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T-Duality and Double Field Theory

Nicholas T. King
Virginia Commonwealth University

T-DUALITY AND DOUBLE FIELD THEORY

A Thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

NICHOLAS KING

B.S. in Physics, Virginia Commonwealth University - August 2011 to May 2013

Director: Dr. Marco Aldi,

Assistant professor, Department of Mathematics

Virginia Commonwewalth University

Richmond, Virginia

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The purpose of this thesis is to study a symmetry of string theory known as T-duality. We focus on a particular example establishing the equivalence between a quantized string moving in a circular space of radius R and a dual string moving in a similar space of radius $\frac{1}{R}$. We will show that this duality implies that the momentum of the string in one picture becomes the number of times the string is wound around the circle in the dual picture. We present two proofs of T-duality. The first reflects the standard interpretation of T-duality as an isomorphism of quantum theories. The second approach is based on Hull's Double Field Theory.

CHAPTER 1

INTRODUCTION

String theory is arguably one of the most fundamental contributions to contemporary theoretical high energy physics. While originally formulated as an unsuccessful attempt to describe the strong force, string theory turned out to offer a solution to a much more ambitious problem: formulating a quantum theory of gravity [1]. While string theory is far from definitive (for example, there are many different ways to get the standard model from string theory and no good reason to pick one over the other)[2], it has a number of appealing features, such as including the graviton. In standard quantum field theory, quantizing gravity is exceedingly difficult because the renormalization methods that are used for other quantum field theories become problematic when applied to gravity. However, in the framework of string theory, quantizing gravity becomes possible due to the assumption that particles are vibrating strings of finite length[3]. More precisely, when looking at the spectrum of the closed bosonic string, one finds that one of the physical states behaves as a massless spin-two particle, just as the graviton is predicted to be[4].

To better understand string theory and the mathematical framework of my thesis, consider a classical string with mass density μ and tension T_0 that is fixed at the ends with transverse oscillations in the y direction traveling along the length of the string in the x direction. The corresponding action [1] is

$$S = \int \left(\frac{1}{2}\mu \left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T_0 \left(\frac{\partial y}{\partial x}\right)^2 \right) dxdt\,.$$

Setting the variation $\delta S = 0$ for all variations of y, and imposing the boundary conditions, one obtains the wave equation

$$\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2-\frac{1}{2}T_0\left(\frac{\partial y}{\partial x}\right)^2=0$$

as the equation of motion. One follows a similar procedure for the closed string, the only difference being that periodic boundary conditions are used instead. The relativistic string, however, has a slightly different set up. To understand why this is, consider first a relativistic point particle. In the interest of keeping our action Lorentz invariant, one integrates over proper time, as this is a quantity that all Lorentz observers agree on. The action for a free relativistic particle is then $S=-mc^2\int d\tau$. Proper time is related to the length of the worldline by $d\tau=\frac{ds}{c}$, where ds is an infinitesimal length in spacetime. The action is then an integral over the worldline of the particle $S=-mc\int ds$ [1] . One can show that this action yields a correct Hamiltonian, conjugate momenta, and equations of motion, which are obtained by minimizing the length of the world-line. While the history of a particle is described by a line in spacetime, the history of a string is a two-dimensional surface, known as the worldsheet. The action is an integral over the worldsheet. The action of a relativistic string is known as the Nambu-Goto action, and is given by

$$S=-\frac{1}{2\pi\alpha'}\int d\tau\int d\sigma\sqrt{-\gamma}\,,$$

where τ is the time component of the worldsheet, σ is the spatial component, α' is a constant known as the "slope parameter", and γ is the determinant of the metric induced on the worldsheet by the spacetime metric i.e. the metric which describes the shape of the worldsheet once it has been embedded in physical spacetime. As a first approximation, the world sheet for a closed string is a tube, and the worldsheet for an open string is a strip. Minimizing the area of the worldsheet, one obtains the equations of motion. The Polyakov action is another form of the action that is written in terms of the "intrinsic" metric $h_{\alpha\beta}$ of the worldsheet. The Polyakov action is given by

$$S = -\frac{1}{4\,\pi\alpha'} \int_{\Sigma} d\,\sigma\, d\,\tau \sqrt{-h} h^{\alpha\beta} g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \,,$$

[1] where $g_{\mu\nu}$ is the metric of the ambient spacetime. In general, h depends on the fields X and is a dynamical variable with its own equations of motion. By varying the action, one can find the equations of motion for h in terms of γ to be [1]

$$\gamma_{\alpha\beta} - \frac{1}{2} h_{\alpha\beta} (h^{\mu\delta} \gamma_{\mu\delta}) = 0 \,.$$

Since the indices μ and δ are summed over, the equation of motion establishes a proportionality between h and γ and one can show that the factor of proportionality is irrelevant when rewriting the Polyakov action in terms of γ. In terms of the induced metric, the Polyakov action becomes

$$S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{-\gamma} \gamma^{\alpha\beta} \gamma_{\alpha\beta} \,.$$

Summing over the indices, we obtain the Nambu-Goto action [1]. Thus, these two actions are classically equivalent. While the geometric interpretation of the Polyakov action is less intuitive, it is much more convenient for deriving the equations of motion. The equation of motion for a relativistic string is again the wave equation. In order to turn this into a quantum system, we impose canonical commutators on solutions of the equation of motion. In order to completely understand the system, we must know how the Hilbert space is constructed, and how the operators that represent observables commute with one another.

This thesis will focus on a particular symmetry of string theory known as T-Duality. First, it is important to briefly discuss the concept of duality in physics. Roughly speaking, a duality is a mathematical equivelance between two a priori distinct physical models. A classic example of duality is found in Maxwell's equations of Electromagnetism without sources. One can show that Maxwell's equations for a charge free space are invariant under the transformation $(E, B) \to (-B, E)$, where B and E are the magnetic and electric field strengths respectively [1]. This effectively interchanges the electric and magnetic fields. While this transformation appears to be changing the physical interpretation of the system in question, the symmetry of Maxwell's equations shows that these two seemingly distinct systems actually describes the same physics. When one considers duality in quantum mechanics, it is important to ensure that the commutation relations are preserved. As an example, consider a simple harmonic oscillator of a mass m attached to a spring of constant k, the Hamiltonian is $H = p^2/(2m) + 1/2kx^2$. The duality transformation $(m, k) \to (1/k, 1/m)$ must be accompanied by a canonical transformation such as $x \to p, p \to -x$ in order to preserve the canonical commmutation relation $[x, p] = i$. [1] Another important example of duality in string theory comes from the AdS/CFT correspondence, which shows that string theory (which includes gravity) on anti-de Sitter space can be equivalently described by a gauge (conformal) field theory in which gravity is not a fundamental force [5]. In its simplest form, T-duality (or Target duality) is a duality relating systems of strings that exist on circular target spaces of different radii, one the reciprocal of the other [6]. My project focuses on a low-energy limit of string theory, known as the Sigma Model. In string theory, the metric of the target space (the physical spacetime in which the string lives), is also a field that can vary with time [1]. However, in the sigma model, these fields decouple from the scalar fields that are maps from the worldsheet to the target manifold. In this low-energy limit, we can focus on our scalar fields and neglect the effect that these fields have on the surrounding spacetime. I will first consider a closed string confined to move on a circular space of radius R. The action for this system is obtained from the Polyakov action, and I will vary it and obtain the equations of motion. Afterwards, I will quantize the system and discuss how the Hilbert space is constructed. Next, I will compare this quantum theory to a closed string confined to move on a circular space of radius $\frac{1}{R}$. I will show that there exists a one-to-one map between the two physical pictures, which implies that the two systems are identical on the quantum level. However, while the two physical pictures do describe the same system, the physical interpretations of some of the quantities will change. For example, the momentum of the quantized string on the circle of radius R will actually correspond to the number of times the quantized string on the circle of radius $\frac{1}{R}$ is wound around the circle. This is particularly interesting because this duality only exists on the quantum level. While the number of times the string is wound around the circle naturally takes integer values, momentum is not classically discrete. Thus, while these two physical pictures do in fact describe the same physics, the physical meaning of some of the quantities will change from one picture to the other.

After the traditional approach to T-duality has been described, I will reinterpret it using Double Field Theory. First discovered by Chris Hull [7], Double Field Theory doubles the degrees of freedom by introducing fields associated with both S^1_R and $S^1_{\frac{1}{R}}$. I will show that in the double circle, T-duality is a manifest symmetry rather than a duality. Dualities are equivalences of distinct theoretical frameworks. Symmetries act on a fixed quantum theory. To obtain a consistent description at the quantum level, I will use a polarization constraint to cut down on the the degrees of freedom, and show that this will give us the Hilbert Space for the single circle. In doing so, this will show that the Hilbert space of a single circle"sits"inside the larger Hilbert Space of the double circle.

CHAPTER 2

THE QUANTUM SIGMA MODEL ON A CIRCLE

We denote circles of radius R by $S^1_R = \mathbb{R}/(2\pi R\mathbb{Z})$. We begin our discussion of T-duality by considering a cylindrical world sheet $\Sigma = S^1_1 \times \mathbb{R}$ with coordinates (s,t). Let $X = X(s,t)$ be bosonic fields thought of as maps from the world sheet to our target space i.e. $X : \Sigma \to S^1_R$.

