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Rotating Supporting Hyperplanes and Snug Circumscribing Simplexes

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ROTATING SUPPORTING HYPERPLANES AND SNUG CIRCUMSCRIBING SIMPLEXES

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy (Systems Modeling and Analysis) at Virginia Commonwealth University.

by

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Abstract

ROTATING SUPPORTING HYPERPLANES AND SNUG CIRCUMSCRIBING SIMPLEXES

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Virginia Commonwealth University.

Virginia Commonwealth University, 2018.

Director: Professor José H. Dulá

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This dissertation has two topics. The first one is about rotating a supporting hyperplane on the convex hull of a finite point set to arrive at one of its facets. We present three procedures for these rotations in multiple dimensions. The first two procedures rotate a supporting hyperplane for the polytope starting at a lower dimensional face until the support set is a facet. These two procedures keep current points in the support set and accumulate new points after the rotations. The first procedure uses only algebraic operations. The second procedure uses LP. In the third procedure we rotate a hyperplane on a facet of the polytope to a different adjacent facet. Similarly to the first procedure, this procedure uses only algebraic operations. Some applications to these procedures include data envelopment analysis (DEA) and integer programming.

The second topic is in the field of containment problems for polyhedral sets. We present three procedures to find a circumscribing simplex that contains a point set in any dimension. The first two procedures are based on the supporting hyperplane rotation ideas from the first topic. The third circumscribing simplex procedure uses polar cones and other geometrical properties to find facets of a circumscribing simplex. One application of the second topic discussed in this dissertation is in hyperspectral unmixing.

CHAPTER 1

INTRODUCTION

This dissertation deals with operations and procedures involving finite point sets in multiple dimensional space. It treats two topics. First topic is about rotating supporting hyperplanes on a convex hull of a finite point set in multidimensional space until they land on a facet. The second topic is about the generation of simplexes that contain convex hulls. The results from the first topic are used to design some of the procedures in the second topic.

In the first topic of the dissertation, we present three procedures to rotate a supporting hyperplane for a polytope defined as the convex hull of a finite point set in $\Re^m$. These procedures start with a supporting hyperplane somewhere on the convex hull and end on a facet. The support set of a supporting hyperplane for a polytope contains a set of extreme points of this polytope that are located in this supporting hyperplane.

In one procedure, the dimension of the rotated hyperplane for the convex hull of a point set increases by at least one at each iteration. So, the support set for the rotated hyperplane will eventually be a facet of the polytope. This procedure uses only linear algebra operations. In the next procedure, we rotate a supporting hyperplane such that the support set is a facet after a single rotation. This procedure uses linear programming (LP). Finally, we introduce a procedure to rotate a hyperplane from one facet to another adjacent facet. This procedure does not rely on LP. This procedure is equivalent to a dual simplex pivot.

In the second topic of the dissertation, the goal is to find a special simplex that contains a point set. In order to achieve this we present three procedures to find a circumscribing simplex with special properties for a given point set. For a full dimensional polytope in m dimensions, the minimum number of facets that contain a point set and makes a polytope is $m+1$. This polytope is called a simplex. Facial decomposition of a simplex, that is, finding its facets given $m+1$ extreme points (and vice-versa), is easy.

In the first topic of the dissertation we study the problem of rotating supporting hyperplanes over convex hulls of a finite point set. We use this in the second part in procedures for enclosing convex hulls of finite point sets in m dimensions with simplexes. We develop procedures that confine a convex hull of a point set in m dimensions with a convex hull of $m+1$ points. These are snug in the sense that they intersect the contained hull in some ways.

Consider a given point set $\mathcal{A}$ such that its convex hull has full dimension. We generate a polytope that contains $\mathcal{A}$ with smallest number of facets such that the convex hull of a point set $\mathcal{A}$ is tight in the sense that m of its facets coincide with m facets of the convex hull of the point set $\mathcal{A}$, and all its facets coincide with all facets of the convex hull of the point set $\mathcal{A}$ when this is a simplex.

1.1 Convex analysis

The backdrop for this dissertation is optimization and computational geometry. These two fields use many different concepts and definitions. We review some of them in this section. In this document we consider the point set $\mathcal{A} = \{a^1, \ldots, a^n\}$ in $\Re^m$.

Definition 1.1. *A set in $\Re^m$ is convex if the line segment between any two arbitrary points in this set lies in the set [1].*

More formally, for the two different points a^p and a^q in the set, then $\lambda a^p + (1 - \lambda)a^q$ is in this set for any $\lambda \in [0, 1]$ [1].

Fig. 1. shows a convex set $\mathcal{P}_1$ and a non-convex set $\mathcal{P}_2$ in $\Re^2$.

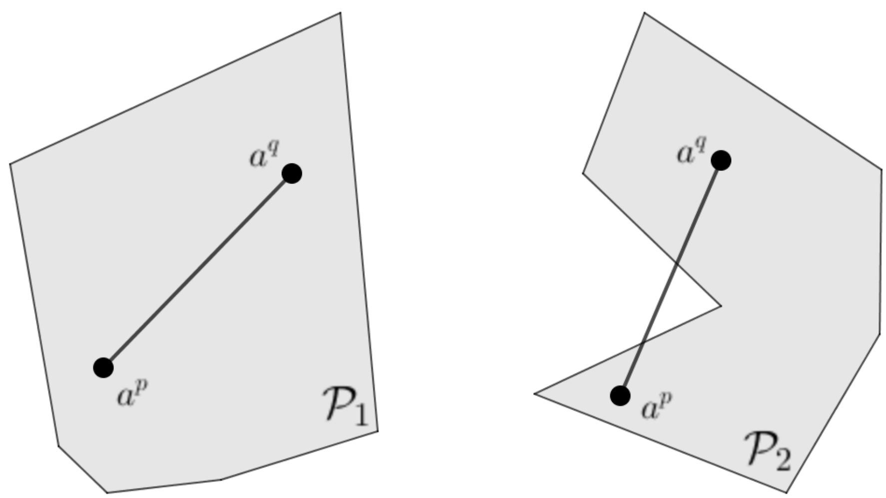

Fig. 1.: Convex set $\mathcal{P}_1$ and non-convex set $\mathcal{P}_2$

Definition 1.2. *A set in $\Re^m$ is affine if the entire line through any two points in this set lies in the set [1].*

So, for the two points a^p and a^q in the set, then $\lambda a^p + (1-\lambda)a^q$ is in this set for any $\lambda \in \Re$ [1].

Definition 1.3. *A subspace is an affine set that contains the origin [2].*

Definition 1.4. *The points $a^1, \ldots, a^n$ in $\Re^m$ are said to be linearly independent if System $\left\{\sum_{j=1}^{n} a^j \lambda_j = 0\right\}$ has a unique solution $\lambda_j = 0$ for $j = 1, \ldots, n$ [2].*

Definition 1.5. *The points $a^1, \ldots, a^n$ in $\Re^m$ are said to be affinely independent if System $\left\{\sum_{j=1}^{n} a^j \lambda_j = 0,\ \sum_{j=1}^{n} \lambda_j = 0\right\}$ has a unique solution $\lambda_j = 0$ for $j = 1, \ldots, n$ [2].*

An important result from these two recent definitions is that the points $a^1, \ldots, a^n$ in $\Re^m$ are affinely independent if and only if the vectors $a^2 - a^1, \ldots, a^n - a^1$ are linearly independent. Moreover, the maximum numbers of linearly and affinely independent points in $\Re^m$ are m and $m+1$ respectively.

Definition 1.6. *The dimension of a point set $\mathcal{A}$ in $\Re^m$ is the number of affinely independent points in $\mathcal{A}$ minus one [2], or is the dimension of the smallest affine set that contains $\mathcal{A}$.*

In $\Re^m$, the dimension of a single point, line, and plane is zero, one, and two respectively. If the dimension of the convex hull of a point set is m in $\Re^m$, then this convex hull has *full dimension*.

1.2 Polyhedral sets

There are two ways to characterize a polyhedral set: 1) as the intersection of finite halfspaces, 2) as a constrained linear vector combination of finite point sets. The former is referred to as an "external" representation the latter is said to be an "internal" representation [2]. Any polyhedral set can be expressed internally or externally. Given a characterization of a polyhedral set as either internal or external, the other characterization can be found. This is known by many names. In [3], it is known by Motzkin's representation theorem. Going from one characterization to another is referred to here as *facial decomposition*.

In the following, linear, affine, conical, and convex hulls definitions are internal representations, and polyhedron, halfspace, and hyperplane definitions are external representations.

1.2.1 Linear hull

Consider a point set $\mathcal{A} = \{a^1, \ldots, a^n\}$ in $\Re^m$. All points y define the linear hull of these n points if y is represented as [4]

$$\mathrm{lin}(\mathcal{A}) = \left\{ y \in \Re^m \;\middle|\; \sum_{j=1}^{n} a^j \lambda_j = y,\ \lambda_j \in \Re;\ j = 1, \ldots, n \right\}. \tag{1.1}$$

When the number of the linearly independent points in $\mathcal{A}$ is one, the linear hull of the point set $\mathcal{A}$ is a line. Furthermore, when the number of the linearly independent points in $\mathcal{A}$ is $m-1$, the linear hull of the point set $\mathcal{A}$ can define a hyperplane that contains the origin.

If the point set $\mathcal{A}$ has m linearly independent points, then the linear hull of the point set $\mathcal{A}$ spans $\Re^m$, so we have $\mathrm{lin}(\mathcal{A}) = \Re^m$. In other words, any point in $\Re^m$ is in the linear hull of these m linearly independent points.

Overall, if the number of the linearly independent points in $\mathcal{A}$ is k, then the dimension of $\text{lin}(\mathcal{A})$ is k. Note that the linear hull of a non-empty point set always contains the origin, therefore it is a subspace.

1.2.2 Affine hull

If sum of the multipliers used to combine the point set is 1, then this hull is said to be affine.

Consider a point set $\mathcal{A} = \{a^1, \ldots, a^n\}$ in $\Re^m$. All points y define the affine hull of these n points if y is represented as [1]

$$\text{aff}(\mathcal{A}) = \left\{ y \in \Re^m \;\middle|\; \sum_{j=1}^{n} a^j \lambda_j = y,\ \sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \in \Re;\ j = 1, \ldots, n \right\}. \tag{1.2}$$

The affine hull of the point set $\mathcal{A}$ is exactly a single point when the number of affinely independent points in $\mathcal{A}$ is one. Note that in this case, there exists just one variable λ_1 such that $\lambda_1 = 1$.

When there exists m affinely independent points in $\Re^m$, the affine hull of the point set $\mathcal{A}$ is a hyperplane. The dimension of a hyperplane is $m-1$ in $\Re^m$. In other words, a subspace of $(m-1)$-dimensional in $\Re^m$ is called a hyperplane. We will later explain some hyperplane's properties in this chapter.

An affine hull of $m+1$ affinely independent points in $\Re^m$ spans $\Re^m$.

If the number of affinely independent points in $\mathcal{A}$ is k, then the dimension of $\text{aff}(\mathcal{A})$ is $k-1$.

Note that the affine hull of the non-empty point set always contains all the points in this point set.

1.2.3 Conical hull

If all the multipliers used to combine the point set are non-negative, then this hull is said to be conical.

Consider a point set $\mathcal{A} = \{a^1, \ldots, a^n\}$ in $\Re^m$. All points y define the conical hull of these n points if y is represented as [3]

$$\text{pos}(\mathcal{A}) = \left\{ y \in \Re^m \;\middle|\; \sum_{j=1}^{n} a^j \lambda_j = y,\ \lambda_j \geq 0,\ \lambda_j \in \Re;\ j = 1, \ldots, n \right\}. \tag{1.3}$$

If the point set $\mathcal{A}$ has only one non-zero point a^1, then the conical hull of $\mathcal{A}$ is a half-line such that starts from the origin and approaches to infinity.

When the point set $\mathcal{A}$ has two affinely independent points but not linearly independent, the conical hull of $\mathcal{A}$ can be either a half-line or a line. If these two points are on the same orthant, then the conical hull of $\mathcal{A}$ is a half-line, otherwise it is a line.

If the point set $\mathcal{A}$ has m linearly independent points, then the conical hull of $\mathcal{A}$ is a cone among m vectors such that each vector contains one of the linearly independent points, and the origin is the vertex of this cone.

If $\text{pos}(\mathcal{A}) = \Re^m$, then the point set $\mathcal{A}$ has $m + 1$ affinely independent points.

Similarity to the linear hull, the conical hull of the non-empty point set always contains the origin and all the points in this point set. Note that the conical hull of a point set is always an unbounded region.

1.2.4 Convex hull

If sum of the multipliers used to combine the point set is one, and all these multipliers are non-negative, then this hull is said to be convex.

Consider a point set $\mathcal{A} = \{a^1, \ldots, a^n\}$ in $\Re^m$. All points y define the convex hull of these n points if y is represented as [1]

$$\text{con}(\mathcal{A}) = \left\{ y \in \Re^m \;\middle|\; \sum_{j=1}^{n} a^j \lambda_j = y,\ \sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geq 0,\ \lambda_j \in \Re;\ j = 1, \ldots, n \right\}. \quad (1.4)$$

Unlikely to a conical hull, the convex hull of a point set is a bounded region always. Furthermore, the convex hull of a point set in $\Re^m$ contains the origin if and only if the conical hull of them spans $\Re^m$.

If the point set $\mathcal{A}$ has $m+1$ affinely independent points, then $\text{con}(\mathcal{A})$ is a full dimension body in $\Re^m$. Overall, when the number of affinely independent points in $\mathcal{A}$ is k, the dimension of $\text{con}(\mathcal{A})$ is $k-1$.

1.2.5 Externally characterized polyhedral sets

A non-empty set $\mathcal{P} \subset \Re^m$ is a polyhedron if there is a system of finitely many inequalities $\langle \pi^j, x \rangle \leq \beta^j$ for $j = 1, \ldots, n$ such that

$$\mathcal{P} = \left\{ x \in \Re^m \;\middle|\; \langle \pi^j, x \rangle \leq \beta^j;\ \pi^j \in \Re^m,\ \beta^j \in \Re : \quad j = 1, \ldots, n \right\}. \quad (1.5)$$

A polyhedron can be bounded or unbounded and can have zero to m dimensions. A bounded polyhedron is called a polytope [5].

Fig. 2. shows two polyhedra, one unbounded polyhedron $\mathcal{P}_1$, and one bounded polyhedron $\mathcal{P}_2$ in $\Re^2$.

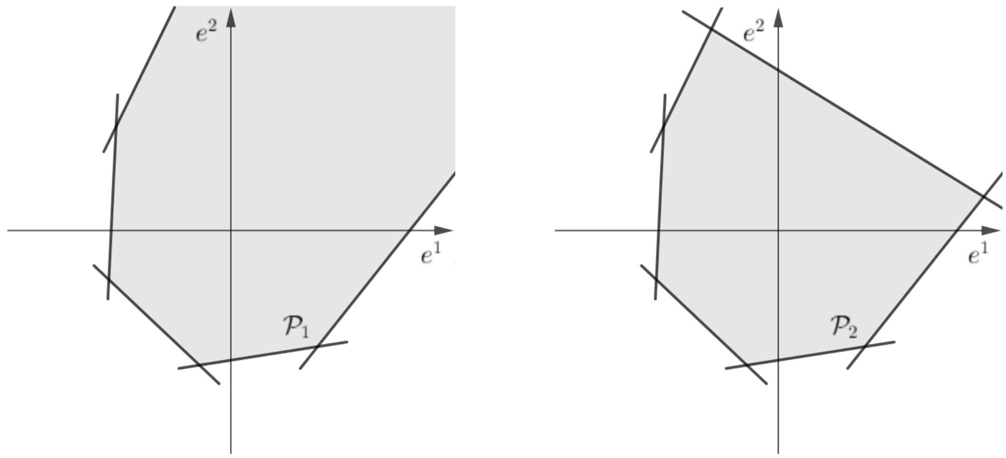

Fig. 2.: Polyhedron: Two polyhedra: one unbounded polyhedron $\mathcal{P}_1$, and one bounded polyhedron $\mathcal{P}_2$ in $\Re^2$.

Note that any polyhedron is convex [6].

1.2.6 Polytope

A bounded polyhedron is called a polytope [7].

The convex hull of a finite point set is a polytope [8].

A Polytope in $\Re^2$ is called *polygon* [7]. Polygon is a bounded region of a plane that is bounded with finite straight lines [5]. A regular polygon is a polygon that all its angles are same, and all its facets are same too [9].

1.2.7 Hyperplane

A hyperplane is an affine set. A hyperplane $\mathcal{H}(\pi, \beta)$ external representation is [1]

$$\mathcal{H}(\pi, \beta) = \left\{ x \in \Re^m \;\middle|\; \langle \pi, x \rangle = \beta;\ \pi \in \Re^m,\ \beta \in \Re \right\}. \tag{1.6}$$

A hyperplane is a point, a line, and a plane in $\Re$, $\Re^2$, and $\Re^3$ respectively. There exists a unique hyperplane that contains m affinely independent points in $\Re^m$. As with any polyhedral set, there is also an internal representation of a hyperplane.

