphere" portion of S_k , and, because the embedding is 2-cell, each handle has at least one edge of G running over it. Select a handle, and draw two disjoint simple closed curves C_1 and C_2 around this handle. Suppose edges x_1, x_2, \dots, x_n run over the handle where $n \geq 1$. Then C_i meets x_j in a point of S_k which we designate by u_{ij} , $i = 1, 2; j = 1, 2, \dots, n$. Consider the points u_{ij} to be vertices of a new pseudograph, with edges determined in the natural manner. Remove the portion of the handle between C_1 and C_2 and "fill in" the two resulting holes (bounded by C_1 and C_2 respectively) with two disks (this is called a capping operation). The result is a

2 – cell embedding of a connected pseudograph in S_{k-1} with parameters p' , q' and r' .

$$p' = p + 2n$$

$$q' = q + 3n$$

$$r' = r + n + 2$$

By the induction Hypothesis:

$$\begin{aligned} 2 - 2(k - 1) &= p' - q' + r' \\ &= (p + 2n) - (q + 3n) + r + n + 2 \\ &= p - q + r + 2 \\ 2 - 2k &= p - q + r \end{aligned}$$

□.

We can see that when we substitute 0 for k in the formula the famous Euler formula follows. This shows us that the sum $n - m + r$ is dependent on the surface and not the graph. For all graphs embedded on the sphere the number (on the right side of the equation) is 2, for the torus (S_1 , $k = 1$) this number is 0, the double torus (S_2 , $k = 2$) it is -2 . These numbers are referred to as the Euler characteristic of the surface.

Corollary 3 *A connected bipartite graph G of order $n \geq 3$ and size m , is minimally embedded, then*

$$\gamma(G) = \frac{m}{4} - \frac{n}{2} + 1$$

We noted previously for a bipartite graph G , $2q = 4r$ or $r = \frac{q}{2}$. Substituting this into Euler's Generalized Polyhedra Formula yields the desired result.

Chapter 4

Voltage graphs and their covers

In this chapter we introduce the concepts of ordinary and permutation voltage graphs and their covering graphs. We will present examples (of both ordinary and permutation voltage graphs) to help us become familiar with the concept of how a voltage graph "lifts" to its covering graph. [6]

Definition 14 A **regular voltage graph** is a pair $\langle G, \alpha : E_G \rightarrow \Gamma \rangle$ such that $G = (V, E)$ is a digraph, Γ is a group and α is a Γ -voltage assignment on G . The **voltage group** of a voltage graph $\langle G, \alpha : E_G \rightarrow \Gamma \rangle$ is the group Γ from where voltages are assigned. Let $G = (V, E)$ be a digraph and let Γ be a group. A **regular voltage assignment** on G in the group Γ is a function α that assigns to every arc (edge) $e \in E(G)$ an element $\alpha(e) \in \Gamma$. The element $\alpha(e) \in \Gamma$ is called the *voltage* on e .

If voltages are assigned in a group Γ to a base graph G , then for every vertex v of G , the set of vertices v_α in the derived graph is called the "fiber" over v . Also, for every edge e of G , the set of edges e_α in the derived graph is called the "fiber" over e .

Theorem 4 Let W be a walk in an ordinary voltage graph $\langle G, \alpha \rangle$ such that the initial vertex of W is u . Then for each vertex u_α in the fiber over u , there is a unique lift of W that starts at u_α .

Theorem 5 Let W be a walk from u to v in an ordinary voltage graph $\langle G, \alpha \rangle$, and let b be the net voltage on W . Then the lift W_a starting at u_a terminates at the vertex v_{ab} .

We show below how K_5 is the covering graph of a regular voltage graph. The voltage graph uses the abelian group Z_5 with the two generators $\gamma_1 = 1$ and $\gamma_2 = 2$. The order of both of these elements in Z_5 is five. That is $\text{ord}_{Z_5}(1) = 5 = \text{ord}_{Z_5}(2)$. Our covering graph has 5 vertices labeled v_0, v_1, v_2, v_3, v_4 and we will have the edges $v_i v_{i+\gamma_1}$ and $v_i v_{i+\gamma_2}$ for $0 \leq i \leq 4$. The single vertex of the base voltage graph will lift to the five vertices of the covering graph and are shared by both of the generators. Starting at vertex 0, and using generator γ_1 we can easily see how the five outside edges are created as the generator moves along its orbit, $(0, 1, 2, 3, 4, 0)$ producing the red pentagon. Generator γ_2 also begins at vertex 0, and has order 5, but the progression is slightly different producing the blue pentagram inside the pentagon. The orbit for γ_2 proceeds as follows; $(0, 2, 4, 1, 3, 0)$. This graph is clearly K_5 . That is, we have obtained K_5 from a humble one-vertex and two-loop voltage graph. The one vertex is covered by 5 vertices and the 2 edges by 10 edges. Figure 4.1 shows both the base voltage graph and its covering graph.

