2015

Domination Numbers of Semi-strong Products of Graphs

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Domination Numbers of Semi-strong Products of Graphs

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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August, 2015

Acknowledgements

There are many to whom I owe my heartfelt gratitude. At the top of that list is Dr. Ghidewon Asmerom, my thesis director, but also my teacher, advisor, mentor and friend. Without his tireless efforts this document would not exist. I would also like to thank Dr. Dewey Taylor for three years of instruction, guidance and advice. In addition let me thank all of the VCU faculty who have helped me, especially those who taught me and/or are serving on the advisory committee.

I would also like to thank my friends and family, most especially my wonderful wife Terri for her patience and forbearing; in particular for the time, when looking at several toilet paper roles with arcane sketches on the desk, and asking, ”Are those necessary?”, was able to bite her tongue when I answered, ”Yes”. It would be remiss of me to fail to thank both my co-workers and students at J. R. Tucker High School for their encouragement and inspiration. Finally let me thank my church family at Seventh Street Christian Church for their prayers and support.

Let me take this opportunity to dedicate this effort to my friend, the late Robin Grayson, who taught me the beauty and importance of friendship.

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Abstract

This thesis examines the domination number of the semi-strong product $\gamma(G \,\overline{\times}\, H)$ where both G and H are simple and connected graphs. The product has an edge set that is the union of the edge set of the direct product $G \times H$ together with $|V(H)|$ copies of G. Unlike the other more common products (Cartesian, direct and strong), the semi-strong product is neither commutative nor associative. The semi-strong product is not supermultiplicative, so it does not satisfy Vizing like conjecture. It is also not submultiplicative so it shares these two properties with the direct product.

After giving the basic definitions related with graphs, domination in graphs and basic properties of the semi-strong product, this paper includes a general upper bound for the domination of the semi-strong product, $\gamma(G \,\overline{\times}\, H) \leqslant 2\gamma(G)\gamma(H)$. Similar general results for perfect-paired domination numbers, $\gamma_{pr}(G \,\overline{\times}\, H) = \gamma_{pr}(G)i(H)$ and $\gamma_{pr}(C_m \,\overline{\times}\, C_n) = 2\lceil m/4 \rceil \lceil n/3 \rceil$ when $m/4 \equiv 0 \bmod 4$, and $n/3 \equiv 0 \bmod 3$. In addition, a result involving $\gamma(C_m \,\overline{\times}\, C_n)$ and when it could be less than, equal to, or greater than $\gamma(C_m)\gamma(C_n)$, depending on the values of m and n is given.

Chapter 1

Introduction

The concept of domination in graphs began in the middle of the 17th century and the queens problem, where the question was how many queens on a chessboard were required so that no square on the chessboard was unavailable to a queen move. The question was first stated mathematically by de Jaenisch in 1862 [6]. The problem was formalized in work by Berge and Ore in 1958 [8], with one of the first treatments by Berge in 1962 [2]. This work heralded a re-awakening of interest in domination in the 1960's and 70's which has continued in multiple forms, and notably in domination numbers of products of graphs, to the present day.

The purpose of this thesis is to investigate domination parameters of graphs. All graphs are simple, connected graphs with at least two vertices. Several common types of graphs will be considered including paths, cycles and complete graphs, and extension to general graphs will be considered in some cases. The domination parameters of the graphs as well as products of these graphs will be studied. Domination parameters of graphs is an area of growing interest in todays society as networks, coverage and protection issues are of vital importance in computer networks, cell phone coverage networks, credit transfer networks, information retrieval, social network theory and land surveying among many other applications. In all of these applications, domination of the graphs representing these situations is an important consideration. The importance of domination may well influence the design of a network. It is this fact that makes the mathematical study of domination an important area of study. It is certainly possible that an as yet undiscovered aspect of domination sets of combinations of graphs may become a key aspect of network design. This also indicates that the study of different methods of creating a network, different products of graphs as an example, could be a productive endeavor.

Chapter 2

Graphs and Domination Sets

The basis of this thesis is graphs and products of graphs. We start by defining graphs as in [3].

Definition 1. *A **Graph** $G = (V, E)$ is a finite nonempty set $V(G)$ of objects called **vertices**, together with a possibly empty set $E(G)$ of distinct unordered pairs of elements of $V(G)$ called **edges**.*

When there is no ambiguity we will write V and E instead of $V(G)$ and $E(G)$ in that order. The **order** of a G is the cardinality of $V(G)$. The **size** of G is the cardinality of $E(G)$. Two vertices are said to be **adjacent** or **neighbors** if there is an edge between them. An edge between two vertices is **incident** to each vertex.

Definition 2. *The open neighborhood of the vertex v consists of the set of vertices adjacent to v, that is,* $N(v) = \{w \in V : vw \in E\}$*, and the closed neighborhood of v is* $N[v] = N(v) \cup \{v\}$.

The **degree deg(v)** of v is the number of edges incident to v or simply the cardinality of $N(v)$. A vertex v such that $N(v) = \emptyset$ is called an **isolate**.

A set S of vertices in a graph G is **independent** or **stable** if no two vertices in the set are adjacent in G. A **maximum independent set** is an independent set of largest possible cardinality. If a set S is a maximum independent set, then it is not a subset of any other independent set; i.e., every edge of G is incident to at least one vertex not in S and every vertex not in S has at least one neighbor in S. The **vertex independence number** or simply independence number $\alpha(G)$ of G is the cardinality of a maximal independent set of G.

A set of edges in a graph G is **independent** if no two edges in the set are adjacent in G. The edges in an independent set of edges of G are called a **matching** in G. A matching of maximum cardinality in G is a **maximum matching** in G. The **edge independence number** $\alpha_1(G)$ of G is the number of edges in a maximum matching of G. If M is a matching of G where every vertex of G is incident with an edge of M, then M is a **perfect matching** in G.

We follow the notation in [8] for the following definitions related to domination in graphs.

Definition 3. *A **domination set** of graph G(V,E), is $S \subseteq V$, such that every vertex in V is either in S or adjacent to a vertex in S.*

This definition gives rise to several equivalent forms of the definition of a domination set from [8]: A set $S \subseteq V$ is a **dominating set** if and only if:

i. for every vertex $v \in V - S$ there exists a vertex $u \in S$ such that v is adjacent to u;

ii. for every vertex $v \in V - S$, the distance between v and S, $d(v, S) \leqslant 1$;

iii. the closed neighborhood of S equals V, $N[S] = V$;

iv. for every vertex $v \in V - S, |N(v) \cap S| \geqslant 1$, that is, every vertex $v \in V - S$ is adjacent to at least one vertex in S;

v. for every vertex $v \in V, |N[v] \cap S| \geqslant 1$.

Note that if S is a domination set of graph $G(V, E)$, then any subset of V that contains S is also a domination set. On the other hand, not every subset of S is necessarily a domination set. For a subset S the graph that includes the vertices of S and the same edges between those vertices as in the original graph G is called the **induced subgraph** of G, or the subgraph of G induced by S, signified by $\langle S \rangle$.

Definition 4. *A domination set S is a* **minimal domination set** *if no proper subset S' is a domination set.*

The set of all minimal domination sets of a graph G is denoted by $MDS(G)$. Figure 2.1 shows a graph with minimal domination sets of cardinality three (graph B), four (graph C), and five (graph D).

