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THE TWO-WAY MIXED MODEL ANALYSIS OF VARIANCE

by

Kenneth Davis Buckley

B.A., William Paterson College, 1972

Thesis

submitted in partial fulfillment of the requirements for the Degree of Master of Science in the Department of Biometry at the Medical College of Virginia, Health Sciences Division, Virginia Commonwealth University, Richmond, Virginia, May, 1974

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This thesis by Kenneth Davis Buckley is accepted in its present form satisfying the thesis requirement for the degree of Master of Science

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DEDICATION

This thesis is dedicated to my parents, Thomas E. Buckley and Evelyn E. Buckley and to Marguerite C. Malmgren whom I love very much.

Abstract of

Two-Way Mixed Model Analysis of Variance

by

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Virginia Commonwealth University
May 1974

The analysis of variance for experiments where the fixed effects or random effects model is appropriate is generally agreed upon with regard to testing procedures and covariance structure. It is only in experiments involving both random and fixed factors, i.e. mixed effects models, that controversy occurs as to the proper analysis. The mixed effect model has been considered by many statisticians, and several techniques have been developed for explaining its structure and performing its analysis for balanced data sets. The relationship of these techniques have been discussed in several papers as well.

The simplest case of the difficulties presented by the mixed effects models occurs in the two-way cross classification model with interaction. The various models for the two-way mixed situation were examined and compared. It was found that Scheffe's model defined the effects in a meaninful way, is completely general, and provides exact tests. In situations where Scheffe's model cannot be applied, it was found that Kempthorne's model or Graybill's model should be used since they define effects in a meaningful way and, under certain assumptions, gives exact tests. Searle's model does not define the effects in the same manner as the former three models. Searle's effects are defined more for mathematical appeal and his

model is designed for easy application to unbalanced cases. Consequently, his model was not found to be desirable in balanced two-way mixed effect designs.

In higher order models, Scheffe's modeling techniques were found not to be practical since his test for fixed effect differences in models with more than two random effects cannot be computed. Kempthorne's models and Graybill's models both, under certain assumptions, provide straightforward tests for all effects. For this reason, their modeling techniques are recommended for higher order mixed models involving balanced data sets. Searle's modeling technique was again found unapplicable for balanced data sets in higher order mixed models for the same reasons as those in the two-way case.

The results of the investigation recommends Scheffe's model for two-way situations, but Kempthorne's modeling technique and Graybill's modeling technique seem the most versatile. Although the task would be very cumbersome, further investigation is suggested in comparing Kempthorne's procedure and Graybill's procedure to Scheffe's procedure for testing fixed effect differences.

CONTENTS

		Page
1.	INTRODUCTION	1
2.	REVIEW OF LITERATURE	4
3.	COMPARISON OF TWO-WAY MIXED MODELS	11
	3.1 Scheffé's Model	11
	3.2 Other Suggested Models	17
4.	ANALYSIS OF THE MIXED TWO-WAY MODEL	25
	4.1 Expected Mean Squares	25
	4.2 Tests of Significance	30
	4.3 Applications to Data	32
5.	CONCLUSIONS AND EXTENSIONS	40
6.	BIBLIOGRAPHY	

1. INTRODUCTION

Approximately fifty years ago, Sir R. A. Fisher developed the analysis of variance technique as a statistical tool for the interpretation of data. Although originally developed for use in agricultural and biological experiments, the technique is now widely used. The analysis of variance (ANOVA) derives its name from its function of subdividing the total variation among the observations into meaningful components associated with identifiable differences in the conditions under which the observations are taken.

The components or effects associated with the differences in the conditions under which observations are taken are classified as either fixed or random depending on how the conditions were selected and on the type of inference that the researcher intends to make. In experiments where observations have been obtained on an entire set of conditions of interest and inferences are to be made to this set of conditions alone, then the conditions or effects are said to be fixed. When the researcher chooses a random sample of conditions from an infinite (large) population and intends to make inferences regarding the variation among the entire population of

conditions or effects rather than the particular sample observed, then the conditions or effects are said to be random.

Frequently experimental situations arise where more than one set of conditions or source of variation is recognized. These types of conditions are called factors. Models for observations are labeled fixed effects models if all factors are fixed, and random effects models if all factors are random. The analysis of variance for experiments where the fixed effects or random effects model is appropriate is generally agreed upon with regard to testing procedures and covariance structure. It is only in experiments involving both random and fixed factors, i.e. mixed effect models, that controversy occurs as to the proper analysis. The mixed effect model has been considered by many statisticans, and several techniques have been developed for explaining its structure and performing its analysis for balanced data sets. The relationship of these techniques have been discussed in several papers as well.

The simplest case of the difficulties presented by the mixed effects models occurs in the two-way cross classification model with interaction. In chapter 3 this case will be examined as to the definition of the effects

and the structure of the variance components and will be illustrated by a hypothetical example. The relationships of different models and their applications to experimental situations will be explained. In chapter 4 the expected mean squares and tests for significance will be examined. The final chapter will contain recommendations on how the results of the investigation of the two-way mixed model can be applied to higher order models.

2. REVIEW OF LITERATURE

Explicit identification of the different types of observational models was first accomplished by Eisenhart (1947). He worked extensively with the analysis of the fixed effects and random effects models, deriving expected means squares and significance tests but doing little with the mixed effect model. On mixed models, Eisenhart (1947:21) stated, "More general methods need to be devised for interpreting "mixed" analysis-of-variance tables, particularly in regard to tests of significance ".

Most of Eisenhart's research was based on earlier results from Daniel (1939) and Crump (1946).

Cornfield and Tukey (1949) introduced the concept of the interaction component. The model they suggested was of the form

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$$

where

μ	denotes the general mean,
α_i	denotes the contribution of the a levels of the fixed factor,
β_j	denotes the contribution of the b levels of the random factor,
$(\alpha\beta)_{ij}$	denotes the interaction components, and
ϵ_{ijk}	denotes the random errors.

One important aspect of this paper was the introduction of the assumption

$$\sum_{i} (\alpha\beta)_{ij} = 0, \text{ for all } j.$$

Schultz (1955) was the first to give rules for determining expected mean squares for general mixed model designs with balanced observation sets. Schultz concluded that for the mixed effects model the interaction variance components should be included in the fixed effect expected mean square but omitted from the random effect expected mean square when the interaction was between a random effect and fixed effect. He reasoned that ". . . such a component does exist as a part of the expectation of the mean square of the fixed effect (since measured over the random variate) but does not exist as a part of the expectation of the random variate (since measured over the fixed effect)" (Schultz, 1955:125). Schultz's rules are particularly useful in higher order designs where there may be many fixed and random effects under consideration. Schultz based some of his work for the mixed model rules on work done by Kempthorne and Wilk (1955) who treated the model as a flexible concept and used it for the development of the expected means square for the two-way ANOVA. Like Tukey, Kempthorne and Wilk (1955) assumed that the interaction term "is not independent

of main effect terms." The model they proposed was very general and then was tailored to fit different designs. In the general case they included a term labeled the "interactive error ". Since there was "no structuring of the experimental units ", (Kempthorne & Wilk, 1955:1149) the term was assumed to equal zero so that their general model reduced to the more familiar model. Further investigation led Kempthorne and Wilk to conclude that there was no interaction term present in the random effect expected mean square, but it was present in the fixed effect expected mean square. They reasoned that the fixed effect was representative of (i.e. included) the entire population tested, while the random effect was a sample from an infinite population.