1.2.8 Halfspace

A halfspace $\mathcal{H}^+(\pi, \beta)$ is the set of all points such that [2]

$$\mathcal{H}^+(\pi, \beta) = \left\{ x \in \Re^m \;\middle|\; \langle \pi, x \rangle \geq \beta;\ \pi \in \Re^m,\ \beta \in \Re \right\}. \tag{1.7}$$

Same way, a halfspace $\mathcal{H}^-(\pi, \beta)$ is the set of all points such that

$$\mathcal{H}^-(\pi, \beta) = \left\{ x \in \Re^m \;\middle|\; \langle \pi, x \rangle \leq \beta;\ \pi \in \Re^m,\ \beta \in \Re \right\}. \tag{1.8}$$

Fig. 3. shows two halfspaces $\mathcal{H}^+(\pi, \beta)$ and $\mathcal{H}^-(\pi, \beta)$.

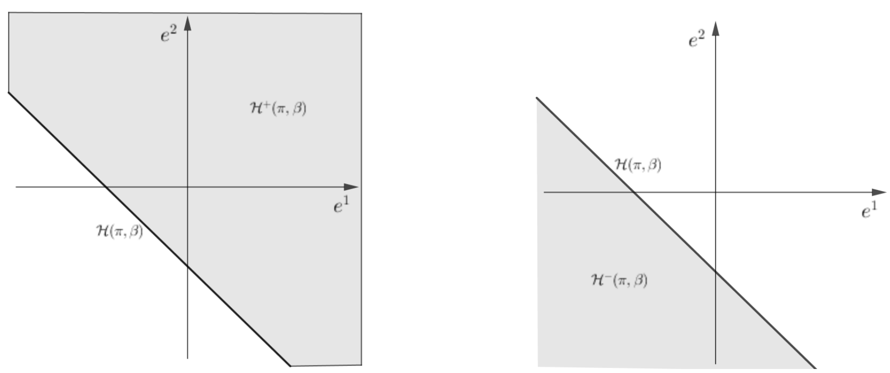

Fig. 3.: Halfspace: The shadow area in the left figure shows the halfspace $\mathcal{H}^+(\pi, \beta)$, and in the right figure shows the halfspace $\mathcal{H}^-(\pi, \beta)$.

One important remark is that any halfspace is convex [10].

1.3 Computational geometry

Computational geometry is the study of polyhedral sets defined either externally or internally. Computational geometry traces its beginnings to the problems that arise from facial decomposition of polyhedral sets especially focused in the case of convex hulls and polytopes. Computational geometry identifies geometrical properties of polyhedral sets to design algorithms and extract information to solve a problem.

The field of computational geometry was shaped by combining two fields of algorithms design and analysis in the 1970's [11], grew quickly in the 1980's and 1990's, and is still developing [12].

There are many applications in the field of computational geometry [13]. Some important applications of computational geometry are LP, computer graphics, numerical analysis, geographic information systems, and robotics ([11], [13], [14])

1.3.1 Separating hyperplane

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two nonempty disjoint convex sets, and $\langle \pi, a^0 \rangle \leq \beta$ for all $a^0 \in \mathcal{P}_1$ and $\langle \pi, a^0 \rangle \geq \beta$ for all $a^0 \in \mathcal{P}_2$. The hyperplane $\mathcal{H}(\pi, \beta)$ is called a separating hyperplane for these two sets $\mathcal{P}_1$ and $\mathcal{P}_2$ [1]. If we have $\langle \pi, a^0 \rangle < \beta$ for all $a^0 \in \mathcal{P}_1$ and $\langle \pi, a^0 \rangle > \beta$ for all $a^0 \in \mathcal{P}_2$, then the hyperplane $\mathcal{H}(\pi, \beta)$ is called a strict separation of the these two sets $\mathcal{P}_1$ and $\mathcal{P}_2$ [1].

Fig. 4. shows two convex sets $\mathcal{P}_1$ and $\mathcal{P}_2$, and their separation hyperplane $\mathcal{H}(\pi, \beta)$.

We conclude that the value of $\langle \pi, a^0 \rangle - \beta$ is non-positive on $\mathcal{P}_1$ and non-negative on $\mathcal{P}_2$ [1].

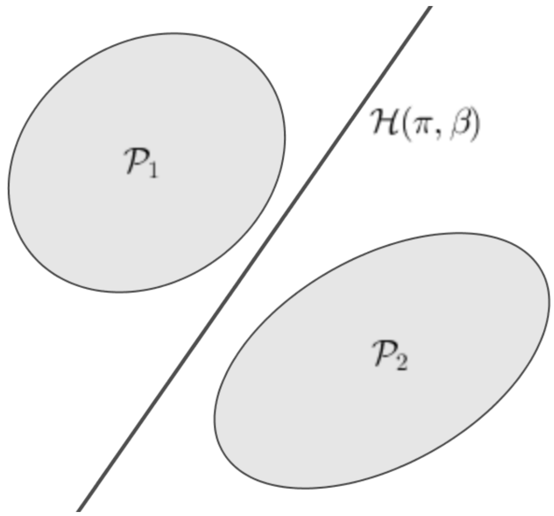

Fig. 4.: Separation hyperplane: Two convex sets $\mathcal{P}_1$ and $\mathcal{P}_2$, and their separation hyperplane $\mathcal{H}(\pi, \beta)$.

For any two nonempty disjoint convex sets $\mathcal{P}_1$ and $\mathcal{P}_2$, there exists a $\pi \neq 0$ and a β such that $\langle \pi, a^0 \rangle \leq \beta$ for all $a^0 \in \mathcal{P}_1$ and $\langle \pi, a^0 \rangle \geq \beta$ for all $a^0 \in \mathcal{P}_2$ [1].

To construct a separating hyperplane between two convex sets, assume the closest point from convex set $\mathcal{P}_1$ to convex set $\mathcal{P}_2$ is a^1, and the closest point from convex set $\mathcal{P}_2$ to convex set $\mathcal{P}_1$ is a^2. The hyperplane $\mathcal{H}(\pi, \beta)$ that is perpendicular to the line segment a^1a^2, and divide it to exactly two parts, is a separating hyperplane for these two sets $\mathcal{P}_1$ and $\mathcal{P}_2$ [1].

Fig. 5. shows two convex sets $\mathcal{P}_1$ and $\mathcal{P}_2$, and their separation hyperplane $\mathcal{H}(\pi, \beta)$ that is perpendicular to a^1a^2.

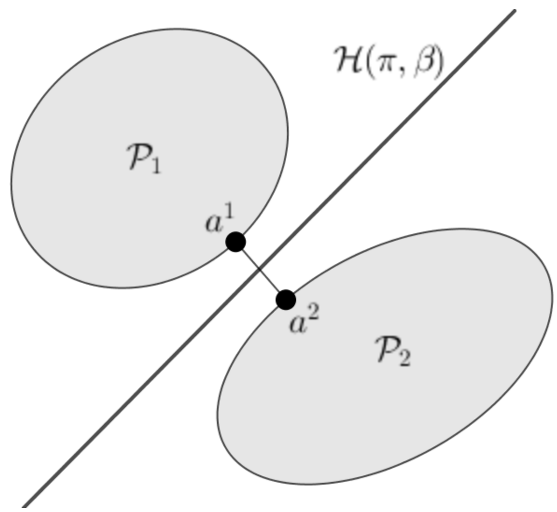

Fig. 5.: Constructing a separation hyperplane: Two convex sets $\mathcal{P}_1$ and $\mathcal{P}_2$, and their separation hyperplane $\mathcal{H}(\pi, \beta)$ that is perpendicular to $a^1 a^2$.

1.3.2 Supporting hyperplane

Consider a convex set $\mathcal{P}$ in $\Re^m$, and a^0 is a point in its boundary. If we have $\langle \pi, a^j \rangle \leq \langle \pi, a^0 \rangle$ for all points a^j in $\mathcal{P}$, then the hyperplane $\mathcal{H}(\pi, \beta)$ is called a supporting hyperplane to $\mathcal{P}$ at the point a^0 where $\beta = \langle \pi, a^0 \rangle$ [1]. In fact, the hyperplane $\mathcal{H}(\pi, \beta)$ separates the point a^0 and the set $\mathcal{P}$, and the hyperplane $\mathcal{H}(\pi, \beta)$ is tangent to $\mathcal{P}$ at a^0 [1]. Note that the halfspace $\mathcal{H}^-(\pi, \beta)$ contains $\mathcal{P}$.

Fig. 6. shows the supporting hyperplane $\mathcal{H}(\pi, \beta)$ at $a^0 \in \mathcal{P}$ for the convex set $\mathcal{P}$.

Consider the point set $\mathcal{A} = \{a^1, \ldots, a^n\}$ in $\Re^m$. The dimension of a supporting hyperplane $\mathcal{H}(\pi, \beta)$ for the convex hull of a point set $\mathcal{A}$ is the number of affinely independent points from $\mathcal{A}$ that are located in the hyperplane $\mathcal{H}(\pi, \beta)$ minus one. More formally, if the the hyperplane $\mathcal{H}(\pi, \beta)$ contains k affinely independent points

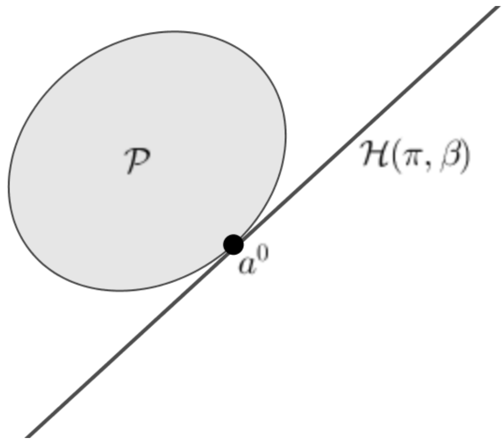

Fig. 6.: Supporting hyperplane: Supporting hyperplane $\mathcal{H}(\pi, \beta)$ at $a^0 \in \mathcal{P}$ for convex set $\mathcal{P}$.

of $\mathcal{A}$, the dimension of the hyperplane $\mathcal{H}(\pi, \beta)$ for the convex hull of a point set $\mathcal{A}$ is $k-1$. Since there are at most m affinely independent points in a hyperplane in $\Re^m$, so we always have $1 \leq k \leq m$. We conclude that the dimension of the hyperplane $\mathcal{H}(\pi, \beta)$ for the convex hull of a point set $\mathcal{A}$ is always between zero to $m-1$ [15].

For any nonempty convex set $\mathcal{P}$, and any boundary point a^0, there exists a supporting hyperplane to $\mathcal{P}$ at a^0 [1].

1.3.3 Recession cone

Consider a non-empty unbounded convex set $\mathcal{P}$ in $\Re^m$ such that the origin is not in $\mathcal{P}$. A cone $\mathcal{C}$ contains all half-lines in $\mathcal{P}$, is called a recession cone [2]. Therefore we have [16]

$$\mathcal{C} = \left\{ x \in \Re^m \;\middle|\; a^0 + \lambda x \in \mathcal{P};\; a^0 \in \mathcal{P},\; \lambda \geq 0 \right\}. \tag{1.9}$$

In Fig. 7. we show an unbounded convex polyhedron $\mathcal{P}$, and the recession cone for three arbitrary points a^1, a^2, a^3 of $\mathcal{P}$ in $\Re^2$.

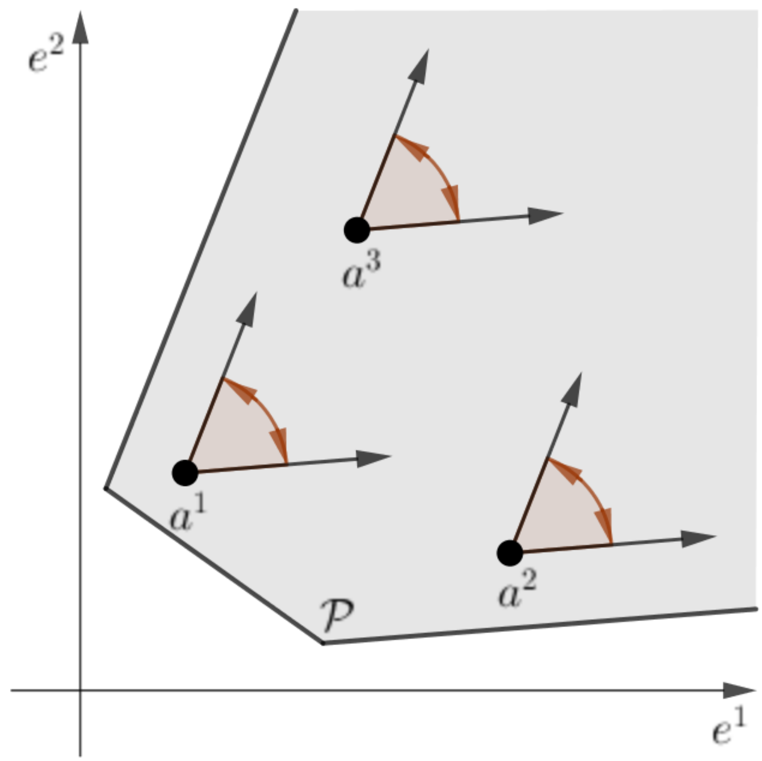

Fig. 7.: Recession cone: An unbounded polyhedron $\mathcal{P}$, and its recession cone for three arbitrary points a^1, a^2, a^3 of $\mathcal{P}$ in $\Re^2$.

Notice that all recession cones in an unbounded convex polyhedral set are the same independently of the cone's vertex.

1.4 Simplexes

A simplex is a polytope with at most $m+1$ affinely independent vertexes in $\Re^m$. A full dimension simplex in $\Re^m$ has $m+1$ facets. A full dimension simplex is a segment, triangle, and tetrahedron in $\Re, \Re^2$, and $\Re^3$ respectively. The following figures show a simplex in $\Re, \Re^2$, and $\Re^3$.

In computational geometry, if we have all vertexes of a polytope, finding its facets of a polytope can be difficult when the number of vertexes is more than $m+1$ in $\Re^m$. It becomes intractable when the number of vertexes is large. On the other hand, by having all facets of a polytope, finding its vertexes is just as difficult. We refer to this as facial decomposition.. If a polytope is a simplex, finding its facets by having its vertexes, or finding its vertexes by having its facets is simple and easy.

Fig. 8. shows three simplexes in $\Re$, $\Re^2$, and $\Re^3$.

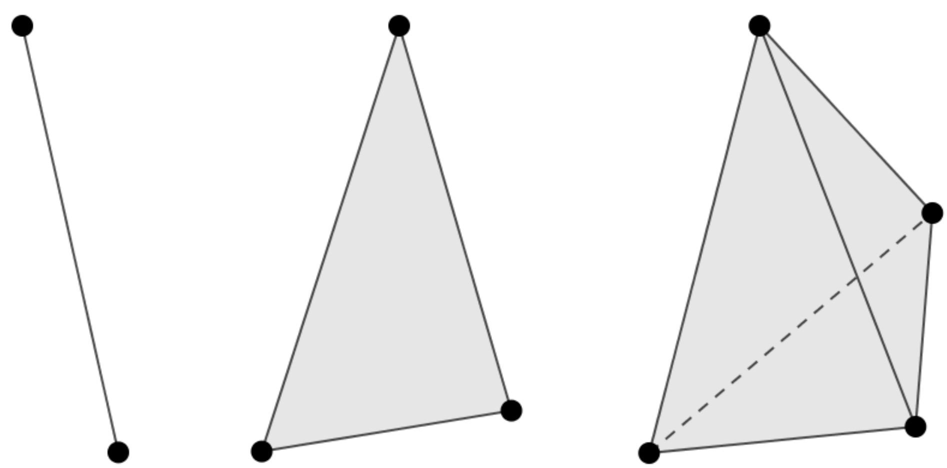

Fig. 8.: Simplex: Three simplexes in $\Re$, $\Re^2$, and $\Re^3$ from the left hand side respectively.

In this dissertation every time we mention a simplex, it refers to a full dimension simplex, unless otherwise specified.

1.4.1 Finding $m+1$ facets of a simplex, by having its vertexes

Consider a simplex with $m+1$ affinely independent vertexes $a^1, \ldots, a^{m+1}$ in $\Re^m$. To find all $m+1$ facets of this simplex, we need to solve $m+1$ systems of equations. There are $m+1$ vertexes, so any m of these vertexes generate a hyperplane that contains a facet of the simplex. Therefore, to find the normal of the hyperplane $\mathcal{H}(\pi^j, \beta)$ where $j = 1, \ldots, m+1$, we construct the following system of equations:

$$\langle a^i, \pi^j \rangle = \beta; \quad i = 1, \ldots, j-1, j+1, \ldots, m+1, \tag{1.10}$$

where β is an arbitrary nonzero scalar. Since these m vertexes $a^1, \ldots, a^{j-1}, a^{j+1}, \ldots, a^{m+1}$ are affinely independent, and this system of equations has m variables and m equations, then it has a unique solution that yields the normal of the hyperplane $\mathcal{H}(\pi^j, \beta)$ where $j = 1, \ldots, m+1$. Here we assume that no facet of this simplex contains origin.