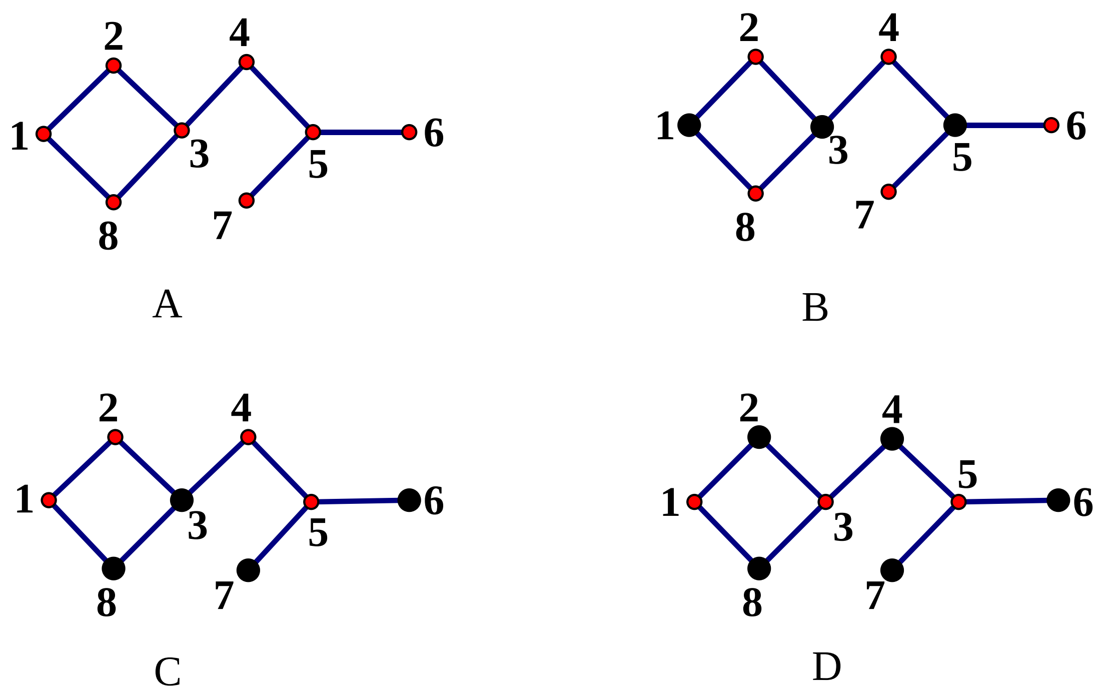

Figure 2.1: Minimal domination sets of differing cardinality

In [11], Ore demonstrated that for any minimal domination set S, every vertex $u \in S$

was either an isolate of S or there was a vertex $v \in V - S$ for which $N(v) \cap S = \{u\}$. All of the minimal domination sets in Figure 2.1 meet this standard. This idea of minimal domination set leads to the following definitions:

Definition 5. *The* **domination number** $\gamma(G)$ *of a graph* G *equals the minimum cardinality of a set in MDS(G), or equivalently, the minimum cardinality of a domination set in G.*

Figure 2.1 shows a graph with domination number of 3, and clearly this graph could not have a domination set of cardinality 2. In more complex graphs, proving the domination number could not be smaller requires more effort.

Definition 6. *The* **upper domination number** $\Gamma(G)$ *equals the maximum cardinality of a set in MDS(G), or equivalently, the maximum cardinality of a minimal domination set of* G.

These definitions of minimal domination and domination number are for the least restrictive form of domination. Other domination parameters are formed by combining domination with other graph theoretical properties. In this work three such properties are considered: First is that the domination set S has all isolated vertices, which leads to independent domination. Second is that S has no isolated vertices, which leads to total domination. Third is that S has a perfect matching, which leads to paired domination.

Independent domination, as defined by Goddard et al. [5]

Definition 7. *An* **independent dominating set** *of* G *is a set that is both dominating and independent in* G.

The **independent domination number** of G, denoted by $i(G)$ is the minimum cardinality of an independent dominating set. In total domination, not only does every vertex in $V - S$ have to be adjacent to a vertex in S, but each vertex in S must also be adjacent to a different vertex in S. This leads to the definitions:

Definition 8. *The set* S_t *is a* **total domination set** *if* $V = N(S_t)$.

As in the case of the domination number, the total domination number is given by the least cardinality of all possible total domination sets.

Definition 9. *The* **total domination number** *is* $\gamma_t(G) = \min\{|S_t| \mid S_t \subseteq V(G)$ *and* $V = N(S_t)\}$.

Since the total domination set is also a domination set, it is clear that $\gamma(G) \leqslant \gamma_t(G)$ for any graph G.

In paired domination, the total domination set must satisfy an additional condition. The adjacent vertices in S must also form a perfect matching. The definition is:

Definition 10. *A* **paired-domination set** S_{pr} *with matching* M *is a domination set* $S_{pr} = \{v_1, v_2, \ldots, v_{2t-1}, v_{2t}\}$ *with independent edge set* $M = \{e_1, e_2, \ldots, e_t\}$ *where each edge* e_i *is incident to two vertices in* S_{pr}.

The perfect matching requires that no two e_i can be incident to the same v_i, which is why $|S| = 2|M|$. This also means that a vertex set with a perfect matching has an even cardinality. The paired-domination number is, as expected:

Definition 11. *The* **paired-domination number** $\gamma_{pr}(G)$ *is the minimum cardinality of a paired-domination set* S_{pr} *in* G.

Both total domination and paired domination require that there be no isolated vertices in S, and every paired-domination set is also a total domination set. In addition $\gamma(G) \leqslant i(G) \leqslant \gamma_t(G) \leqslant \gamma_{PR}(G)$.

As clarification, consider the same graph as in figure 2.1 but with different domination sets, see figure 2.2.

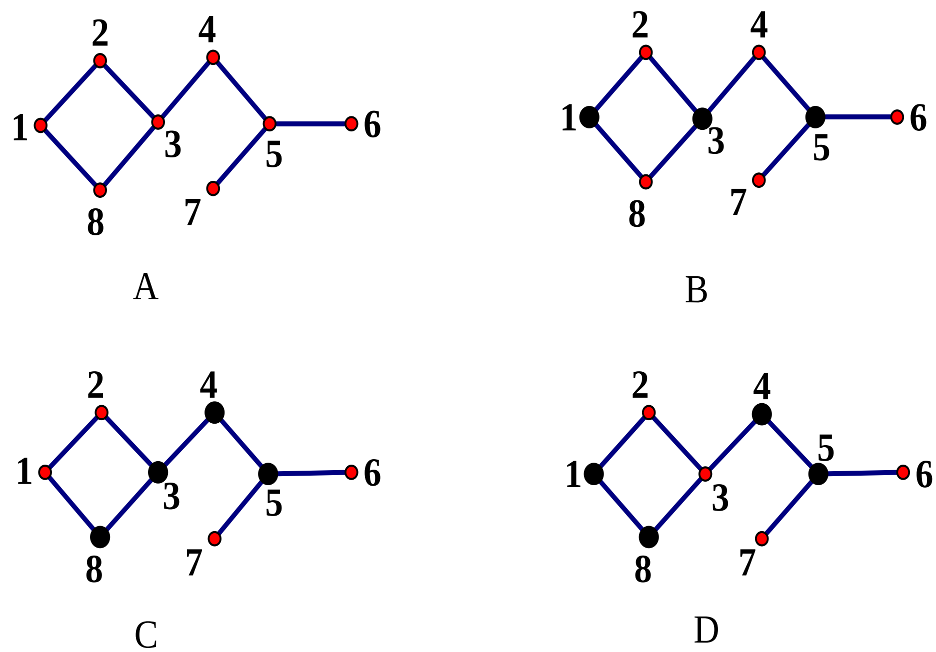

Figure 2.2: Graph G, $\gamma(G) = i(G) = 3$ in B, $\gamma_t(G) = 4$ in C, and $\gamma_{pr}(G) = 4$ in D

Chapter 3

Graph Products and Domination

In graph theory, products of graphs have been an area of interest and several different types of products have been studied. Three common types of products that have been previously studied are the Cartesian product, Direct product, and Strong product. All of these products are both commutative and associative, which is evident in the definitions.

The definitions of these three types of graph products follow:

Definition 12. *The* **Cartesian product** *of two graphs* $G(V,E)$ *and* $H(V,E)$ *is the graph* $G\Box H$ *with vertices* $V(G\Box H) = V(G) \times V(H)$*, and edges*
$E(G\Box H) = \big\{(g_1,h_1)(g_2,h_2) \mid g_1g_2 \in E(G) \textit{ and } h_1 = h_2, \textit{ or } g_1 = g_2 \textit{ and } h_1h_2 \in E(H)\big\}$.