Scheffé (1956a) not only recognized the interaction term was not independent of the main effect, but proposed covariance expressions for it and the covariance of the interaction term. Until Scheffé's work there had been general agreement on the fact that all tests generated by consideration of ANOVA expected mean squares in the conventional way followed an F distribution. However, in Scheffé's article it was noted that the fixed main effect did not have an exact F distribution and that Hotelling's T^2-test is appropriate. The relationship of the usual

F-test and T^2-test will be examined later. Scheffé also disputed the assumption that the variance-covariance matrix yielded equal correlation coefficients. It was noted, though, that Scheffé achieved the same expected mean squares as others except that the variances were defined differently. Later Scheffé (1956b) discussed the changes in testing terms and expected mean square expressions under different assumptions. He noted that when the interaction term was judged as purely random, then it was included in the expected mean square for the random main effect. This in turn changed the testing term for random main effects from error to mean square interaction, the same as used for the fixed main effect. This possibility was also discussed in an article by Johnson (1948). Although Johnson made no specific reference to the mixed model, he assumed random interaction for both purely fixed and purely random effects cases. For this reason the interaction term was included in both the fixed and random effect expected mean square. It should be noted, however, that this assumption of purely random interaction when fixed effects are involved does not appear reasonable in most situations.

Searle and Hartley (1969) offered a significant challenge to the accepted expected mean square situation

in the two-way mixed model. They drew attention to the fact that there exists". . . a discontinuity between customary analyses of balanced and of unbalanced data concerning the occurrence of certain interaction variance components in the expected mean squares" (Searle and Hartley , 1969:573). In their paper they explained that in the unbalanced two-way ANOVA expectation of Mean Square Blocks (random effect) there occurs an interaction component. Thus if all the n_{ij} are set equal to n then the balanced data expected mean square for block effects should also contain an interaction term. They do not make any assertion to the correctness of their result but ". . . merely to emphasize its existence" (Searle and Hartley, 1969:575). Searle (1971) first proposes a model that is general and then proceeds to add restrictions to it in order to show its relationship to the "classical" model for the mixed effects case. He shows that the restricted model under the $\sum_{i}(\alpha\beta)_{ij} = 0$ assumption will lead to the usual accepted expected mean squares for the random main effect, but that this assumption is not always the case in the "real world situations ". Searle was not the only person to include an interaction component in the expected mean square for the random main effect. Others that include the interaction component with the

random effect are Henderson (1969), Steele and Torrie (1960), and Kirk (1968). Kirk stated in his book that "If the block and treatment effects are not additive, then the interaction component . . . appears in . . . the denominator of MS_s/MS_{res}" (Kirk, 1968:137),where "s" represented a random population. Kirk's main effect expected mean square differs from Searle's, but his testing term was identical. He points out, though, that this leads to a negatively biased test.

Two recent papers have attempted to clarify the mixed model controversy. Kim and Carter (1972a, 1972b) conducted an "empirical" study to determine whether the interaction variance component was present in the expected mean square of the random effect. They generated a fixed factor A such that $\sum_i a_i = 0$, and a random factor $\beta \sim N(0, \sigma_\beta^2)$. Their study concluded that the interaction variance component should be absent from the expected mean square of the random effect. Hocking (1973) compared three basic models; Scheffé's, Searle's, and Graybill's. The models were described in two fashions, the first in standard model form and the second "described the data by specifying the first two moments of the observations " (Hocking, 1973:148). Hocking made suggestions pertaining to experimental situations which applied to each model.

He also compared the various variance-covariance matrices of the "true" means of the observations for the different models to that of Scheffé's.

3. COMPARISON OF TWO-WAY MIXED MODELS

In this chapter the various two-way mixed models will be presented and comparisons made between them to point out their differences. Since differences in assumptions amongst the various models are often difficult to visualize, they will be illustrated in terms of a "real life" situation. As used by Scheffé (1956a, 1959), a fixed population of I machines will be operated by a random sample of J workers each of whom will operate each machine K times. The I X J X K responses will be measured in terms of output, i.e. piece work.

3.1 Scheffé's Model

Scheffé's model appears to be the most general so his approach to deriving expected mean squares and variance components will be used as a basis for all comparisons. The k^{th} output of worker j on machine i is structured as

$$Y_{ijk} = \mu_{ij} + e_{ijk}$$

The "errors" $\{e_{ijk}\}$ are assumed to be "... independently distributed with zero means and variance σ^2". (Scheffé, 1959:261). The errors are also distributed independently of the "true" means μ_{ij}. μ_{ij} is the "true" mean of worker j using machine i. The distribution of μ_{ij} is of primary interest.

The workers selected for this experiment are to be representative of all workers capable of using the machinery to be tested. The population of all such workers has a distribution $\mathscr{P}_v$, from which each worker is selected. The J workers are randomly selected so that m_{ij} can be denoted as a random variable m(i,v) where v represents worker v randomly selected according to $\mathscr{P}_v$. There are I random variables m(i,v) for each worker v which can be used as components of a "vector random variable ", m(v), where

$$m(v) = [m(1,v),\ m(2,v),\ldots,\ m(I,v)]. \qquad (3.2)$$

Since there will be J of these vectors, the resulting I X J matrix is formed

$$Q = [m(1)',\ m(2)',\ \ldots\ m(J)']. \qquad (3.3)$$

To find the "true" mean of machine i, it is necessary to take the expected value of m(i,v) with respect to $\mathscr{P}_v$. The "true" mean of machine i will be denoted as

$$\mu_i = m(i,\cdot). \qquad (3.4)$$

By taking the arithmetic average of the I "true" means, the general mean is found and denoted as

$$\mu = \frac{\sum \mu_i}{I} = m(\cdot,\cdot). \qquad (3.5)$$

The effect of machine i is defined as the amount μ_i exceeds the general mean. By letting α_i represent this

effect, then

$$\alpha_i = \mu_i - \mu = m(i,.) - m(.,.). \qquad (3.6)$$

The "true" mean for worker v is found by taking the average of m(i,v) over the i machines. The effect of worker v would be a random variable measuring his excess over the general mean. Using b(v) as the random variable measuring worker v's effect, then

$$b(v) = m(.,v) - \mu . \qquad (3.7)$$

The response, however, does not necessarily depend only on the worker and machine effects but on the interaction resulting from the particular worker and machine combination, also. To measure this interaction effect for worker v on machine i, the m(i,v)'s excess is measured over α_i, b(v), and μ. Letting $c_i(v)$ represent the interaction effect, then

$$\begin{aligned} c_i(v) &= m(i,v) - \alpha_i - b(v) - \mu \\ &= m(i,v) - [m(i,.) - \mu] - [m(.,v) - \mu] - \mu \\ &= m(i,v) - m(i,.) - m(.,v) + \mu . \end{aligned} \qquad (3.8)$$

The "true" mean m_{ij} now explained in terms of m(i,v) broken into effects, is

$$\begin{aligned} Y_{ijk} &= m_{ij} + e_{ijk} \\ &= \mu + \alpha_i + b(v) + c_i(v) + e_{ijk} . \end{aligned} \qquad (3.9)$$

Now that the model has been defined in terms of the effects, the properties of these effects will next be examined.