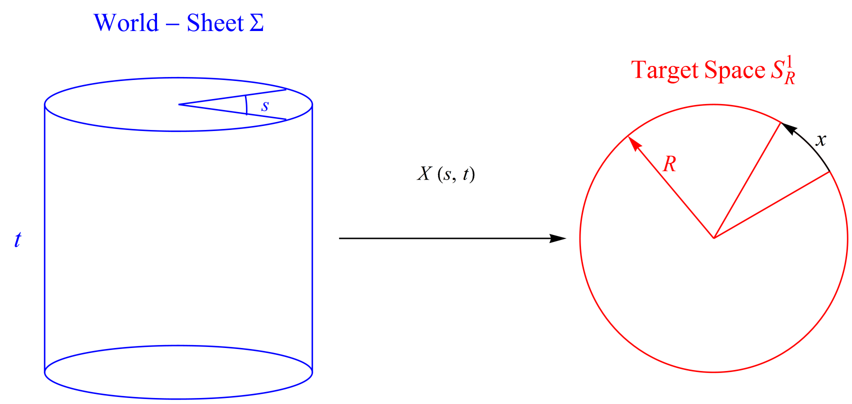

Fig. 1. The field $X(s,t)$ maps points on the worldsheet Σ to points on the target space S^1_R

In order to obtain the action for our system, we first consider the Polyakov action

The Polyakov Action

The Polyakov action is

$$S = -\frac{1}{4\,\pi\alpha'\hbar\,c^2}\int_\Sigma d\,\sigma\,d\,\tau\sqrt{-h}h^{\alpha\beta}g_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu\,. \tag{2.1}$$

Here

$$\alpha' = \frac{1}{2\,\pi\,T\,\hbar\,c}\,, \tag{2.2}$$

$h^{\alpha\beta}$ is the inverse of the intrinsic metric of the worldsheet, and $g_{\mu\nu}$ is the metric of the target manifold (in this case, the background space in which the string propagates). While in principle, the target manifold can have an arbitrary number of dimensions, we choose to focus on a one-dimensional target, namely a circle of radius R. With this specific choice of a target space in mind, our action becomes

$$S = -\frac{1}{4\,\pi\alpha'\hbar\,c^2}\int_\Sigma d\,s\,d\,t\sqrt{-h}h^{\alpha\beta}\left(\partial_\alpha X\partial_\beta X\right)\,. \tag{2.3}$$

Taking the radius of the worldsheet to be r, and keeping all the constants, a line element on the worldsheet metric h is given by

$$d\,s^2 = h_{\alpha\,\beta}d\,x^\alpha\,d\,x^\beta = r^2\,d\,s^2 - c^2 d\,t^2\,. \tag{2.4}$$

Thus, the worldsheet metric $h_{\alpha\beta}$ is given by

$$h_{\alpha\,\beta} = \begin{pmatrix} r^2 & 0 \\ 0 & -c^2 \end{pmatrix}\,, \tag{2.5}$$

with inverse

$$h^{\alpha\,\beta} = \begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & -\frac{1}{c^2} \end{pmatrix}\,. \tag{2.6}$$

Notice that these constants ensure that we have the appropriate units of length squared for each term. Substitution into the action gives

$$S = -\frac{r}{4\,\pi\alpha'\hbar\,c}\int_\Sigma d\,s\,d\,th^{\alpha\beta}\left(\partial_\alpha X\partial_\beta X\right)\,. \tag{2.7}$$

Summing over the indices α, β gives

$$S = -\frac{r}{4\,\pi\alpha'\hbar\,c}\int_\Sigma d\,s\,d\,t\left\{\frac{1}{r^2}\left(\partial_s X\right)^2 - \frac{1}{c^2}\left(\partial_t X\right)^2\right\}\,. \tag{2.8}$$

We want to make sure this has the appropriate units for an action. The action we obtained is

$$S_1 = -\frac{r}{4\,\pi\alpha'\hbar\,c}\int_\Sigma d\,s\,d\,t\left\{\frac{1}{r^2}\,(\partial_s X)^2 - \frac{1}{c^2}\,(\partial_t X)^2\right\}\,. \tag{2.9}$$

Unit analysis on the slope parameter yields

$$\alpha' \sim \frac{1}{\frac{J}{m}\,(J\,s)\,\left(\frac{m}{s}\right)} = \frac{1}{J^2}\,. \tag{2.10}$$

Thus, the term in front of the action has units

$$\frac{r}{4\,\pi\alpha'\hbar\,c} \sim \frac{m}{\left(\frac{1}{J^2}\right)(J\,s)\left(\frac{m}{s}\right)} = \frac{m}{\left(\frac{1}{J}\right)(m)} = J\,. \tag{2.11}$$

The components of the fields X are measured in meters. Keeping this in mind, the action itself has the expected units

$$S_1 \sim (J)(s)\left(\frac{m^2}{m^2}\right) = J\,s\,. \tag{2.12}$$

In this thesis, since we are not concerned with phenomenological applications, we choose a system of units where $\hbar = c = 1$, choose $\alpha' = 1$, and choose the radius of the worldsheet to be 1. We have

$$\boxed{S = -\tfrac{1}{4\pi}\int_\Sigma\ d\,s\,d\,t\left\{(\partial_s X)^2 - (\partial_t X)^2\right\}}\,. \tag{2.13}$$

The Action and its Variation

We now wish to use the action we obtained to calculate our equation of motion. The action is

$$S = \frac{1}{4\,\pi}\int_\Sigma dtds\left\{(\partial_t X)^{\,2} - (\partial_s X)^{\,2}\right\}\,. \tag{2.14}$$

A small variation of the field X gives rise to the following variation of the action

$$S + \delta S = \frac{1}{4\,\pi} \int_\Sigma dtds \left\{ \left(\partial_t \left(X + \delta \mathrm{X}\right)\right)^2 - \left(\partial_s \left(X + \delta \mathrm{X}\right)\right)^2 \right\} \, . \tag{2.15}$$

Squaring the first two terms and dropping the terms quadratic in δx gives

$$S + \delta S = \frac{1}{4\,\pi} \int_\Sigma dtds \left\{ \left(\partial_t X\right)^2 + 2\left(\partial_t X \partial_t \delta\, X\right) - \left(\partial_s X\right)^2 - 2\left(\partial_s X \partial_s \delta X\right) \right\} \, . \tag{2.16}$$

Using the Leibniz rule, we note that

$$\partial_t \left(\partial_t X \delta X\right) = \partial_t^2 X + \partial_t X \partial_t \delta X, \tag{2.17}$$

and

$$\partial_s \left(\partial_s X \delta X\right) = \partial_s^2 X + \partial_s X \partial_s \delta X \, . \tag{2.18}$$

Solving for the terms that appear in our action, we get

$$\partial_t \left(\partial_t X \delta\, X\right) - \delta\, X \partial_t^2 X = \partial_t X \partial_t \delta X \, . \tag{2.19}$$

and

$$\partial_s \left(\partial_s X \delta\, X\right) - \delta\, X \partial_s^2 X = \partial_s X \partial_s \delta X \, . \tag{2.20}$$

Substitution into the action yields

$$\begin{aligned} S + \delta S = \frac{1}{4\,\pi} \int_\Sigma dtds \Big\{ & \left(\partial_t X\right)^2 + 2\left(\partial_t \left(\partial_t X \delta\, X\right) - \delta\, X \partial_t^2 X\right) \\ & - \left(\partial_s X\right)^2 - 2\left(\partial_s \left(\partial_s X \delta\, X\right) - \delta\, X \partial_s^2 X\right) \Big\} . \end{aligned} \tag{2.21}$$

$\partial_t \left(\partial_t X \delta\, X\right)$ is a total time derivative because X depends on the independent variables s, t. Dropping the total time derivatives (since we assume them to be 0 at $\pm\infty$), we have

$$S+\delta S=\frac{1}{4\,\pi}\int_{\Sigma}dtds\left\{\left(\partial_{t}X\right)^{2}-2\left(\delta X\partial_{t}^{2}X\right)-\left(\partial_{s}X\right)^{2}-2\left(\partial_{s}\left(\partial_{s}X\delta X\right)-\delta X\partial_{s}^{2}X\right)\right\}\,. \tag{2.22}$$

Thus, the change in the action is given by the terms with $\delta\,X$. We have,

$$\delta S=\frac{1}{2\,\pi}\int_{\Sigma}dtds\left\{-\delta X\partial_{t}^{2}X-\partial_{s}\left(\partial_{s}X\delta X\right)+\delta X\partial_{s}^{2}X\right\}\,. \tag{2.23}$$

Now, recall that our world-sheet Σ is a cylinder. The total derivative with respect to s will turn into a boundary term, but since we are integrating over a closed string, the end points will have the same value and the corresponding term integrates to zero. This leaves us with

$$\delta S=\frac{1}{2\,\pi}\int_{\Sigma}dtds\left\{\delta\,X\left(\partial_{s}^{2}X-\partial_{t}^{2}X\right)\right\}\,. \tag{2.24}$$

Since this must be zero for an arbitrary variation, this gives us the wave equation

$$\boxed{\partial_{s}^{2}X-\partial_{t}^{2}X=0}\,. \tag{2.25}$$

Field Expansions

Now, recall that our spatial coordinate has periodicity $s=s+2\pi$. Because of this, we can expand solutions to the wave equation as Fourier series

$$X(s,t)=x_{0}(t)+R\,w\,s+\sum_{n\neq 0}\frac{x_{n}(t)}{n}e^{i\,n\,s}\,. \tag{2.26}$$

Here w is the winding number, which corresponds to the number of times the string is wrapped around the circle. It is important to note that the periodicity of $X(s,t)$ is

$$X(s+2\pi,t)-X(s,t)=2\pi Rw. \tag{2.27}$$

Thus, the field $X(s,t)$ is not well defined, which we will account for later when we construct the Hilbert space. Imposing the wave equation on our solution, we obtain

$$x_n(t) = x_n^+ \, e^{i\,n\,t} + x_n^- \, e^{-i\,n\,t} \qquad \text{for all} \quad n \neq 0\,. \tag{2.28}$$