Definition 13. *The* **direct product** *of two graphs* $G(V,E)$ *and* $H(V,E)$ *is the graph* $G \times H$ *with vertices* $V(G \times H) = V(G) \times V(H)$*. and edges*
$E(G \times H) = \big\{(g_1,h_1)(g_2,h_2) \mid g_1g_2 \in E(G) \textit{ and } h_1h_2 \in E(H)\big\}$

The strong product is the graph that has the same vertex set as the Cartesian and direct products and whose edge set is the union of the edge sets of both.

Definition 14. *The* **strong product** *of two graphs* $G(V,E)$ *and* $H(V,E)$ *is the graph* $G \boxtimes H$ *with vertices* $V(G \boxtimes H) = V(G) \times V(H)$ *and edges*
$E(G\boxtimes H) = \big\{\{(g_1,h_1)(g_2,h_2) \mid g_1g_2 \in E(G) \textit{ and } h_1 = h_2 \textit{ or } g_1 = g_2 \textit{ and } h_1h_2 \in E(H)\}\bigcup\{(g_1,h_1)(g_2,h_2) \mid g_1g_2 \in E(G) \textit{ and } h_1h_2 \in E(H)\}\big\}$

These products are illustrated in the following figures:

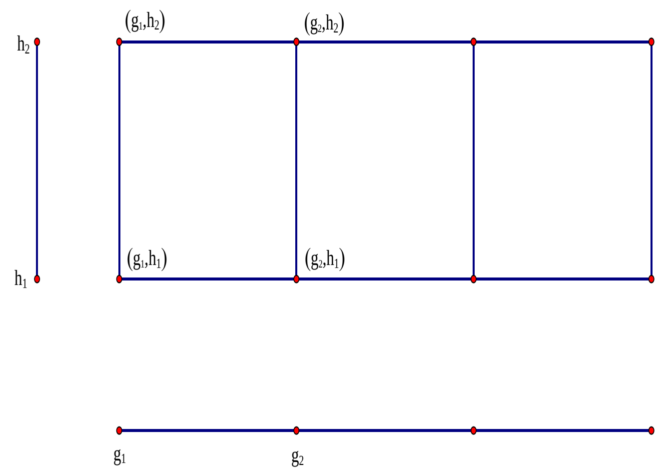

Figure 3.1: Graph of $G\Box H$ where $G = P_4, H = K_2$

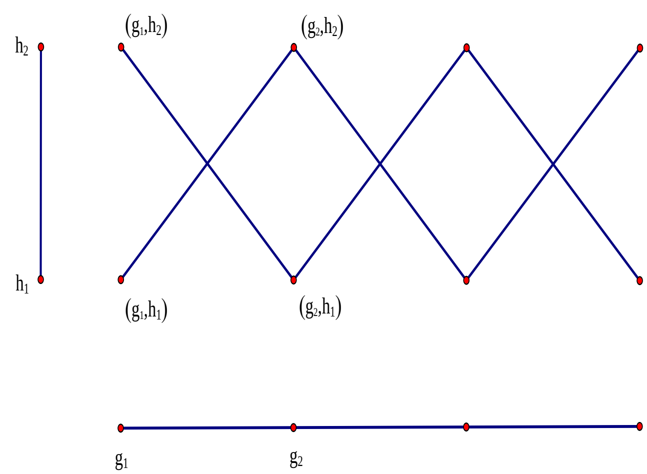

Figure 3.2: Graph of $G \times H$ where $G = P_4, H = K_2$

The strong product has an edge set that is the union of the edge sets of both the Cartesian and direct products.

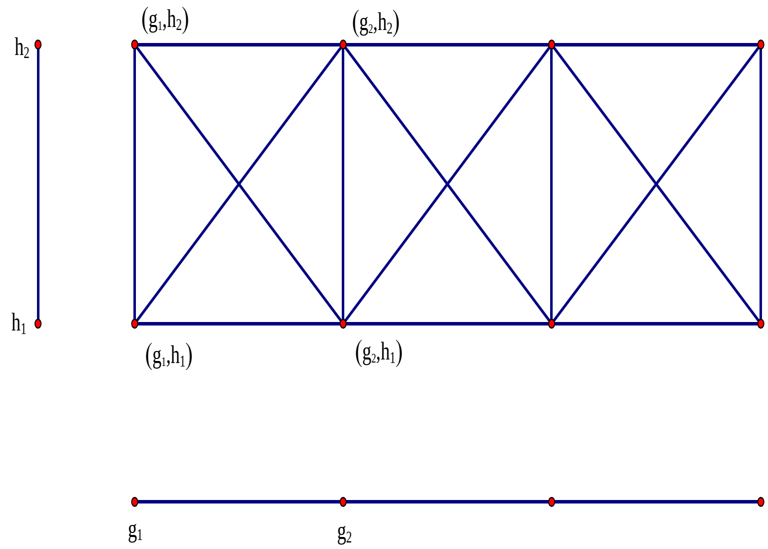

Figure 3.3: Graph $G \boxtimes H$ where $G = P_4, H = K_2$

It is worth noting that these graph products use variations of standard mathematical notation, and actually use the same notation for different types of products. The "$\times$" symbol is typically used for the Cartesian product, and is used in that fashion in all three definitions for the vertex sets, which are true Cartesian products of two sets producing ordered pairs. The graph products are distinctly different types of products. Here the product symbol for the Cartesian product, "$\square$", differs from the standard symbol. Similarly the direct product symbol for the graph product, "$\times$", is the standard symbol for the Cartesian product, which this is definitely not, instead of the standard symbol, "$\bigotimes$" typically used for direct, or also refered to as tensor or Knocker, products. In looking at the figures the purpose of these symbols can be seen, as the edges produced by adjacent vertices in each graph result in edges of the product graph that resemble the symbol. The strong product symbol "$\boxtimes$" shows the edges created in similar fashion to the direct and Cartesian products.

One of the earliest and still unsolved probelms in domination theory of graph products was first asked by V. G. Vizing [14] in 1963, and five years later offered as a conjecture [15].

Vizing's Conjecture: The domination number of the Cartesian product of any two graphs is at least as large as the product of their domination numbers.

Proof of Vizing's conjecture for the general case is still an open question, however much work has been done on some classes and types of graphs which do satisfy Vizing's conjecture. Barcalkin and German [1] discovered a method of partitioning graphs into subgraphs of a particular type which yielded a large class of graphs, called BG-graphs, which satisfy Vizing's conjecture. A graph G is said to *satisfy Vizing's conjecture* if $\gamma(G\Box H) \geqslant \gamma(G)\gamma(H)$ holds for every graph H. In 2000 Clark and Suen [4] proved that for all graphs G and H, $\gamma(G\Box H) \geqslant \frac{1}{2}\gamma(G)\gamma(H)$.

Although Vizings conjecture is specific to Cartesian products it has become the standard of comparison for other products as well. In addition, where the conjecture sets a lower bound for the product domination number, attempts have been made to set upper bounds for the domination number of different types of products.

As stated by Rall [13], a graph product is **submultiplicative** if $\gamma(G \cdot H) \leqslant \gamma(G) \cdot \gamma(H)$. A graph product is **supermultiplicative** if $\gamma(G \cdot H) \geqslant \gamma(G) \cdot \gamma(H)$. In the same work Rall shows by way of counter-example that the direct product is neither submultiplicative nor supermultiplicative for any two graphs G and H. Later Brešar, Klavžar and Rall [12] proved three significant results for direct products:

$$\gamma(G \times H) \leqslant 3\gamma(G)\gamma(H)$$

$$\Gamma(G \times H) \geqslant \Gamma(G)\Gamma(H)$$

$$\gamma_{pr}(G \times H) \leqslant \gamma_{pr}(G)\gamma_{pr}(H)$$

These bounds will be of interest in the study of semi-strong products.

For strong products, in *Handbook of Product Graphs* Hammack et al. [7] give a similarly significant result:

$$\gamma(G \boxtimes H) \leqslant \gamma(G)\gamma(H)$$

These results have some significance to the work of this thesis as semi-strong products also study a graph product which, like the strong product, uses a combination of some of the edges in a Cartesian product and the edges of the direct product.