Property 1: Since the expected value of m(i,v) has been

taken over $\mathscr{P}_v$, then $m(i,.) - \mu$ is a constant. Since the general mean is the average of I $m(i,.)$'s, i.e. the I machine means, then the sum of the machine effects is equal to zero. Thus property 1 is

$$\sum_i \alpha_i = m(.,.) - m(.,.) = 0.$$

Property 2: If the expected value of $b(v)$ is defined as $b(.)$, then

$$E(b(v)) = b(.) = m(.,.) - \mu = 0.$$

Property 3: The interaction effect is "fixed in one direction and random in the other." (Schultz, 1955:125) so that

$$\sum_i c_i(v) = \sum_i \left[m(i,v) - m(i,.) - m(.,v) + \mu \right]$$
$$= I\, m(.,v) - I\, m(.,.) - I\, m(.,v) + I\mu = 0,$$

and

$$E\left[c_i(v) \right] = c_i(.) = m(i,.) - m(i,.) - m(.,.) + \mu = 0.$$

It is next helpful to investigate the variance-covariance structure of the system of "true" means. Denote the covariance between machine i and machine i′ operated by worker v as $\sigma_{ii'}$. Then

$$\begin{aligned}
\sigma_{ii'} &= \mathrm{Cov}\left[m(i,v),\ m(i',v) \right] \\
&= E\Big\{ \Big[\Big(\mu + \alpha_i + b(v) + c_i(v) \Big) - E\Big(\mu + \alpha_i + b(v) + c_i(v) \Big) \Big] \\
&\quad \Big[\Big(\mu + \alpha_{i'} + b(v) + c_{i'}(v) \Big) - E\Big(\mu + \alpha_{i'} + b(v) + c_{i'}(v) \Big) \Big] \Big\} \\
&= E\Big\{ \left[b(v) + c_i(v) \right] \left[b(v) + c_{i'}(v) \right] \Big\} \\
&= E\Big\{ b^2(v) + c_i(v)\, b(v) + c_{i'}(v) b(v) + c_i(v)\, c_{i'}(v) \Big\}
\end{aligned} \tag{3.11}$$

The variance of the worker effects can now be expressed as

$$\text{Var}\left[b(v)\right] = \text{Cov}\left[m(.,v) - \mu,\ m(.,v) - \mu\right]$$
$$= \text{Cov}\left[\frac{\sum_i m(i,v) - I}{I},\ \frac{\sum_{i'} m(i',v) - I}{I}\right]$$

$$= 1/I^2\ E\left\{\left[\sum_i (\mu + a_i + b(v) + c_i(v)) - I\mu\right]\left[\sum_{i'} (\mu + a_{i'} + b(v) + c_{i'}(v) - I\mu\right]\right\}$$

$$= 1/I^2\ E\left\{\left[\sum_i (b(v) + c_i(v))\right]\left[\sum_{i'} (b(v) + c_{i'}(v)\right]\right\}$$

$$= 1/I^2\ E\left\{\sum_i \sum_{i'} (b^2(v) + c_i(v)b(v) + c_{i'}(v)b(v) + c_i(v)c_{i'}(v))\right\}$$

$$= \sum_i \sum_{i'} \frac{\sigma_{ii'}}{I^2} = \sigma_{..} \ . \qquad (3.12)$$

The covariance between the interaction effects of worker v using machine i and i′ can be found as

$$\text{Cov}\left[c_i(v),\ c_{i'}(v)\right] =$$
$$\text{Cov}\left[(m(i,v) - m(i,.) - m(.,v) + \mu),\ (m(i',v) - m(i',.) - m(.,v) + \mu)\right].$$

Expressing each of the terms of this last expression in terms of the appropriate effects and carrying out the indicated operations results in

$$\text{Cov}\left[c_i(v),\ c_i(v)\right] = \sigma_{ii'} - \frac{\sum_i \sigma_{ii'}}{I} - \frac{\sum_{i'} \sigma_{ii'}}{I} + \sum_i \sum_{i'} \frac{\sigma_{ii'}}{I^2}$$
$$= \sigma_{ii'} - \sigma_{.i'} - \sigma_{i.} + \sigma_{..} \ . \qquad (3.13)$$

The matrix $\{\sigma_{ii'}\}$ denoting $\text{Cov}\left[m(i,v),\ m(i',v)\right]$ is an I X I symmetric matrix such that $\sigma_{ii'} = \sigma_{i'i}$.

Thus

$$\text{Var}\left[c_i(v)\right] = \text{Cov}\left[c_i(v),\ c_i(v)\right] = \sigma_{ii} - \sigma_{i.} - \sigma_{.i} + \sigma_{..}$$
$$= \sigma_{ii} - 2\sigma_{.i} + \sigma_{..} \ .$$

The last covariance to be evaluated is between the worker v, machine i interaction effect and the worker v

effect, i.e.

$$\text{Cov}\left[b(v),\ c_i(v)\right] = \text{Cov}\left[m(.,v) - \mu, (m(i,v) - m(i,.) - m(.,v) + \mu)\right] .$$

Again expressing the terms of this with the appropriate effects leads to

$$\text{Cov}\left[b(v)\ ,\ c_i(v)\right] = \sum_i \frac{\sigma_{ii'}}{I} - \sum_{i'}\sum_{i''} \frac{\sigma_{i'i''}}{I^2}$$

$$= \sigma_{i.} - \sigma_{..} \quad . \tag{3.14}$$

Scheffé "adopts" the following definitions of "variances" of the three effects:

The variance of the worker effect is taken to be

$$\text{Var}\left[b(v)\right] \text{ and is denoted by } \sigma_B^2 , \tag{3.15}$$

the variance of machine effects is taken to be

$$\sum_i \frac{\alpha_i^2}{(I-1)} \text{ and is denoted by } \sigma_A^2, \tag{3.16}$$

and the variance of the machine x worker interaction effects is defined to be

$$\sum_i \frac{\text{Var}\ (\ c_i(v))}{(I-1)} \text{ and is denoted by } \sigma_{AB}^2 . \tag{3.17}$$

The quantities (3.15) and (3.17) can be expressed in terms of the covariance matrix as

$$\sigma_B^2 = \sigma_{..} \text{ and} \tag{3.18}$$

$$\sigma_{AB}^2 = \sum_i \frac{\text{Var}(c_i(v))}{(I-1)}$$

$$= \sum_i \frac{(\sigma_{ii} - 2\sigma_{i.} + \sigma_{..})}{(I-1)}$$

$$= \frac{\sum_i \sigma_{ii} - 2\sum_i \sum_{i'} \frac{\sigma_{ii'}}{I} + \sum_i \sigma_{..}}{(I-1)}$$

$$= \frac{\sum_i \sigma_{ii} - 2I\sigma_{..} + \sum_i \sigma_{..}}{(I-1)}$$

$$= \frac{\sum_i \sigma_{ii} - \sum_i \sigma_{..}}{(I-1)} \qquad = \frac{\sum_i (\sigma_{ii} - \sigma_{..})}{(I-1)} \quad . \qquad (3.19)$$

3.2 Other Suggested Models

The covariance between the random main effects and interaction effects pointed out in Scheffé's model was first discussed by Tukey (1949). Kempthorne (1952), Searle (1971), Graybill (1961), Mood (1950), and others implicitly imposed restrictions on the matrix $\{\sigma_{ii'}\}$ such that

$$\sigma_{ii'} = \sigma^2 \qquad \text{if} \qquad i = i',$$
$$= p\sigma^2 \qquad \text{if} \qquad i \neq i'$$

in their models. Note that these restrictions imply homogenity of variances of the I machine means and homogenity of covariances between all pairs of machine means. The imposition of this assumption leads to independence of the random main effects and the interaction effects as well, i.e.

$$\text{Cov}\left[b(v), c_i(v)\right] = \sigma_{i.} - \sigma_{..} = \frac{\sum_{i'} \sigma_{ii'}}{I} - \frac{\sum_{ii'} \sigma_{ii'}}{I^2} =$$

$$(\sigma^2 + (I-1)\, p\sigma^2)/I - (I\sigma^2 + [I(I-1)]\, p\sigma^2)/I^2 = 0. \qquad (3.20)$$

The simplifications resulting from the restriction are appealing, but obviously should be used only in situations where valid. In the example used earlier it is very

probable that worker v will not be independent of his interaction with a machine. In fact the output depends almost entirely upon the correlation between the worker effect and his interaction with the machine effect. The worker might be independent of the interaction effect if perhaps he could not tell any difference between the machines' individual performances, but suppose that the worker "feels" more confident with machine i than with machine i'. This would surely cause a covariance between himself and the interaction. It is certainly plausible to assume that each worker v would react differently operating different machines, even of the same make. However, the more similar the machines, the less pronounced the covariance would be. It is again up to the researcher himself whether he thinks that a correlation is possible. In an experiment involving plants and fertilizer it might be entirely erroneous to assume a covariance between the plant effect and the fertilizer by plant interaction, since one could assume equal correlation between plants and fertilizers. This raises the question of determining which model fits a particular situation.