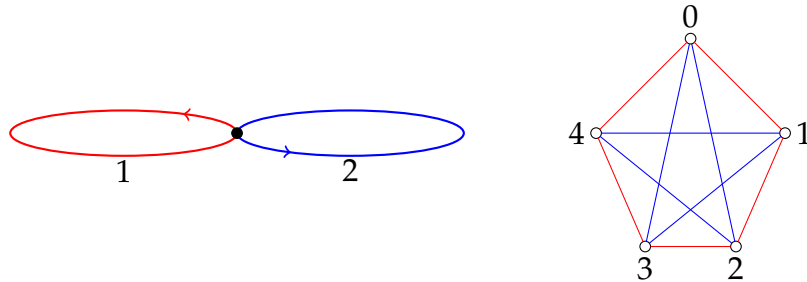


Figure 4.1:

Theorem 6 Let C be a k -cycle in the base graph of an ordinary voltage graph $\langle G, \alpha \rangle$ such that the net voltage on C has order m in the voltage group Γ . Then each component of the preimage $p^{-1}(C)$ is a km - cycle, and there are $\frac{|\Gamma|}{m}$ such components.

This theorem and its permutation voltage version (Theorem 9) are instrumental in our work in Chapter 5 and show the reader how the covering graph is lifted from the voltage graph.

In a second example, here is how a regular voltage graph would work and how its cover would lift from this voltage representation. In figure 4.2 the five letters (A, B, C, D, E) represent the five regions of the voltage graph. Each digon (two sided region) in the voltage graph will lift to regions in the cover in the following way. The group we have chosen for this voltage graph is $\mathbb{Z}_2 \times \mathbb{Z}_2$. The generators we have chosen from our group are (10) and (01). Consider the region labeled A. The sides of the digon are labeled with voltages 00 and 01. $01 + 00 = 01$ and the order of 01 in $\mathbb{Z}_2 \times \mathbb{Z}_2$ is 2. Now we have a $2 - \text{gon} \times 2$ (the order of 01 in $\mathbb{Z}_2 \times \mathbb{Z}_2$) which gives us a quadrilateral or a 4 - gon in the cover. The order of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is 4, and this is divided by the order of the net voltage of the digon, 01. The result is 2, and this tells us we have two copies of this quadrilateral from the region labeled A.

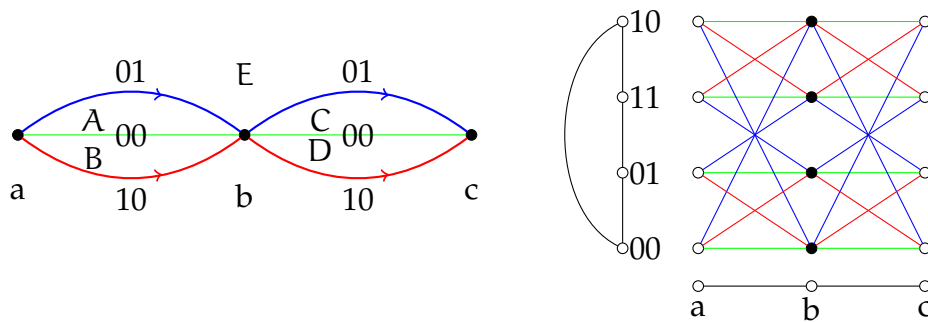


Figure 4.2:

Similarly, the region labeled D would also result in two copies of a 4 - gon. This leaves us with three more regions of our voltage graph to consider. The regions labeled C and B would also lift to identical regions in the cover as follows. $00 + 10 = 10$ and the order of 10 in $\mathbb{Z}_2 \times \mathbb{Z}_2$ is 2. So we have a $2 - \text{gon} \times 2$ (the order of 10 in $\mathbb{Z}_2 \times \mathbb{Z}_2$) which results in a 4 - gon. Since the order of $\mathbb{Z}_2 \times \mathbb{Z}_2$ divided by the order of 10 is 2

we have two copies of this quadrilateral. The region labeled B would follow similarly leaving us with 8 copies of a 4 – gon in the cover. Finally we need to consider the region labeled E. The voltage would net out as follows: $01 + 01 + 10 + 10 = 00$. The order of the identity element in $\mathbb{Z}_2 \times \mathbb{Z}_2$ is 1, so our four sided region would lift to a four sided region. Dividing the order of the entire group by the order of 00 equals 4, and that tells us that we have 4 copies of this quadrilateral in the cover.