Chapter 4

Semi-strong Products of Graphs

In this Chapter we study the semi-strong products of graphs and the domination numbers associated with them. The definition of a semi-strong product is:

Definition 15. *The* **semi-strong product** *of two graphs* $G(V, E)$ *and* $H(V, E)$ *is the graph* $G \overline{\underline{\times}} H$ *with vertices* $V(G \overline{\underline{\times}} H) = V(G) \times V(H)$ *and edges* $E(G \overline{\underline{\times}} H) = \big\{(g_1, h_1)(g_2, h_2) \mid g_1g_2 \in E(G)$ *and* $h_1 = h_2;$ *or* $g_1g_2 \in E(G)$ *and* $h_1h_2 \in E(H)\big\}$.

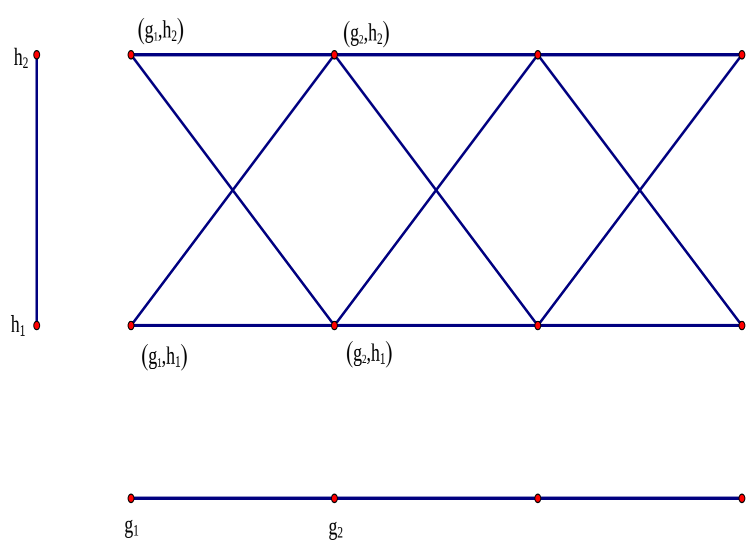

Figure 4.1: Graph of semi-strong product of P_4 and K_2

4.1 Basic Properties of the Semi-strong Product

The same notational discrepancies occur in this definition as in the three previous graph products, and again the symbol for the semi-strong product mimics the edges produced by the product. Apart in the study of domination of graphs, this product has an interest on its own. In support of this we present the following basic observations.

While there are a number of similarities between the semi-strong product and the three others, there is one very significant difference - the semi-strong product is neither associative nor commutative.

This is demonstrated using two graphs, K_2 and C_4, and the fact that $K_2 \overline{\times} K_2 = C_4$, as seen Figure 4.2 (A). In addition we have $C_4 \overline{\times} K_2$ in Figure 4.2 (B) and $K_2 \overline{\times} C_4$ in Figure 4.2 (C).

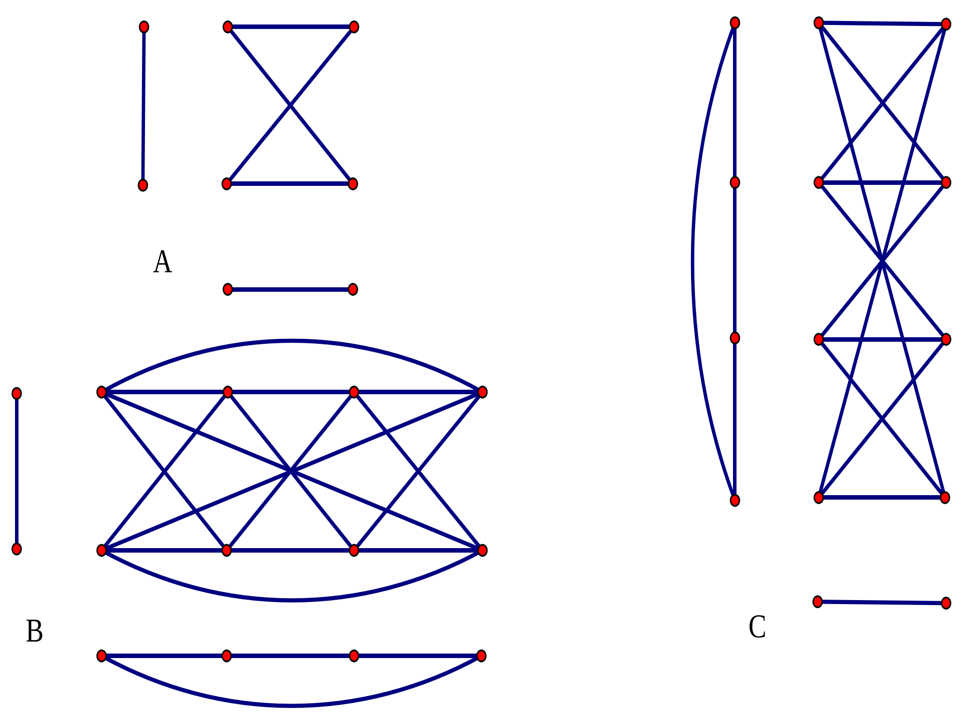

Figure 4.2: Graphs showing non-commutativity of semi-strong products

We can see that the product in Figure 4.2 (B), $C_4 \overline{\times} K_2$, has 16 edges, while $K_2 \overline{\times} C_4$

in Figure 4.2 (C) has only 12 edges. In other words, the first graph is the 4-regular bipartite graph $K_{4,4}$ and the second one the 3-regular bipartite graph Q_3 or the 3-cube. This means that $C_4 \,\underline{\times}\, K_2 \neq K_2 \,\underline{\times}\, C_4$ and semi-strong products are not necessarily commutative. Next consider $(K_2 \,\underline{\times}\, K_2) \,\underline{\times}\, K_2 = C_4 \,\underline{\times}\, K_2$ as seen in part A of Figure 4.2, while $K_2 \,\underline{\times}\, (K_2 \,\underline{\times}\, K_2) = K_2 \,\underline{\times}\, C_4$. As was noted $C_4 \,\underline{\times}\, K_2 \neq K_2 \,\underline{\times}\, C_4$ so semi-strong products are also not associative.

Proposition 1. *The degree of* $(g, h) \in V(G \,\underline{\times}\, H)$ *is given by* $\deg((g, h)) = \deg_G(g)\,(\deg_H(h) + 1)$.

Proof. From the definition of the edge set of a semi-strong product the direct product edges will contribute $\deg_G(g)\deg_H(h)$ to the degree of any given vertex, while the edges repeated from G will give $\deg_G(g)$. □

Proposition 2. *Given the graphs* G *with* n_1 *vertices and* m_1 *edges, and* H *with* n_2 *vertices and* m_2 *edges, then* $G \,\underline{\times}\, H$ *has* n_1n_2 *vertices and* $m_1n_2 + 2m_1m_2$ *edges.*

Proof. The vertex set in the semi-strong is the Cartesian product of the vertex set of the factors and hence the result n_1n_2, while the number of edges is derived from the definition of the edges of the semi-strong product, with m_1n_2 edges with the n_2 copies of G in $G \,\underline{\times}\, H$, while $2m_1m_2$ represents the pair of edges produced by the direct product. Thus $|E(G \,\underline{\times}\, H)| = n_2|E(G)| \cup |E(G \times H)| = m_1n_2 + 2m_1m_2$. □

Proposition 3. *If* $H \neq K_1$ *then* $G \,\underline{\times}\, H$ *is connected if, and only if,* G *and* H *are both connected and* $G \neq K_1$. *Furthermore, if* $G = \bigcup_{i=1}^{r} G_i$ *and* $H = \bigcup_{j=1}^{s} H_j$, *with no* $G_i = K_1$, *then*

$$G \,\underline{\times}\, H = \bigcup_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant s}} G_i \,\underline{\times}\, H_j.$$

Proof. Assume both G and H are connected and $G \neq K_1$. Consider any two vertices (g_1, h_1) and (g_2, h_2) of $G \,\underline{\times}\, H$. If $h_1 = h_2$, then because G is connected, and (g_1, h_1) and (g_2, h_2) are on the same copy of G we can use the g_1–g_2 path in the connected graph G to connect (g_1, h_1) and (g_2, h_1). Otherwise, the g_1–g_2 path in G and the h_1–h_2 path in H can be used to get a $(g_1, h_1) - (g_2, h_2)$ path. This means $G \barwedge H$ is connected.