The zero mode is

$$x_0(t) = x_0 + p_0 t\,. \tag{2.29}$$

Separating left and right movers, we obtain

$$X(s,t) = X^+(s+t) + X^-(s-t)\,, \tag{2.30}$$

where

$$X^\pm(s \pm t) \;= x_0^\pm + p_0^\pm(s \pm t) \;+ \sum_{n\neq 0} \frac{x_n^\pm}{n}\, e^{i\,n\,(s\pm t)}\,. \tag{2.31}$$

Matching the terms linear in t and s we obtain

$$p_0^+(s+t) \;+ p_0^-(s-t) = p_0 t + R\,w\,s\,. \tag{2.32}$$

and thus,

$$p_0^+ - p_0^- = p_0\,, \qquad p_0^+ + p_0^- = R\,w\,. \tag{2.33}$$

Adding and subtracting these two equations gives

$$p_0^\pm(s \pm t) = \frac{R\,w \pm p_0}{2}\,, \tag{2.34}$$

which implies

$$X^\pm(s \pm t) \;= x_0^\pm + \frac{R\,w \pm p_0}{2}(s \pm t) \;+ \sum_{n\neq 0} \frac{x_n^\pm}{n}\, e^{i\,n\,(s\pm t)}\,. \tag{2.35}$$

For later use, we calculate the time derivatives to be

$$\partial_t X^\pm(s \pm t) \;= \pm\frac{R\,w \pm p_0}{2} \;\pm i \;\sum_{n\neq 0} x_n^\pm\, e^{i\,n\,(s\pm t)}\,. \tag{2.36}$$

In terms of $X^{\pm}$, the Lagrangian associated with our action is

$$L = \frac{1}{4\,\pi}\int\left\{\left(\partial_t\left(X^{+}\right)\right)^2 - \left(\partial_s\left(X^{+}\right)\right)^2 + \left(\partial_t\left(X^{-}\right)\right)^2 - \left(\partial_s\left(X^{-}\right)\right)^2\ \right\} ds\,, \tag{2.37}$$

from which we calculate the conjugate momenta to be

$$P^{\pm} = \frac{\partial L}{\partial\left(\partial_t X^{\pm}\right)} = \frac{1}{2\,\pi}\partial_t X^{\pm} = \frac{1}{2\,\pi}\left(\pm\frac{R\,w \pm p_0}{2} \pm\ i\sum_{n\neq 0} x_n^{\pm}\, e^{i\,n\,(s\pm t)}\right)\,. \tag{2.38}$$

Canonical Commutators

It is at this stage that we would like to quantize our system by promoting classical observables to operators acting on a suitable Hilbert space. In standard Quantum Mechanics, one usually starts by imposing the commutator of the canonically conjugate position and momenta $[x,p] = i\hbar$. By analogy, we will impose commutation relations between our fields and their canonically conjugate momenta. We quantize our theory by imposing

$$\left[X^{\pm}\left(s'\pm t\right), P^{\pm}(s\pm t)\right] = i\hbar\delta\left(s - s'\right)\,, \tag{2.39}$$

where

$$\delta(s) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{ins}$$

is the delta function located at $s = 0$. For simplicity, we set $\hbar = 1$ in the remainder of the thesis. Substituting our expression for the momenta, we have

$$\left[X^{\pm}\left(s'\pm t\right), \partial_t X^{\pm}(s\pm t)\right] = 2\,\pi i\,\delta\left(s - s'\right)\,. \tag{2.40}$$

Let us focus on the + case. Substituting our expression for the field, this becomes

$$\left[x_0^+ + \frac{R\,w+p_0}{2}\,(s'+t) + \sum_{m\neq 0}\frac{x_m^+}{m}\,e^{i\,m\,(s'+t)}, \frac{R\,w+p_0}{2} + i\sum_{n\neq 0} x_n^+\,e^{i\,n\,(s+t)}\right]$$
$$= i\sum_{n\in\mathbb{Z}} e^{i\,n\,(s-s')}\,. \tag{2.41}$$

Taking the derivative with respect to s' on both sides, we have

$$\left[\frac{R\,w+p_0}{2} \; + i\sum_{m\neq 0} x_m^+\,e^{i\,m\,(s'+t)}, \frac{R\,w+p_0}{2} \; + i\sum_{n\neq 0} x_n^+\,e^{i\,n\,(s+t)}\right] = \sum_{n\neq 0} n\,e^{i\,n\,(s-s')}\,. \tag{2.42}$$

By linearity of the commutators, we obtain

$$i\sum_{n\neq 0}\left[\frac{R\,w+p_0}{2}\;, x_n^+\right] e^{i\,n\,(s+t)} + i\sum_{m\neq 0}\left[x_m^+, \frac{R\,w+p_0}{2}\right]\; e^{i\,m\,(s'+t)}$$
$$-\sum_{n,m\neq 0}\left[x_m^+, x_n^+\right]\; e^{i\,(m\,(s'+t)+n(s+t))} = \sum_{m\neq 0} n\,e^{i\,n\,(s-s')}\,. \tag{2.43}$$

Since the RHS has no time dependence, the LHS vanishes as a Fourier series in t . Therefore,

$$\left[\frac{R\,w+p_0}{2}\;, \sum_{n\neq 0} x_n^+\;\right]\left(e^{i\,n\,s} - e^{i\,m\,s'}\right) \; + \sum_{m\neq 0}\left[x_m^+\,, x_{n-m}^+\;\right]\; e^{i\,m\,s'}\,e^{i\,(n\,-m\,s)} = 0\,, \tag{2.44}$$

for all $m \neq 0$ and for all s, s' . The variability of the coefficient of this Fourier series in s and s' yields

$$\left[\frac{R\,w+p_0}{2}, x_n^+\right] = 0\,, \qquad \left[x_n^+, x_{m\neq -n}^+\right] = 0 \qquad \text{for all}\, m \neq 0\,. \tag{2.45}$$

Plugging this back into our canonical commutation relation, we obtain, for all integers n, m

$$\left[x_n^+, x_{-n}^+\right] = \delta_{n,-m} n\,. \tag{2.46}$$

Similarly for the X^- ,

$$\left[X^-(s'-t)\,,P^-(s-t)\right] = i\,\delta\,(s-s')\ , \tag{2.47}$$

we obtain

$$\left[\frac{R\,w-p_0}{2}, x_m^-\right] = 0 \qquad \text{for all}\, m \neq 0\,, \tag{2.48}$$

and

$$\left[x_n^-, x_m^-\right] = \delta_{n,-m} n \qquad \text{for all}\, n, m \in \mathbb{Z}\,. \tag{2.49}$$

By assumption, X^+ and X^- are decoupled, i.e.

$$\left[X^+(s'+t), X^-(s-t)\right] = 0\,. \tag{2.50}$$

If we take the derivative with respect to s and s', we would have

$$\left[\partial_{s'} X^+\,(s'+t)\,, \partial_s X^-(s-t)\right] = 0. \tag{2.51}$$

Substitution gives

$$\left[\frac{R\,w+p_0}{2} \ + i\sum_{n\neq 0} x_n^+\, e^{i\,n\,(s'+t)}, \frac{R\,w-p_0}{2} \ + i\sum_{m\neq 0} x_m^-\, e^{i\,m\,(s-t)}\right] = 0. \tag{2.52}$$

Since this must be true for all s, s' and t, we have

$$\left[x_n^+\,, x_m^-\ \right] = 0,\, n, m \neq 0\,, \tag{2.53}$$

and

$$\left[\frac{R\,w+p_0}{2}\ , \frac{R\,w-p_0}{2}\right] \ = \left[\frac{p_0 \pm R\,w}{2}, x_m^{\mp}\right] = 0,\, m \neq 0. \tag{2.54}$$

From these relations, we obtain

$$\boxed{[p_0, x_m^{\pm}] = [w, x_m^{\pm}] = [p_0, w] = 0\ }. \tag{2.55}$$

Taking only the s derivative, we find

$$\left[X^{+}\left(s'+t\right),\partial_s X^{-}(s-t)\right]=0. \tag{2.56}$$

Substitution gives us

$$\left[x_0^{+}+\frac{R\,w+p_0}{2}\left(s'+t\right)\;+\sum_{n\neq 0}\frac{x_n^{+}}{n}\,e^{i\,n\,(s'+t)},\frac{R\,w-p_0}{2}\;+i\sum_{m\neq 0}x_m^{-}\,e^{i\,m\,(s-t)}\right]=0. \tag{2.57}$$

Using the previous results, this reduces to

$$\left[x_0^{+},\frac{R\,w-p_0}{2}\right]+i\sum_{m\neq 0}\left[x_0^{+},x_m^{-}\right]e^{i\,m\,(s-t)}=0. \tag{2.58}$$

Since this must be true for all s,t, we find

$$\left[x_0^{+},\frac{R\,w-p_0}{2}\right]=\left[x_0^{+},x_m^{-}\right]=0,\ m\neq 0\,. \tag{2.59}$$

Similarly by taking the derivative with respect to s', one can show

$$\left[x_0^{-},\frac{R\,w+p_0}{2}\right]=\left[x_0^{-},x_m^{+}\right]=0,\ m\neq 0\,. \tag{2.60}$$

Substituting the commutators obtained so far in the canonical commutator (2.39) , we find

$$\left[x_0^{+},p_0^{+}\right]-i\left[x_0,\sum_{m\neq 0}x_m^{+}\,e^{i\,m\,(s+t)}\right]=\left[x_0^{+},p_0^{+}\;-i\sum_{m\neq 0}x_m^{+}\,e^{i\,m\,(s+t)}\right]=i\ . \tag{2.61}$$