Our cover graph has twelve vertices (the three from the voltage graph times the order of the group, which is 4), and the edges total 24. We had calculated 8 copies of a 4 – gon from the regions labeled A, B, C, D, and four more 4 – gons from the outside for a total of 12 quadrilaterals. In short, the covering graph of the voltage graph has $m = 12$, $n = 24$, and $r = 12$. Furthermore, the covering graph is bipartite because the product graph is also bipartite. Using corollary 3 in Chapter 3 we can determine the minimum embedding of this covering graph, by plugging the number of edges and vertices into the formula for bipartite graphs, and the result is 1.

$$\gamma(G) = \frac{m}{4} - \frac{n}{2} + 1$$

We could have used the Generalized Euler Formula for calculating the genus of the graph. Because the bipartite graph $P_3 \overline{\times} C_4$ is embedded with all regions as quadrilaterals, minimally, we could have used the Generalized Euler formula to calculate the genus as 1. If we substitute the number of edges and vertices in the covering graph into the Generalized Euler Formula, the result is 1.

$$n - m + r = 2 - 2k.$$

The following theorems and proofs can be found in [6] and [5] and are essential to understanding the permutation voltage graph H^* covering or lift to the product graph $H \overline{\times} G$.

Definition 15 A **permutation voltage graph** is the subscripted pair $\langle G, \alpha \rangle_n$ where G is a graph and $\alpha : G^* \rightarrow S_n$ where $G^* = (u, v) | uv \in E(G)$ and S_n is the symmetric group of degree n , is a function that satisfies $\alpha(e^{-1}) = (\alpha(e))^{-1}$.

Definition 16 The **permutation voltage derived (covering) graph** G^α for $\langle G, \alpha \rangle_n$ is the graph whose vertex set is the cartesian product $V(G) \times \{1, \dots, n\}$ and whose edge set is the cartesian product $E(G) \times \{1, \dots, n\}$. The incidence structure on G^α is described in the following way: if the directed edge e in $\langle G, \alpha \rangle_n$ runs from the vertex u to the vertex v and if it is assigned permutation voltage π then e determines n edges $(e, i), i = 1, \dots, n$ in derived graph G^α each running to (u, i) to $(v, \pi(i))$.

Definition 17 Let W be a walk in a permutation voltage graph $\langle G, \alpha \rangle_n$ and let π_1, \dots, π_r be the successive permutation voltages encountered on the walk W . The **net permutation voltage** on W is the permutation $\pi = \pi_r \pi_{r-1} \dots \pi_1$.

Theorem 7 Let $\langle G, \alpha \rangle_n$ be a connected permutation voltage graph and let u be a vertex of the base graph. The number of components of the derived graph equals the number of orbits induced by the local group at u on the set $\{1, \dots, n\}$.

Definition 18 A **Bouquet graph** is a pseudograph consisting of a single vertex with n self-graph loops. A order n bouquet graph denoted B_n .

The following two results are found in [6] and provide an explanation of a base graph and how it lifts to its cover. Permutation voltage graphs work differently from regular voltage graphs. The net permutation of a cycle determines the length and number, as the following theorems discuss.

Theorem 8 Let G be a connected regular graph of degree $2k$. Then G is the covering space of the bouquet B_k .

Theorem 9 *Let C be the boundary walk of a l – sided face in the embedding of a permutation voltage graph $\langle G, \alpha \rangle_n$ in an orientable surface S . If the net permutation voltage on C has the cycle structure (c_1, c_2, \dots, c_n) then there are $c_1 + c_2 + \dots + c_n$ faces of the derived embedding $G^\alpha \rightarrow S^\alpha$ corresponding to the face C including for $j = 1, \dots, n$ exactly c_j faces with lj – sides.*

In figure 4.3 we see a permutation graph and its cover and we will discuss how the base graph lifts to its cover. We choose S_4 , the symmetric group on four elements, as our group and a B_4 (Bouquet) as its voltage representative. To each loop of B_4 we assign the following voltages, the black loop = (1234), the blue loop = (1324), the green loop = (1243) and the red loop = (1342). The lift works in the following manner: the single vertex in the voltage graph lifts to four vertices in the derived graph labeled v_1, v_2, v_3 , and v_4 . We mentioned in the definition of the permutation voltage graph that a vertex labeled v_i is joined to $v_{\sigma(i)}$, so that $(v_i, v_{\sigma(i)}) \in E(CG)$. In other words the edge $(v_i, v_{\sigma(i)})$ is an element in the covering graph. In our example the black loop creates the following edges: vertex joins vertex v_1 to v_2 , v_2 to v_3 , v_3 to v_4 and v_4 to v_1 . This is the black square in the covering graph in the figure [4.3]. Similarly, the blue loop from the bouquet will join v_1 to v_3 , v_3 to v_2 , v_2 to v_4 and v_4 to v_1 giving the 4 – cycle shown in blue. It is a similar action for the green and red loops from the voltage graph to get the derived graph on the right of figure [4.4]. It has four vertices and each vertex is 8-regular. Note this 4 vertex, 16 edge covering graph is lifted from the 1 vertex, 4 loop voltage graph in a 4 : 1 manner.