Conversely, if $G = K_1$ then $G \barwedge H$ will only have $|V(H)|$ isolated vertices, hence if $G \barwedge H$ is connected $G \neq K_1$. Assume one or both of G or H are not connected, but $G \barwedge H$ is connected. If we take G to be not connected. Take two vertices g_1 and g_2 that belong to two different components of G and the vertices (g_1, h_1) and g_2, h_2. Since $G \barwedge H$ is connected there exists a $(g_1, h_1) - g_2, h_2$ path in $G \barwedge H$, leading to g_1–g_2 path in G, which is a contradiction. Similar contradiction arises if we assume H is not connected. Therefore if $G \barwedge H$ is connected, then G and H are both connected. The second result follows from component by component product and the proof above. □

Let $\chi(G)$ be the **chromatic number** of G, the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color.

Proposition 4. $\chi(G \barwedge H) = \chi(G)$. *Moreover, if* G *is bipartite, then* $G \barwedge H$ *is also bipartite.*

Proof. The vertices $(g_1, h_1), (g_2, h_2) \in V(G \barwedge H)$ are adjacent, if and only if $g_1 g_2 \in E(G)$. This means the color class of (g_1, h_1) and (g_2, h_2) is determined by g_1 and g_2, i.e., $\chi(G \barwedge H) = \chi(G)$. The second result easily follows from the fact that for a bipartite graph B, $\chi(B) = 2$. □

4.2 Domination and the Semi-strong Product

The following theorem shows the relationship between domination numbers of direct products, semi-strong products and strong products.

Theorem 1. *For any graphs* G *and* H, $\gamma(G \boxtimes H) \leqslant \gamma(G \barwedge H) \leqslant \gamma(G \times H) \leqslant 3\gamma(G)\gamma(H)$.

Proof. Since the edge set of $G \barwedge H$ is a subset of the edge set of $G \boxtimes H$, any domination set of $G \boxtimes H$ must also be a domination set for $G \barwedge H$, while the converse is not necessarily true, hence $\gamma(G \boxtimes H) \leqslant \gamma(G \overline{\times} H)$. Similarly, $E(G \times H) \subset E(G \overline{\times} H)$, giving $\gamma(G \overline{\times} H) \leqslant \gamma(G \times H)$. The result $\gamma(G \times H) \leqslant 3\gamma(G)\gamma(H)$ follows from the work of Rall et al. [12]. $\square$

Let P_3 be the path on 3 vertices. In similar fashion to the direct product, the semi-strong product is not submultiplicative since $\gamma(P_3 \overline{\times} K_2) = 2 > 1 = \gamma(P_3)\gamma(K_2)$.

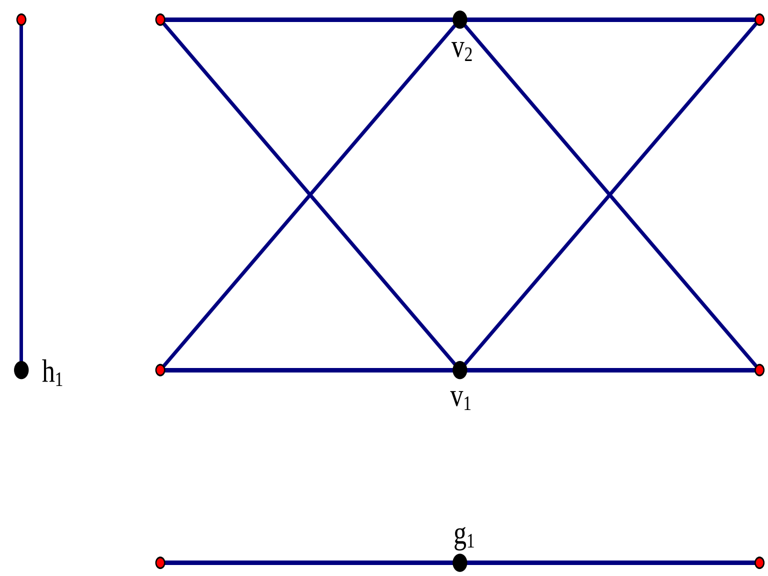

Figure 4.3: Graph of P_3, K_2 and their semi-strong product with labeled domination sets

Note that C_n is the cycle on n vertices and that C_n a connected 2-regular graph, i.e., $deg(v) = 2$ for every vertex $v \in V(C_n)$. This is: $N(v) = 2$ and $N[v] = 3$ for every vertex v in C_n. What this means is that any cycle C_n contains at most $\lfloor \frac{n}{3} \rfloor$ independent neighborhoods. If we choose the central vertex of each such neighborhood as a subset of $V(C_n)$ then we have a minimal dominating set of C_n, or $\gamma(C_n) = \lfloor \frac{n}{3} \rfloor$, if $\frac{n}{3} \equiv 0 \pmod 3$. If $\frac{n}{3} \equiv 1, 2 \pmod 3$, then the domination set of C_n must include one addtional vertex meaning $\gamma(C_n) = \lfloor \frac{n}{3} \rfloor + 1$, or equivalently $\gamma(C_n) = \lceil \frac{n}{3} \rceil$. This yields $\gamma(C_6) = 2$, while $\gamma(C_7) = 3$.

In the following Lemma, Berge [2], we have a general lower bound for the domination number of a graph in terms of the maximum degree $\Delta(G)$ of a graph G.

Lemma 1. *Let the maximum degree of* G *with* n *vertices be* $\Delta(G)$*, then any dominating set of* G *has at least* $\frac{n}{\Delta(G)+1}$ *vertices.*

Proof. For any vertex $v \in V(G)$, $N[v]$ has at most $\Delta(G)+1$ vertices. Thus the number of vertices any vertex v of G can dominate is at most $\Delta(G)+1$, therefore every dominating set must have at least $\frac{n}{\Delta(G)+1}$ vertices. $\square$

Corollary 1. *If* G *is r-regular, then any dominating set of* G *has at least* $\frac{n}{r+1}$ *vertices.*

Corollary 2. *Every total dominating set and every perfect-paired dominating set of* G *must have at least* $\frac{n}{\Delta(G)}$ *vertices.*

Figure 4.4 below shows the case of the $C_7 \overline{\times} C_7$ drawn on the torus; $C_7 \overline{\times} C_7$ is a 6-regular graph with 49 vertices, thus any dominating set of this product must have at least 7 vertices. Figure 4.4 also shows the 7 vertices indicated in **black** as a dominating set; hence we have $\gamma(C_7 \overline{\times} C_7) = 7$.

The above argument shows that the semi-strong product is not supermultiplicative since $\gamma(C_7 \overline{\times} C_7) = 7 < 9 = \gamma(C_7)\gamma(C_7)$.

When the product $C_{7j} \overline{\times} C_{7k}$ where j, k are positive integers, is similarly drawn on a torus we observe the same pattern seen in the $C_7 \overline{\times} C_7$ graph. We can summarize the general result showing that the submultiplicativity is not necessarily so, as seen in the following theorem.