Hocking's (1973) approach seems suitable to answer this question, provided Kempthorne's model is considered as a fourth alternative. The models are:

(1) Scheffé's model.

$Y_{ijk} = \mu + \alpha_i + b_j + c_{ij} + e_{ijk}$ $c_{ij} = c_i(v)$ and $b_j = b(v)$, where $\sum_i \alpha_i = 0$, $\sum_i c_{ij} = 0 \ \forall_j$, the $\{e_{ijk}\}$ are NID $(0,\sigma^2)$ and distributed independently of $\{b_j\}$ and $\{c_{ij}\}$, and the $\{b_j\}$ and $\{c_{ij}\}$ are normally distributed with zero means and the following variances and covariances as defined in terms of an I X I covariance matrix with elements $\{\sigma_{ii'}\}$:

$\text{Var}(b_j) = \sigma_{..}$,

$\text{Cov}(c_{ij'}, c_{ij}) = \sigma_{ii'} - \sigma_{i.} - \sigma_{.i'} + \sigma_{..}$,

$\text{Var}(c_{ij}) = \dfrac{\sum_i [\sigma_{ii} - \sigma_{..}]}{(I-1)}$, and

$\text{Cov}(b_j, c_{ij}) = \sigma_{i.} - \sigma_{..}$.

(2) Searle's model (although used by others such as Plackett (1960) and Mood (1950)).

$Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + e_{ijk}$,

where $\{\beta_j\}$, $\{(\tau\beta)_{ij}\}$, and $\{e_{ijk}\}$ are normally distributed with zero means, are uncorrelated, and have the following variances:

$\text{Var}(e_{ijk}) = \sigma^2$,

$\text{Var}(\beta_j) = \sigma_\beta^2$, and

$\text{Var}(\tau\beta_{ij}) = \sigma_{\tau\beta}^2$.

(3) Graybill's model (used by many other statisticans and statistics textbooks).

$Y_{ijk} = \mu + \alpha_i + \gamma_j + (\alpha\gamma)_{ij} + e_{ijk}$

where $\sum_i \alpha_i = 0$, $\sum_i (\alpha\gamma)_{ij} = 0 \quad \forall j$ and $\{\gamma_j\}$, $\{(\alpha\gamma)_{ij}\}$, and $\{e_{ijk}\}$ are normally distributed with zero means and the following covariances and variances:

$\text{Var}\ (e_{ijk}) = \sigma^2$,

$\text{Var}\ (\gamma_j) = \sigma_\gamma^2$,

$\text{Var}\ ((\alpha\gamma)_{ij}) = \frac{(I-1)}{I} \sigma_{\alpha\gamma}^2$, and

$\text{Cov}\left[(\alpha\gamma)_{ij}, (\alpha\gamma)_{i'j}\right] = -1/I\ \sigma_{\alpha\gamma}^2$, $i \neq i'$.

All other covariances are zero.

(4) Kempthorne's model.

Kempthorne's two-way model is of the form

$Y_{ijk} = \mu + a_i + b_j + (ab)_{ij} + e_k + n_{ijk}$.

where the a_i are a random sample of size i* from a finite population of size I, the b_j are a random sample of size j* from a finite population of size J, and the $(ab)_{ij}$ are interaction components associated with each (ij) combination. He assumes furthermore that

$\sum_i^I a_i = 0, \sum_j^J b_j = 0, \sum_i^I (ab)_{ij} = 0$, and $\sum_j^J (ab)_{ij} = 0$

In this model e_k is the "additive error of the k^{th} unit" and is measured as $(Y..._k - Y...)$. n_{ijk} is called the "interactive error of the k^{th} unit and treatment (ij)". $n_{ijk} = (Y_{ijk} - Y_{ij\cdot} - Y..._k + Y...)$ and this "unit

treatment additivity" is assumed to be zero since the experimental units are not structured in most cases. Kempthorne and Wilk (1955:1149) state that "...even if untrue, it will not, in many cases, affect the interpretation of the analysis of variance too heavily...". To obtain the two-way mixed model we assume i* = I and $J \to \infty$. In this case the model reduces to

$$Y_{ijk} = \mu + a_i + b_j + (ab)_{ij} + e_k$$

where $\sum_{i}^{i*} a_i = 0$ and $\sum_{i}^{i*} (ab)_{ij} = 0$.

Kempthorne expresses variances and covariances for the general case. In the mixed model these become

$$\text{Var}(b_j) = \lim_{J\to\infty} \frac{(J-1)}{J}\sigma_b^2 = \sigma_b^2,$$

$$\text{Var}\left[(ab)_{ij}\right] = \lim_{J\to\infty} \frac{(J-1)(I-1)}{JI}\sigma_{ab}^2 = \frac{(I-1)}{I}\sigma_{ab}^2, \text{ and}$$

$$\text{Cov}\left[(ab)_{ij}, (ab)_{i'j}\right] = \lim_{J\to\infty} -\frac{(J-1)}{IJ}\sigma_{ab}^2 = \frac{-1}{I}\sigma_{ab}^2.$$

Thus although developed from a general point of view, this model reduces to Graybill's form.

All of these models are said to describe the two way mixed model with interaction, yet there are several differences. Kempthorne's model is fairly close to Graybill's with the exception of the error term and variances. Hocking (1973) suggests relating the models by specifying and comparing the first two moments. As in Hocking's paper, relationships among the variances

will be examined.

Basically each model structures the variance-covariance matrix $\{\sigma_{ii'}\}$ in different ways. Where Scheffé placed only the requirement of positive definiteness and symmetry on the matrix, the other models require equal correlation and homogenity of variance. Suppose the structure of $\{\sigma_{ii'}\}$ is pI + qH where I is an I X I identity matrix and H is an I X I matrix of all ones. Then

$$\sigma_{ii'} = p+q \quad \text{if } i=i', \text{ and}$$
$$= q \quad \text{if } i\neq i'.$$

Now using this restriction and imposing it upon (3.14), (3.18), and (3.19) the following quantities are obtained for Scheffé's model

$$\text{Cov}(b_j, c_{ij}) = \sigma_{i.} - \sigma_{..} = 0, \text{ as indicated in (3.20),}$$

$$\sigma_B^2 = \sigma_{..} = \sum_{ii'} \frac{\sigma_{ii'}}{I^2} = I(p+q)/I^2 + I(I-1)\,q/I^2$$
$$p/I + q .$$

$$\text{Var}\left((ab)_{ij}\right) = \sigma_{ii} - \sigma_{..}$$
$$= p+q - (p/I + q) = (I-1)p/I, \text{ and}$$

$$\sigma_{AB}^2 = \sum_i \frac{(\sigma_{ii} - \sigma_{..})}{(I-1)} = I(p+q)/(I-1) - I\,(p/I+q)/I-1$$
$$= p .$$

Thus as a result of the restrictions, the elements of $\{\sigma_{ii'}\}$ in terms of variance components are

$$\sigma_{ii'} = \sigma_B^2 + \frac{(I-1)}{I}\sigma_{AB}^2 \quad \text{if } i=i', \text{ and}$$
$$= \sigma_B^2 - \sigma_{AB}^2/I \quad \text{if } i\neq i'.$$

Searle's model also imposes a similar structure on $\{\sigma_{ii'}\}$, but as Hocking (1973:150) points out Searle is even more restrictive by making $p = \sigma_{\tau\beta}^2$ and $q = \sigma_\beta^2$.