This is a very simple example, but it demonstrates the power behind the method and how a simple voltage graph can represent a large derived graph.

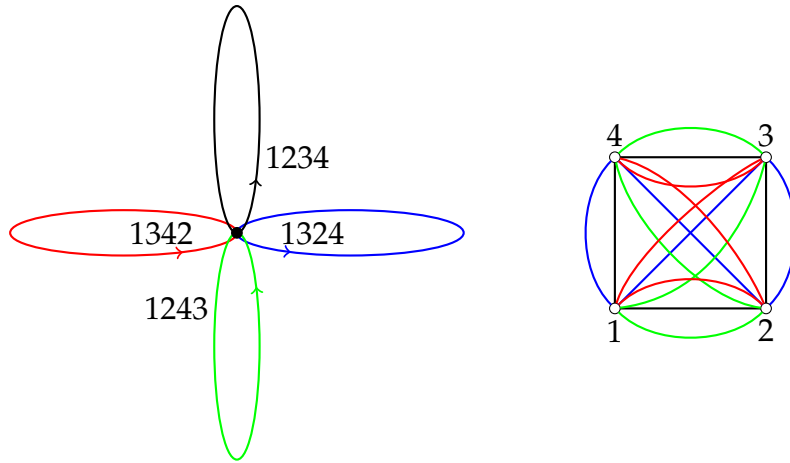


Figure 4.3:

Chapter 5

Results

These results follow the lead of Pisanski [9] and his work with the tensor product and regular graphs and Abay-Asmerom [1] on the direct tensor product. We will show the embeddings of the Semi-Strong product where the second factor is a regular graph.

This next Theorem 10 is an extension of the construction used by Arthur White [12] and by Abay-Asmerom [2] to depict the Semi-Strong product as a covering graph of a voltage graph. In their work G was a cayley graph and in our work G will be a regular graph and it is the key to all of the following results. It explains how our regular graph G has been generated by a voltage graph, some bouquet, and produces along with H our Semi-Strong product.

Theorem 10 *Let G be a $2k$ -regular graph with a vertex set $\{1, \dots, n\}$ and the set of generators $\{\pi_1, \dots, \pi_k\}$, where $\pi_i \in S_n$ (a permutation group). Then for any factor H that is embedded in an orientable surface, the Semi-Strong product graph $H \bar{\times} G$ is equal to the permutation derived graph for the permutation voltage graph H^* . H^* is obtained from H by first adding directions to the edges of H and then replacing each directed edge of H with $2k + 1$ equally directed edges and assigning the permutation voltages $e, \pi_1, \dots, \pi_k, \pi_1^{-1}, \dots, \pi_k^{-1}$ to these edges in any order.*

Proof.

The proof follows from the definition of the Semi-Strong product shown directly below and the rules governing the permutation voltages of H^* :

$$V(H \overline{\times} G) = \{(h, g) \mid h \in V(H) \text{ and } g \in V(G)\}$$

$$E(H \overline{\times} G) = \{(h_1, g_1) (h_2, g_2) \mid (g_1 g_2) \in E(G) \text{ and } (h_1 h_2) \in E(H)$$

or $g_1 = g_2$ and $(h_1 h_2) \in E(H)\}$.

For vertices (h, g) in the graph $H \overline{\times} G$, the permutation voltages π_n, π_n^{-1}, e generate the following moves within the graph:

If $h_1 h_2$ is an edge of $E(H)$, then the edge labeled e on H^* lifts to the edges: $(h_1, g_i)(h_2, g_{e(i)}) = (h_1, g_i)(h_2, g_i)$ this gives us the edges of $E(H \overline{\times} G)$ that come from : $(h_1, g_1)(h_2, g_2)$ if $g_1 = g_2$ and $h_1 h_2 \in E(H)$.

Similarly, the edge of H^* between h_1 and h_2 labeled by π_n lifts to the edges $(h_1, g_i)(h_2, g_{\pi_n(i)})$. This is equivalent to the edge in $H \overline{\times} G$ that comes from $h_1 h_2 \in E(H)$ and $g_i g_{\pi_n(i)} \in E(G)$. The latter is possible because G is the permutation derived graph of the original voltage graph.

For all of the following results we will assume that G is the covering graph generated by the permutation voltage graph B_k . The voltage assignments (generators) $\sigma_1, \sigma_2, \dots, \sigma_k$ of this permutation voltage graph arise from the symmetric group S_n . Based on the assumptions, G will be a $2k -$ regular graph of order n as described earlier in Chapter 4 by theorem 8.