1.4.2 Finding $m+1$ vertexes of a simplex, by having its facets

Facial decomposition finds the normal and the level of the hyperplanes that contains the facets of a polyhedron for a given collection of $m+1$ affinely independent points. The converse is also referred to as facial decomposition.

Consider a full dimension simplex in $\Re^m$. Assume $m+1$ different hyperplanes $\mathcal{H}(\pi^j, \beta^j)$ for $j = 1, \ldots, m+1$ contain the facets of a simplex. Any m different hyperplanes in $\Re^m$ make a pointed cone, so to find their intersection, it is enough to solve a system of equations. To find the vertexes a^j for $j = 1, \ldots, m+1$, we construct and solve the following system of equations.

$$\langle a^j, \pi^i \rangle = \beta^i; \quad i = 1, \ldots, j-1, j+1, \ldots, m+1. \tag{1.11}$$

1.4.3 Volume of a simplex

Consider a simplex with $m+1$ affinely independent vertexes $a^1, \ldots, a^{m+1}$ in $\Re^m$. The volume of this simplex is the absolute value of $\mathcal{V}(x)$ that is calculated as follows [17].

$$\mathcal{V}(x) = (\frac{1}{m!}) \left(det \begin{bmatrix} a_1^1 & \ldots & a_m^1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ a_1^{m+1} & \ldots & a_m^{m+1} & 1 \end{bmatrix}_{(m+1)\times(m+1)} \right). \tag{1.12}$$

Note that the $m+1$ vertexes of the simplex should be affinely independent, otherwise the value of $\mathcal{V}(x)$ is zero.

1.4.4 Polar cone

Consider k vectors $d^1, \ldots, d^k$ in $\Re^m$ the positive hull of which define a cone $\mathcal{C}$. So we have

$$\mathcal{C} = \left\{ y \in \Re^m \;\middle|\; \sum_{i=1}^{k} \lambda_i d^i = y,\ \lambda_i \geq 0;\ i = 1, \ldots, k \right\}. \tag{1.13}$$

The polar cone of $\mathcal{C}$ is given by $\mathcal{C}^*$ and defined as [2]

$$\mathcal{C}^* = \left\{ x \in \Re^m \;\middle|\; \langle x, y \rangle \leq 0,\ \forall y \in \mathcal{C} \right\}. \tag{1.14}$$

Fig. 9. shows two polar cones for a singleton vector d^1 and an obtained cone by

two vectors d^1 and d^2.

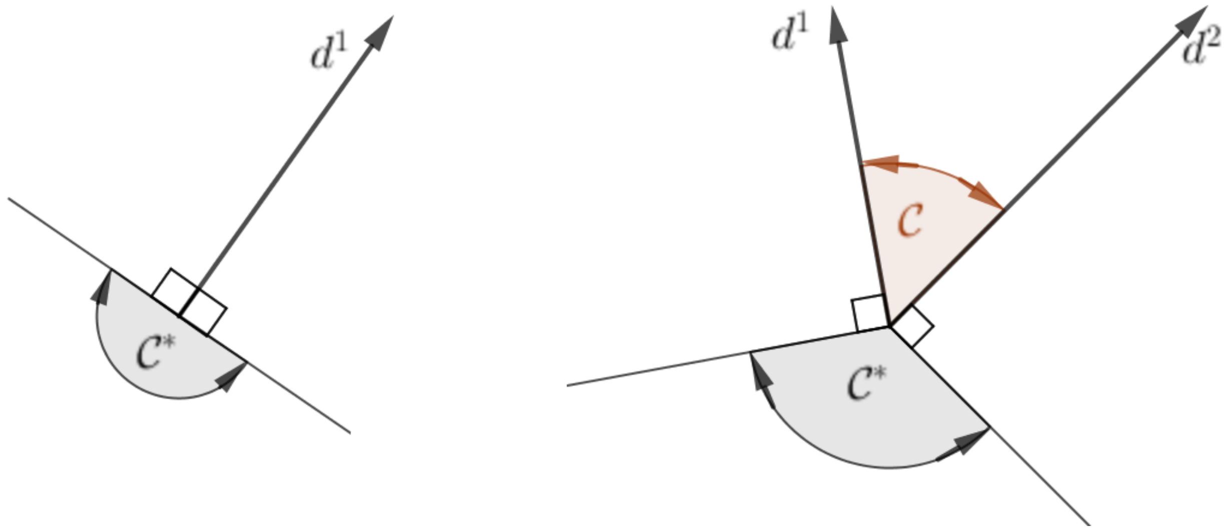

Fig. 9.: Polar Cone: Two cones and their polars in 2D: the first one for a singleton vector and the second generated by two vectors.

1.5 Degenerate facet and vertex

There needs to be m affinely independent points in $\Re^m$ to define a unique hyperplane. A facet of a polytope with more than m extreme points, is said to be degenerate. Moreover, the intersection of m non-parallel hyperplanes in $\Re^m$ defines a unique point. If an extreme point of a polytope contacts more than m facets, then this is a degenerate vertex.

1.6 Assumptions and properties

The procedures in this dissertation make use of the following assumptions. Consider a point set $\mathcal{A} = \{a^1, \ldots, a^n\}$ in $\Re^m$.

- **Assumption 1.** There exist $m + 1$ affinely independent points in $\mathcal{A}$. This means that con($\mathcal{A}$) has full dimension. This also means $n \geq m + 1$.
- **Assumption 2.** All the points of $\mathcal{A}$ are extreme for con($\mathcal{A}$) (the set $\mathcal{A}$ is its own frame). Finding the frame of a point set is a relatively easy operation.
- **Assumption 3.** There are no duplicate points in $\mathcal{A}$.
- **Assumption 4.** No face of con($\mathcal{A}$) with one or more dimensions is parallel to an axis of $\Re^m$. See remark below.
- **Assumption 5.** No two faces of con($\mathcal{A}$) with one or more dimensions are parallel. See remark below.
- **Assumption 6.** The polytope con($\mathcal{A}$) has no degenerate face.

Remark. Two affine sets are "parallel" if their uniquely defined subspaces are such that one is a subset of the other. Notice that two parallel affine sets never meet unless one is a subset of the other. However, this definition allows affine sets with different dimensions to be compared regarding this property.

1.7 Conclusion

In this chapter, we reviewed definitions and concepts that we will be using in this dissertation. This document has two main topics and both of them are under the field of computational geometry.

We described four polyhedral sets, because polyhedral sets play a central role in understanding our procedures in this document. Moreover, we explain the properties of simplexes, because the second topic of the dissertation is about finding a special circumscribing simplex such that contains a given finite point set.

Finally, we presented five assumptions to make sure that all procedures work correctly.

CHAPTER 2

LITERATURE REVIEW

In this chapter we review previous works in the two topics that we present in this document. The first section of this chapter is about the background and literature review of rotating a hyperplane. Next, we present the literature review of containment problems, and specialty snug circumscribing simplexes.

2.1 Rotating hyperplanes

Perhaps the oldest example of hyperplane rotation of a supporting hyperplane on a polyhedron comes from LP. The way we think about LP is identifying vertexes of a polyhedron and then performing operations where we move from one vertex to an adjacent vertex, until we find the vertex where the optimal solution is located. The dual version of this algorithm, known as a dual simplex proceeds in an analogous, but not identical way [18].

The dual simplex instead of generating a sequence of adjacent vertexes, generates a sequence of adjacent facets of a different polyhedron. The dual simplex algorithm starts with a full facet of the polyhedron, and we rotate this facet to an adjacent facet, and continue rotating supporting hyperplanes from one facet to an adjacent facet until the optimal solution is reached.

We will discuss in the next chapter how Procedure Facet To Facet (`FTF`) rotation can be a procedure itself, and separate from a dual simplex pivot.

López and Dulá [19] introduce a procedure about rotating a hyperplane, and use it to add and remove an attribute in a special hull of a finite point set used in Data Envelopment Analysis (DEA). They name it `HyperClimb`. `HyperClimb` procedure uses linear algebra operations to uncover extreme points after a new dimension has been added.

Consider $\mathcal{A} = \{a^1, \ldots, a^n\}$ is a point set in $\Re^m$. We define a hyperplane $\mathcal{H}(\pi, \beta) = \{y \mid \langle \pi, y \rangle = \beta\}$ where $\pi = (\pi_1, \ldots, \pi_m)$ is a non-zero vector in $\Re^m$, and $\beta \in \Re$ [1], and consider the supporting hyperplane $\mathcal{H}(\pi, \beta)$ for the convex hull of a point set $\mathcal{A}$, con($\mathcal{A}$), contains the extreme efficient decision-making unit, DMU, a^k, and the hyperplane $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$ in $\Re^{m+1}$, where $\tilde{\pi}^T = [\pi \quad \gamma]$ and $\gamma \in \Re$.

Construct the system of equations

$$\langle \pi, a^k \rangle + \gamma a^k_{m+1} = \beta \tag{2.1}$$

$$\langle \pi, a^j \rangle + \gamma a^j_{m+1} \leq \beta; \quad j = 1, \ldots, n. \tag{2.2}$$

From (2.1) and (2.2), we get

$$\gamma \leq -\frac{\langle \pi, a^j \rangle - \langle \pi, a^k \rangle}{a^j_{m+1} - a^k_{m+1}}; \quad j = 1, \ldots, n, \tag{2.3}$$

when $a^i_{m+1} - a^k_{m+1} > 0$. The maximum value of γ yields a rotation of that removes the point a^k from the support set, and add the point a^j to the support set where

$$\gamma = -\frac{\langle \pi, a^j \rangle - \langle \pi, a^k \rangle}{a^j_{m+1} - a^k_{m+1}}. \tag{2.4}$$

If there does not exist a maximum for γ, so the maximum rotation is obtained from the hyperplane $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$ supports $\text{con}(\mathcal{A})$ at a face that is orthogonal to the last axis. We figure out the level of the new supporting hyperplane from $\tilde{\beta} = \langle \tilde{\pi}, a^j \rangle$.

This procedure is applicable for unbounded polyhedra. Furthermore, the new support set does not contain all points that are in the last support set.

Example 2.1. Consider the given point set $\mathcal{A} = \{a^1 = (1,2)^T,\ a^2 = (4,5)^T,\ a^3 = (5,1)^T,\ a^4 = (2,1)^T\}$ in $\Re^2$, and the supporting vector $\mathcal{H}(\pi, 21)^T$ where $\pi^1 = (-1,5)^T$. This supporting vector contains the point a^2. Suppose we add a new attribute, and get the new set $\tilde{\mathcal{A}} = \{a^1 = (1,2,5)^T,\ a^2 = (4,5,1)^T,\ a^3 = (5,1,2)^T,\ a^4 = (2,1,4)^T\}$ that is in $\Re^3$. The new supporting hyperplane is $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$ where $\tilde{\pi} = (-1,5,\gamma)^T$. Hence, we have

$$\gamma \leq -\frac{\langle \pi, a^1 \rangle - \langle \pi, a^2 \rangle}{a^1_3 - a^2_3} = 3, \tag{2.5}$$

$$\gamma \leq -\frac{\langle \pi, a^3 \rangle - \langle \pi, a^2 \rangle}{a^3_3 - a^2_3} = 21, \tag{2.6}$$

$$\gamma \leq -\frac{\langle \pi, a^4 \rangle - \langle \pi, a^2 \rangle}{a^4_3 - a^2_3} = 6. \tag{2.7}$$

We conclude that the maximum value for γ is 3. So, we get $\tilde{\pi} = (-1,5,3)^T$, and we have

$$\langle \tilde{\pi}, a^1 \rangle = 24, \tag{2.8}$$

$$\langle \tilde{\pi}, a^2 \rangle = 24, \tag{2.9}$$

$$\langle \tilde{\pi}, a^3 \rangle = 6, \tag{2.10}$$

$$\langle \tilde{\pi}, a^4 \rangle = 15. \tag{2.11}$$

Therefore, the new plane $\mathcal{H}(\tilde{\pi}, 24)$ is a supporting plane for $\text{con}(\tilde{\mathcal{A}})$, and contains two points a^1 and a^2.

Furthermore, Dulá and Helgason [20] demonstrate how to rotate a supporting hyperplane in any dimension to identify the extreme rays of the conical hull. Let vector set $\mathcal{A} = \{a^1, \ldots, a^n\}$ in $\Re^m$ makes a pointed cone such that all vectors are non-zero, and no vector is a scaler multiple of any vector. Consider the vector set $\mathcal{A}^t = \{a^1, \ldots, a^m\}$, the average vector a^l for $\mathcal{A}^t$ is

$$a^t = \frac{1}{m} \sum_{j=1}^{m} a^j, \tag{2.12}$$

and $\tilde{\mathcal{A}}^t = \{-a^t, a^1, \ldots, a^m\}$. For a $i = m+1, \ldots, n$, solve

$$\begin{aligned} \max \quad & \langle \pi, a^i \rangle \\ \text{s.t.} \quad & -\langle \pi, a^t \rangle \leq 1, \\ & \langle \pi, a^j \rangle \leq 0; \quad j = 1, \ldots, m, \end{aligned} \tag{2.13}$$

and assume the optimal solution is π^*. Suppose σ is a vector in the vector set $\mathcal{A}^t$.

The last step is to find

$$a = \operatorname{argmax} \frac{\langle \sigma, a^j \rangle}{\langle \pi^*, a^j \rangle}; \quad j = m+1, \ldots, n. \tag{2.14}$$

The obtained vector a, is an extreme ray of the conical hull $\mathcal{A}$. Technically, a hyperplane that contains this vector, is a rotation of the hyperplane that contains some vectors of $\mathcal{A}^t$.

Example 2.2. Consider seven vectors $a^1 = (3, 1, 2)^T, a^2 = (1, 2, 1)^T, a^3 = (-4, 6, 4)^T$, $a^4 = (-1, 2/3, 2/3)^T, a^5 = (-1, -1/4, 1/2)^T, a^6 = (-9/2, -6, 3)^T,\ a^7 = (5, -15/2, 5)^T$, a vector set $\mathcal{A}^t = \{a^1, a^2, a^3\}$, and $\sigma = (1, -1, -3)^T$ is in the polar of this cone. If we solve LP

$$\begin{aligned} \max \quad & \langle \pi, a^4 \rangle \\ \text{s.t.} \quad & -\langle \pi, a^4 \rangle \leq 1, \\ & \langle \pi, a^j \rangle \leq 0, \quad j = 1, 2, 3, \end{aligned} \tag{2.15}$$

then we get $\pi^* = (7/6, -2/3, -1/3)^T$. For $i = 4, \ldots, 7$, we have

$$a = \operatorname{argmax} \left\{ \frac{36}{7},\ \frac{5}{4},\ 0,\ -\frac{27}{11} \right\} = a^4. \tag{2.16}$$

Another application of rotating hyperplanes is in the field of mixed integer program (MIP). Consider the following MIP.

$$\begin{aligned} \max \quad & cx && (2.17) \\ \text{s.t.} \quad & Ax \geq b, \\ & x \in \{0,1\}^m, \end{aligned}$$

where matrix A is $n \times m$, matrix C is $1 \times m$, and matrix b is $n \times 1$. There are 2^m disjunctive polyhedra, and the optimal solution is in one of them. By relaxing the binary requirement for x, an optimal solution is in an extreme point of the union of these 2^m polyhedra. If all elements of this point are integers, the MIP optimal solution is this point. Otherwise, this point cannot be the optimal solution of this MIP. If we cut this point off from the union of polyhedra, without missing any integer points, then it makes a tighter polyhedra that contains the MIP optimal solution. Perregaard and Balas [21] present a hyperplane rotation procedure to cut off this point. They use a linear transformation to rotate a hyperplane to generate a new facet of the convex hull of the union of polyhedra.

Consider a supporting hyperplane $\mathcal{H}(\pi^1, \beta^1)$ for the union of polyhedra $\mathcal{P}$ with $\mathcal{S}_1$ in the support set, and a hyperplane $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$ is tight for $\mathcal{S}_1$, and some points of $\mathcal{P}$ are not in a halfspace defined by this hyperplane. If the optimal solution of LP relaxation is fractional, by solving a separation problem, the hyperplane $\mathcal{H}(\pi^1, \beta^1)$ is rotated to a facet of the convex hull of the integer points in $\mathcal{P}$, and the fractional point will be cut off. Solving additional separation problem increases 1 the dimension of the support set for separating hyperplane. Thus, there are needed to solve $m-1$ separation problems in m dimensions to cut off the fractional optimal solution of the

LP relaxation. In the following example we demonstrate this procedure in $\Re^3$.

Example 2.3. Consider the system of equations $\{-x_1 - x_2 + x_3 \geq -1,\ -x_1 - x_2 - x_3 \geq -2,\ x \in \{0,1\}^3\}$ [15]. There are eight polyhedra the union of which is shown below [15].

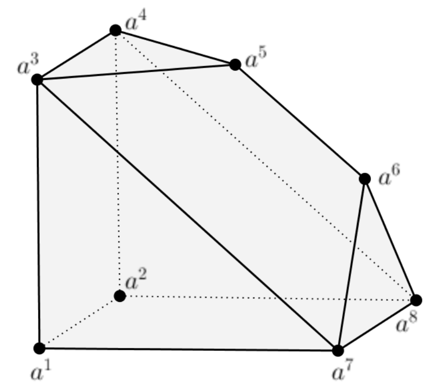

Fig. 10.: Union of polyhedra $\mathcal{P}$.