This relates Searle's model to Scheffé's model as follows

$$\sigma_{\beta}^2 = \sigma_{\beta}^2 + 1/I\ \sigma_{\tau\beta}^2, \text{ and } \sigma_{AB}^2 = \sigma_{\tau\beta}^2 .$$

In Searle's model the effects are defined differently when compared to Scheffé's model. The following relationships exist as pointed out by Hocking (1973:150)

$$b_j = \beta_j + (\overline{\tau\beta})_{\cdot j}, \text{ and } c_{ij} = (\tau\beta)_{ij} - (\overline{\tau\beta})_{\cdot j} .$$

Recalling how Scheffé developes his effects, it is difficult to assign a meaning to Searle's model terms in the same manner. Where Scheffé defines his worker effect as

$$b_j = m(\cdot, v) - \mu,$$

Searle's worker effect would be

$$\beta_j = m(\cdot, v) - \mu - (\overline{\tau\beta})_{\cdot j} .$$

The value $(\overline{\tau\beta})_{\cdot j}$ will be a random variable depending upon worker j. Thus Searle's worker effect is measured as the difference of the "true" mean of the worker, minus the general mean, minus the average interaction effect of the worker on the I machines. Again in his interaction effect, Searle adjusts for an average interaction. Searle has "defined" all covariances between interaction effects and main effects out of existence since, as shown, re-defining Searle's effects in terms of Scheffé's effects will lead to covariances. The variance-covariance

matrix $\{\sigma_{ii'}\}$ used by Searle is even more restrictive than Graybill's or Kempthorne's since variances are attributed to worker effect and interaction effect variances, but covariances are attributed only to worker effect variances.

The definition of terms and restrictions on $\{\sigma_{ii'}\}$ make Searle's model mathematically appealing but not very practical for experiments, since the model terms have little intuitive meaning. The effects of Graybill's and Kempthorne's model make sense in experimental situations but still the assumption of homogenity of variance and equal covariance is questionable. Scheffé's model is the most "flexible" or "practical" model for the two-way mixed effects case since the definition of effects and matrix $\{\sigma_{ii'}\}$ does not create any bounds on an experiment. Although Searle's, Kempthorne's, or Graybill's model is applicable in certain situations, Scheffé's model is less ambiguous in its applications.

4. ANALYSIS OF THE MIXED TWO-WAY MODEL

In chapter 3 the mixed two-way cross classification models proposed by Scheffé, Searle, Kempthorne, and Graybill were stated and examined. Comparisons between them pointed out the differences in assumptions and in the meaning of the model terms. In the analysis of experimental data generated according to each of these models, the ANOVA calculations of sums of squares are the same. However, the model selected to represent the experimental situation shapes the analysis of the data both in the interpretation of the model terms and through the construction of significance tests. In this chapter differences in the expected mean squares under the various model forms are first examined. Next the tests of significance including the multivariate test of the fixed main effects proposed by Scheffé are presented. Finally the various significance tests and methods of estimating variance components are applied to a data set taken from literature.

4.1 Expected Mean Squares

The significance tests of recognized sources of variation in the ANOVA are typically constructed by considering the expected mean squares. The ANOVA expected mean squares under the four models considered

are given in Table 4.1. Kempthorne's and Graybill's

Table 4.1 Expected Mean Squares For the Two-Way Cross Classification Models With Interaction

	Fixed Effect A	Random Effect B	A B Interaction
Scheffe's Model	$JK\sigma_A^2 + K\sigma_{AB}^2 + \sigma^2$	$IK\sigma_B^2 + \sigma^2$	$K\sigma_{AB}^2 + \sigma^2$
Searle's Model	$\frac{JK\sum\gamma_i^2}{(I-1)} + K\sigma_{\gamma\beta}^2 + \sigma^2$	$IK\sigma_\beta^2 + K\sigma_{\gamma\beta}^2 + \sigma^2$	$K\sigma_{\gamma\beta}^2 + \sigma^2$
Kempthorne's Model	$\frac{JK\sum\alpha_i^2}{(I-1)} + K\sigma_{ab}^2 + \sigma^2$	$IK\sigma_b^2 + \sigma^2$	$K\sigma_{ab}^2 + \sigma^2$
Graybill's Model	$\frac{JK\sum\alpha_i^2}{(I-1)} + K\sigma_{\alpha\gamma}^2 + \sigma^2$	$IK\sigma_\gamma^2 + \sigma^2$	$K\sigma_{\alpha\gamma}^2 + \sigma^2$

models have essentially the same ANOVA expected mean squares and consequently the same significance tests. Scheffe's model results in ANOVA expected mean squares which, again, are essentially the same as Kempthorne's and Graybill's but the more general structure of this model results in a multivariate test for the fixed main effects. Tests of other effects are the same. Searle's model, though, has a different expected mean square for the random effect which causes this to be tested by the interaction rather than by the error mean square as in the former three models.

The conflict over the proper expected mean square for the random effects has been discussed in numerous papers. Kim and Carter (1973a), (1973b) attempted to

resolve the controversy by an "empirical study." Using Monte Carlo simulation 1200 data sets were generated for each of two sets of selected variance components. It should be noted that these data sets were generated with correlated interactions consistent with Graybill's, Kempthorne's, and Scheffé's models. (In this section Scheffé's model will be used as including Kempthorne's and Graybill's models). The χ^2 values under Searle's expected mean squares and Scheffé's expected mean squares were examined and tested for significance. However Kim and Carter failed to recognize that Searle's random effect is not defined the same way as Scheffé's. Table 4.2 appears in Kim and Carter (1973a:8) using $\sigma^2 = 1$, $\sigma^2_{AB} = 1/6$ and $\sigma^2_B = 1$.