Theorem 2. *Given the semi-strong product graph* $C_m \overline{\times} C_n$ *with* $m = 7j, n = 7k$ *for some positive integers j and k, then:*

a. $\gamma(C_m \overline{\times} C_n) < \gamma(C_m)\gamma(C_n)$ *when* $j = 1, 7k \equiv 1, 2 \pmod 3$ *(or conversely* $7j \equiv 1, 2 \pmod 3$ *and* $k = 1$*).*

b. $\gamma(C_m \overline{\times} C_n) = \gamma(C_m)\gamma(C_n)$ *when* $j = 1, 7k \equiv 0 \pmod 3$ *(or conversely* $7j \equiv 0 \pmod 3$ *and* $k = 1$*).*

c. $\gamma(C_m \overline{\times} C_n) > \gamma(C_m)\gamma(C_n)$ *when* $j > 1$ *and* $k > 1$*.*

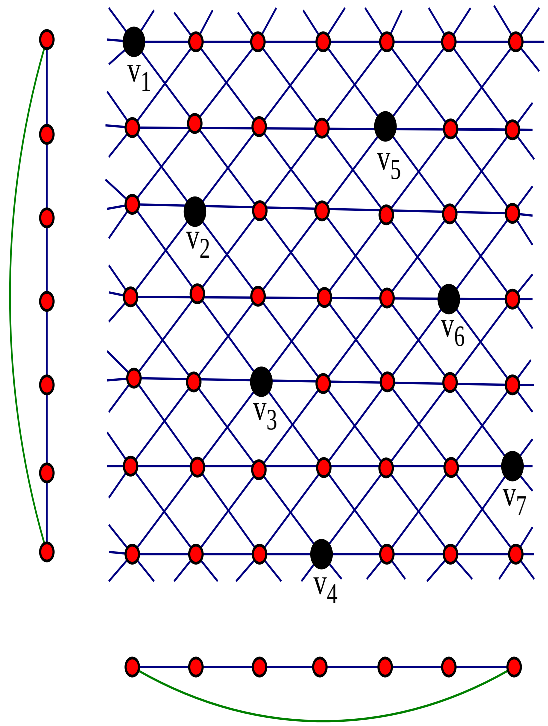

Figure 4.4: Graph of semi-strong product of C_7, C_7 with labeled domination set

Proof. Let $C_m \barwedge C_n$ with $m = 7j, n = 7k$ for some positive integers j and k. Using Corollary 1 above and the facts that $C_m \barwedge C_n$ is a 6-regular graph, we see that any dominating set of the product must have at least 7jk vertices, i.e., $\gamma(C_m \barwedge C_n) \geqslant 7jk$. However, if we draw the graph $C_m \barwedge C_n$ on the torus we can see that it consists of jk blocks, each of which is of the same pattern as a $C_7 \barwedge C_7$ and we can get a dominating set of 7jk vertices following the pattern of a minimal dominating set for a $C_7 \barwedge C_7$ block, ensuring $\gamma(C_m \barwedge C_n) \leqslant 7jk$. Thus, $\gamma(C_m \barwedge C_n) = 7jk$.

a. Let $j = 1$ and $7k \equiv 1, 2 \pmod 3$. $\gamma(C_m) = \gamma(C_7) = 3$ and $\gamma(C_n) = \lceil 7k/3 \rceil$, hence $\gamma(C_m)\gamma(C_n) = 3 \cdot (\lceil 7k/3 \rceil)$. But if $7k \equiv 1, 2 \pmod 3$, by definition $\lceil 7k/3 \rceil = (7k + r)/3$ where either $r = 1$ or $r = 2$ so that (7k + r) is divisible by 3, and $3(\lceil 7k/3 \rceil) = 3((7k + r)/3) = 7k + r > 7k$ and therefore $\gamma(C_m \barwedge C_n) < \gamma(C_m)\gamma(C_n)$.

b. Let $j = 1$ and $7k \equiv 0 \pmod 3$; again $\gamma(C_m) = \gamma(C_7) = 3$, $\gamma(C_n) = \lceil 7k/3 \rceil$ and again $\gamma(C_m)\gamma(C_n) = 3(\lceil 7k/3 \rceil)$. But if $7k \equiv 0 \pmod 3$, by definition $\lceil 7k/3 \rceil = 7k/3$ and $3 \cdot (\lceil 7k/3 \rceil) = 3(7k/3) = 7k$, therefore $\gamma(C_m \,\overline{\times}\, C_n) = \gamma(C_m)\gamma(C_n)$.

c. Let $j > 1$ and $k > 1$. $\gamma(C_m) = \gamma(C_{7j}) = \lceil 7j/3 \rceil$ and $\gamma(C_n) = \gamma(C_{7k}) = \lceil 7k/3 \rceil$. Let $7j + r_1 \equiv 0 \pmod 3$ where $r_1 = 0, 1$ or 2 and $7k + r_2 \equiv 0 \pmod 3$ where $r_2 = 0, 1$ or 2. Hence $\gamma(C_m)\gamma(C_n) = \lceil 7j/3 \rceil \lceil 7k/3 \rceil = \frac{(7j+r_1)(7k+r_2)}{9} = \frac{49jk+7(r_1k+r_2j)+r_1r_2}{9}$. This product is minimum when both r_1 and r_2 are 0. This yields $\gamma(C_m)\gamma(C_n) = (49jk)/9 = (7/9)(7jk) < 7jk = \gamma(C_m \,\overline{\times}\, C_n)$. Therefore $\gamma(C_m \,\overline{\times}\, C_n) > \gamma(C_m)\gamma(C_n)$. That is, when $j > 1$ and $k > 1$, $\gamma(C_m \,\overline{\times}\, C_n) > \gamma(C_m)\gamma(C_n)$.

□

This pattern which allows the neighborhoods of the vertices of the domination set to all be independent thus provides an example of a group of semi-strong product graphs for which the domination number of the product is at times less than, or other times equal to, and yet other times greater than the product of the domination numbers of the two graphs.

Two special cases are now considered. Both deal with **complete graphs**. A *complete* graph K_n on n vertices is a graph where every vertex is adjacent to every other vertex of the graph. Thus if v is any vertex in $V(K_n)$, then $N[v] = V(K_n)$ and by definition $\{v\}$ is a domination set for K_n which means $\gamma(K_n) = 1$ for all $n \in \mathbb{Z}^+$.

Before considering these cases, we present a lemma and corollary which will be used in the sequel.

Lemma 2. *Given the semi-strong product* $G \,\overline{\times}\, H$ *and a vertex* $(g_1, h_1) \in V(G \,\overline{\times}\, H)$, *any vertex* $(g, h) \in V(G \,\overline{\times}\, H)$ *where* $g \in N(g_1)$ *and* $h \in N(h_1)$ *must be in* $N((g_1, h_1)) \subseteq V(G \,\overline{\times}\, H)$.

Proof. Let, $(g, h) \in V(G \,\overline{\times}\, H)$; from the definition of the semi-strong product, if $g \in N(g_1)$ then $gg_1 \in E(G)$ and $(g, h_1)(g_1, h_1) \in E(G \,\overline{\times}\, H)$. If h is also in $N(h_1)$, then $(g, h)(g_1, h_1) \in E(G \,\overline{\times}\, H)$. Therefore $(g, h) \in N((g_1, h_1))$. □

Corollary 3. *Given* $(g_d, h_d) \in V(G \boxtimes H)$ *where* $g_d \in D_G$ *where* D_G *is a minimal domination set of* G *and* $h_d \in D_H$ *where* D_H *is a minimal domination set of* H*, then* $(g, h) \in N(D_G \times D_H)$ *whenever* $h \in N(h_d)$ *and* $g \in N(g_d)$.