The extreme points of $\mathcal{P}$ are $a^1 = (0,0,0)^T,\ a^2 = (0,0,1)^T,\ a^3 = (0,1,0)^T,\ a^4 = (0,1,1)^T,\ a^5 = (0.5,1,0.5)^T,\ a^6 = (1,0.5,0.5)^T,\ a^7 = (1,0,0)^T$, and $a^8 = (1,0,1)^T$.

If we solve LP

$$\begin{aligned} \max_{x_1,x_2,x_3} \quad & 5x_1 + 3x_2 + 2x_3 && (2.18)\\ \text{s.t.} \quad & -x_1 - x_2 + x_3 \geq -1, \\ & -x_1 - x_2 - x_3 \geq -2, \\ & 0 \geq -x_j \geq -1, && j = 1,2,3, \end{aligned}$$

then an optimal solution is at the point $a^6 = (1, 0.5, 0.5)^T$. This non-integer point must be cut off. To do this, consider the supporting plane $\mathcal{H}(\pi^1, \beta^1)$, where $\pi^1 =$

$(4, -3, 1)^T$ and $\beta^1 = -3$. We have the initial set $\mathcal{S}_1 = \{a^3\}$ from the support set. To rotate this plane through a facet of the polyhedron $\mathcal{P}$, two separation problems should be solved. In the first LP, we take $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$, where $\tilde{\pi} = (-2, -1, 4)^T$ and $\tilde{\beta} = -1$. The separation LP problem is as follows.

$$
\begin{aligned}
\gamma^* = \max \quad & 3x_0 + 4x_1 - 3x_2 + x_3 && \qquad (2.19)\\
\text{s.t.} \quad & x_0 - x_1 - x_2 + x_3 \geq 0, \\
& 2x_0 - x_1 - x_2 - x_3 \geq 0, \\
& x_0 - x_j \geq 0, && j = 1, 2, 3, \\
& x_0 - 2x_1 - x_2 + 4x_3 = -1, \\
& (-x_1 \geq 0) \vee (-x_0 + x_1 \geq 0).
\end{aligned}
$$

The optimal value of objective function is $\gamma^* = 7$. The rotated plane $\mathcal{H}(\pi^2, \beta^2)$ is obtained by rotating the plane $\mathcal{H}(\pi^1, \beta^1)$, where $\pi^2 = \pi^1 + \gamma^* \tilde{\pi} = (-10, -10,\ 29)^T$, and $\beta^2 = \beta^1 + \gamma^* \tilde{\beta} = -10$. The support set for the new plane includes $\mathcal{S}_2 = \{a^3, a^7\}$.

Next, we need to solve one more separation LP problem to get final plane. We take $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$, where $\tilde{\pi} = (3,\ 3, -1)^T$ and $\tilde{\beta} = 3$. We formulate the second separation

LP as follows:

$$\gamma^* = \max \quad -10x_0 - 10x_1 - 10x_2 + 29x_3 \tag{2.20}$$

$$\begin{aligned} \text{s.t.} \quad & x_0 - x_1 - x_2 + x_3 \geq 0, \\ & 2x_0 - x_1 - x_2 - x_3 \geq 0, \\ & x_0 - x_j \geq 0, \qquad\qquad\qquad\qquad j = 1, 2, 3, \\ & -3x_0 + 3x_1 + 3x_2 - x_3 = -1, \\ & (-x_1 \geq 0) \vee (-x_0 + x_1 \geq 0). \end{aligned}$$

The optimal value of objective function is $\gamma^* = 29$. The rotated plane $\mathcal{H}(\pi^3, \beta^3)$ is obtained by rotating the plane $\mathcal{H}(\pi^2, \beta^2)$, where $\pi^3 = \pi^2 + \gamma^* \tilde{\pi} = (77, 77, 0)^T$, and $\beta^2 = \beta^1 + \gamma^* \tilde{\beta} = 77$. The support set for the this plane includes $\mathcal{S}_2 = \{a^3, a^4, a^7, a^8\}$. All these planes are shown in the following figures.

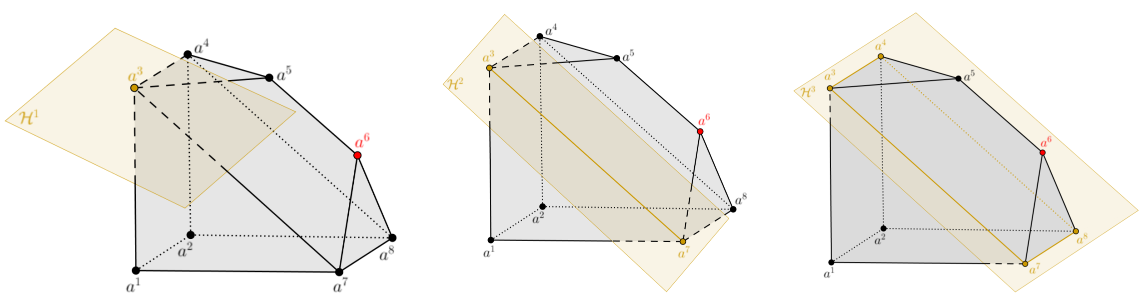

Fig. 11.: Rotating the planes: Rotating the plane $\mathcal{H}(\pi^1, \beta^1)$ through $\mathcal{H}(\pi^2, \beta^2)$, then $\mathcal{H}(\pi^2, \beta^2)$ through $\mathcal{H}(\pi^3, \beta^3)$.

The plane $\mathcal{H}(\pi^3, \beta^3)$ is a facet-defining inequality for a convex hull of the integer points of $\mathcal{P}$. If we cut off the points that are not located in a half space defined by this plane, then we get a polyhedron that all elements of its extreme points are integer. The next figure shows this polyhedron.

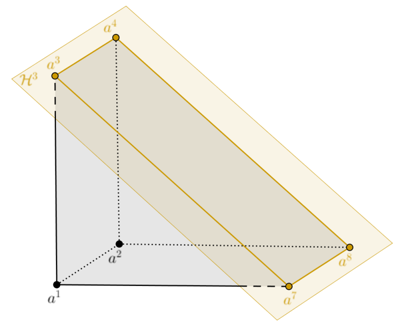

Fig. 12.: The obtained polyhedron after using cutting plane $\mathcal{H}(\pi^3, \beta^3)$.

We should emphasize that Perregaard and Balas use polyhedra defined by the intersection of halfspaces. In the current work, the polyhedron will be defined by the convex hull of a point set,

Avis and Fukuda [22] presents a procedure based on pivoting solving some important enumeration problems in the field of computational geometry in any dimension. They find the facets or vertexes of the convex hull of a point set under the assumption non-degeneracy facets.

A polytope can be defined by finite number of hyperplanes or the convex hull of its vertexes. Bremner, Fukuda, and Marzetta [23] say that the transformation from one of these two representations to the other is called vertex enumeration problem, or facet enumeration problem. They also extend two procedures: *Raindrop Algorithm* and *Dual Raindrop Algorithm* that indeed they use hyperplane rotation to solve vertex and facet enumeration problems. They use the equivalency between primal and dual problems [23].

Chand and Kapur [24] were the first to explain how to rotate from a facet of a polytope to an adjacent facet. They observed that two adjacent facets share $m-1$ extreme points. This rotation involves linear algebra operations. This sort of hyperplane rotation from one facet to another adjacent one is the principle behind *gift wrapping algorithms* for facial decomposition [24]. Seidel makes the observation that pivoting operations are dually related to *gift-wrapping* [25] which is equivalent facet to facet rotation.

Chan [26] presents a procedure in $\Re^2$ and $\Re^3$ to rotate a supporting hyperplane containing a facet of a polytope to its adjacent facet.

Hyperplane rotation has applications in different fields. Natwichai and Li [27] use hyperplane rotation to construct a procedure for drifting in data streams. Bounsiaret al. [28] to find the best separating hyperplane according to minimum error use hyperplane rotation. Hyperplane rotation is also used to construct the attainable region (AR). Ming et al. [29] build a procedure to use the plane rotation for constructing the AR.

In this paper, we present three procedures to rotate a supporting hyperplane for the convex hull of a point set. One of these procedures is based on LP. In Procedure Axis Rotation Hyperplane (`ARH`), we rotate the supporting hyperplane such that the dimension of its support set is increased one at a time. In the second procedure, Procedure Full Rotation Hyperplane (`FRH`), a supporting hyperplane is rotated once such that the final support set is a facet of the convex hull of a point set. In addition to these procedures, we derive a procedure to rotate a supporting hyperplane for the convex hull of a point set from one facet to another adjacent facet. We refer to this procedure as Procedure Facet To Facet (`FTF`). We prove the result of Procedure `FTF` is equivalent to a dual simplex pivot (DSP).

To perform a rotation by applying any of these three procedures, it is not needed to have the convex hull of a point set, or its frame. However, having the frames would make the procedures run faster, because of the reduction in number of constraints.

2.2 Snug circumscribing simplexes

The problem of finding a volume that contains, or is contained, in a body is a topic in computational geometry and is called the "containment problem". Usually this volume is required to be convex. Finding the smallest convex volume containing a body, or the biggest convex volume contained in a body is the objective. Depending on the shape of the body, finding a convex volume with this property can be hard. When a containment problem is the optimal solution to a mathematical program, the container or the contained set is called "extreme". If an optimal solution to the mathematical program is too difficult, an approximation may be practical [30]. The problem of finding a containing volume is known by several names. It is called the *Circumbody problem* by Gritzmann and Klee [30].

Graham and Oberman [31] approximate a convex hull of a point set with small number of the extreme points.

Sartipizadeh and Vincent [32] present an algorithm to approximate the convex hull of a point set that the dimension of the point set does not affect to their algorithm. This algorithm can be efficient to find (or approximate) the convex hull of a point set in high dimensions.

Finding or approximating an enclosing polyhedron for a point set has many applications [33]. There is some related work to find enclosing bodies such that ellipsoids, spheres, boxes, or simplexes [33]. Bādoiu and others demonstrate finding a circumscribing sphere has applications in clustering, and present a procedure to find it [34]. Panigrahy shows a procedure to find an enclosing polyhedron using only translations, and not allowing rotations [35]. In this topic of the dissertation we present procedures for the case where the contained body is the convex hull of a point set in $\Re^m$ and the containing volume is a special type of polyhedron in that space; namely a simplex.

Consider for a moment relaxing Assumption 2 in Chapter 1 and allow the point set $\mathcal{A}$ not to be its own frame. If we denote the frame of $\mathcal{A}$ with $\mathcal{F}$, the cardinality of $\mathcal{F}$ is at most n, and at least $m+1$. Finding the frame $\mathcal{F}$ of the finite point set becomes an interesting problem with applications in computational geometry, LP, stochastic optimization, DEA, statistics, and etc [36].

There are several procedures to find the frame of a finite point set. For the point set $\mathcal{A}$, a direct approach to find the frame solves n LPs such that in each, one point is tested. Suppose we score an arbitrary point $b \in \Re^m$. Suppose further for our purposes here that $\mathrm{con}(\mathcal{A})$ contains the origin in its strict interior. Consider the following LP [36].

$$\begin{aligned} f(b) = \min_{\lambda \in \Re^n} \ & \sum_{j=1}^{n} \lambda_j & & \textbf{Gauge LP} \qquad (2.21) \\ \text{s.t.} \ & \sum_{j=1}^{n} a^j \lambda_j = b, & & \\ & \lambda_j \geq 0; & & j = 1, \ldots, n. \end{aligned}$$

Theorem 2.1. *Gauge LP is feasible and bounded.*

Proof. Carathéodory's theorem says if a point in $\Re^m$ is inside a polytope, then this point can be defined as the convex combination of the at most $m+1$ extreme points of this polytope [2]. All λ_j for $j = 1, \ldots, n$ are nonnegative, so zero is a lower bound for Gauge LP. If b is in con($\mathcal{A}$), then according to Carathéodory's theorem, Gauge LP is feasible. Otherwise, we can scale down the point b such that b falls inside con($\mathcal{A}$), because zero is in con($\mathcal{A}$). Here again we invoke Carathéodory's theorem to find a feasible solution for this scaled point b. Next, we scale back b and multiply the optimal solution of Gauge LP by the scaling constant. ■

We refer to this LP as Gauge LP, because its solution allows us to determine whether the point b is inside, outside, or on the boundary of the convex hull of a point set. In the first two of these cases, the objective function value serves to assess the proximity of the point to the boundary. The function $f(b)$ in (2.21) is a *gauge* function in the sense of [2]. The optimal dual solution of the Gauge LP yields a facet's normal of con($\mathcal{A}$). The dual of Gauge LP is as follows.

$$\begin{aligned} f(b) = \max_{\pi \in \Re^m} \; & \langle \pi, b \rangle & & \textbf{Dual Gauge LP} \\ \text{s.t.} \; & \langle \pi, a^j \rangle \leq 1; & & j = 1, \ldots, n. \end{aligned} \tag{2.22}$$

For example, in the following figure, consider the convex hull of the points $\{a^1, \ldots, a^8\}$ in $\Re^2$. All these points are extreme for their convex hull. The optimal solution of the Gauge LP for three points b^1, b^2, b^3 identifies three supporting hyperplanes $\mathcal{H}(\pi^1, \beta^1), \mathcal{H}(\pi^2, \beta^2)$, and $\mathcal{H}(\pi^3, \beta^3)$ for the convex hull of these eight points, respectively. In addition, $f(b)$ is less than, equal to, and greater than one respectively. These planes are shown in Fig. 13.

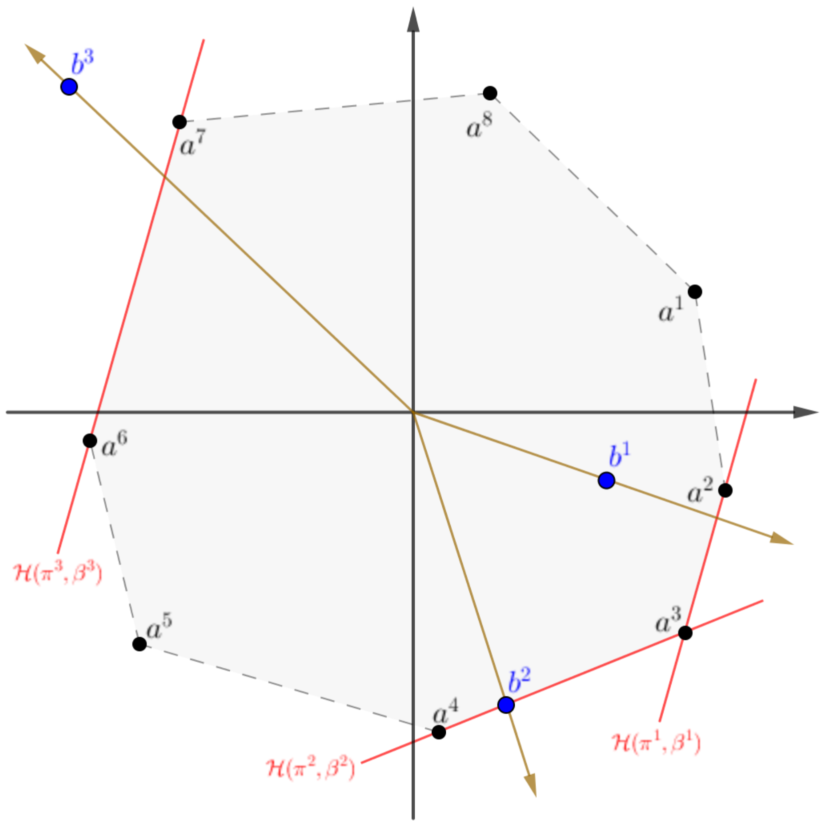

Fig. 13.: Gauge LP: The convex hull of a point set, and its relation to the Gauge LP and its dual.

Nine important observations about Gauge LP are as follows:

Remark 1. The point b is inside of $\text{con}(\mathcal{A})$ if and only if $f(b) < 1$.

Remark 2. The point b is on the boundary of $\text{con}(\mathcal{A})$ if and only if $f(b) = 1$.

Remark 3. The point b is outside of $\text{con}(\mathcal{A})$ if and only if $f(b) > 1$.

Remark 4. Any optimal basic solution to this LP yields a facet's normal of $\text{con}(\mathcal{A})$

with level value of 1.

Remark 5. For any point $a^j \in \mathcal{F}$, when we set $b = a^j$, then we get $f(b) = 1$.

Remark 6. If a^j for $j = 1, \ldots, n$ is interior to convex hull, it does not appear in any basic feasible solution (BFS).

Remark 7. Any BFS will be composed of points on the boundary.

Remark 8. As $f(b)$ gets closer to one, the scored point is in some sense closer to the boundary of $\text{con}(\mathcal{A})$.

Remark 9. An optimal value close to zero means the scored point is close to the origin.

We will use the Gauge LP in a procedure to find a snug circumscribing simplex for a point set.

The Gauge LP will play an important role in our work to find snug circumscribing simplexes. In this part of the dissertation, we design procedures to generate simplexes that contain a point set using just $m+1$ affinely independent points.