Theorem 11 *Let H be a simple bipartite (p, q) graph with a quadrilateral embedding, and let G be as described above. Then find a 1 – 1 mapping π from the set $1, \dots, 2k$ to the set $\sigma_1, \sigma_2, \dots, \sigma_k, \sigma_1^{-1}, \dots, \sigma_k^{-1}$, where for each $i = 1, \dots, 2k$ we denote $\pi_i = \pi(i)$. Finally, with the following conditions:*

$$\begin{aligned}\pi_k &= \pi_k^{-1} \\ \pi_{k+1} &= \pi_{k+1}^{-1} \\ (\pi_i \pi_{i+1}^{-1})^2 &= 1, \text{ except for } i = k \\ (\pi_1 \pi_{2k}^{-1})^2 &= 1\end{aligned}$$

$H \underline{\times} G$ has a quadrilateral embedding and

$$\gamma(H \underline{\times} G) = \frac{mq(2k+1)}{4} - \frac{mp}{2} + 1.$$

Proof:

We replaced each edge in H with $2k+1$ edges as discussed previously in Theorem 10 and as demonstrated below in figure 5.1. The bipartite property of H easily accommodates this replacement for all edges of H . Now as a result, H^* become the permutation voltage graph obtained from H as described above, and H^* is exactly $H \underline{\times} G$. We placed conditions on the permutation voltages π_1, \dots, π_{2k} , so that the net voltage for each digon (two sided region) in H^* has the following cycle structure $(0, \frac{m}{2}, 0, 0, \dots, 0)$. Therefore, by Theorem 9 all digons in H^* lift to quadrilateral faces in $H \underline{\times} G$. The cycle structure created these quadrilaterals by having net permutation voltages of order two and each region in the graph H^* being two-sided. $H \underline{\times} G$, which has $(2k+1)mq$ edges and pm vertices. Since $H \underline{\times} G$ is bipartite with a quadrilateral embedding, the embedding is minimal and we have:

$$\gamma(H \underline{\times} G) = \frac{mq(2k+1)}{4} - \frac{mp}{2} + 1$$

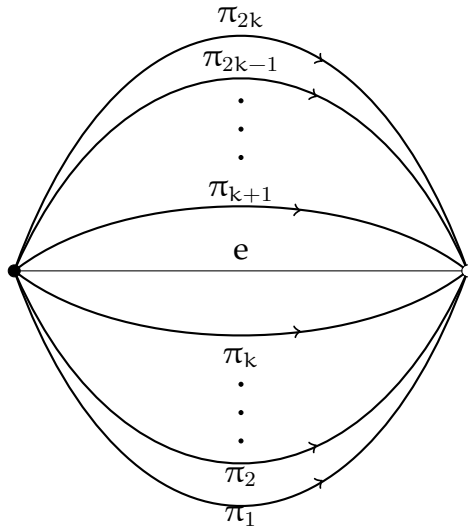


Figure 5.1:

This second result discusses a 1 – factor, so we will start with a definition and example before proceeding to the theorem.

Definition 19 A matching in a graph G is a subset m of $E(G)$ such that no two edges of m are adjacent. A **Perfect – Matching**, (1 – factor), is a matching in which every vertex in a graph G is incident to exactly one edge in the matching. Below are three different 1 – factors for the 3 – cube. The three sets of colored edges, blue, red and green, are all 1 – factors.

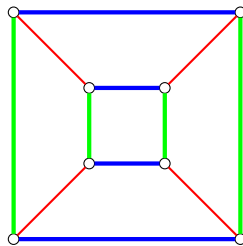


Figure 5.2: The one factors of Q_3

Theorem 12 *Let H be a simple bipartite graph with an orientable quadrilateral embedding having p vertices and q edges. Assume G as stated above with the exception that it is a $(2k + 1)$ -regular graph having a 1-factor F . Let σ_{k+1} be the generator for the graph which is created from F by adding a parallel edge to each edge of F , rendering G $2k + 2$ regular. Find a set of generators $\{\sigma_1, \dots, \sigma_k\}$ for $G - F$ such that there is an $1 - 1$ mapping π from the set $\{1, 2, \dots, 2k + 2\}$ to the set $\{\sigma_1, \dots, \sigma_{k+1}, \sigma_1^{-1}, \dots, \sigma_{k+1}^{-1}\}$, where for $i = 1, \dots, 2k + 2$ we denote $\pi_i = \pi(i)$. In this mapping if $\pi_i = \sigma_{k+1}$ then either π_{i-1} or $\pi_{i+1} = \sigma_{k+1}^{-1}$ and the following conditions must hold:*

$$\begin{aligned}
(\pi_1 \pi_2^{-1})^2 &= 1 \\
(\pi_2 \pi_3^{-1})^2 &= 1 \\
&\vdots \\
(\pi_{2k+1} \pi_{2k+2}^{-1})^2 &= 1 \\
(\pi_{2k+2} \pi_1^{-1})^2 &= 1 \\
\pi_{k+1} &= \pi_{k+1}^{-1} \\
\pi_{k+2} &= \pi_{k+2}^{-1}.
\end{aligned}$$