Since the RHS is constant, and this must be true for all s,t, we have

$$\left[x_0,i\sum_{m\neq 0}x_m^{+}\,e^{i\,m\,(s+t)}\right]=0\,. \tag{2.62}$$

Together with a similar calculation for the X^{-}, we obtain

$$\left[x_0^{+},p_0^{+}\right]=i\,,\qquad \left[x_0^{-},p_0^{-}\right]=-i\ . \tag{2.63}$$

Since $x_0 = x_0^+ + x_0^-$ and $p_0 = p_0^+ - p_0^-$, we conclude that

$$[x_0, p_0] = 2i\,. \tag{2.64}$$

Construction of the Hilbert Space

To fully understand the quantum sigma model we need to explain how the commutators calculated so far can be redefined concretely as commutators of operators acting on a concrete Hilbert Space i.e. a complete vector space equipped with a hermitian inner product in which the state vectors (which represent the various quantum configurations of the system) live. We consider the Hilbert space of our theory and the operators that act on it a representation of the entire physical system on the quantum level. Let H be the Hilbert space of the sigma model with target space S^1_R. Since w and p_0 commute and have discrete eigenvalues, we obtain the decomposition

$$H_{\boldsymbol{w},\boldsymbol{p}} = \bigoplus_{\boldsymbol{w},\boldsymbol{p}\in\mathbb{Z}} H_{\boldsymbol{w},\boldsymbol{p}}\,. \tag{2.65}$$

$H_{\boldsymbol{w},\boldsymbol{p}}$ are eigenstates with eigenvalues $\boldsymbol{w}$ and $\boldsymbol{p}$ for w and p_0 respectively . Now, recall that $[x_0, p_0] = 2i$. The Hilbert space of the above system is constructed as a suitable completion of the space of polynomials in $x^{\pm}_{-n}$, $n > 0$ with each sector labeled by the eigenvalues of p_0 and w.

Since x_0 is not a well-defined operator, we wish to work with the well defined operator $e^{\frac{ikx_0}{R}}$ instead. Let us consider the operators $p_0, e^{\frac{ikx_0}{R}}$, $k \in \mathbb{Z}$ acting on the vacuum $|0,0\rangle \in H_{0,0}$.

By construction, p_0 acts with eigenvalue 0 on the vacuum, and thus

$$p_0 e^{\frac{ikx_0}{R}} |0,0\rangle = \left[p_0, e^{\frac{ikx_0}{R}}\right] |0,0\rangle\,. \tag{2.66}$$

If we expand $e^{\frac{ikx_0}{R}}$ in a Taylor series, we get

$$\begin{aligned} p_0 e^{\frac{ikx_0}{R}} |0,0\rangle &= \left[p_0, 1 + e^{\frac{ikx_0}{R}} + \frac{1}{2}\left(\frac{i\,k\,x_0}{R}\right)^2 + ...\right] |0,0\rangle \\ &= \left\{[p_0, 1] + \left[p_0, \frac{i\,k\,x_0}{R}\right] + \frac{1}{2!}\left[p_0, \left(\frac{i\,k\,x_0}{R}\right)^2\right] + ...\right\} |0,0\rangle \,. \end{aligned} \tag{2.67}$$

Now, for all integers n

$$[p_0, {x_0}^n] = p_0 {x_0}^n - {x_0}^n p_0 \tag{2.68}$$

$$\begin{aligned} &= [p_0, x_0]\, {x_0}^{n-1} - x_0 p_0 {x_0}^{n-1} - {x_0}^n p_0 \\ &= [p_0, x_0]\, {x_0}^{n-1} - x_0\, [x_0, p_0]\, {x_0}^{n-2} + {x_0}^2 p_0 {x_0}^{n-2} - x^n p_0 + x_0\, [\ p_0, x_0]\, {x_0}^{n-2} \\ &= 2\, [p_0, x_0]\, {x_0}^{n-1} \,. \end{aligned} \tag{2.69}$$

Applying this result to (2.67), and substituting $[p_0,\, x_0] = -2i$, we get

$$p_0 e^{\frac{i\,k\,x_0}{R}} |0,0\rangle = \left[p_0, e^{\frac{i\,k\,x_0}{R}}\right] |0,0\rangle = \frac{i\,k}{R}\frac{R}{2}\, [p_0,\, x_0]\, e^{\frac{i\,k\,x_0}{R}} |0,0\rangle = \,k e^{\frac{i\,k\,x_0}{R}} |0,0\rangle \,. \tag{2.70}$$

Therefore, we obtain the identification $e^{\frac{i\,k\,x_0}{R}}|0,0\rangle = |0,k\rangle$. More generally, since $[w, p_0] = 0$,

$$e^{\frac{i\,k\,x_0}{R}} |\boldsymbol{w}, 0\rangle = |\boldsymbol{w}, k\rangle \,. \tag{2.71}$$

Each $H_{\boldsymbol{w},\boldsymbol{p}}$ is a completed space of polynomials constructed from the $x^{\pm}_{-n}$. That is,

$$H_{\boldsymbol{w},\boldsymbol{p}} = \mathbb{C}\left[x^{\pm}_{-1}, x^{\pm}_{-2}, x^{\pm}_{-3}, ...\right] |\boldsymbol{w}, \boldsymbol{p}\rangle \,. \tag{2.72}$$

For simplicity, we may write

$$H_{\boldsymbol{w},\boldsymbol{p}} = H^{+}_{\boldsymbol{w},\boldsymbol{p}} \otimes H^{-}_{\boldsymbol{w},\boldsymbol{p}} \,. \tag{2.73}$$

Where $H^{\pm}_{\boldsymbol{w},\boldsymbol{p}}$ is generated by $x^{\pm}_{-n}\ \forall n > 1$, respectively. The vacuum $|\boldsymbol{w}, \boldsymbol{p}\rangle$ is defined such that $x^{+}_{m}|\boldsymbol{w}, \boldsymbol{p}\rangle = 0\ \forall m > 0$.

Here it is important to note that x^{+}_{n} commutes with all the other x^{+}_{m} unless

$m = -n$. For instance, the result of acting by the operator x_8^+ on the vector $x_{-1}^+ x_{-2}^+ x_{-3}^+ |\boldsymbol{w}, \boldsymbol{p}\rangle$ is

$$x_8^+ (x_{-1}^+ x_{-2}^+ x_{-3}^+) |\boldsymbol{w}, \boldsymbol{p}\rangle = x_{-1}^+ x_{-2}^+ x_{-3}^+ (x_8^+ |\boldsymbol{w}, \boldsymbol{p}\rangle) = 0 \,. \tag{2.74}$$

However,

$$x_8^+ \left(x_{-1}^+ x_{-2}^+ x_{-8}^+ |\boldsymbol{w}, \boldsymbol{p}\rangle \right) = \left[x_8^+ , x_{-8}^+ \right] \left(x_{-1}^+ x_{-2}^+ |\boldsymbol{w}, \boldsymbol{p}\rangle \right) = 8 x_{-1}^+ x_{-2}^+ |\boldsymbol{w}, \boldsymbol{p}\rangle \,. \tag{2.75}$$

At this point in our discussion, we make use of the State-Field Correspondence. This states that, for every state v in the Hilbert space $H^+_{\boldsymbol{w},\boldsymbol{p}}$, there exists a field Y_v such that

$$Y_v(z) |\boldsymbol{w}, \boldsymbol{p}\rangle = v + O \,. \tag{2.76}$$

where O represents higher order terms in z . Or

$$\left[Y_v(z) |\boldsymbol{w}, \boldsymbol{p}\rangle \right]_{z=0} = v \; + O \,. \tag{2.77}$$

Now, let $z \; = e^{-i(s+t)}$. Writing X^+ in terms of the above, we have

$$X^+(z) \; = x_0^+ + \frac{R\, w + p_0}{2} \ln(z) + \sum_{n \neq 0} \frac{x_n^+}{n} \, z^{-n} \,. \tag{2.78}$$

Now, taking the derivative with respect to z. we have

$$\partial_z X^+(z) \; = \frac{R\, w + p_0}{2} \frac{1}{z} - \sum_{n \neq 0} x_n^+ \, z^{-n-1} \,. \tag{2.79}$$

We impose the following

$$(R\, w + p_0) \, |\boldsymbol{w}, \boldsymbol{p}\rangle = 0 \,, \tag{2.80}$$

$$x_n^+ |\boldsymbol{w}, \boldsymbol{p}\rangle = 0 \;\; n > 0 \,. \tag{2.81}$$

Then applying our field $\partial_z X^+(z)$ to the vacuum gives us

$$\partial_z X^+(z)\ |\boldsymbol{w},\boldsymbol{p}\rangle = \left(\frac{R\,w+p_0}{2}\frac{1}{z} - \sum_{n\neq 0} x_n^+\, z^{-n-1}\right)|\boldsymbol{w},\boldsymbol{p}\rangle\,. \tag{2.82}$$

Applying the conditions (2.80) and (2.81), we have

$$\begin{aligned}\partial_z X^+(z)|\boldsymbol{w},\boldsymbol{p}\rangle &= -\sum_{n\neq 0} x_n^+\, z^{-n-1}|\boldsymbol{w},\boldsymbol{p}\rangle \\ &= -\left(x_{-1}^+ + x_{-2}^+ z + x_{-3}^+ z^2 + ...\right)|\boldsymbol{w},\boldsymbol{p}\rangle\,. \end{aligned} \tag{2.83}$$

Thus, if we take this when$z=0$ after the field is applied to the vacuum, we have

$$\left[\partial_z X^+(z)\ |\boldsymbol{w},\boldsymbol{p}\rangle\right]_{z=0} = -\,x_{-1}^+|\boldsymbol{w},\boldsymbol{p}\rangle\,. \tag{2.84}$$