A simplex is snug in the sense that it intersects the hulls in some way. The objective is a snug circumscribing simplex that is also in some sense tight around the point set. Furthermore, finding such a polytope with fewer facet-defining inequalities is also valuable in some fields such that data reduction, dimension reduction, and non-negative factorization matrix. The simplest shape among all full-dimensional polyhedra is a simplex. A simplex is defined by the convex hull of the $m+1$ vertexes or by the intersection of the $m+1$ halfspaces. Going from one representation to the other in either direction (facial decomposition) is easy when it involves simplexes.

In this part of the dissertation, we develop three procedures to find an m-

dimensional polytope with $m+1$ extreme points such that its facets coincide with at least m facets of the convex hull of the point set, and $m+1$ facets if the convex hull is also a simplex. Indeed, the two will be the same polytope in the latter case. We refer to this as a snug circumscribing simplex. The first procedure is called "Axis Rotation Snug" (`ARS`), and is based on an application of Procedure `ARH` for rotating hyperplanes. We identify the facets of the polytope using this rotation procedure which starts at a vertex and visits faces of ever increasing dimension until a facet is reached. We apply this process m times to identify m facets of the polytope. The last facet of the simplex, called a "cap", is found using specialized procedures that involve LP. A cap will intersect the hull but not necessarily at a facet. The second procedure is called Procedure "Full Rotation Snug" (`FRS`) and is based on the optimal solutions of an LP. It achieves full facet support in one rotation using LP. Here, a cap may be necessary here, as well, to identify the last facet of the simplex such that it is bounded. Finally, the last procedure is called Procedure "Breakout Snug" (`BOS`). It identifies $m+1$ facets of $\text{con}(\mathcal{A})$ such that the $m+1$ normals are each in the polars of the normals of the previous facets and they positively span the space assuring boundedness.

The snug procedures are initialized with a large simplex that contains the convex hull of the point set. Then, its facets are translated towards the polytope until they eventually make contact. Next by rotating facets of the outside simplex, we increase the contact between two polyhedra to complete facets.

One application of finding a snug circumscribing simplex for a point set is in non-negative matrix factorization (NMF). When a point set is contained in a snug circumscribing simplex, any point of this point set can be written as a product of a matrix with the extreme points of this simplex with another matrix containing the multipliers needed to represent the point set.

NMF is a fundamental problems in multivariate analysis is to find a suitable representation of data, which makes the implicit structure of data explicit and reduces the dimensionality as well [37]. NMF is developed in the last two decades [38]. NMF has applications in data analysis, clustering, and neural network ([37] and [38]). NMF is popular linear dimensionality reduction technique which has the desired properties that we mentioned [39]. Moreover, NMF usually generates a sparse representation of a given point set [37].

NMF is formally defined as follows: for a given non negative matrix V, find two non negative matrix W and H such that $V \approx WH$ where V is an $n \times m$ matrix, W is an $n \times r$ matrix, and H is a $r \times m$ matrix.

It is usually desired for r to be smaller than m or n, hence W and H will have smaller dimension than the matrix V. In the context of multivariate analysis, where NMF have been extensively applied, the columns of matrix V are our observations (or point sets) which are vectors of dimension n. Then the goal is to find a compressed representation of matrix V using smaller matrixes W and H [40].

2.3 Conclusion

In this chapter we presented some previous works and applications in rotating hyperplanes and containment problems.

Rotating hyperplanes have applications in integer programming to find a facet-defining inequality. Another application of rotating hyperplanes is in DEA. Furthermore, we explained how a pivot in the dual simplex method can be related to rotating hyperplanes.

In the second section of this chapter, we presented the concept of the containment problems. We presented how Gauge LP can be related to the direct procedure of finding the frame of the convex hull of the finite point set. In addition, we discussed some remarks on Gauge LP and Dual Gauge LP.

CHAPTER 3

ROTATING SUPPORTING HYPERPLANES

We present three procedures to rotate a supporting hyperplane for a polytope defined as the convex hull of a finite point set in $\Re^m$. The first two procedures rotate a supporting hyperplane for the polytope at a lower dimensional face until it supports this polytope at a facet. These two procedures keep current extreme points in the support set and accumulate new points after the rotations. In other words, the support set of the rotated hyperplane includes all the extreme points in the support set of the supporting hyperplane that get rotated, in addition to whatever new extreme points join this support set. In the first procedure, the dimension of the rotated hyperplane for the convex hull of a point set increases one at a time. So, the support set for the rotated hyperplane will eventually be a facet of the polytope. This procedure does not rely on LP using only linear algebra operations. In the second procedure, we rotate a supporting hyperplane for a polytope with a lower dimensional support set and in one iteration the new hyperplane will have a facet as its support set. The second procedure uses LP. Finally, we develop a procedure to rotate a hyperplane on a facet of the polytope to another adjacent facet. Similarity to the first procedure, this procedure does not rely on LP. This procedure is equivalent to a dual simplex pivot.

Consider a supporting hyperplane $\mathcal{H}(\pi^k, \beta^k)$ for the polytope con($\mathcal{A}$). Recall that we assumed all the points in $\mathcal{A}$ are the extreme points for con($\mathcal{A}$). Assume, without loss of generality (wlog), the support set of the hyperplane $\mathcal{H}(\pi^k, \beta^k)$ includes $\mathcal{S}_k = \{a^1, \dots, a^k\}$ where $k \leq m-1$. Hence we have

$$\langle \pi^k, a^j \rangle = \beta^k; \quad j = 1, \dots, k. \tag{3.1}$$

The other points $a^{k+1}, \dots, a^n$ of $\mathcal{A}$ are located in a halfspace defined by the hyperplane $\mathcal{H}(\pi^k, \beta^k)$. Under our assumption, this halfspace defines the inequalities

$$\langle \pi^k, a^j \rangle \leq \beta^k; \quad j = k+1, \dots, n. \tag{3.2}$$

In the remainder of this chapter, the support set of the supporting hyperplane $\mathcal{H}(\pi^k, \beta^k)$ includes $\mathcal{S}_k = \{a^1, \dots, a^k\}$ from the point set $\mathcal{A} = \{a^1, \dots, a^n\}$ in $\Re^m$.

3.1 Procedure Axis Rotation Hyperplane (ARH)

In this procedure we rotate a supporting hyperplane $\mathcal{H}(\pi^k, \beta^k)$ such that at each iteration, the dimension of the support set is increased by one. Recall that we assume there do not exist degenerate facets. We define the new hyperplane as $\mathcal{H}(\pi^{k+1}, \beta^{k+1})$. The procedure is designed around the specification that two vectors π^k and π^{k+1} will share $m-k$ elements. Assume, wlog, that the common components are $\pi^k_{k+1}, \dots, \pi^k_m$, and at least one of them is not zero. This set up makes the first k components of the new vector free to take on values. Define $\pi^{k+1} = (\gamma_1, \ldots, \gamma_k, \pi^k_{k+1}, \ldots, \pi^k_m)$ where $\gamma_1, \ldots, \gamma_k$ are scalar parameters.

The rotation of a supporting hyperplane of $\text{con}(\mathcal{A})$ such that only k components of its normal are allowed to change can be interpreted geometrically. Components that are allowed to change will be referred to as *free* and the rest as *fixed*. The rotation restricts the normal of the supporting hyperplane to reside in a $k+1$ dimensional cone defined by the combinations of vectors in the subspace of the free axes (k dimensions) and the perpendicular vector with the original values for the fixed components and zeros for the rest. The normal is further restricted to lie in the strict interior of this cone which now becomes a pointed "circular" $k+1$ dimensional cone. This cone is pointed only because rotation vectors cannot be any of the free axes – or on the subspace defined by them – since this would require ∞ or $-\infty$ as a value for a free component. In the case of $k = 1$, the rotation maintains the normal of the hyperplane in a two-dimensional (very flat but still pointed) cone. When $k = 2$, this is a circular pointed cone above the plane defined by two free axes. Next, we will see how the rotation vector will be further restricted by the shape of the polytope below the supporting hyperplane in the following figure.

In order to keep the points that are currently on the hyperplane $\mathcal{H}(\pi^k, \beta^k)$ in the support set of the rotation, we force them to have the same level value. This is done by constructing the system of $k-1$ equations

$$\langle \pi^{k+1}, a^j \rangle = \langle \pi^{k+1}, a^1 \rangle; \quad j = 2, \ldots, k, \tag{3.3}$$

where a^1 is used, wlog, in lieu of any $a^j \in \mathcal{S}_k$.

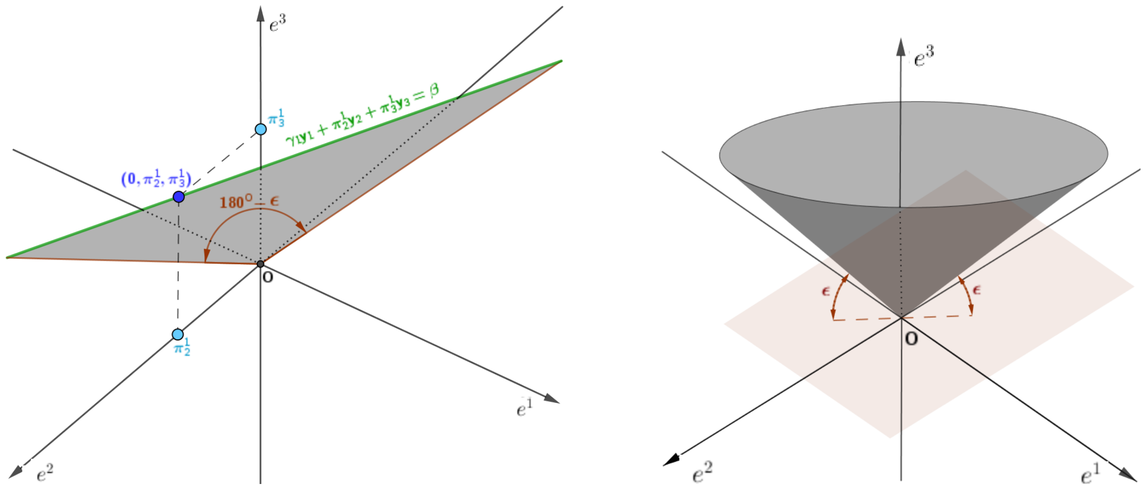

Fig. 14.: Procedure ARH in $\Re^3$: Left figure; $k = 1$, right figure; $k = 2$.

The points $a^{k+1}, \ldots, a^n$ will be located in a halfspace defined by the hyperplane $\mathcal{H}(\pi^{k+1}, \beta^{k+1})$. This generates the $n - k$ inequalities

$$\langle \pi^{k+1}, a^j \rangle \leq \langle \pi^{k+1}, a^1 \rangle; \quad j = k+1, \ldots, n. \tag{3.4}$$

The solutions to System (3.3 - 3.4) define a polyhedral region. Any feasible solution represents a supporting hyperplane for $\text{con}(\mathcal{A})$ with a support set containing $\mathcal{S}_k$. System (3.3 - 3.4) has $k-1$ equations and $n-k$ inequalities, where $1 \leq k \leq m-1$; all in k variables. There are at least two inequalities. When $k = 1$, System (3.3) is empty and the whole thing reduces to a system of inequalities with one variable.

In the following two subsections we present Procedure ARH in the cases of $k = 1$ and $k \geq 2$ separately.

3.1.1 Procedure ARH when $k = 1$

In the case of $k = 1$, System (3.3) is empty. We thus need to find an upper or lower bound for γ_1 from the system of inequalities:

$$\langle \pi^{k+1}, a^j \rangle \leq \langle \pi^{k+1}, a^1 \rangle; \quad j = 2, \ldots, n. \tag{3.5}$$

We define the following sub-vectors based on truncation after the first element.

$$\hat{a}^j \quad = (a_2^j, \ldots, a_m^j); \qquad j = 1, \ldots, n, \tag{3.6}$$

$$\hat{\pi}^{k+1} = (\pi_2^{k+1}, \ldots, \pi_m^{k+1}) = (\pi_2^k, \ldots, \pi_m^k) = \hat{\pi}^k. \tag{3.7}$$

Then, we rewrite System (3.5) using these truncated vectors as follows.

$$a_1^j \gamma_1 + \langle \hat{\pi}^{k+1}, \hat{a}^j \rangle \leq a_1^1 \gamma_1 + \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle; \qquad j = 2, \ldots, n. \tag{3.8}$$

After that, we manipulate System (3.8) algebraically. First by using the sub-vectors to separate the inner products:

$$a_1^j \gamma_1 - a_1^1 \gamma_1 \leq -\langle \hat{\pi}^{k+1}, \hat{a}^j \rangle + \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle; \qquad j = 2, \ldots, n, \tag{3.9}$$

and distributing after substituting:

$$(a_1^j - a_1^1)\gamma_1 \leq -\langle \hat{\pi}^{k+1}, \hat{a}^j \rangle + \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle; \qquad j = 2, \ldots, n. \tag{3.10}$$

The value of $a_1^j - a_1^1$ cannot be zero, according to Assumption 4 from Chapter 1, for $j = 2, \ldots, n$. If for one j we have $a_1^j - a_1^1 = 0$ where $j = 2, \ldots, n$, then we have $a_1^j = a_1^1$. Hence, the face of $\mathrm{con}(\mathcal{A})$ that contains just these two points a^1 and a^j is perpendicular to the first axes, and therefore is parallel to another axes. This violates our assumptions.

A solution to System (3.10) generates four possibilities on the values of γ_1:

1. There are no bounds.

2. There is only one finite upper bound.

3. There is only one finite lower bound.

4. There are two different and finite bounds: one lower and one upper.

Case 1 can be discarded using geometric arguments. System (3.10) is feasible since there is always a hyperplane that supports con($\mathcal{A}$) at exactly any extreme point and has complete freedom to rotate some nontrivial amount in any direction. Notice that, no rotation can span 360° or more without contradicting the assumption that the polytope has full dimension.

We will illustrate how Cases 2, 3, and 4 are possible.

If $a_1^j - a_1^1 > 0$ for $j = 2, \ldots, n$, then we can rewrite System (3.10) as follows.

$$\gamma_1 \leq -\frac{\langle \hat{\pi}^{k+1}, \hat{a}^j \rangle - \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle}{a_1^j - a_1^1}; \qquad j = 2, \ldots, n. \tag{3.11}$$

System (3.11) yields an upper bound for γ_1. This is Case 2. An upper bound for γ_1 that satisfies all inequalities in System (3.11) is

$$\gamma_1 = \min \left\{ -\frac{\langle \hat{\pi}^{k+1}, \hat{a}^j \rangle - \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle}{a_1^j - a_1^1}; \quad j = 2, \ldots, n \right\}. \tag{3.12}$$

The following example illustrates Case 2 in $\Re^3$.

Example 3.1. Consider the given point set $\mathcal{A} = \{a^1 = (2,8,4)^T, a^2 = (3,7,1)^T, a^3 = (5,4,9)^T, a^4 = (9,5,8)^T, a^5 = (1,2,3)^T, a^6 = (7,1,5)^T, a^7 = (6,3,2)^T\}$ in $\Re^3$.

Fig. 15. shows con($\mathcal{A}$) in three views.

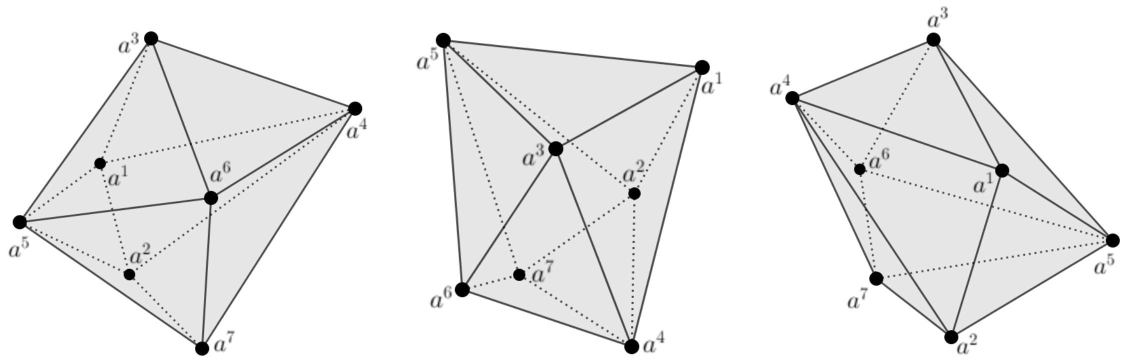

Fig. 15.: Three views of con($\mathcal{A}$).

Let the supporting plane $\mathcal{H}(\pi^1, -1)$ where $\pi^1 = (-9, 1, 2)^T$, with the support set $\mathcal{S}_1 = \{a^5\}$.

Fig. 16. shows con($\mathcal{A}$) and the plane $\mathcal{H}(\pi^1, -1)$ in three views.