$H \overline{\times} G$ has a quadrilateral embedding and

$$\begin{aligned}
\gamma(H \overline{\times} G) &= \frac{qm(2k + 2)}{4} - \frac{mp}{2} + c \\
&= \frac{qm(k + 1)}{2} - \frac{mp}{2} + c
\end{aligned}$$

Proof:

The proof follows very similarly to the proof of Theorem 11. The graph G is a subgraph of the graph G' created by adding the parallel edge to the 1 - factor of G . We now evaluate the genus of $H \overline{\times} G'$ as we did the genus of $H \overline{\times} G$ in theorem 11. By Theorem 10, replacing each edge of H with the voltage directed edges of G creating H^* (see figure 5.3) is exactly $H \overline{\times} G$. The required condition that if $\pi_i = \sigma_{k+1}$ then π_{i-1} or $\pi_{i+1} = \sigma_{k+1}^{-1}$

was essential so that all digons of H^* have net voltages of order 2 resulting in a lift to quadrilaterals in the embedding of the product graph. The edge added to the $1 -$ factor can be easily eliminated and by propositions and theorem,

$$\begin{aligned} \gamma(H \overline{\times} G') &= \frac{qm(2k+3) - qm}{4} - \frac{mp}{2} + c. \\ &= \frac{qm(2k+2)}{4} - \frac{mp}{2} + c. \end{aligned}$$

If either G or H were disconnected, then c is the number of components of $H \overline{\times} G$.

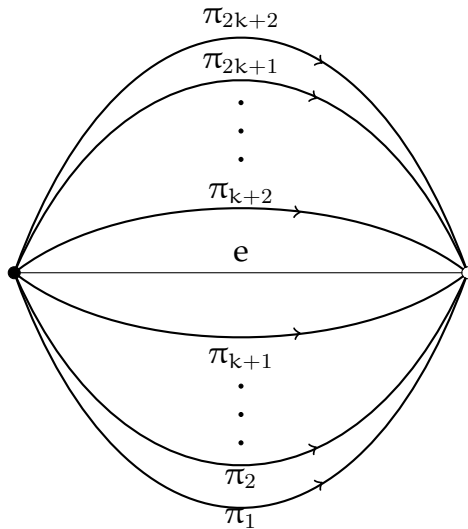


Figure 5.3: Theorem 12

Theorem 13 *Let H be a simple connected non-bipartite (p, q) graph with a quadrilateral embedding and a bipartite dual (the embedding is bichromatic or two-colorable) and let G be described as above and bipartite. Find a set of generators $\{\sigma_1, \dots, \sigma_k\}$ for G such that has a 1-1 mapping π from the set $\{1, \dots, 2k\}$ to the set $\{\sigma_1, \sigma_2, \dots, \sigma_k, \sigma_1^{-1}, \dots, \sigma_k^{-1}\}$ where for the $i = 1, \dots, 2k$ we denote $\pi_i = \pi(i)$. If the following conditions hold:*

$$\begin{aligned} \pi_1^4 &= 1 \\ \pi_{2k}^4 &= 1 \\ (\pi_1 \pi_2^{-1})^2 &= 1 \\ (\pi_2 \pi_3^{-1})^2 &= 1 \\ &\vdots \\ (\pi_{2k} \pi_1^{-1})^2 &= 1 \\ \pi_k &= \pi_k^{-1} \\ \pi_{k+1} &= \pi_{k+1}^{-1} \end{aligned}$$

Then $H \overline{\times} G$ has a quadrilateral embedding and

$$\gamma(H \overline{\times} G) = \frac{mq(2k+1)}{4} - \frac{mp}{2} + 1.$$

Proof

The conditions set for H , that it is connected with a bipartite dual embedding, provides us with a very simple two coloring of the regions of H . Each edge of H is then altered with $2k+1$ edges with half in one color and the other half in the second color as demonstrated in figure 5.3. This can be done easily and consistently due to the two different colors of the regions and the conditions set upon the orientation of the H embedding. Then H^* is the permutation voltage graph which Theorem 10 proves is precisely $H \overline{\times} G$.

For each $(\pi_i \pi_{i+1}) = 1$ for $i = 1, 2, \dots, 2k-1$, every 2-sided region of H^* lifts to a

quadrilateral in $H \overline{\times} G$.

The bichromatic nature of the embedding of H divides the quadrilaterals of H^* into two different classes. One class will have quadrilaterals with all four sides charged with a voltage label of π_1 and the other class with all four sides charged with a voltage label of π_{2k} . The conditions set in the theorem require these two charges to have order 4, $\pi_1^4 = 1$ $\pi_{2k}^4 = 1$, which now easily lift to quadrilaterals in the embedding of $H \overline{\times} G$.