Table 4.2 The χ^2 Test With $\sigma^2 = 1$, $\sigma^2_B = 1$

χ^2	χ^2 Test	D.F.	Z Value
SSA/E(MSA)	1251.79	1200	1.0562
SSB/($\sigma^2 + 16\sigma^2_B$)	1157.57	1200	-.8637
SSB/($\sigma^2 + 8\sigma^2_{AB} + 16\sigma^2_B$)	1073.38	1200	-2.6464*
SSAB/E(MSAB)	1256.23	1200	1.1448
SSE/E(MSE)	33521.77	3600	-.3000

*denotes ($p < .01$)

Z value = $\sqrt{2\chi^2} - \sqrt{2n-1}$

Since SSB = 1157.57 ($\sigma^2 + 16\sigma^2_B$) then it is found that SS_B = 1157.57 (1 + 16) = 19678.79. To prevent confusion,

Searle's random effect variance component will be noted as σ^2_θ and the interaction variance component as $\sigma^2_{\lambda\theta}$. In chapter 3, section 3.2, it was found that $\sigma^2 B = \sigma^2_\theta + 1/A\,\sigma^2_{\lambda\theta}$ and that $\sigma^2_{AB} = \sigma^2_{\lambda\theta}$ Numerically then, Searle's variance components in this case are valued as

$$\sigma^2_{\lambda\theta} = 1/6, \text{ and } \quad \sigma^2_\theta = \frac{11}{12}$$

Kim and Carter give the χ^2 test of Searle's model as

$$\chi^2 = \frac{19678.69}{\sigma^2 + 8\sigma^2 AB + 16\sigma^2 B} = \frac{19678.69}{1 + \frac{8\cdot 1}{6} + 16\cdot 1} = \frac{19678.69}{\frac{110}{6}} = 1073.38^*$$

The χ^2 test of Searle's model should be

$$= \frac{19678.69}{\sigma^2 + 8\sigma^2_{\lambda\theta} + 16\sigma^2_\theta} = \frac{19678.69}{1 + 8\cdot\frac{1}{6} + 16\cdot\frac{11}{12}}$$

$$\frac{19678.69}{17} = 1157.57 \text{ n.s.}$$

Thus the χ^2 test of Searle's model produces the same "non-significant" value as the test of Scheffe's.

In using $\sigma^2 = .0833$, $\sigma^2 AB = .1667$, and $\sigma^2_B = .3334$ Table 4.3 appears in Kim and Carter (1973a:9).

Table 4.3 The χ^2 Test With $\sigma^2 = .0833$, $\sigma_\beta^2 = .334$

χ^2	χ^2 Test	D.F.	Z Value
$SS_A/E(MS_A)$	1215.32	1200	.3218
$SS_B/(\sigma^2 + 16\sigma_\beta)$	1252.12	1200	1.0629
$SS_B/(\sigma^2 + 8\sigma_{A\beta} + 16\sigma_\beta^2)$	1004.83	1200	-4.1503*
$SS_{AB}/E(MS_{AB})$	1224.21	1200	.5019
$SS_E/E(MS_E)$	33775.33	3600	.6770

* denotes ($p < .01$)

Z value = $\sqrt{2\chi^2} - \sqrt{2n-1}$

Using similar methods as before, it was found that $SS_B = 6783.6105$ and $\sigma_\theta = .2301$. The χ^2 test of Searle's model was found to be

$$\chi^2 = \frac{6783.6105}{\sigma^2+8\sigma^2_{A\beta}+16\sigma^2_\beta} = \frac{6783.6105}{.0833+1.333+5.3244} = \frac{6783.6105}{6.7410} =$$

1004.83* .

Actually the χ^2 test of Searle's model is

$$\chi^2 = \frac{6783.6105}{\sigma^2+8\sigma^2_{\zeta\theta}+16\sigma^2_\theta} = \frac{6783.6105}{.0833+1.333+4.0112} =$$

$$\frac{6783.6105}{5.417} = 1252.12 \text{ n.s.}$$

The statement by Kim Carter (1973b:5) "Under Assumption I [Scheffe], the χ^2 values are not statistically significant while under Assumption II [Searle], the χ^2 values are statistically significant" is clearly in error. In examining the power of the two tests, Kim and Carter repeat their mistake. Clearly, it is not a question of which expected mean squares are correct, but rather for which model do the terms have the meaning which

the experimenter attributes to them. The model selection then determines the expected mean squares.

4.2 Tests of Significance

Usually tests of significance of the sources of variation in analyses of variance are constructed through consideration of the expected mean squares. A source of variation is generally tested by taking its mean square as the numerator component of an F statistic and by selecting another mean square from the ANOVA which has the same expectation under the null hypothesis. Thus using Scheffe's model the random effect would be tested by error while in Searle's model it would be tested by interaction. Obviously Searle's test is more conservative than Scheffé's, but it should not be forgotten that in each model the effects to be tested are defined in a different manner. Again, it is not a question of which model is better than the other, but rather which model fits a specific experimental situation best.

Ironically, it is the significance test of the fixed effect that is not exact, yet it is only Scheffé who discusses the situation. Aside from Searle and a few others, most statisticans define SSA and SSAB in terms of effects as

$$SSA = JK \sum_{i=1}^{I} (\alpha_i + c_{i\cdot} + e_{i\cdot\cdot} - e_{\cdot\cdot\cdot})^2, \text{ and} \quad (4.1)$$

$$SSAB = K \sum_{i=1}^{I} \sum_{j=1}^{J} (c_{ij} - c_{i\cdot} + e_{ij\cdot} - e_{i\cdot\cdot} - e_{\cdot j\cdot} + e_{\cdot\cdot\cdot})^2. \quad (4.2)$$

Except for Searle's model, most other models used by statisticans recognize a covariance between c_{ij} and $c_{i'j}$ For this reason, under the hypothesis

$$H_o: \alpha_i = 0 \qquad i=1,2...I.$$

the F test can't be used because SSA and SSAB are not distributed as a constant times a central (or non-central) χ^2 random variable. The proper test for fixed effects is Hotelling's T^2 statistic. The drawback to Hotelling's T^2 statistic is that it is cumbersome to calculate. In order to use it, extensive matrix manipulations are required. Most statisticans ignore the T^2 statistic and use an F-test with (I-1) and (J-1)(I-1) degrees of freedom. Scheffé (1956:36) remarks of this practice, "A justification of this would be welcomed by the practicioner because the computations are simpler and more familiar than those of Hotelling's T^2, but until numerical investigations are made which indicate the errors involved are tolerable, the practice should be suspect in the present case ".

Searle's model does not have covariance existing between interaction effects. Since he defines SSA and SSAB in terms of effects as

$$SSA = JK\sum_{i=1}^{I}(\bar{c}_{i} - \bar{c}_{..})^2 = JK\sum_{i=1}^{I}(\alpha_i + \bar{e}_{i.} - \bar{e}_{..})^2, \qquad (4.3)$$

and

$$SSAB = K\sum_{i=1}^{I}\sum_{j=1}^{J}(c_{ij} - \bar{c}_{i.} - \bar{c}_{.j} + \bar{c}_{..})^2 = K\sum_{i=1}^{I}\sum_{j=1}^{J}((\alpha\beta)_{ij} - (\alpha\beta)_{i.} - (\alpha\beta)_{.j} + (\alpha\beta)_{..} + e_{ij} - e_{i..} - e_{.j.} + e_{...})^2 \qquad (4.4)$$

then SSA and SSAB are distributed as χ^2 random variables and an F-test can be used. Mathematically Searle's modeling technique is more appealing than Scheffé's but it is considerably more restrictive in its application to experimental situations.

4.3 Applications to Data

To investigate the worth of the T^2 statistic and to illustrate the analysis resulting from different model assumptions, the following two-way mixed model experimental data is used from Anderson and Bancroft (1952). Nine sprays are tested for their ability to help hold fruit on cherry trees. The number of fruit in four one-pound random samples of the crop from 81 trees is counted.

Table 4.4 Cell Totals in Fruit Per Four Pounds

	Treatments								
Reps	1	2	3	4	5	6	7	8	9
1	506	471	580	438	497	514	468	455	494
2	444	464	718	478	483	484	515	451	507
3	452	417	638	485	474	526	495	445	506
4	453	443	503	437	500	539	476	457	469
5	468	459	596	417	493	516	462	436	470
6	427	428	559	457	531	496	442	479	430
7	460	468	583	482	509	427	470	468	462
8	395	506	571	414	457	452	475	418	489
9	455	454	718	429	515	511	406	425	484

In analyzing the experimental data in table 4.4, two models will be used. The first model will be Scheffé's

model (effectwise equivalent to Kempthorne's and Graybill's).

$$Y_{ijk} = \mu + \alpha_i + b_j + c_{ij} + e_{ijk}$$

where μ = the general mean,

α_i = the spray effect 1,2,...,9,

b_j = the tree replication effect j=1, ...9

c_{ij} = spray x tree effect, and

e_{ijk} = error in sampling k=1, ...4.