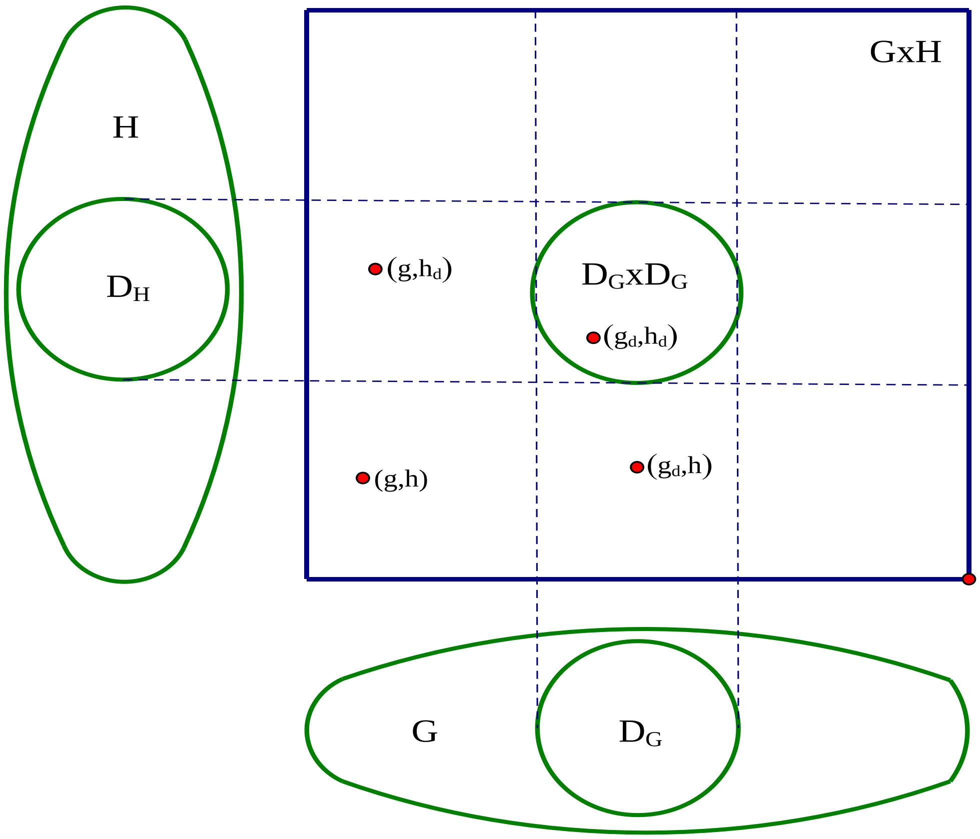

Figure 4.5: Illustration of Corollary 3

N. B. Corollary does NOT say (g, h_d) or (g_d, h) are in $N(D_G \times D_H)$

Theorem 3. *Let* K_n *be a complete graph of order* n*. For any graph* H*, then* $\gamma(K_n \boxtimes H) \leqslant 2\gamma(H)$.

Proof. Let $\{v_1\}$ be a minimal domination set for K_n and $D \subseteq V(H)$ be a minimal domination set for H. Consider the set of vertices $D_p = \{(v_1, h_d) \mid (v_1, h_d) \in V(K_n \boxtimes H)\}$ where h_d ranges over D . Clearly $|D_p| = |D|$, and for all $(v_i, h_f) \in V(K_n \boxtimes H)$ one of the following holds:

i. $(v_1, h_d) \in D_p$ by definition of D_p.

ii. For $v_i, i \neq 1$ then $(v_i, h_d) \in N((v_1, h_d))$ since $v_1 v_i \in E(K_n \boxtimes H)$ by the semi-strong definition.

iii. For $v_i, i \neq 1$ and $h_e \neq h_d$, then $(v_i, h_e) \in N((v_1, h_d))$ by Corollary 3.

iv. For $h_e \notin \{h_d\}$, then $(v_1, h_e) \notin N[(v_1, h_d)]$.

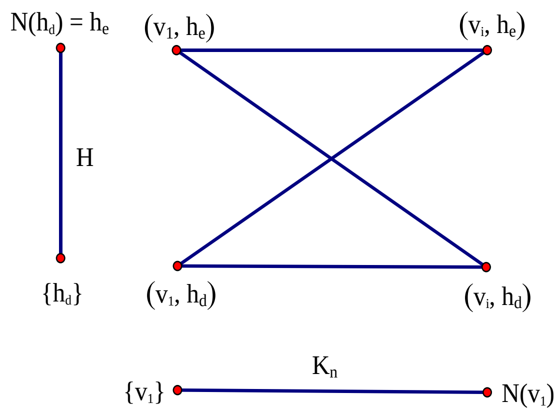

Figure 4.6: Illustration of four cases in Theorem 3

Now choose $v_2 \neq v_1$ and define another set of vertices, D_{p_2} in exactly the same manner as D_p except using v_2 instead of v_1. Then for all $(v_i, h_f) \in V(K_n \boxtimes H)$, either $(v_i, h_f) \in D_p$ or $(v_i, h_f) \in D_{p_2}$. This means $D_p \cup D_{p_2}$ must form a domination set for $K_n \boxtimes H$. Since $|D_{p_2}| = |D_p| = |\gamma(H)|$, the cardinality of a minimal domination set for the product must be less than or equal to the cardinality of $D_p \cup D_{p_2}$, that is, $\gamma(K_n \boxtimes H) \leqslant 2\gamma(H)$. $\square$

Corollary 4. *Let K_n be a complete graph. For any graph H, then $\gamma_t(K_n \boxtimes H) \leqslant 2\gamma(H)$.*

Noting the semi-strong product is not commutative, we reverse the order of factors we had in Theorem 3 to obtain the following result which has the complete graph K_n as a second factor rather than the first. What that means here is the semi-strong product will have n isomorphic copies of H as subgraph.

Theorem 4. *Let K_n be a complete graph. For any graph H, then $\gamma(H \overline{\times} K_n) \leqslant 2\gamma(H)$.*

Proof. Let $D \subseteq V(H)$ be a minimal domination set for H and $\{v_1\}$ be a minimal domination set for K_n. Consider the set of vertices $D_p = \{(h_d, v_1) \mid (h_d, v_1) \in V(H \overline{\times} K_n)\}$ where h_d ranges over D. Clearly $|D_p| = |D|$ and for all $(h_i, v_f) \in V(H \overline{\times} K_n)$ one of the following holds:

i. $(h_d, v_1) \in D_p$ by definition of D_p.

ii. For $h_e \notin D$ then $(h_e, v_1) \in N((h_d, v_1))$ for some $h_d \in D$ by definition of domination and the edges of the semi-strong product.

iii. For $h_e \notin D$ and $v_i, i \neq 1$, then $(h_e, v_i) \in N((h_d, v_1))$ for some $h_d \in D$ by Corollary 3.

iv. $(h_d, v_i), i \neq 1$

For the vertices in Case iv, there are two possibilities, if there is a $h'_d \in N(h_d)$ in D, then $(h_d, v_i) \in N((h_d, v_1))$ by Lemma 2. If no such h'_d exists (h_d is an isolate in D), then $(h_d, v_i) \notin N((h_d, v_1))$. For each isolate $h_d \in D$ there must be some $h'_e \notin D$ such that $h'_e \in N(h_d)$ then $(h'_e, v_i) \in N((h_d, v_1))$ by Lemma 2. Let $D' = D \cup \{h_e \mid h_e \text{ is a necessary } h'_e\}$, then let $D_e = \{(h_i, v_1) \mid h_i \in D'\}$. D_e is a domination set for $H \overline{\times} K_n$.

If every $h_d \in D$ is an isolate, and $d(h_{d_1}, h_{d_2}) = 2$ for each $h_{d_1}, h_{d_2} \in D$, then every $h_d \in D$ will require a distinct h'_e. In this case $|D'| = 2|D|$ and $\gamma(H \overline{\times} K_n) = 2\gamma(H)$. Otherwise $|D'| < 2|D|$ and $\gamma(H \overline{\times} K_n) < 2\gamma(H)$. This along with the definition of a minimal domination set yields. Therefore, $\gamma(H \overline{\times} K_n) \leqslant 2\gamma(H)$. □

Corollary 5. *Let K_n be a complete graph and H a graph with minimal total domination set $D_t \subset V(H)$. Then $\gamma_t(H \overline{\times} K_n) = \gamma_t(H)$.*

Next let's consider the more general case for $\gamma(C_m \overline{\times} C_n)$. Let's start with $C_4 \overline{\times} C_3$; because this graph does not fit the pattern of $C_7 \overline{\times} C_7$ it is not possible to create maximal independent neighborhoods of each vertex, however the domination set shown in the figure is a perfect pairing with a maximal neighborhood for such a perfect pairing.