$H \overline{\times} G$ is a quadrilateral embedding and since G is bipartite, $H \overline{\times} G$ is also bipartite. This product graph, $H \overline{\times} G$, has pm vertices and $2mkq$ edges and by corollary [2], from chapter 3.

$$\gamma(H \overline{\times} G) = \frac{mq(2k+1)}{4} - \frac{mp}{2} + 1.$$

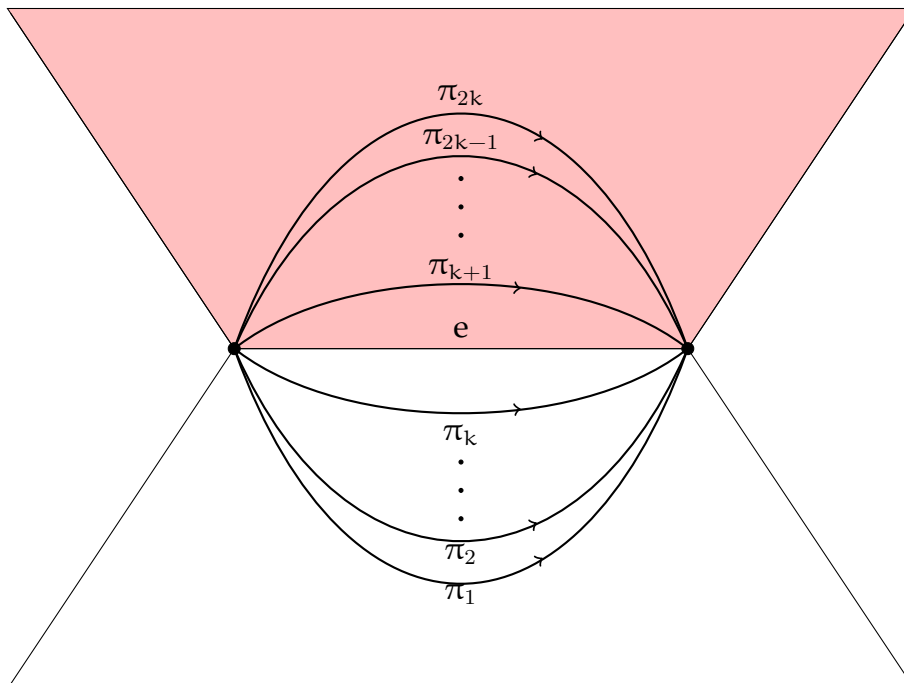


Figure 5.4: The bichromatic nature of H^*

The following is a similar theorem for a different class of graphs. Let H be a simple connected (p, q) graph with a quadrilateral embedding, and let there be a binary relation on the edges of the dual of H (that is 4-regular). This binary relation will operate in the following manner. Let e and f be edges in the quadrilateral embedding of H . Then $e \sim f$ if e and f are opposite sides of a quadrilateral in the embedding of H . This is an equivalence relation as it satisfies all of the properties (reflexivity, symmetry and transitivity). H is said to have what Pisanski calls a "Straight Ahead" embedding if all of the equivalence classes under this relation, \sim , are cycles in the dual H .

Theorem 14 *Let H be a simple bipartite (p, q) graph with a quadrilateral embedding, which also has this "Straight Ahead" embedding and let G again be as stated above. Find a 1-1 mapping π from the set $1, 2, \dots, 2k$ to the set $\sigma_1, \dots, \sigma_k, \sigma_1^{-1}, \dots, \sigma_k^{-1}$ and where for $i = 1 \dots 2k$ we denote $\pi_i = \pi(i)$. If the following system of equations hold:*

$$\begin{aligned} (\pi_1 \pi_2^{-1})^2 &= 1 \\ (\pi_2 \pi_3^{-1})^2 &= 1 \\ &\vdots \\ (\pi_{2k-1} \pi_{2k}^{-1})^2 &= 1 \\ (\pi_1^2 \pi_{2k}^{-2})^2 &= 1 \end{aligned}$$

then $H \overline{\times} G$ has a quadrilateral embedding and

$$\gamma(H \overline{\times} G) = \frac{mq(2k+1)}{4} - \frac{mp}{2} + 1.$$

Proof:

We take each equivalence class C and choose one of two possible orientations (which orientation is not important). Any edges of H that are intersected by C are then assigned an orientation according to the direction of C . If C goes left to right, then the edges of H intersected by C go left to right, and so on. See figure 5.5.

As we have in the previous theorems, we replace each edge of H with the $2k + 1$ directed edges assigned the voltages $e, \pi_1, \dots, \pi_{2k}$ precisely as we did before. We have now created H^* which by Theorem 10 is exactly $H \bar{\times} G$. The proof then follows similarly to Theorem [11] where all of the digons have net permutation of order two lifting to quadrilaterals, and the four-sided regions in the interior of the H^* will have one of the following four voltage permutations, $(\pi_1 \pi_1 \pi_{2k}^{-1} \pi_{2k}^{-1})$, $(\pi_1 \pi_1^{-1} \pi_{2k}^{-1} \pi_{2k})$, $(\pi_1 \pi_{2k} \pi_{2k}^{-1} \pi_1^{-1})$, $(\pi_1 \pi_{2k}^{-1} \pi_{2k}^{-1} \pi_1)$. In each case the net permutation voltage will be equal to the identity and the four-cycle with net permutation of order 1 lifts to a quadrilateral in the product embedding. See figure 5.6.