Therefore, we can see that $\partial_z X^+(z)$ is the field corresponding to the state x_{-1}^+. The second derivative applied to the vacuum gives us

$$\begin{aligned}\partial_z^2 X^+(z)\ |\boldsymbol{w},\boldsymbol{p}\rangle &= \left(-\frac{R\,w+p_0}{2}\frac{1}{z^2} - \sum_{n\neq 0} -(n+1)x_n^+\, z^{-n-2}\right)|\boldsymbol{w},\boldsymbol{p}\rangle \\ &= -\left(x_{-2}^+ \;\; + 2x_{-3}^+\, z + ...\right)|\boldsymbol{w},\boldsymbol{p}\rangle\,. \end{aligned} \tag{2.85}$$

Taking $z=0$ after the field is applied to the vacuum

$$\left[\partial_z^2 X^+(z)|\boldsymbol{w},\boldsymbol{p}\rangle\right]_{z=0} = -x_{-2}^+|\boldsymbol{w},\boldsymbol{p}\rangle\,. \tag{2.86}$$

Therefore, we can see that $\partial_z^2 X^+(z)$ is the field corresponding to state $x_{-2}^+|\boldsymbol{w},\boldsymbol{p}\rangle$. More generally, we can see that $\partial_z^m X^+(z)$ is the field corresponding (up to scaling) to the state $x_{-m}^+|\boldsymbol{w},\boldsymbol{p}\rangle$. We now know that we can represent any particular $x_{-m}^+|\boldsymbol{w},\boldsymbol{p}\rangle$ with a field $\partial_z^m X^+(z)$, but what about general monomials? What about, for example, $x_{-1}^+ x_{-2}^+|\boldsymbol{w},\boldsymbol{p}\rangle$? If we represent each individual term with the corresponding field, this gives us

$$\partial_z X^+(z)\partial_z^2 X^+(z)|\boldsymbol{w},\boldsymbol{p}\rangle = -\partial_z X^+(z)\partial_z^2 X^+(z)|\boldsymbol{w},\boldsymbol{p}\rangle\,. \tag{2.87}$$

Substituting the expressions for the fields, we get

$$\begin{aligned}
&-\left(\partial_z X^+(z)\right)^2 \partial_z{}^2 X^+(z)|\boldsymbol{w},\boldsymbol{p}\rangle \\
&= -\left[\left(\frac{R\,w+p_0}{2}\frac{1}{z} - i\sum_{n\neq 0} x_n^+\,z^{-n-1}\right)\left(-\frac{R\,w+p_0}{2}\frac{1}{z^2} - i\sum_{m\neq 0} -(m+1)x_m^+\,z^{-m-2}\right)\right]|\boldsymbol{w},\boldsymbol{p}\rangle.
\end{aligned} \tag{2.88}$$

Taking the product of our two terms, we get

$$\begin{aligned}
&\Bigg[\left(\frac{R\,w+p_0}{2}\right)^2\frac{1}{z^3} + \sum_{n\neq 0} x_n^+\,z^{-n-1}\left(\frac{R\,w+p_0}{2}\right)\frac{1}{z^2} + \left(\frac{R\,w+p_0}{2}\right)\sum_{m\neq 0}(m+1)x_m^+\,z^{-m-2} \\
&-\sum_{n\neq 0} x_n^+\,z^{-n-1}\sum_{m\neq 0}(m+1)x_m^+\,z^{-m-2}\Bigg]|\boldsymbol{w},\boldsymbol{p}\rangle\,.
\end{aligned} \tag{2.89}$$

Splitting this up, we get

$$\begin{aligned}
&\left(\frac{R\,w+p_0}{2}\right)^2\frac{1}{z^3}|\boldsymbol{w},\boldsymbol{p}\rangle + \sum_{n\neq 0} x_n^+\,z^{-n-1}\left(\frac{R\,w+p_0}{2}\right)\frac{1}{z^2}|\boldsymbol{w},\boldsymbol{p}\rangle \\
&+\left(\frac{R\,w+p_0}{2}\right)\sum_{m\neq 0}(m+1)x_m^+\,z^{-m-2}|\boldsymbol{w},\boldsymbol{p}\rangle - \sum_{n\neq 0} x_n^+\,z^{-n-1}\sum_{m\neq 0}(m+1)x_m^+\,z^{-m-2}|\boldsymbol{w},\boldsymbol{p}\rangle\,.
\end{aligned} \tag{2.90}$$

In general, this is not well defined, as the order in which the operators are arranged is ambiguous. In order to account for this, we normally order, so that the creation operators are to the left of the anihilation operators. Leaving the last term alone and normal ordering the rest gives us

$$
\begin{aligned}
&\frac{1}{z^3}\left(\frac{R\,w+p_0}{2}\right)^2|\boldsymbol{w},\boldsymbol{p}\rangle+\frac{1}{z^2}\sum_{n\neq 0}x_n^+\,z^{-n-1}\left(\frac{R\,w+p_0}{2}\right)|\boldsymbol{w},\boldsymbol{p}\rangle\\
&+\frac{1}{z^2}\sum_{n\neq 0}x_n^+\,z^{-n-1}\left(\frac{R\,w+p_0}{2}\right)|\boldsymbol{w},\boldsymbol{p}\rangle+\sum_{m\neq 0}(m+1)x_m^+\,z^{-m-2}\left(\frac{R\,w+p_0}{2}\right)|\boldsymbol{w},\boldsymbol{p}\rangle\\
&-\sum_{n\neq 0}x_n^+\,z^{-n-1}\sum_{m\neq 0}(m+1)x_m^+\,z^{-m-2}|\boldsymbol{w},\boldsymbol{p}\rangle\,. \qquad (2.91)
\end{aligned}
$$

These operators then kill the vacuum and leave us with

$$
-\sum_{n\neq 0}x_n^+\,z^{-n-1}\sum_{m\neq 0}(m+1)x_m^+\,z^{-m-2}|\boldsymbol{w},\boldsymbol{p}\rangle\,. \qquad (2.92)
$$

Now we recall that x_m^+ will kill the vacuum if $m>0$. Thus, we have to split up the sums. We have

$$
\begin{aligned}
&-\sum_{n\neq 0}x_n^+\,z^{-n-1}\sum_{m\neq 0}(m+1)x_m^+\,z^{-m-2}|\boldsymbol{w},\boldsymbol{p}\rangle\\
&=-\sum_{n\neq 0}x_n^+\,z^{-n-1}\sum_{m>0}(m+1)x_m^+\,z^{-m-2}|\boldsymbol{w},\boldsymbol{p}\rangle-\sum_{n\neq 0}x_n^+\,z^{-n-1}\sum_{m<0}(m+1)x_m^+\,z^{-m-2}|\boldsymbol{w},\boldsymbol{p}\rangle\\
&=-\sum_{n>0}x_n^+\,z^{-n-1}\sum_{m>0}(m+1)x_m^+\,z^{-m-2}|\boldsymbol{w},\boldsymbol{p}\rangle-\sum_{n<0}x_n^+\,z^{-n-1}\sum_{m>0}(m+1)x_m^+\,z^{-m-2}|\boldsymbol{w},\boldsymbol{p}\rangle\\
&-\sum_{n>0}x_n^+\,z^{-n-1}\sum_{m<0}(m+1)x_m^+\,z^{-m-2}|\boldsymbol{w},\boldsymbol{p}\rangle-\sum_{n<0}x_n^+\,z^{-n-1}\sum_{m<0}(m+1)x_m^+\,z^{-m-2}|\boldsymbol{w},\boldsymbol{p}\rangle\,.
\end{aligned}
\qquad (2.93)
$$

Normal ordering this expression gives

$$
\begin{aligned}
&-\sum_{n>0}x_n^+\,z^{-n-1}\sum_{m>0}(m+1)x_m^+\,z^{-m-2}|\boldsymbol{w},\boldsymbol{p}\rangle\;-\sum_{n<0}x_n^+\,z^{-n-1}\sum_{m>0}(m+1)x_m^+\,z^{-m-2}|\boldsymbol{w},\boldsymbol{p}\rangle\\
&-\sum_{m<0}(m+1)x_m^+\,z^{-m-2}\sum_{n>0}x_n^+\,z^{-n-1}|\boldsymbol{w},\boldsymbol{p}\rangle-\sum_{n<0}x_n^+\,z^{-n-1}\sum_{m<0}(m+1)x_m^+\,z^{-m-2}|\boldsymbol{w},\boldsymbol{p}\rangle\,.
\end{aligned}
\qquad (2.94)
$$

Finally, we are left with

$$-\sum_{n<0} x_n^+ \, z^{-n-1} \sum_{m<0} (m+1) x_m^+ \, z^{-m-2} |\boldsymbol{w}, \boldsymbol{p}\rangle = -\sum_{n,m<0} (m+1) x_n^+ \, x_m^+ \, z^{-(m+n)-3} |\boldsymbol{w}, \boldsymbol{p}\rangle \,. \tag{2.95}$$

Thus, taking $z = 0$ the only term that survives is when $m = -2, n = -1$. We have

$$-\sum_{n,m<0} (m+1) x_n^+ \, x_m^+ \, z^{-(m+n)-3} |\boldsymbol{w}, \boldsymbol{p}\rangle = -\left((-2+1) x_{-1}^+ \, x_{-2}^+\right) |\boldsymbol{w}, \boldsymbol{p}\rangle = \left(x_{-1}^+ \, x_{-2}^+\right) |\boldsymbol{w}, \boldsymbol{p}\rangle \,. \tag{2.96}$$

Thus, by imposing normal ordering, we can represent any state vector in the Hilbert space as a field. Since the operators acting on the Hilbert space are also constructed from the $X^{\pm}$(the calculation for X^- is analogous), this means that the entire physical system can be represented by these fields. Thus, our discussing of the state fields correspondence has shown us that our entire quantum theory is generated by taking derivatives of these fields. It is worth noting that vertex operators must be used in order to avoid ambiguous eigenvalues. We will now use this fact to construct a map between the Hilbert spaces of our physical system to another.