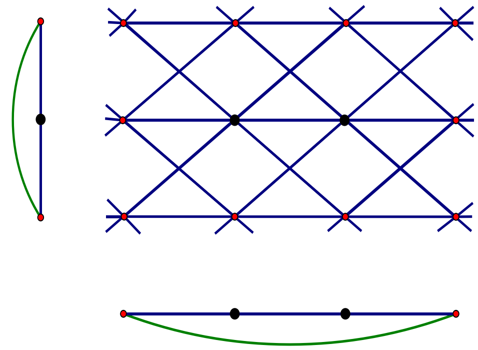

Figure 4.7: Semi-strong product graph of C_4 and C_3 with domination sets

Theorem 5. *Given a semi-strong product of the form* $C_m \,\overline{\times}\, C_n$ *where* $m = 4j, n = 3k$ *for some positive integers j and k, then* $\gamma_{pr}(C_m \,\overline{\times}\, C_n) = 2\lceil \frac{m}{4} \rceil \cdot \lceil \frac{n}{3} \rceil = 2 \cdot j \cdot k$.

Proof. If $j = k = 1$ then the graph is $C_4 \,\overline{\times}\, C_3$ which has 12 vertices with a regular degree of 6. By corollary 2 the minimum required vertices for a perfect-paired domination set is two. Since the diagram shows this the two domination set vertices must be a minimal perfect-paired domination set. Now consider the product $C_m \,\overline{\times}\, C_n = C_{4j} \,\overline{\times}\, C_{3k}$ which contains $j \cdot k$ blocks identical to $C_4 \,\overline{\times}\, C_3$ which will have a domination set of $2jk$ vertices in perfect pairing. The product contains $mn = 4j \cdot 3k = 12jk$ vertices. Since the graph is r-regular of degree 6, the minimal perfect-paired domination set must have a minimum of $12jk/6 = 2 \cdot j \cdot k$ vertices. Therefore $\gamma_{pr}(C_m \,\overline{\times}\, C_n) = (C_{4j} \,\overline{\times}\, C_{3k}) = 2 \cdot j \cdot k$.

Another way to state this result is: given $m \equiv 0 \pmod 4$, and $n \equiv 0 \pmod 3$ we have $\gamma_{pr}(C_m \,\overline{\times}\, C_n) = 2\lceil m/4 \rceil \lceil n/3 \rceil = \gamma_{pr}(C_m)\gamma(C_n)$. □

Corollary 6. *Given semi-strong product of the form* $P_m \,\overline{\times}\, P_n = P_{4k} \,\overline{\times}\, P_{3k}$, *then* $\gamma_{pr}(P_m \,\overline{\times}\, P_n) = 2 \cdot j \cdot k$.

The other problem that is conveniently solved by this pattern occurs consistently in semi-strong products. In the semi-strong product $G \,\overline{\times}\, H$, a vertex $g \in V(G)$ continues to be adjacent to any other vertices adjacent to it in $V(G)$, and also adjacent to the product vertices of these G vertices adjacent in H, but is not adjacent to those product vertices

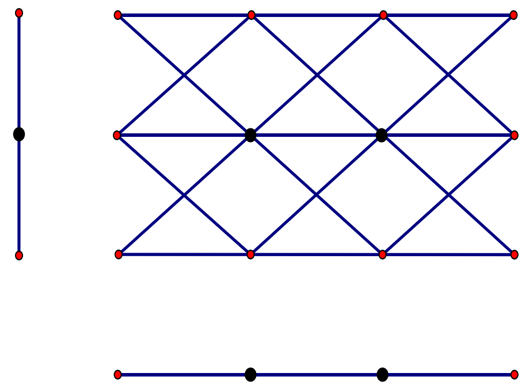

Figure 4.8: Graph illustrating Corollary 6

which are adjacent in H, but have the same g vertex. When two vertices adjacent in V(G) are included in the domination set for $G \,\overline{\times}\, H$, this problem is solved for them both.

This leads to the following conjectures:

Conjecture 1. *For the semi-strong product* $C_m \,\overline{\times}\, C_n$*, then* $\gamma(C_m \,\overline{\times}\, C_n) \leqslant 2\lceil m/4 \rceil \cdot \lceil n/3 \rceil$

Conjecture 2. *For the semi-strong product* $P_m \,\overline{\times}\, P_n$*, then* $\gamma(P_m \,\overline{\times}\, P_n) \leqslant 2\lceil m/4 \rceil \cdot \lceil n/3 \rceil$.

Although it is believed these conjectures are true the author was unable to devise acceptable proofs.

This treatment concludes with an attempt at an upper boundary for the domination number of a general semi-strong product.

Theorem 6. *For any graphs* G *and* H*,* $\gamma(G \,\overline{\times}\, H) \leqslant 2\gamma(G)\gamma(H)$.

Proof. Let $g_d \in D_G$ where D_G is a minimal domination set for G, and $h_d \in D_H$ where D_H is a minimal domination set for H. Consider $D_G \times D_H \subseteq V(G \,\overline{\times}\, H)$. By definition $(g_d, h_d) \in D_G \times D_H$ and for all $(g', h') \in V((G \,\overline{\times}\, H)) \backslash (D_G \times D_H)$, we have $(g', h') \in N(D_G \times D_H)$ by Corollary 3.

Two cases remain, first is (g', h_d) for $g' \in V(G) \backslash D_G$ but by the first part of the definition of a semi-strong product $gg' \in E(G \,\overline{\times}\, H)$ and hence $(g', h_d) \in N(D_G \times D_H)$. For all of these thus far, $D_G \times D_H$ acts as a domination set. The second case is

(g_d, h') for $h' \in V(H) \backslash D_H$. If g_d is adjacent to D_G, meaning g_d is not an isolate of D_G, then $(g_d, h') \in N(D_G \times D_H)$. If g_d is an isolate in D_G then there must exist an $g' \in V(G) \backslash D_G$ adjacent to g_d such that (g', h') is adjacent to (g_d, h') in $V(G \overline{\underline{\times}} H)$. Let $D_p = \{(g', h') \mid g'$ is a necessary such vertex to make the adjacency needed $\}$. Then $(D_G \times D_H) \cup (D_p \times D_H)$ forms a domination set for $G \overline{\underline{\times}} H$.

Clearly $|D_p| \leqslant |D_G|$ depending on how many g' vertices are needed to include the isolates of D_G. Since the cardinality of a minmal domination set is always less than or equal to the cardinality of a domination set,
$\gamma(G \overline{\underline{\times}} H) \leqslant \gamma(G)\gamma(H) + \gamma(G)\gamma(H) = 2\gamma(G)\gamma(H)$. □

Corollary 7. *For any graphs* G *and* H, $\gamma_t(G \overline{\underline{\times}} H) \leqslant \gamma_t(G)\gamma(H)$.

Corollary 8. *For any graphs* G *and* H, $\gamma_{pr}(G \overline{\underline{\times}} H) = \gamma_{pr}(G)i(H)$.

4.3 Further Questions for Research

The first task is to settle the conjectures in Chapter 4; second is come up with the smallest possible constant c that gives a tighter bound, $\gamma(G \overline{\underline{\times}} H) \leqslant c\gamma(G)\gamma(H)$ than the three that naturally comes as a corollary to the result from direct products, and if possible characterize those factors that would attain this tighter bound.

There are also many variations of domination, for example independent domination, semi-total domination, fair domination, roman domination, fractional domination, edge domination, etc that are addressed for other product graphs. These domination parameters should be examined for the semi-strong product as well.

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Vita

Stephen Cheney was born and raised in Atlanta, Georgia. Having spent nine years of off-and-on work in pursuit of an undergraduate degree, finally receiving a B.S. in Mathematics from University of Georgia, it should come as no surprise that he is finally earning a Masters in his sixties.

After beginning with a career as a swimming coach, Steve changed to high school teaching, which has been his work for the last 30 years. This degree supports his view of learning as a lifelong pursuit, and he still has so much to learn!

Steve has been married to his beautiful wife, Terri, for 27 years, and has two sons, Grayson and Chris, of whom he is inordinately proud, a daughter-in-law Stephanie and two grandchildren, Annabelle Grace and Isaac Grayson, who are the light of his life!