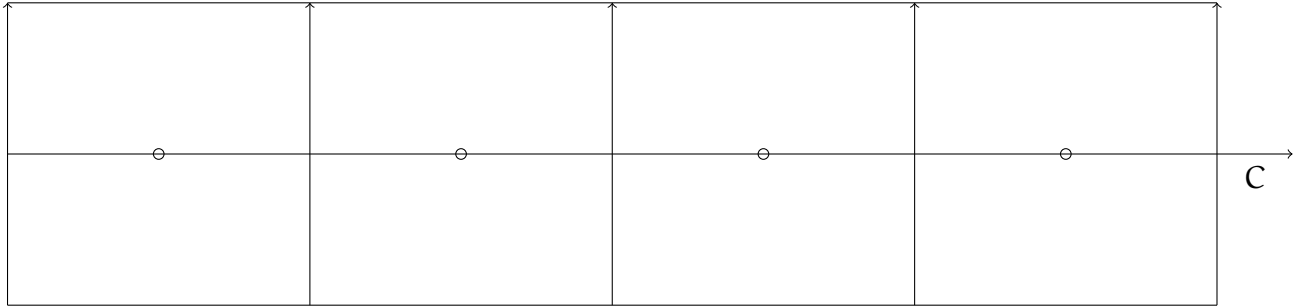


Figure 5.5: Equivalence class orientation.

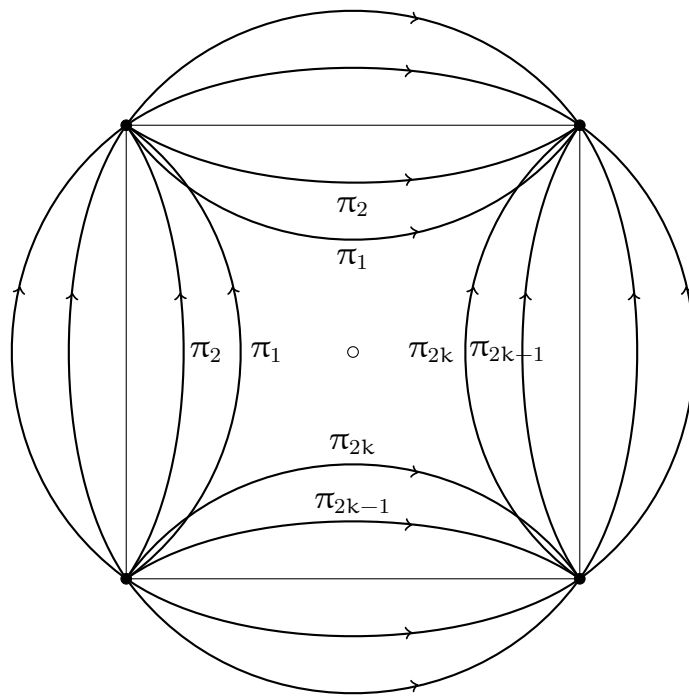


Figure 5.6: Cycle orientation in H^*

Chapter 6

Conclusion

In this thesis we have looked at the Semi-Strong product and its properties, noting that its main difference from the other three main products is non-commutativity. Using permutation voltage graphs and their covering graphs we were able to compute the genus of several products. The first factor, H , was a bipartite graph with a quadrilateral embedding, a non-bipartite graph with a quadrilateral embedding with a bipartite dual and finally a bipartite graph with quadrilateral embedding and a "straight ahead" embedding of the dual. Our second factor, G , was always a regular graph. Abay-Asmerom discussed this technique using the Tensor product, where the second factor was a Cayley graph, and Pisanski followed this also using the Tensor product with the second factor a regular graph.

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Vita

Eric Brooks was born in 1959 spending his formative years in Ohio, where he graduated from Perrysburg High School in 1977. He graduated from The Ohio State University in 1982 with a dual marketing and finance major. He spent the next 18 years in various marketing related jobs with Marathon Petroleum Company. In 2000 he took an indefinite leave to stay home with his two sons. During this time he began to study one of his former passions, mathematics. He also coached youth ice hockey developing a passion for teaching and mentoring. He attended Oakland University in Rochester, Michigan with a goal of becoming a math teacher, and after a family move to Richmond, Virginia, he enrolled in a graduate mathematics program at Virginia Commonwealth University. While a student at VCU, he was given an opportunity to teach undergraduate mathematics which has confirmed that teaching will be his new vocation. After graduation he intends to teach and inspire young men and women in the joys and usefulness of mathematics.