CHAPTER 3

T-DUALITY AND THE DOUBLE CIRCLE

Now that we understand the structure of the Hilbert space of our system, we will compare it to a sigma model in which the same worldsheet is mapped to a target space $S^1_{\frac{1}{R}}$. We will begin this discussion by first constructing a map from one field to another such that the commutation relations are preserved.

T-Duality

For the circle of radius R, we have

$$X^{\pm}(z) \ = x_0^{\pm} + \frac{R\,w \pm p_0}{2}\ln(z) + \sum_{n\neq 0}\frac{x_n^{\pm}}{n}\,z^{-n}\,. \tag{3.1}$$

For a string moving on a circle with radius $\frac{1}{R}$, the field expression is (using identical steps to the ones shown in the previous chapter)

$$X^{*\pm}(z) = x_0^{*\pm} + \frac{\frac{1}{R}\,w^* \pm {p_0}^*}{2}\ln(z) + \sum_{m\neq 0}\frac{x_m^{*\pm}}{m}\,z^{-m}\,. \tag{3.2}$$

We know that in such a system, we can get the states of the Hilbert space (which represent the physical states of the system) by taking derivatives of these fields. We now wish to construct a one-to-one map from the fields on S^1_R to $S^1_{\frac{1}{R}}$. By doing this, we will show that these different fields actually describe the same Hilbert space. We begin by considering again the action for the string on S^1_R associated with the Lagrangian given in (2.37). By analogy, the action for a circle of radius $\frac{1}{R}$ would simply be

$$S=\frac{1}{4\pi}\int_{\Sigma}dtds\left\{\left(\partial_t\left(X^{+*}\right)\right)^2-\left(\partial_s\left(X^{+*}\right)\right)^2+\left(\partial_t\left(X^{-*}\right)\right)^2-\left(\partial_s\left(X^{-*}\right)\right)^2\right\}. \tag{3.3}$$

In order to construct an appropriate map between the two physical systems, we define it on the level of the fields by declaring $X^{+}\mapsto\ X^{*+}$, and $X^{-}\mapsto -X^{*-}$. While the minus sign makes no difference on the level of the action, it will ensure that the commutation relations are preserved under this map. Applying this map to our action for the string on S^1_R, we end up with

$$S'=\frac{1}{4\pi}\int_{\Sigma}dtds\left\{\left(\partial_t\left(X^{+*}\right)\right)^2-\left(\partial_s\left(X^{+*}\right)\right)^2+\left(\partial_t\left(X^{-*}\right)\right)^2-\left(\partial_s\left(X^{-*}\right)\right)^2\right\}. \tag{3.4}$$

This gives us the action for the circle of radius $\frac{1}{R}$. Let us see what the consequences of such a map are. The expressions for the fields are given by (3.1) and (3.2). By comparing the fields, we can see that $x_n^{+}\mapsto x_n^{*+}, x_n^{-}\mapsto -x_n^{*-}$. Similarly, we also have $x_0^{+}\mapsto x_0^{*+}$and $x_0^{-}\mapsto -x_0^{*-}$, which implies that $x_0^{+}-x_0^{-}\mapsto x_0^{*}$. Lastly, we have $p_0^{+}\mapsto p_0^{*+}$and $p_0^{-}\mapsto -p_0^{*-}$. This final map has a peculiar consequence, which is that the winding and momenta have been interchanged i.e. the momenta of the string on one target space maps to the winding of the string on the other. More specifically, we have

$$p_0\mapsto\ \frac{w^*}{R}\,,\quad R\,w\mapsto\ {p_0}^*\,.$$

This shows us that, while we have constructed a map showing that the two physical systems are identical, the physical quantities in one picture may have a different interpretation on the other. Now, what happens if we apply this map to the commutation relations? Recall the relations (2.55), (2.46) and (2.49). Let us apply the map to each commutator and see if the commutation relations map appropriately to what they should be for the circle of radius $\frac{1}{R}$. We have,

$$\left[p_0, x_m^{\pm}\right] = 0 \mapsto \left[\frac{w^*}{R}, \pm\; x_m^{*\pm}\right] = 0\,. \tag{3.5}$$

This simply gives us $[\, w^*,\; x_m^{*\pm}] = 0$. This is exactly what one would find if one worked out the commutators for $S^1_{\frac{1}{R}}$. Similarly, we have

$$\left[w, x_m^{\pm}\right] = 0 \mapsto \left[\frac{{p_0}^*}{R}, \pm\, x_m^{*\pm}\right] = 0\,. \tag{3.6}$$

When simplified, this gives us $[{p_0}^*, x_m^{*\pm}] = 0$. Yet again, this is exactly what one would find for $S^1_{\frac{1}{R}}$. For the higher modes, we have

$$\left[x_n^{\pm}, x_m^{\pm}\right] = \delta_{n,-m} n \;\mapsto \left[\pm x_n^{*\pm}, \pm\, x_m^{*\pm}\right] = \left[\, x_n^{*\pm},\;\; x_m^{*\pm}\right] = \delta_{n,-m} n\,. \tag{3.7}$$

Simplifying, we have

$$\left[\, x_n^{*\pm},\;\; x_m^{*\pm}\right] = \delta_{n,-m} n\,. \tag{3.8}$$

This is exactly what one would find for $S^1_{\frac{1}{R}}$. Next, we have $\left[x_0^{\pm}, p_0^{\pm}\right] = \pm i$. Applying the map, we obtain

$$\left[x_0^{+}, p_0^{+}\;\right] \mapsto \left[x_0^{*+}, p_0^{*+}\;\right]\,, \tag{3.9}$$

and

$$\left[x_0^{-}, p_0^{-}\;\right] \;\mapsto \left[-x_0^{*-}, -p_0^{*-}\;\right] = \left[x_0^{*-}, p_0^{*-}\;\right]\,. \tag{3.10}$$

This gives us the commutation relations $\left[x_0^{*\pm}, p_0^{*\pm}\;\right] = \pm i$. Using the steps from the previous section, we obtain

$$[x_0^*, p_0^*] = 2i\,, [x_0^*, w^*\;] = 0\,. \tag{3.11}$$

Thus, under this map, all of our commutation relations are preserved. With that, we know that these two physical pictures describe exactly the same thing.

The Double Circle

Now we consider bosonic fields on the worldsheet $\Sigma = S^1 \times \mathbb{R}$ and values in $S^1 \times S^1$ where the first circle has radius $\frac{1}{R}$ and the second has radius R. The target space is shown below.

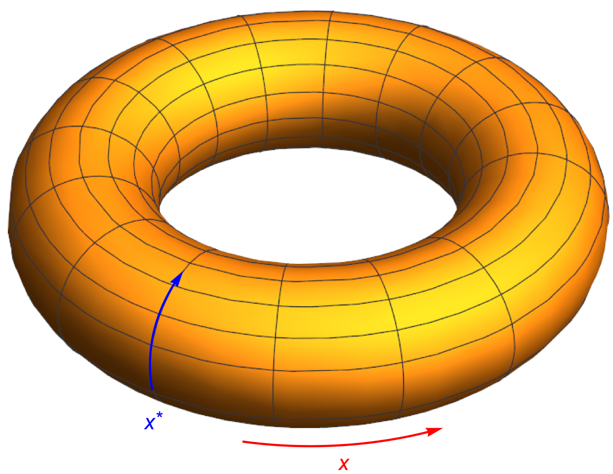

Fig. 2. Target space $S^1 \times S^1$ with target coordinates x and x^*

We start with the action

$$S = \frac{1}{8\pi} \int_{\Sigma} dtds \left\{ \left((\partial_t X)^2 - (\partial_s X)^2 \right) - \left((\partial_t X^*)^2 - (\partial_s X^*)^2 \right) \right\} . \tag{3.12}$$

The variation of the action is similar to the previous section, and gives us two wave equations

$$\left(\partial_s^2 - \partial_t^2\right) X = 0 \,, \quad \left(\partial_s^2 - \partial_t^2\right) X^* = 0 \,. \tag{3.13}$$

Decoupling $X(s,t) = X^+(s+t) + X^-(s-t)$ and $X^*(s,t) = X^{*+}(s+t) + X^{*-}(s-t)$, we obtain

$$S = \frac{1}{8\pi}\int_{\Sigma} dtds\Big\{ \left(\left(\partial_t\left(X^{+}\right)\right)^2 - \left(\partial_s\left(X^{+}\right)\right)^2 + \left(\partial_t\left(X^{-}\right)\right)^2 - \left(\partial_s\left(X^{-}\right)\right)^2\right)$$
$$- \left(\left(\partial_t\left(X^{*+}\right)\right)^2 - \left(\partial_s\left(X^{*+}\right)\right)^2 + \left(\partial_t\left(X^{*-}\right)\right)^2 - \left(\partial_s\left(X^{*-}\right)\right)^2\right)\Big\}. \quad (3.14)$$

The Fourier expansion of left and right moving solutions to the wave equation is

$$X^{\pm}(s \pm t) \; = \frac{R\,w \pm p_0}{2}(s \pm t) \; + \sum_{n \neq 0} \frac{x_n^{\pm}}{n}\, e^{i\,n\,(s \pm t)}\,, \quad (3.15)$$

$$X^{*\pm}(s \pm t) \; = \frac{\frac{w^*}{R} \pm {p^*}_0}{2}(s \pm t) \; + \sum_{n \neq 0} \frac{x_n^{*\pm}}{n}\, e^{i\,n\,(s \pm t)}\,. \quad (3.16)$$