The other model used will be Searle's.

$$Y_{ijk} = \mu + \zeta_i + \beta_j + (\zeta\beta)_{ij} + e_{ijk}$$

where μ = the general mean,

ζ_i = the spray effect as defined by Searle i=1, ...9,

β_j = the tree replication effect as defined by Searle j=1, ...9,

$(\zeta\beta)_{ij}$ = spray x tree effect, and

e_{ijk} = error in sampling k=1, ..4.

The resulting ANOVA of this data is given in Table 4.5.

Table 4.5 ANOVA of Fruit Trees

Sources of Variation	D.F.	S.S.	M.S.
Treatment (Fixed)	8	45,326	5,665.7
Replication (Random)	8	2,804	350.5
Treatment x Replication	64	19,722	308.1
Error	243	4,610	18.97
Total	323	72,462	

The first hypothesis to be tested is

$H_o: \sigma^2AB = 0$ or $H_o: \sigma^2_{\zeta\beta} = 0$

$H_a: \sigma^2AB \neq 0$ $H_a: \sigma^2_{\zeta\beta} \neq 0$

The F-test for both models is $F = \frac{308.1}{18.97} = 16.24 > F_{64,243,.05} \approx 1.30$. The null hypothesis of $\sigma^2_{AB} = 0$ or $\sigma^2_{\gamma\beta} = 0$ is rejected. Normally analysis would turn to differences between individual treatment means, but analysis in this case will continue as if the null hypothesis was not rejected in order to indicate other test differences.

The next hypothesis to be considered is

$H_o: \sigma^2_B = 0$ $H_a: \sigma^2_B \neq 0$ or $H_o: \sigma^2_\beta = 0$ $H_a: \sigma^2_\beta \neq 0$

To test Scheffe's, Kempthorne's, and Graybill's random effect, the F test is

$$F = \frac{350.5}{18.97} = 19.002 > F_{8,243,.05} \approx 1.96$$

so that H_o is rejected. To test Searle's random effect, the F test is

$$F = \frac{350.5}{308.1} = 1.13 < F_{8,64,.05} = 2.10.$$

Here we fail to reject H_o.

The last hypothesis to be tested is

$H_o: \alpha_i = 0 \; \forall_i = 1,2,..9,$ $H_a: \alpha_i \neq 0$ for some i. or $H_o: \gamma_i = 0 \; \forall_i = 1,2,...9$ $H_a: \gamma_i \neq 0$ for some i.

Since Searle does not assume any correlation among his $(\gamma\beta)_{ij}$ and his SSA and SSAB are equations (4.3) and (4.4) as opposed to equations (4.1) and (4.2) then the F-test can be used to test the null hypothesis on the fixed

effects. For this example the test is

$$F = \frac{5,665.7}{308.1} = 18.52 > F_{8,64,.025} = 2.41.$$

Thus the null hypothesis is rejected. Kempthorne and Graybill use the identical F-test for their fixed effect. It should be noted here that $\zeta_i = \alpha_i + 1/9\ c_i$. . Which is not the same fixed effect as defined by Scheffé, Kempthorne and Graybill. Now to use the T^2 statistic, assuming Scheffé's model it is necessary to examine the data in Table 4.4 and calculate d_{rj} where

$$d_{rj} = Y_{rj\cdot} - Y_{Ij\cdot} \qquad (4.5)$$

In other words the last cell total of each row is subtracted from each cell total of that row. A 9x8 matrix results, which is given in Table 4.6.

Table 4.6 The Matrix Resulting from $d_{rj}=Y_{rj\cdot} - Y_{Ij\cdot}$

35	-23	86	-56	3	20	-26	-39	
-63	-43	211	-29	-24	-23	8	-56	
-54	-89	132	-21	-32	20	-11	-61	
-16	-26	34	-32	31	70	7	-12	
-2	-11	226	-53	23	46	-8	-34	
-3	-2	129	27	101	66	12	49	
-2	6	121	20	47	-35	8	6	
-91	20	85	-72	-29	-34	-11	-68	
-29	-30	234	-55	31	27	-78	-59	
-225	-192	1258	-271	151	157	-99	-274	total
-25	-22	-139.77	-30.11	16.77	17.44	-11	-30.44	mean

Next the (I-1) means and ½ (I-1)(I) sums of products are

computed. The resultant A matrix is (I-1) x (I-1), i.e.

8x8, and symmetric with

$$A_{rr'} = \sum_{j=1}^{J} (d_{rj} - d_{r.}) (d_{r'j} - d_{r'.}) \qquad (4.6)$$

This is given in Table (4.7).

Table 4.7 The A Matrix

11880.0	1242.0	-2107.0	2869.9	9573.3	5965.0
1242.0	8080.0	-3345.0	132.9	4616.4	-1955.0
-2107.0	-3345.0	38995.5	-1980.1	-1860.2	-1955.1
2869.9	32.9	-1980.1	9428.8	6872.7	1221.4
9573.3	4616.4	-1860.2	6872.7	14777.5	7480.3
5965.0	-1955.0	-1955.1	1221.4	7480.3	13072.2
-180.0	1105.0	-6403.0	4240.0	851.8	-295.0
6607.0	3740.0	-5553.8	8608.5	11760.3	5406.7

-180.0	6607.0
1105.0	3740.0
-6403.0	-5553.0
4240.0	8608.5
851.8	11760.3
-295.0	5406.7
6298.0	4397.0
4397.0	11878.2

There are then nine vectors $\underline{d}^j$, where

$$\underline{d}^j = (d_{1j} \quad d_{2j} \quad d_{3j} \ldots d_{8j})'$$

and each is normally and independently distributed

$N(\underline{\xi}, \Sigma_d)$ where an unbiased estimate of Σ_d is

$$\Sigma_d = (J-1)^{-1} \quad A = 1/8 \; A.$$

The vector $\bar{\underline{d}}' = (d_{1.}, d_{2.}, d_{3.}, \ldots, d_{8.})'$ and Hotelling's

T^2 statistic is

$$T^2 = J(J-1) \; (\bar{\underline{d}} - \underline{\xi})' \; A^{-1} \; (\bar{\underline{d}} - \underline{\xi}) \qquad (4.7)$$

in this case J = 9 and $\underline{\xi} = 0$ so equation (4.7) becomes

$$T^2 = 9(9-1) \quad \bar{\underline{d}}' A^{-1} \bar{\underline{d}} \qquad (4.8)$$

This T^2 statistic is equivalant to the following F statistic

$$\left[\frac{(J-1)\,(I-1)}{(J-I+1)}\right] F_{(I-1),(J-1+1)} = 64\, F_{8,1}$$

which changes (4.8) to

$$\frac{72}{64}\ \bar{\underline{d}}'\, A^{-1}\, \bar{\underline{d}} \cap F_{8,1}. \qquad (4.9)$$

Scheffe recommends a shortcut for calculations using

$$\frac{72}{64}\ \bar{\underline{d}}'\, A^{-1}\, \bar{\underline{d}} = \frac{72}{64}\left[\frac{|A + \bar{\underline{d}}'\,\bar{\underline{d}}|}{|A|} - 1\right]. \qquad (4.10)$$

Thus the F statistic becomes from this data

$$\frac{72}{64}\left[\frac{.7965 \times 10^{29}}{.1586 \times 10^{27}} - 1\right] = \frac{72}{64}\ [506.79] = 571.468 > F_{8,1} =$$

239 at $\alpha = .05$.