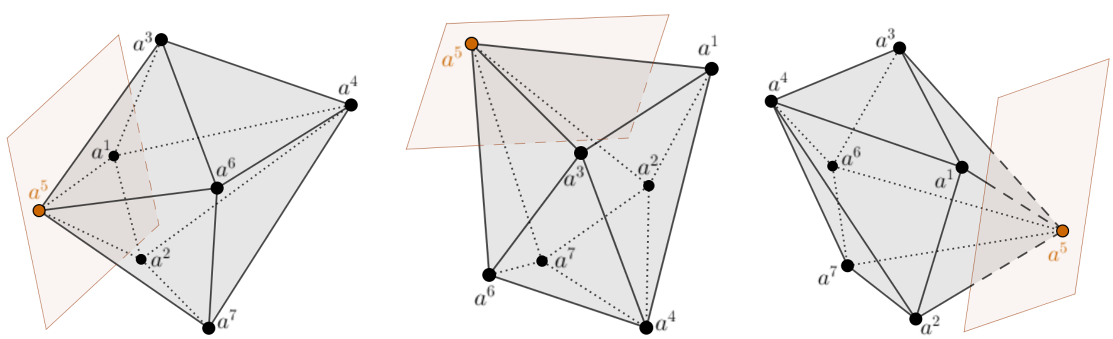

Fig. 16.: Three views of the supporting plane $\mathcal{H}(\pi^1, -1)$ where $\pi^1 = (-9, 1, 2)^T$ for con($\mathcal{A}$).

We define the new plane $\mathcal{H}(\pi^2, \beta^2)$ where $\pi^2 = (\gamma_1, 1, 2)^T$ and $\beta^2 = \langle \pi^2, a^5 \rangle$. So we have a system with six inequalities and a single variable:

$$\gamma_1 \leq -\frac{\langle \hat{\pi}^2, \hat{a}^1 \rangle - \langle \hat{\pi}^2, \hat{a}^5 \rangle}{a_1^1 - a_1^5} = -\frac{16-8}{2-1} \Longrightarrow \gamma_1 \leq -\ 8; \tag{3.13}$$

$$\gamma_1 \leq -\frac{\langle \hat{\pi}^2, \hat{a}^2 \rangle - \langle \hat{\pi}^2, \hat{a}^5 \rangle}{a_1^2 - a_1^5} = -\frac{9-8}{3-1} \Longrightarrow \gamma_1 \leq -\ \frac{1}{2}; \tag{3.14}$$

$$\gamma_1 \leq -\frac{\langle \hat{\pi}^2, \hat{a}^3 \rangle - \langle \hat{\pi}^2, \hat{a}^5 \rangle}{a_1^3 - a_1^5} = -\frac{22-8}{5-1} \Longrightarrow \gamma_1 \leq -\ \frac{7}{2}; \tag{3.15}$$

$$\gamma_1 \leq -\frac{\langle \hat{\pi}^2, \hat{a}^4 \rangle - \langle \hat{\pi}^2, \hat{a}^5 \rangle}{a_1^4 - a_1^5} = -\frac{21-8}{9-1} \Longrightarrow \gamma_1 \leq -\frac{13}{8}; \tag{3.16}$$

$$\gamma_1 \leq -\frac{\langle \hat{\pi}^2, \hat{a}^6 \rangle - \langle \hat{\pi}^2, \hat{a}^5 \rangle}{a_1^6 - a_1^5} = -\frac{11-8}{7-1} \Longrightarrow \gamma_1 \leq -\ \frac{1}{2}; \tag{3.17}$$

$$\gamma_1 \leq -\frac{\langle \hat{\pi}^2, \hat{a}^7 \rangle - \langle \hat{\pi}^2, \hat{a}^5 \rangle}{a_1^7 - a_1^5} = -\frac{7-8}{6-1} \Longrightarrow \gamma_1 \leq \quad \frac{1}{5}. \tag{3.18}$$

Hence -8 is an upper bound for γ_1. Therefore, the normal of the rotated plane is $\pi^2 = (-8, 1, 2)^T$, and we have: $\langle \pi^2, a^1 \rangle = 0$, $\langle \pi^2, a^2 \rangle = -15$, $\langle \pi^2, a^3 \rangle = -18$, $\langle \pi^2, a^4 \rangle = -51$, $\langle \pi^2, a^5 \rangle = 0$, $\langle \pi^2, a^6 \rangle = -45$, $\langle \pi^2, a^7 \rangle = -41$. So $\beta^2 = 0$, and the points a^1, a^5 are on the hyperplane $\mathcal{H}(\pi^2, 0)$ and the other a^2, a^3, a^4, a^6, a^7 are below of this hyperplane. The plane $\mathcal{H}(\pi^2, 0)$ is a supporting hyperplane for $\mathrm{con}(\mathcal{A})$ with the points $\mathcal{S}_2 = \{a^1, a^5\}$ in the support set.

Fig. 17. shows $\mathrm{con}(\mathcal{A})$ and the plane $\mathcal{H}(\pi^2, 0)$ in three views.

In this example, when we rotate the plane $\mathcal{H}(\pi^1, -1)$, the reason there is no lower bound is that the rotation towards more negative values of γ_1 is limited by $-\infty$. At this limit, the normal of the rotating hyperplane becomes for all intents and purposes the first coordinate (with the strange appearance $(-\infty, 1, 2)$) and is halted from further rotation by the fact that these operations cannot transcend infinity. This is a direct consequence of the fact that the initializing point is the

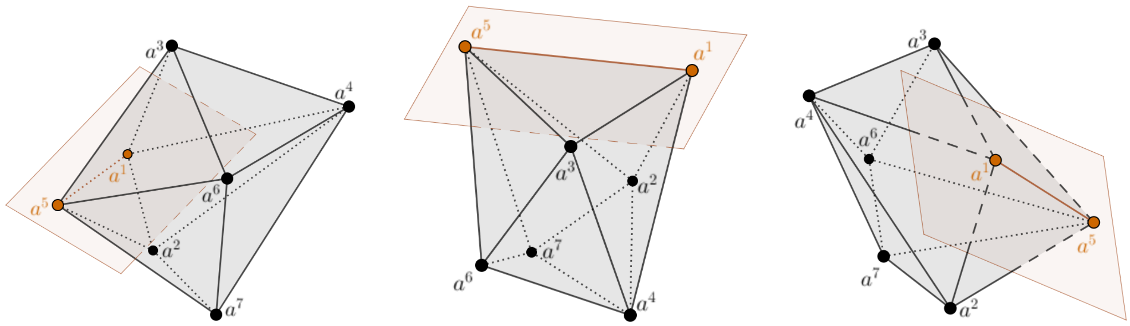

Fig. 17.: Three views of the supporting plane $\mathcal{H}(\pi^2, 0)$ where $\pi^2 = (-8, 1, 2)^T$ for con($\mathcal{A}$).

only one in the support set of a hyperplane with normal $(1, 0, 0)$. Recall that the initializing point, a_1^5 has the smallest first coordinates of all seven points in this set. Rotation towards more negative values of γ_1 eventually ends with $(-\infty, *, *)$ which is precisely this hyperplane and this will be the last supporting hyperplane these algebraic operations will allow. It is possible to resume rotation by "coming back" from ∞ by reversing the inequalities in System (3.5).

For the case when $k = 1$, it is possible to extract a second supporting hyperplane in a rotation that starts with a supporting hyperplane on an extreme point by a^5. This is the purpose for the following theorem.

Theorem 3.1. *The value for γ_1 in System (3.11) obtained from*

$$\gamma_1 = max \left\{ -\frac{\langle \hat{\pi}^{k+1}, \hat{a}^j \rangle - \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle}{a_1^j - a_1^1}; \quad j = 2, \ldots, n \right\} \tag{3.19}$$

defines a supporting hyperplane $\mathcal{H}(-\pi^{k+1}, \beta^{k+1})$*, which contains* a^1, a^t *where*

$$t = argmax \left\{ -\frac{\langle \hat{\pi}^{k+1}, \hat{a}^j \rangle - \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle}{a_1^j - a_1^1}; \quad j = 2, \dots, n \right\}. \tag{3.20}$$

Proof. We first show $\langle -\pi^{k+1}, a^t \rangle = \beta^{k+1}$. We have;

$$\begin{aligned}
\langle -\pi^{k+1}, a^t \rangle - \langle -\pi^{k+1}, a^1 \rangle &= \Big(\langle -\hat{\pi}^{k+1}, \hat{a}^t \rangle - \gamma_1 a_1^t \Big) - \Big(\langle -\hat{\pi}^{k+1}, \hat{a}^1 \rangle - \gamma_1 a_1^1 \Big) \\
&= \Big(\langle -\hat{\pi}^{k+1}, \hat{a}^t \rangle - \langle -\hat{\pi}^{k+1}, \hat{a}^1 \rangle \Big) - \gamma_1 \left(a_1^t - a_1^1 \right) \\
&= \Big(\langle -\hat{\pi}^{k+1}, \hat{a}^t \rangle - \langle -\hat{\pi}^{k+1}, \hat{a}^1 \rangle \Big) - \left(-\frac{\langle \hat{\pi}^{k+1}, \hat{a}^t \rangle - \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle}{a_1^t - a_1^1} \right) \left(a_1^t - a_1^1 \right) \\
&= \Big(\langle -\hat{\pi}^{k+1}, \hat{a}^t \rangle - \langle -\hat{\pi}^{k+1}, \hat{a}^1 \rangle \Big) - \Big(-\langle \hat{\pi}^{k+1}, \hat{a}^t \rangle + \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle \Big) \\
&= \Big(\langle -\hat{\pi}^{k+1}, \hat{a}^t \rangle - \langle -\hat{\pi}^{k+1}, \hat{a}^1 \rangle \Big) - \Big(\langle -\hat{\pi}^{k+1}, \hat{a}^t \rangle - \langle -\hat{\pi}^{k+1}, \hat{a}^1 \rangle \Big) \\
&= 0.
\end{aligned} \tag{3.21}$$

We thus have $\langle \pi^{k+1}, a^t \rangle = \langle \pi^{k+1}, a^1 \rangle$, and then $\langle \pi^{k+1}, a^t \rangle = \beta^{k+1}$.

We need to show all the points $a^2, \dots, a^n$ are located in a half space defined by the hyperplane $\mathcal{H}(-\pi^{k+1}, \beta^{k+1})$. We will therefore show $\langle -\pi^{k+1}, a^j \rangle \leq \beta^{k+1}$ for $j = 2, \dots, n$. Recall that we have;

$$\gamma_1 \geq -\frac{\langle \hat{\pi}^{k+1}, \hat{a}^j \rangle - \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle}{a_1^j - a_1^1} \Longrightarrow -\gamma_1 \leq \frac{\langle \hat{\pi}^{k+1}, \hat{a}^j \rangle - \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle}{a_1^j - a_1^1}. \tag{3.22}$$

Moreover, we have $a^j - a^1 > 0$. We thus have;

$$
\begin{aligned}
\langle -\pi^{k+1}, a^j \rangle - \langle -\pi^{k+1}, a^1 \rangle &= \Big(\langle -\hat{\pi}^{k+1}, \hat{a}^j \rangle - \langle -\hat{\pi}^{k+1}, \hat{a}^1 \rangle \Big) + (-\gamma_1) \left(a_1^j - a_1^1 \right) \\
&\leq \Big(\langle -\hat{\pi}^{k+1}, \hat{a}^j \rangle - \langle -\hat{\pi}^{k+1}, \hat{a}^1 \rangle \Big) + \left(\frac{\langle \hat{\pi}^{k+1}, \hat{a}^j \rangle - \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle}{a_1^j - a_1^1} \right) \left(a_1^j - a_1^1 \right) \\
&= 0. \qquad (3.23)
\end{aligned}
$$

Hence we have $\langle -\pi^{k+1}, a^j \rangle \leq \langle -\pi^{k+1}, a^1 \rangle$. So, $\langle -\pi^{k+1}, a^j \rangle \leq \beta^{k+1}$, for $j = 2, \ldots, n$. ■

Example 3.2. According to Theorem 3.1, the other bound for γ_1 in Example 3.1 is $\frac{1}{5}$. So the hyperplane $\mathcal{H}(\pi^2, -41)$ where $\pi^2 = (-1, -5, -10)^T$, with $\mathcal{S}_2 = \{a^5, a^7\}$ in the support set, supports $\mathrm{con}(\mathcal{A})$.

Fig. 18. shows $\mathrm{con}(\mathcal{A})$ and the plane $\mathcal{H}(\pi^2, -41)$ in three views.

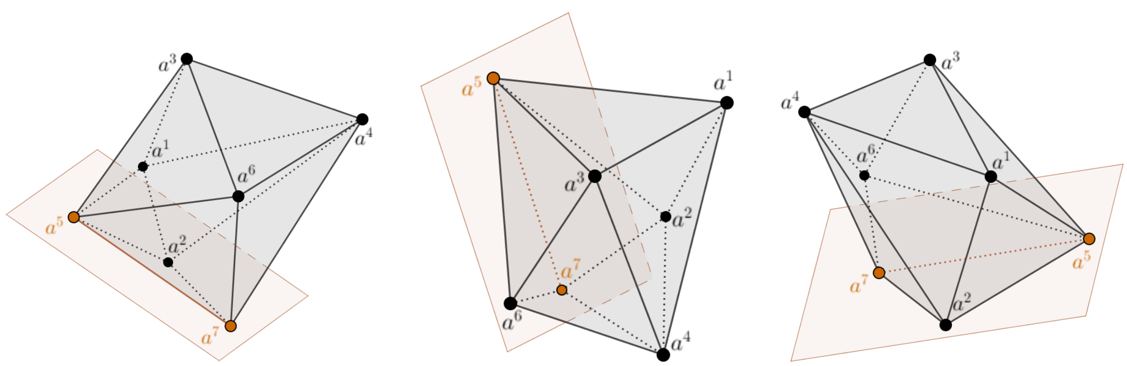

Fig. 18.: Three views of the supporting plane $\mathcal{H}(\pi^2, -41)$ where $\pi^2 = (-1, -5, -10)^T$ for $\mathrm{con}(\mathcal{A})$.

Case 3 happens when $a_1^j - a_1^1 < 0$ for $j = 2, \ldots, n$. We thus rewrite System

(3.10) as follows.

$$\gamma_1 \geq -\frac{\langle \hat{\pi}^{k+1}, \hat{a}^j \rangle - \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle}{a_1^j - a_1^1}; \qquad j = 2, \ldots, n. \tag{3.24}$$

System (3.24) yields only a lower bound for γ_1. The following example illustrates this case in $\Re^3$.

Example 3.3. Consider the point set $\mathcal{A}$ from Example 3.1, $\mathcal{A} = \{a^1 = (2,8,4)^T, a^2 = (3,7,1)^T, a^3 = (5,4,9)^T, a^4 = (9,5,8)^T, a^5 = (1,2,3)^T, a^6 = (7,1,5)^T, a^7 = (6,3,2)^T\}$ in $\Re^3$, and the supporting plane $\mathcal{H}(\pi^1, 77)$ where $\pi^1 = (3,2,5)^T$, with the support set $\mathcal{S}_1 = \{a^4\}$.

Fig. 19. shows the supporting plane $\mathcal{H}(\pi^1, 77)$ for con($\mathcal{A}$) in three views.

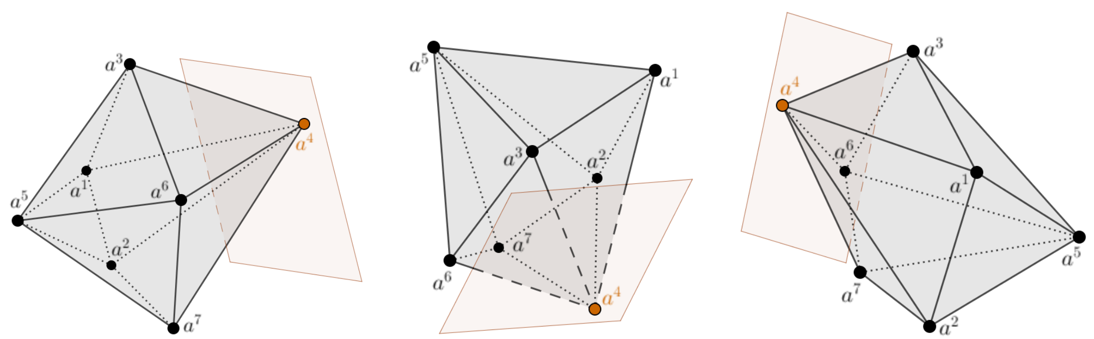

Fig. 19.: Three views of the supporting plane $\mathcal{H}(\pi^1, 77)$ where $\pi^1 = (3,2,5)^T$ for con($\mathcal{A}$).

We first define the new plane $\mathcal{H}(\pi^2, \beta^2)$ where $\pi^2 = (\gamma_1, 2, 5)^T$ and $\beta^2 = \langle \pi^2, a^4 \rangle$. Then, we have a system with six inequalities and a single variable. Similar way to Example 3.1, the minimum feasible ratio is $\frac{3}{4}$ for γ_1. The normal of the rotated plane is $\pi^2 = (3, 8, 20)^T$, and $\beta^2 = 227$. The plane $\mathcal{H}(\pi^2, 227)$ is a supporting hyperplane for con($\mathcal{A}$) with $\mathcal{S}_2 = \{a^3, a^4\}$ in the support set.