With corresponding conjugate momenta

$$P^{\pm}(s \pm t) \; = \frac{1}{4\pi}\left(\pm\frac{R\,w \pm p_0}{2} - i\sum_{n \neq 0} \frac{x_n^{\pm}}{n}\, e^{i\,n\,(s \pm t)}\right)\,, \quad (3.17)$$

$$P^{*\pm}(s \pm t) \; = \frac{1}{4\pi}\left(\pm\frac{{p^*}_0 \pm \;\frac{w^*}{R}}{2} \; - i\sum_{n \neq 0} \frac{x_n^{*\pm}}{n}\, e^{i\,n\,(s \pm t)}\right)\,. \quad (3.18)$$

Let us return to the action of the double circle.

$$S = \frac{1}{8\pi}\int_{\Sigma} dtds\left\{\left((\partial_t X)^2 - (\partial_s X)^2\right) - \left((\partial_t X^*)^2 - (\partial_s X^*)^2\right)\right\}\,. \quad (3.19)$$

We now wish to impose a polarization constraint in order to obtain the number of degrees of freedom that we had for the sigma model with target space S^1_R . We impose the constraints $\partial_t X = \partial_s X^*$ and $\partial_s X = \partial_t X^*$. We find

$$
\begin{aligned}
S &= \frac{1}{8\pi}\int_\Sigma dtds \left\{\left((\partial_t X)^2 - (\partial_s X)^2\right) - \left((\partial_s X)^2 - (\partial_t X)^2\right)\right\} \\
&= \frac{1}{8\pi}\int_\Sigma dtds \left\{(\partial_t X)^2 - (\partial_s X)^2 - (\partial_s X)^2 + (\partial_t X)^2\right\} \\
&= \frac{1}{8\pi}\int_\Sigma dtds\, 2\left\{(\partial_t X)^2 - (\partial_s X)^2\right\} \\
&= \frac{1}{4\pi}\int_\Sigma dtds\, \left\{(\partial_t X)^2 - (\partial_s X)^2\right\}\,.
\end{aligned} \tag{3.20}
$$

This condition gives us the exact same action as the action for the circle of radius R. Thus, we can recover the action for the individual cases by imposing these constraints. This demonstrates that the Hilbert space of the individual circles "sits" in the larger Hilbert space of the double circle.

Now, let us see if this is preserved on the Quantum Mechanical level. To do this, consider the field expansion for X and X^* We have

$$X(s,t) = x_0 + p_0 t + R\,w\,s + \sum_{n\neq 0}\frac{x_n^+}{n}e^{i\,n(s+t)} + \sum_{n\neq 0}\frac{x_n^-}{n}e^{i\,n\,(s-t)}\,, \tag{3.21}$$

$$X^*(s,t) = x^*{}_0 + p^*{}_0 t + \frac{1}{R}\,w^*\,s + \sum_{n\neq 0}\frac{x_n^{*+}}{n}e^{i\,n(s+t)} + \sum_{n\neq 0}\frac{x_n^{*-}}{n}e^{i\,n\,(s-t)}\,. \tag{3.22}$$

As before, these fields are not well defined. When going around the circle, the periodicity of these fields are

$$X(s+2\pi,t) - X(s,t) = 2\pi R w\,, \tag{3.23}$$

$$X^*(s+2\pi,t) - X^*(s,t) = 2\pi\frac{1}{R}w\,. \tag{3.24}$$

Taking the s derivatives of each, we have

$$\partial_s X = R\,w\ + i\ \sum_{n\neq 0} x_n^+\, e^{i\,n(s+t)} + i\sum_{n\neq 0} x_n^-\, e^{i\,n\,(s-t)}\,, \tag{3.25}$$

$$\partial_s X^* = \frac{1}{R}\, w^* \, + i \sum_{n\neq 0} x_n^{*+}\, e^{i\,n(s+t)} + i \sum_{n\neq 0} x_n^{*-}\, e^{i\,n\,(s-t)} \,. \tag{3.26}$$

Taking the t derivatives of each, we have

$$\partial_t X = p_0 + i \sum_{n\neq 0} x_n^{+}\, e^{i\,n(s+t)} - i \sum_{n\neq 0} x_n^{-}\, e^{i\,n\,(s-t)} \,, \tag{3.27}$$

$$\partial_t X^* = {p^*}_0 + i \sum_{n\neq 0} x_n^{*+}\, e^{i\,n(s+t)} - i \sum_{n\neq 0} x_n^{*-}\, e^{i\,n\,(s-t)} \,. \tag{3.28}$$

The polarization condition $\partial_s X = \partial_t X^*$ gives us

$$R\,w + i \sum_{n\neq 0} x_n^{+}\, e^{i\,n(s+t)} + i \sum_{n\neq 0} x_n^{-}\, e^{i\,n\,(s-t)} = {p^*}_0 + i \sum_{n\neq 0} x_n^{*+}\, e^{i\,n(s+t)} - i \sum_{n\neq 0} x_n^{*-}\, e^{i\,n\,(s-t)} \,. \tag{3.29}$$

The polarization condition $\partial_t X = \partial_s X^*$ gives us

$$p_0 + i \sum_{n\neq 0} x_n^{+}\, e^{i\,n(s+t)} - i \sum_{n\neq 0} x_n^{-}\, e^{i\,n\,(s-t)} = \frac{1}{R}\, w^* + i \sum_{n\neq 0} x_n^{*+}\, e^{i\,n(s+t)} + \sum_{n\neq 0} x_n^{*-}\, e^{i\,n\,(s-t)} \,. \tag{3.30}$$

If we match the modes of the Fourier series on each side (noting that this relationship must be true for all t and s , this implies that $w^* = R\,p_0$ and ${p^*}_0 = R\,w$).

Now, let us see what commutation relations this extra structure implies . Recall that

$$[X(s,t),\ \partial_t X\,(s',t)] = 2\pi i\,\delta\,(s-s')\ . \tag{3.31}$$

Using the polarization constraint, we obtain

$$[X(s,t),\ \partial_{s'} X^*\,(s',t)] = 2\pi i\,\delta\,(s-s')\ . \tag{3.32}$$

Expanding the delta function again gives

$$[X(s,t),\ \partial_{s'} X^*\,(s',t)] = i \sum_{k\in\mathbb{Z}} e^{i\,k\,(s-s')} \,. \tag{3.33}$$

If we integrate both sides with respect to s', we obtain

$$[X(s,t), X^*(s',t)] = -\frac{i\sum_{k\neq 0} e^{i\,k\,(s-s')}}{i\,k} + c + a(s-s')\,, \tag{3.34}$$

where the linear term comes from the fact that this must be skewsymmetric. Simplifying gives

$$[X(s,t), X^*(s',t)] = -\sum_{k\neq 0} \frac{e^{i\,k\,(s-s')}}{k} + c + a\,(s-s')\ . \tag{3.35}$$

Substituting the expressions for the fields, we obtain

$$\begin{aligned}
&\Bigg[x_0 + p_0 t + R\,w\,s + \sum_{n\neq 0} \frac{x_n^+}{n} e^{i\,n(s+t)} + \sum_{n\neq 0} \frac{x_n^-}{n} e^{i\,n\,(s-t)},\\
&{x^*}_0 + {p^*}_0 t + \frac{1}{R}\,w^*\,s' + \sum_{n\neq 0} \frac{x_n^{*+}}{n} e^{i\,n(s'+t)} + \sum_{n\neq 0} \frac{x_n^{*-}}{n} e^{i\,n\,(s'-t)}\Bigg]\\
&= -\sum_{k\neq 0} \frac{e^{i\,k\,(s-s')}}{k} + c + a\,(s-s')\ .
\end{aligned} \tag{3.36}$$

Matching the modes of the Fourier series as before, we find

$$[x_0, {x^*}_0] = c, [R\,w, {x^*}_0] = a, \left[x_0, \frac{1}{R} w^*\right] = -a\,. \tag{3.37}$$

and

$$[p_0, {x^*}_0] = [x_0, {p^*}_0] = [p_0, {p^*}_0] = [w\ \ , {p^*}_0] = \left[p_0, \frac{1}{R}\,w^*\ \ \right] = \left[R\,w\ \ , \frac{1}{R}\,w^*\right] = 0\,. \tag{3.38}$$

We impose that the left and right movers obey

$$\left[X^\pm(s,t), X^{*\pm}(s',t)\right] = -\frac{\sum_{k\neq 0} e^{i\,k\,(s-s')}}{k} + c + a\,(s-s')\ . \tag{3.39}$$

Let us focus on the X^+. The expressions for these fields are

$$X^+(s+t)\ = x_0^+ + \frac{R\,w + p_0}{2}(s+t)\ + \sum_{n\neq 0} \frac{x_n^+}{n}\, e^{i\,n\,(s+t)}\,, \tag{3.40}$$

$$X^{*+}(s'+t) \;= x_0^{*+} + \frac{\frac{w^*}{R} + {p^*}_0}{2}(s'+t) \;+ \sum_{n\neq 0} \frac{x_n^{*+}}{n}\, e^{i\,n\,(s'+t)}\,. \tag{3.41}$$

Substitution gives

$$\left[x_0^+ + \frac{R\,w + p_0}{2}(s+t) \;+ \sum_{n\neq 0} \frac{x_n^+}{n}\, e^{i\,n\,(s+t)}, x_0^{*+} + \frac{\frac{w^*}{R} + {p^*}_0}{2}(s'+t) \;+ \sum_{n\neq 0} \frac{x_m^{*+}}{m}\, e^{i\,m\,(s'+t)}\right]$$
$$= -\sum_{k\neq 0} \frac{e^{i\,k\,(s-s')}}{k} + c + a\,(s-s')\;. \tag{3.42}$$