Thus H_o is rejected at $\alpha = .05$ but at $\alpha = .025$ H_o is not rejected since $F_{8,1.025} =$ 957, and $571.468 < 957$. The large difference suggests that at $\alpha \approx .035$ H_o will not be rejected. Searle's, Kempthorne's and Graybill's test for fixed effects is more liberal than Scheffé's test.

Whether the added calculations are worth the accuracy is up to the experimenter. Most texts do not mention Hotelling's T^2 statistic. Instead they use an F test equivalent to Searle's. If Searle's model is used there is no worry about having an exact test. The advantage to using Hotelling's T^2 statistic is that the power can be readily calculated. (For reference on power calculations and

contrasts see Scheffé (1959)). Since most researchers have access to computers, Hotelling's T^2 statistic does not pose that much of a problem in computation and should be used. However the T^2 statistic can only be used when $J \leq I$, and in fact J should be greater than I so as to "deflate" the F value.

In investigating variance component estimation Table 4.8 will be useful. To estimate σ^2_{AB} and $\sigma^2_{\tau\beta}$, the same

Table 4.8 Scheffé's and Searle's Expected Mean Squares

M.S.	Scheffe's EMS	Searle's EMS
5665.7	$\frac{36}{8}\sum\alpha_i^2 + 4\sigma^2_{AB} + \sigma^2$	$\frac{36}{8}\sum\tau_i^2 + 4\sigma^2_{\tau\beta} + \sigma^2$
350.5	$36\sigma^2_{\beta} + \sigma^2$	$36\sigma^2_{\beta} + 4\sigma^2_{\tau\beta} + \sigma^2$
308.1	$4\sigma^2_{AB} + \sigma^2$	$4\sigma^2_{\tau\beta} + \sigma^2$
18.97	σ^2	σ^2

formulas or combinations of M.S. are used

$$\sigma^2_{AB} = \sigma^2_{\tau\beta} = \tfrac{1}{4}\left[\text{MS interaction} - \text{MSE}\right]$$
$$= \tfrac{1}{4}\left[308.1 - 18.97\right]$$
$$= \frac{289.13}{4} = 72.26.$$

To estimate σ^2_{β} the following linear combinations of M.S. is used

$$\sigma^2_{\beta} = 1/36\left[\text{MSB} - \text{MSAB}\right]$$
$$= 1/36\left[350.5 - 308.1\right]$$
$$= 1/36\ (42.2) = 1.18.$$

σ^2_B is estimated as

$$\begin{aligned}\sigma_B^2 &= 1/36\ [MS\,B - MSE] \\ &= 1/36\ [350.5 - 18.97] \\ &= 1/36\ [331.53] = 9.21.\end{aligned}$$

It is now interesting to verify the relationship of σ_B^2 and σ_β^2

$$\sigma_B^2 = \sigma_\beta^2 + 1/9\ \sigma_{\tau\beta}^2$$

$$\begin{aligned}9.21 &= 1.18 + 1/9\ [72.26] \\ &= 1.18 + 8.03 \\ &= 9.21.\end{aligned}$$

In estimating the variance components, Scheffé's, Kempthorne's, and Graybill's models will not vary. Searle's model differs in variance component estimation and testing procedures but this is because of the way Searle defines his effects. Searle's model eliminates inexact F-tests and provides a means of calculating power for each hypothesis, but as mentioned earlier it is a very restrictive model and perhaps leads to an unmeaningful interpretation of the model components. Scheffé's model is the most "realistic" model and provides the most meaningful tests. Although Hotelling's T^2 statistic must be used to test the significance of the fixed effects, the procedure can be easily programed on a computer.

5. CONCLUSIONS AND EXTENSIONS

In chapter 3 the various models for the two-way mixed model with interaction were introduced and compared. Although developed as a special case of the finite population model, it was found that Kempthorne's model was equivalent to Graybill's model. In chapter 4 the analysis of variance expected mean squares and tests of significance were examined for Scheffé's model, Kempthorne's and Graybill's model, and Searle's model. It was found that Scheffé's model has effects defined in a meaningful way, is completely general, and gives exact tests. The test for fixed effects, though, is somewhat tedious to compute and, in fact, can not be applied when the number of levels sampled for the random effect is less than that of the fixed factor. In two-way situations, the T^2 statistic is not overly difficult to calculate so that whenever possible Scheffé's model should be used.

In situations where Scheffé's model can not be applied, then Kempthorne's and Graybill's model should be used, since, like Scheffé, the effects are defined in a meaningful way. Although the assumptions on the variance-covariance structure $\{\sigma_{ii'}\}$ makes their model somewhat restrictive, the tests of significance are straightforward and simple for all sources of variation. If the assumptions

do not hold, then at least the test for fixed effect differences must be considered as approximate. The example of section 4.3 provides an indication as to the adequacy of the approximation where it was found to be only slightly more liberal than Hotelling's T^2 statistic. Little work has been done in comparing the F-test to the T^2 statistic since the task grows more cumbersome as the number of effects and the model complexity increases.

As in Kempthorne's and Graybill's model, Searle's model places assumptions on the variance-covariance structure $\{\sigma_{ii'}\}$, but additionally the "effects" are defined in a way which is conceptually difficult to visualize. Provided the assumptions hold, the tests resulting from consideration of expected mean squares are exact and easy to perform, the problem being that the tests constructed are for mathematical simplicity. The primary appeal of Searle's model lies in its consistency with models often used in unbalanced situations. Consequently, his model should not be used where the alternative models of Scheffé or Kempthorne and Graybill can be used.

In modeling higher order experiments, Scheffé's technique again offers meaningful definitions of effects and is general, but tests of significance for fixed effects become so numerically complex that Hotelling's

T^2 statistic is no longer practical. As Scheffé (1959:288-289) remarks, " ... if in a mixed model two or more of the factors have random effects the use of Hotelling's T^2 statistic ... is unlikely ever to be applied in practice.".

Kempthorne's and Graybill's modeling techniques are much more adaptable to higher order situations than Scheffé's. All effects are meaningfully defined, but similar assumptions as used in the two-way model are applied to restrict the variance-covariance matrix in higher order models. Expected means squares are calculated according to the rules proposed by Schultz (1955). Under the assumptions on the variance-covariance structure, all tests are exact or approximate F-tests based upon Satterthwaite's (1946) procedure.

Searle's modeling techniques are also easily extendable to higher order models, but as in the two-way model, there is difficulty in interpreting the definitions of effects. The same restrictive assumptions imposed upon the variance-covariance structure by the two-way model are extended to higher order models. Expected mean squares are based upon rules proposed by Henderson (1959, 1969). All tests, provided the assumptions hold, are exact F-tests or, as in Kempthorne and Graybill, approximate F-tests based upon Satterthwaite's (1946) work.

While Scheffé's modeling techniques are preferred for two-way modeling situations, they can not be reasonably applied to higher order situations since the resulting tests on the fixed effects are too complex. Kempthorne's and Graybill's modeling techniques have similar restrictive assumptions in order for resulting tests to be exact. It is felt that the approximations resulting from either case do not have large discrepencies from exact tests where the restrictions are not imposed, as seen in the two-way model example of section 4.3. Searle's model is consistent with models often used in the analysis of unbalanced data sets. However, the Kempthorne-Graybill modeling technique is recommended for the analysis of higher order models with balanced data sets because of the advantage in having meaningful definition of each effect.

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