Fig. 20. shows the rotated supporting plane $\mathcal{H}(\pi^2, 227)$ for con($\mathcal{A}$) in three views.

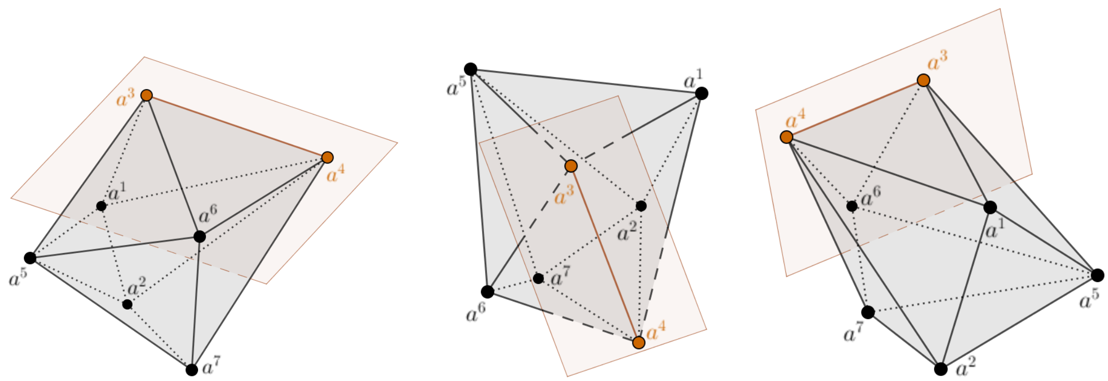

Fig. 20.: Three views of the supporting plane $\mathcal{H}(\pi^2, 227)$ where $\pi^2 = (3, 8, 20)^T$ for con($\mathcal{A}$).

In Example 3.3, when we rotate the plane $\mathcal{H}(\pi^1, 77)$, there is no upper bound. The reason is that the rotation towards more positive values of γ_1 is limited by ∞ for analogous reasons as in Example 1. Here again hyperplane rotation can be found by taking the smallest value in the list of inequalities for γ_1 and using the negative for the normal.

Theorem 3.2. *The value for γ_1 in System (3.24) obtained from*

$$\gamma_1 = min\left\{ -\frac{\langle \hat{\pi}^{k+1}, \hat{a}^j \rangle - \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle}{a_1^j - a_1^1}; \quad j = 2, \ldots, n \right\} \tag{3.25}$$

defines a supporting hyperplane $\mathcal{H}(-\pi^{k+1}, \beta^{k+1})$, *which contains* a^1, a^t *where*

$$t = argmin \left\{ -\frac{\langle \hat{\pi}^{k+1}, \hat{a}^j \rangle - \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle}{a_1^j - a_1^1}; \quad j = 2, \ldots, n \right\}. \tag{3.26}$$

Proof. The proof is similar to the proof of Theorem 3.1. ■

Example 3.4. Based on Theorem 3.2, the other bound for γ_1 in Example 3.3 is $-\frac{23}{2}$. So the hyperplane $\mathcal{H}(\pi^2, 107)$ where $\pi^2 = (23, -4, -10)^T$, with $\mathcal{S}_2 = \{a^4, a^6\}$ in the support set, supports con($\mathcal{A}$).

Fig. 21. shows con($\mathcal{A}$) and the plane $\mathcal{H}(\pi^2, 107)$ in three views.

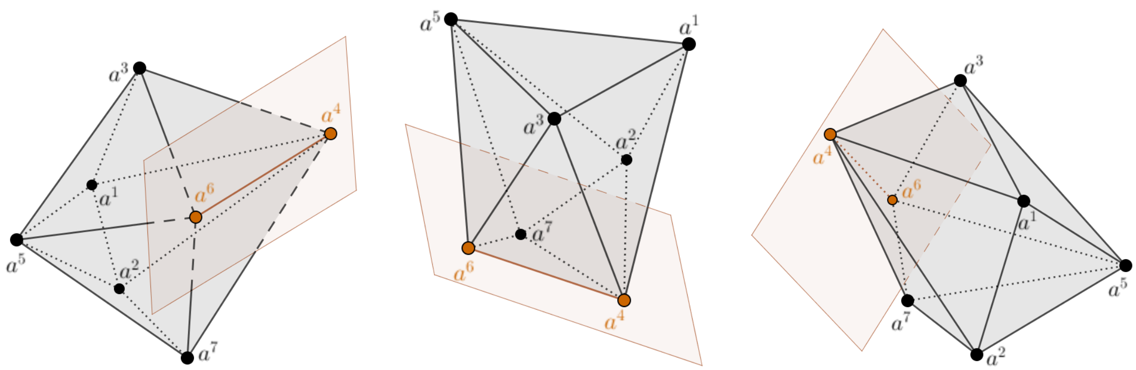

Fig. 21.: Three views of the supporting plane $\mathcal{H}(\pi^2, 107)$ where $\pi^2 = (23, -4, -10)^T$ for con($\mathcal{A}$).

Finally, when the values of $a_1^j - a_1^1$ are negative for some js, and positive for others where $j = 2, \ldots, n$, we face with Case 4. Assume, wlog, we have $a_1^j - a_1^1 > 0$ for $j = 2, \ldots, t$, and $a_1^j - a_1^1 < 0$ for $j = t+1, \ldots, n$, where $2 \leq t \leq n-1$. We thus rewrite System (3.10) as follows.

$$\gamma_1 \leq -\frac{\langle \hat{\pi}^{k+1}, \hat{a}^j \rangle - \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle}{a_1^j - a_1^1}; \qquad j = 2, \ldots, t; \tag{3.27}$$

$$\gamma_1 \geq -\frac{\langle \hat{\pi}^{k+1}, \hat{a}^j \rangle - \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle}{a_1^j - a_1^1}; \qquad j = t+1, \ldots, n. \tag{3.28}$$

System (3.27 - 3.28) yields an upper bound and a lower bound for γ_1. The following example illustrates this case in $\Re^3$.

Example 3.5. Consider the point set $\mathcal{A}$ from Example 3.1, $\mathcal{A} = \{a^1 = (2,8,4)^T, a^2 = (3,7,1)^T, a^3 = (5,4,9)^T, a^4 = (9,5,8)^T, a^5 = (1,2,3)^T, a^6 = (7,1,5)^T, a^7 = (6,3,2)^T\}$ in $\Re^3$, and the supporting plane $\mathcal{H}(\pi^1, 33)$ where $\pi^1 = (1,-2,4)^T$, with the support set $\mathcal{S}_1 = \{a^3\}$.

Fig. 22. shows the supporting plane $\mathcal{H}(\pi^1, 33)$ for con($\mathcal{A}$) in three views.

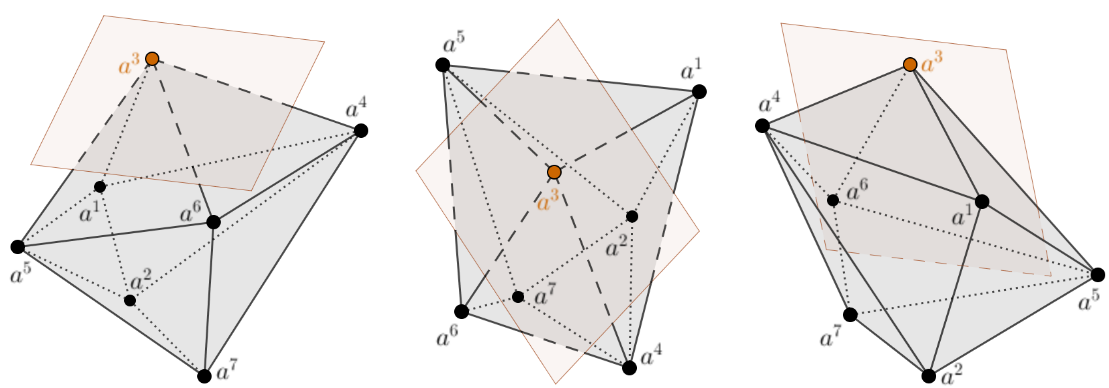

Fig. 22.: Three views of the supporting plane $\mathcal{H}(\pi^1, 33)$ where $\pi^1 = (1,-2,4)^T$ for con($\mathcal{A}$).

We define the new plane $\mathcal{H}(\pi^2, \beta^2)$ where $\pi^2 = (\gamma_1, -2, 4)^T$ and $\beta^2 = \langle \pi^2, a^3 \rangle$.

We thus have a system with six inequalities and a single variable:

$$\gamma_1 \geq -\frac{\langle \hat{\pi}^2, \hat{a}^1 \rangle - \langle \hat{\pi}^2, \hat{a}^3 \rangle}{a_1^1 - a_1^3} = -\frac{0 - 28}{2 - 5} \implies \gamma_1 \geq -\frac{28}{3}; \tag{3.29}$$

$$\gamma_1 \geq -\frac{\langle \hat{\pi}^2, \hat{a}^2 \rangle - \langle \hat{\pi}^2, \hat{a}^3 \rangle}{a_1^2 - a_1^3} = -\frac{-10 - 28}{3 - 5} \implies \gamma_1 \geq -19; \tag{3.30}$$

$$\gamma_1 \leq -\frac{\langle \hat{\pi}^2, \hat{a}^4 \rangle - \langle \hat{\pi}^2, \hat{a}^3 \rangle}{a_1^4 - a_1^3} = -\frac{22 - 28}{9 - 5} \implies \gamma_1 \leq \frac{3}{2}; \tag{3.31}$$

$$\gamma_1 \geq -\frac{\langle \hat{\pi}^2, \hat{a}^5 \rangle - \langle \hat{\pi}^2, \hat{a}^3 \rangle}{a_1^5 - a_1^3} = -\frac{8 - 28}{1 - 5} \implies \gamma_1 \geq -5; \tag{3.32}$$

$$\gamma_1 \leq -\frac{\langle \hat{\pi}^2, \hat{a}^6 \rangle - \langle \hat{\pi}^2, \hat{a}^3 \rangle}{a_1^6 - a_1^3} = -\frac{18 - 28}{7 - 5} \implies \gamma_1 \leq 5; \tag{3.33}$$

$$\gamma_1 \leq -\frac{\langle \hat{\pi}^2, \hat{a}^7 \rangle - \langle \hat{\pi}^2, \hat{a}^3 \rangle}{a_1^7 - a_1^3} = -\frac{2 - 28}{6 - 5} \implies \gamma_1 \leq 26. \tag{3.34}$$

This generates the following bounds for γ_1; $-5 \leq \gamma_1 \leq \frac{3}{2}$.

If we set $\gamma_1 = -5$, then the normal of the rotated plane is $\pi^2 = (-5, -2, 4)^T$, and $\beta^2 = 3$. The plane $\mathcal{H}(\pi^2, 3)$ is a supporting hyperplane for $\text{con}(\mathcal{A})$ with $\mathcal{S}_2 = \{a^3, a^5\}$ in the support set. Fig. 23. shows the rotated supporting plane $\mathcal{H}(\pi^2, 3)$ for $\text{con}(\mathcal{A})$ in three views.

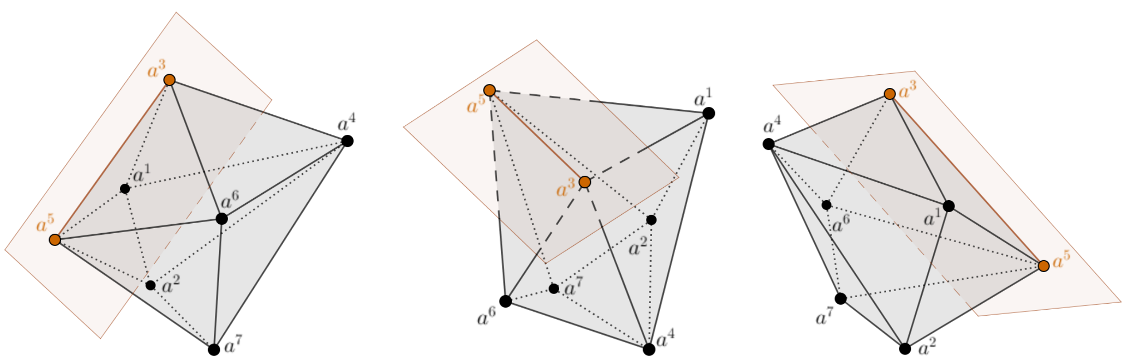

Fig. 23.: Three views of the rotated supporting plane $\mathcal{H}(\pi^2, 3)$ where $\pi^2 = (-5, -2, 4)^T$ for $\text{con}(\mathcal{A})$.

The maximum feasible ratio is $\frac{3}{2}$. If we set $\gamma_1 = \frac{3}{2}$, then the normal of the rotated plane is $\pi^2 = (3, -4, 8)^T$, and $\beta^2 = 71$. The plane $\mathcal{H}(\pi^2, 71)$ is a supporting hyperplane for con($\mathcal{A}$) with $\mathcal{S}_2 = \{a^3, a^4\}$ in the support set. Fig. 24. shows the rotated supporting plane $\mathcal{H}(\pi^2, 71)$ for con($\mathcal{A}$) in three views.

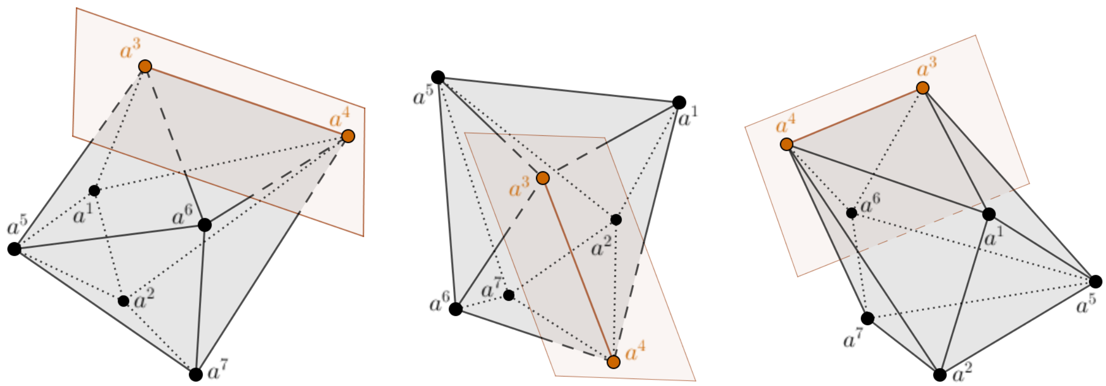

Fig. 24.: Three views of the rotated supporting plane $\mathcal{H}(\pi^2, 71)$ where $\pi^2 = (3, -4, 8)^T$ for con($\mathcal{A}$).

Note that if there are no ties in determining an upper or lower bound then the rotation will land on an edge of the polytope. When a tie occurs, the dimension of the support set for the rotation will increase by the number of the points participating in the tie. By our assumptions, this means the rotation landed on a face with more than one dimension. Under the assumption of no degenerate facet, a tie does not occur.

3.1.2 Matrixes of affinely independent points

We will start this section with some results relating to System (3.3). The system is a consequence of the concatenation of k extreme points of the convex hull to create a $k \times m$ matrix where $k \leq m$. This is the matrix:

$$A = \begin{bmatrix} a_1^1 & \dots & a_k^1 & \dots & a_m^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_1^k & \dots & a_k^k & \dots & a_m^k \end{bmatrix}_{k \times m} . \tag{3.35}$$

In this section we present results that establish properties about this matrix. One observation about the matrix (3.35) is that the k rows are affinely independent. This follows from the fact that they represent extreme points of a face of the convex hull. Because of this, the translation of the matrix by any one of these assures it has full rank.

We start with a lemma about convex hulls when they have fewer than m dimensions. Denote by $e^1, \dots, e^m$ the m axes of m-dimensional Euclidean space. Recall Assumption 4 From Chapter 1: no face with one or more dimensions of $\text{con}(\mathcal{A})$ is parallel to any axis.

Lemma 3.1. Consider a point set $\mathcal{B} = \{a^1, \dots, a^\ell\}$ in $\Re^m$ such that all the points $a^1, \dots, a^\ell$ are extreme for $\text{con}(\mathcal{B})$, $\text{con}(\mathcal{B})$ does not have full dimensional, and $\text{con}(\mathcal{B})$ does not have a face paralleling to any of the axes. For every extreme point in $\mathcal{B}$, there exists a supporting hyperplane for $\text{con}(\mathcal{B})$ that is parallel to an axis e^i and supports $\mathcal{B}$ only at that point.

Proof. The cases when $\ell = 1, 2$ are trivial. When $\ell \geq 3$, the number of the points in

$\mathcal{B}$ can be more than m. Consider, wlog, a point $\hat{a} \in \mathcal{B}$ and the last axis e^m. The point $\hat{a}$ is extreme for con($\mathcal{B}$), so there exists a hyperplane $\mathcal{H}(\bar{\pi}, \bar{\beta})$ which supports con($\mathcal{B}$) only at $\hat{a}$. Suppose $\bar{\pi}_m \neq 0$. The polytope con($\mathcal{B}$) does not have full dimension, so it can be contained in a hyperplane. Let this hyperplane be $\mathcal{H}(\hat{\pi}, \hat{\beta})$. Notice that $\mathcal{H}(-\hat{\pi}, -\hat{\beta})$ also contains con($\mathcal{B}$).