Now, let us focus on the higher modes. Knowing that the higher modes must have an exponential of the form $e^{i\,k\,(s-s')}$, this leaves us with

$$\left[\sum_{n\neq 0} \frac{x_n^+}{n}\,, \sum_{m\neq 0} \frac{x_m^{*+}}{m}\right] e^{i(n\,s+m\,s')} e^{i\,(n+m)\,t} = -\sum_{k\neq 0} \frac{e^{i\,k\,(s-s')}}{k}\,. \tag{3.43}$$

for $n=-m$ this gives us

$$-\left[\sum_{n\neq 0} \frac{x_n^+}{n}\,, \sum_{n\neq 0} \frac{x_{-n}^{*+}}{n}\right] e^{i\,n(\,s-\,s')} = -\sum_{k\neq 0} \frac{e^{i\,k\,(s-s')}}{k}\,. \tag{3.44}$$

This leaves us with

$$\sum_{n\neq 0} \frac{1}{n^2}\left[x_n^+\,, x_{-n}^{*+}\right] e^{i\,n(\,s-\,s')} = \sum_{k\neq 0} \frac{e^{i\,k\,(s-s')}}{k}\,. \tag{3.45}$$

Since n and k are summed over the same values, we have

$$\left[x_n^+\,, x_{-n}^{*+}\right] = \;\; n\,. \tag{3.46}$$

Similarly for X^-, we have

$$\left[x_n^-\,, x_{-n}^{*-}\right] = -n\,. \tag{3.47}$$

We are now going to apply the polarization condition to the commutators given to see if they are consistent with T-duality. For convenience, we list the commutation relations below

$$
\begin{aligned}
&\left[x_n^{\pm}, x_m^{*\pm}\right] = \pm\delta_{m,-n}n, [x_0, {x^*}_0] = c,\\
&[\,w, {x^*}_0] = \frac{1}{R}a, [x_0, w^*] = -R\,a,\\
&[p_0, {x^*}_0] = 0, [x_0, {p^*}_0] = 0,\\
&[p_0, {p^*}_0] = 0, [w \;\; , {p^*}_0] \;\; = 0,\\
&[p_0, \; w^* \;\;] = 0, [w, w^*] = 0.
\end{aligned}
\tag{3.48}
$$

When we imposed the polarization constraint on the fields, we found that $w^* = R\,p_0$ and ${p^*}_0 = R\,w$, $x_n^{+} = x_n^{*+}$ and $x_n^{*-} = -x_n^{-}$. Applying this to our commutators, we have

$$
\begin{aligned}
&\left[x_n^{+}, x_m^{+}\right] = \;\; \delta_{m,-n}n, \left[x_n^{-}, x_m^{-}\right] = \delta_{m,-n}n,\\
&[x_0, {x^*}_0] = c, \left[\frac{1}{R}{p^*}_0, {x^*}_0\right] = \frac{1}{R}a,\\
&[x_0, \; R\,p_0] = -R\,a, \left[\frac{1}{R}w^*, {x^*}_0\right] = 0,\\
&[x_0, R\,w] = 0, [p_0, R\,w] = 0,\\
&[w \;\; , R\,w] = 0, [p_0, \; R\,p_0] = 0, [w, R\,p_0] = 0.
\end{aligned}
\tag{3.49}
$$

Simplifying, we have

$$
\begin{aligned}
&\left[x_n^{+}, x_m^{+}\right] = \;\; \delta_{m,-n}n, \left[x_n^{-}, x_m^{-}\right] = \delta_{m,-n}n,\\
&[x_0, {x^*}_0] = c, [\,{p^*}_0, {x^*}_0] = a,\\
&[x_0, \;\; p_0] = -\,a, [w^*, {x^*}_0] = 0,\\
&[x_0, w] = 0, [p_0, \; w] = 0,\\
&[w \;\; , w] \;\; = 0, [p_0, \; p_0] = 0, [w, p_0] = 0.
\end{aligned}
\tag{3.50}
$$

The Hilbert Space of the Double Circle

Now the Hilbert Space decouples with respect to windings as

$$H = \bigoplus_{\boldsymbol{w},\boldsymbol{w^*}\in\mathbb{Z}} H_{\boldsymbol{w},\boldsymbol{w^*}} \, . \tag{3.51}$$

Our Hilbert space is a space of polynomials constructed from both the $x^{\pm}_{-n}$ and the $x^{*\pm}_{-n}$. Thus,

$$H_{\boldsymbol{w},\boldsymbol{w^*}} = \mathbb{C}\left[x^{\pm}_{-1}, x^{*\pm}_{-1}, x^{\pm}_{-2}, x^{*\pm}_{-2}, x^{\pm}_{-3}, x^{*\pm}_{-3}, \ldots\right] |w, w^*\rangle \, . \tag{3.52}$$

With more degrees of freedom, this Hilbert space is much larger than the Hilbert space of S^1_R. However, when we apply the polarization condition, we will see that the resulting Hilbert space will be the Hilbert space of S^1_R. In the previous section, we showed that the Hilbert space could be generated by taking derivatives of the left and right movers. Thus, these fields represent the entire Hilbert space of S^1_R. Similarly, the left and right movers of X and X^* represent the Hilbert space of the double circle.

The first polarization constraint is $\partial_s X = \partial_t X^*$. In terms of the left and right movers, this gives us $\partial_s X^{\pm} = \partial_t X^{*\pm}$. Substituting our expressions for the fields, we obtain

$$R\,w + i\sum_{n\neq 0} x^+_n\, e^{i\,n(s+t)} + i\sum_{n\neq 0} x^-_n\, e^{i\,n(s-t)} = \; {p_0}^* + i\sum_{m\neq 0} x^{*+}_m\; e^{i\,m(s+t)} - i\sum_{m\neq 0} x^{*-}_m\; e^{i\,m(s-t)} \, . \tag{3.53}$$

Similarly, the polarization condition $\partial_t X = \partial_s X^*$ gives us

$$p_0 + i\sum_{n\neq 0} x^+_n\, e^{i\,n(s+t)} - i\sum_{n\neq 0} x^-_n\, e^{i\,n\,(s-t)} = \frac{1}{R}\,w^* + i\sum_{n\neq 0} x^{*+}_n\, e^{i\,n(s+t)} + i\sum_{n\neq 0} x^{*-}_n\, e^{i\,n\,(s-t)} \, . \tag{3.54}$$

These constraints imply that

$$R\,w = {p_0}^*, x^+_n = x^{*+}_n, x^-_n = -x^{*-}_n, p_0 = \frac{1}{R}\,w^* \, . \tag{3.55}$$

Focusing on the X^{*+} components of X^*(the calculation for the X^{*-} being analagous), we obtain

$$X^{*+}(z) = x_0^{*+} + \frac{R\,w + p_0}{2}\ln(z) + \sum_{m\neq 0}\frac{x_m^+}{m}z^{-m}\,. \tag{3.56}$$

The state-field correspondence told us that any state v in the Hilbert space of the string on S^1_R could be identified with a field such that

$$Y_v(z)|0\rangle = v + O\,. \tag{3.57}$$

From this, we saw that ${\partial_z}^m X^+(z)$ is the field corresponding to state x^+_{-m}. Similarly, for the string on $S^1_{\frac{1}{R}}$, ${\partial_z}^m X^{*+}(z)$ is the field corresponding to state x^{*+}_{-m}. States in the double circle are represented by derivatives of both ${\partial_z}^m X^+(z)$ and ${\partial_z}^m X^{*+}(z)$. Recall that for X^+, $z = e^{-i(s+t)}$. Thus, using the chain rule, we can rewrite the s and t derivatives of X^+ in terms of the z derivative as

$$\partial_s X^+ = -i\,z\partial_z X^+,\ \partial_t X^+ = -i\,z\partial_z X^+. \tag{3.58}$$

Similarly for the X^{*+} we have

$$\partial_s X^{*+} = -i\,z\partial_z X^{*+},\ \partial_t X^{*+} = -i\,z\partial_z X^{*+}. \tag{3.59}$$

Applying the polarization constraint $\partial_s X^+ = \partial_t X^{*+}$ gives

$$-\,i\,z\partial_z X^+ = -i\,z\partial_z X^{*+}. \tag{3.60}$$

Simplifying, we obtain

$$\partial_z X^+ = \partial_z X^{*+}. \tag{3.61}$$

Similarly, from the polarization constraint $\partial_t X^+ = \partial_s X^+$, we obtain

$$\partial_z X^+ = \partial_z X^{*+}. \tag{3.62}$$

Thus, the polarization constraint implies that the states represented by $\partial_z{}^m X^+(z)$ are the same states that are represented by $\partial_z{}^m X^{*+}(z)$. These results are consistent with T-duality, where we have $X^+ \to X^{*+}$. Applying the polarization constraint to the fields that represent states in the double circle has identified states generated by the X^{*+} with states generated by the X^+, and in doing so, we have successfully recovered the Hilbert space for the string in S^1_R.

Conclusion

In this thesis, we have shown that strings confined to move on S^1_R are identical to strings confined to move on $S^1_{\frac{1}{R}}$ on the quantum mechanical level. It has been shown that the winding and momenta are interchanged when going from one theory to another. The Hilbert space is constructed from the $x^{\pm}_n, n < 0$ .The map that we used when going from one theory to another successfully gave us the Hilbert space of the other theory. We then used Double Field Theory to represent T-duality as a symmetry. When an appropriate polarization constraint is used, we have shown that we recover the action and Hilbert space that corresponds to a theory of a single circle, which shows that it sits inside the larger Hilbert space of the double circle.

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VITA

Nicholas Theodore King is an American citizen born on August 21, 1990 in Philadelphia, PA. He graduated from Atlee High school in 2008. He recieved his Bachelor of science in Physics from Virginia commonwealth University in 2013. After he recieved his Bachelor of science, he began attending graduate school and working as a Teaching Assistant at Virginia Commonwealth University.