Consider the cone $\mathcal{C}_1$ defined by the vectors $\bar{\pi}, \hat{\pi}$, and the second cone $\mathcal{C}_2$ between vectors $\bar{\pi}, -\hat{\pi}$. Any vector strictly inside $\mathcal{C}_1$ or $\mathcal{C}_2$, not on the border, corresponds to a hyperplane that supports con($\mathcal{B}$) at only the point $\hat{a}$. The polytope con($\mathcal{B}$) does not have a face parallel to any axes, therefore $\hat{\pi}_m \neq 0$. If $\hat{\pi}_m > 0$, then $-\hat{\pi}_m < 0$, and vise versa. Assume, wlog, $\bar{\pi}_m > 0$ and $\hat{\pi}_m < 0$. A vector $\tilde{\pi} \in \mathcal{C}_1$ can be obtained by taking a positive combination of the two vectors $\bar{\pi}, \hat{\pi}$ such that $\tilde{\pi}_m = 0$. This vector corresponds to the hyperplane $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$ where $\tilde{\beta} = \langle \tilde{\pi}, \hat{a} \rangle$ and all other points in $\mathcal{B}$ are below it. The hyperplane $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$ is parallel the e^m axis, and is supporting con($\mathcal{B}$) only at the point $\hat{a}$. ■

Fig. 25. illustrates an example of Lemma 3.1. when $\mathcal{B} = \{a^1, \ldots, a^8\}$ in $\Re^3$. The polytope con($\mathcal{B}$) is two dimensional. The hyperplane $\mathcal{H}(\hat{\pi}, \hat{\beta})$ contains the points $a^1, \ldots, a^8$, and as a result contains con($\mathcal{B}$). Consider the point a^1 and the axis e^3. The hyperplane $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$ is parallel to the axis e^3, and is supporting con($\mathcal{B}$)only at a^1. Note that, here we have $\tilde{\pi}_3 = 0$.

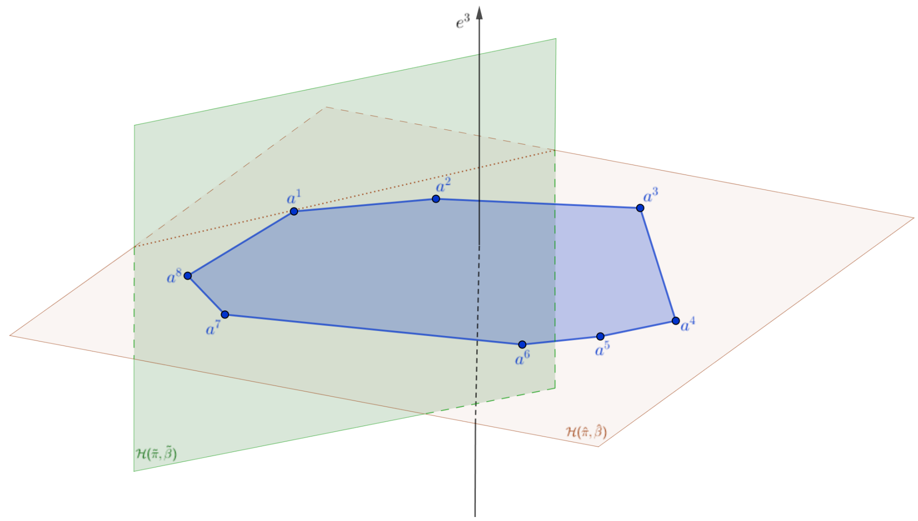

Fig. 25.: An example of Lemma 3.1. when $\mathcal{B} = \{a^1, \ldots, a^8\}$ in $\Re^3$.

Consider the $k \times m$ matrix A in (3.35). Note that $2 \leq k \leq m$. Here, we present an important result of this topic of the dissertation.

Theorem 3.3. *The rows of any $k \times \ell$ submatrix where $k \leq \ell \leq m$ of the matrix (3.35) are affinely independent.*

Proof. Is it enough to show that if any column from (3.35) is removed the remaining rows will be affinely independent. Suppose, wlog, it is the last column. We thus have the $k \times (m-1)$ submatrix:

$$\bar{A} = \begin{bmatrix} a_1^1 & \ldots & a_{m-1}^1 \\ \vdots & \ddots & \vdots \\ a_1^k & \ldots & a_{m-1}^k \end{bmatrix}_{k \times (m-1)} . \tag{3.36}$$

Indeed, one interpretation of the $k \times (m-1)$ submatrix (3.36) is that it is the projection of the original k points on to the subspace defined by the first $m-1$ axes.

The original k points $a^1, \ldots, a^k$ are affinely independent and they are the extreme points of a $(k-1)$-dimensional simplex in $\Re^m$.

Let $a^\ell \in \mathcal{B}$. According to Lemma 3.1., there exists a supporting hyperplane $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$ for con$(\mathcal{B})$ that is parallel to the axis e^m such that includes only point a^ℓ. Since $\tilde{\pi}$ is parallel to e^m, We have the vector $\tilde{\pi} = (\tilde{\pi}_1, \ldots, \tilde{\pi}_{m-1}, 0)^T$. So,

$$\langle \tilde{\pi}, a^\ell \rangle = \tilde{\beta}, \tag{3.37}$$

$$\langle \tilde{\pi}, a^j \rangle < \tilde{\beta}; \quad j = 1, \ldots, \ell-1, \ell+1, \ldots, k. \tag{3.38}$$

Expanding System (3.37 - 3.38) yields the system:

$$\sum_{i=1}^{m-1} a_i^\ell \tilde{\pi}_i = \tilde{\beta}, \tag{3.39}$$

$$\sum_{i=1}^{m-1} a_i^j \tilde{\pi}_i < \tilde{\beta}; \quad j = 1, \ldots, \ell-1, \ell+1, \ldots, k. \tag{3.40}$$

Since $\tilde{\pi}_m = 0$ for each of these supporting hyperplanes the above relations System (3.39 - 3.40) can be replaced with each of the points in $\mathcal{B}$ truncated at the last component and the hyperplane normals using only the first $m-1$ components.

Notice that such "vertical" supporting hyperplanes exist for each point in $\mathcal{B}$, they are each an extreme point of the convex hull of their projection in the lower dimensional subspace. The projection is thus also a $(k-1)$-dimensional simplex making the k points affinely independent. Assume $\tilde{a}^j = (a_1^j, \ldots, a_{m-1}^j)^T$ for $j = 1, \ldots, k$. The k points $\tilde{a}^1, \ldots, \tilde{a}^k$ are in $\Re^{m-1}$. So, the convex hull of these k points does not have full dimension in $\Re^{m-1}$, because $k \leq m-1$. These k points are affinely independent if all these k points are extreme for the convex hull of these k points

$\tilde{a}^1, \ldots, \tilde{a}^k$. ■

We proved that any $k \times (m-1)$ submatrix of the matrix (3.35) has affinely independent rows. We can use the same argument sequentially to show that any $k \times i$ submatrix of the matrix (3.35) has affinely independent rows as long as the number of columns that remain is at least k: $k \leq i$.

Fig. 26. shows the projections of a line segment and a triangle onto a subspace containing two axes e^1, e^2.

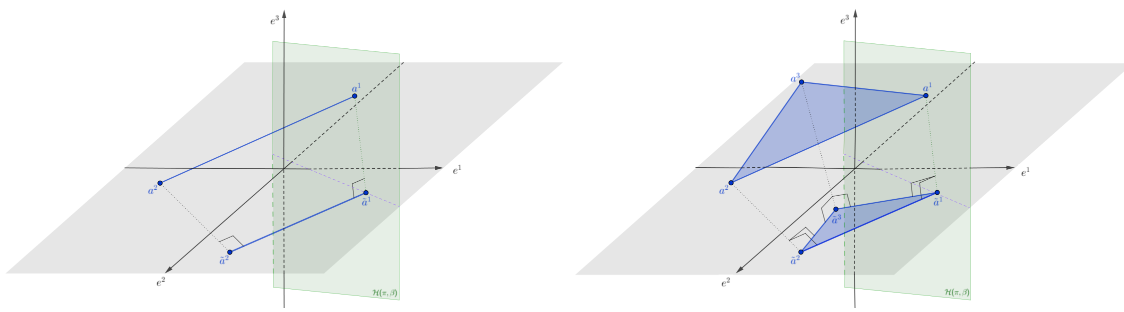

Fig. 26.: The projections of two lower dimensional simplexes: a line segment and a triangle onto e^1, e^2 subspace.

There are four important observations about Theorem 3.3 as follows:

Remark 1. Any of the submatrixes obtained from removing (at most) $m-k$ columns from matrix (3.35) can be translated by one of its rows to generate a full rank matrix.

Remark 2. This property that any subset has full rank, has all elements of a matroid.

Remark 3. If a face of the polytope $\text{con}(\mathcal{A})$ contains the points $a^1, \ldots, a^k$ and is parallel to an axis, then there exists a $(k-1) \times (k-1)$ submatrix of (3.35) without full rank.

Remark 4. The $k \times m$ matrix (3.35) has full rank even there exists the faces paralleling to some axes.

3.1.3 Procedure ARH when $k \geq 2$

Recall that k is both the number of the free dimensions and the cardinality of the extreme points of $\mathcal{A}$ in the support set $\mathcal{S}_k$. In the case of $k \geq 2$, System (3.3) is no longer empty and the solutions to this system define an affine set with 1 dimension. The intersection of this affine set, that is a line, with the polyhedron of feasible solutions to System (3.4) generates the normals of the supporting hyperplanes for con($\mathcal{A}$) that contains $\mathcal{S}_k$.

We can rewrite System (3.3) as follows:

$$\langle \pi^{k+1}, a^j \rangle - \langle \pi^{k+1}, a^1 \rangle = 0; \quad j = 2, \ldots, k. \tag{3.41}$$

The columns of the left hand side of System (3.41) is shown in the $(k-1) \times m$ matrix:

$$\begin{bmatrix} a_1^2 - a_1^1 & \ldots & a_k^2 - a_k^1 & \ldots & a_m^2 - a_m^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_1^k - a_1^1 & \ldots & a_k^k - a_k^1 & \ldots & a_m^k - a_m^1 \end{bmatrix}_{(k-1)\times m}. \tag{3.42}$$

We here develop System (3.3 - 3.4). We define the following sub-vectors based on a partition after the k^{th} element, and assuming that the first k columns of $(k-1) \times m$

matrix (3.42) have full rank.

$$\begin{aligned}
\bar{a}^j &= (a_1^j, \ldots, a_k^j); && j = 1, \ldots, n, && (3.43)\\
\hat{a}^j &= (a_{k+1}^j, \ldots, a_m^j); && j = 1, \ldots, n, && (3.44)\\
\bar{\pi}^{k+1} &= (\pi_1^{k+1}, \ldots, \pi_k^{k+1}) = (\gamma_1, \ldots, \gamma_k); && && (3.45)\\
\hat{\pi}^{k+1} &= (\pi_{k+1}^{k+1}, \ldots, \pi_m^{k+1}) = (\pi_{k+1}^k, \ldots, \pi_m^k). && && (3.46)
\end{aligned}$$

Then, we rewrite System (3.3 - 3.4) using these truncated vectors as follows.

$$\langle \bar{\pi}^{k+1}, \bar{a}^j \rangle + \langle \hat{\pi}^{k+1}, \hat{a}^j \rangle = \langle \bar{\pi}^{k+1}, \bar{a}^1 \rangle + \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle; \qquad j = 2, \ldots, k, \tag{3.47}$$

$$\langle \bar{\pi}^{k+1}, \bar{a}^j \rangle + \langle \hat{\pi}^{k+1}, \hat{a}^j \rangle \leq \langle \bar{\pi}^{k+1}, \bar{a}^1 \rangle + \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle; \qquad j = k+1, \ldots, n. \tag{3.48}$$

After that, we manipulate System (3.47 - 3.48) algebraically. First by using the sub-vectors to separate the inner products

$$\langle \bar{\pi}^{k+1}, \bar{a}^j \rangle - \langle \bar{\pi}^{k+1}, \bar{a}^1 \rangle = \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle - \langle \hat{\pi}^{k+1}, \hat{a}^j \rangle; \qquad j = 2, \ldots, k, \tag{3.49}$$

$$\langle \bar{\pi}^{k+1}, \bar{a}^j \rangle - \langle \bar{\pi}^{k+1}, \bar{a}^1 \rangle \leq \langle \hat{\pi}^{k+1}, \hat{a}^1 \rangle - \langle \hat{\pi}^{k+1}, \hat{a}^j \rangle; \qquad j = k+1, \ldots, n, \tag{3.50}$$

and redistributing after substituting $\bar{v}^j = (\bar{a}^j - \bar{a}^1), \hat{v}^j = (\hat{a}^j - \hat{a}^1)$ where $j = 2, \ldots, n$.

$$\langle \bar{\pi}^{k+1}, \bar{v}^j \rangle = \langle \hat{\pi}^{k+1}, -\hat{v}^j \rangle; \qquad j = 2, \ldots, k, \tag{3.51}$$

$$\langle \bar{\pi}^{k+1}, \bar{v}^j \rangle \leq \langle \hat{\pi}^{k+1}, -\hat{v}^j \rangle; \qquad j = k+1, \ldots, n. \tag{3.52}$$

Let's deal with System (3.51) first. The left-hand side is a set of $k-1$ linear combinations of the k variables $\gamma_1, \ldots, \gamma_k$. The right-hand sides are constants which we will denote by $b_j = \langle \hat{\pi}^{k+1}, -\hat{v}^j \rangle$ for $j = 2, \ldots, k$. This defines the following system with $k-1$ equations and k variables.

$$\sum_{\ell=1}^{k} \bar{v}_{\ell}^{j}\gamma_{\ell} = b_j; \quad j = 2\ldots,k. \tag{3.53}$$

The left-hand side's coefficients of System (3.53) is the $(k-1)\times k$ matrix

$$\begin{bmatrix} \bar{v}_1^2 & \bar{v}_2^2 & \ldots & \bar{v}_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{v}_1^k & \bar{v}_2^k & \ldots & \bar{v}_k^k \end{bmatrix}_{(k-1)\times k}, \tag{3.54}$$

which is obtained by truncating the full rank $(k-1)\times m$ matrix (3.42). Therefore, the affine set that they define has a single dimension. Hence it is a line. This result demonstrates that a rotation is possible provided we can find a non-singular square $(k-1)\times(k-1)$ matrix which is guaranteed to exist.

Let us go back to System (3.3 - 3.4). There are five possibilities for the set of solutions to System (3.3 - 3.4):

1. The set is empty.
2. The set is a point.
3. The set is a complete line.
4. The set is a half-line.
5. The set is a line segment.

The first possibility can be excluded right out since there exists at least one solution always; namely, the current supporting hyperplane with $\mathcal{S}_k$ in the support set. This means there is at least one non-trivial solution. The second possibility can also be excluded, because the number of hyperplanes that contain a k-dimensional face of a polytope in $\Re^m$ and is a supporting hyperplane for this polytope is infinity. Recall that $k \leq m-1$. The third possibility can be discarded too. If it were possible then it would be possible to rotate 360° which is impossible when the original convex hull has full dimension. The set of solutions to System (3.3 - 3.4) is either a line segment with two endpoints or a half line in $\Re^k$.

Fig. 27. depicts two Cases 4 and 5 when $k = 2$. In both figures the obtained region by System (3.4) is shown with dark unbounded area R. The left hand side figure shows how the obtained half-line v from System (3.3) intersects the boundary of Region R at γ^*. If we set $\gamma = \gamma^*$, then the vector π^{k+1} corresponds to the normal of the hyperplane $\mathcal{H}(\pi^{k+1}, \beta^{k+1})$ such that $\mathcal{S}_{k+1}$ contains $\mathcal{S}_k$ and one more point from the point set $\mathcal{A}$. This is Case 4. The right hand side figure shows how Case 5 happens. In this figure, the obtained line segment l from System (3.3) intersects the boundary of Region R at two points γ_1^* and γ_2^*. If we set either $\gamma = \gamma_1^*$ or $\gamma = \gamma_2^*$, then it yields two different vectors π^{k+1}. Either of these two vectors π^{k+1} yields the normal of the hyperplane $\mathcal{H}(\pi^{k+1}, \beta^{k+1})$ that $\mathcal{S}_{k+1}$ contains the set $\mathcal{S}_k$ and one more point from the point set $\mathcal{A}$.

The following two Examples 3.6 and 3.7 demonstrate how these two cases in $\Re^3$ happen.

Example 3.6. Consider the point set $\mathcal{A}$ from Example 3.1, $\mathcal{A} = \{a^1 = (2,8,4)^T, a^2 = (3,7,1)^T, a^3 = (5,4,9)^T, a^4 = (9,5,8)^T, a^5 = (1,2,3)^T, a^6 = (7,1,5)^T, a^7 = (6,3,2)^T\}$ in $\Re^3$, and the supporting plane $\mathcal{H}(\pi^2, 54)$ where $\pi^2 = (1,1,5)^T$, with $\mathcal{S}_2 = \{a^3, a^4\}$ in the support set.

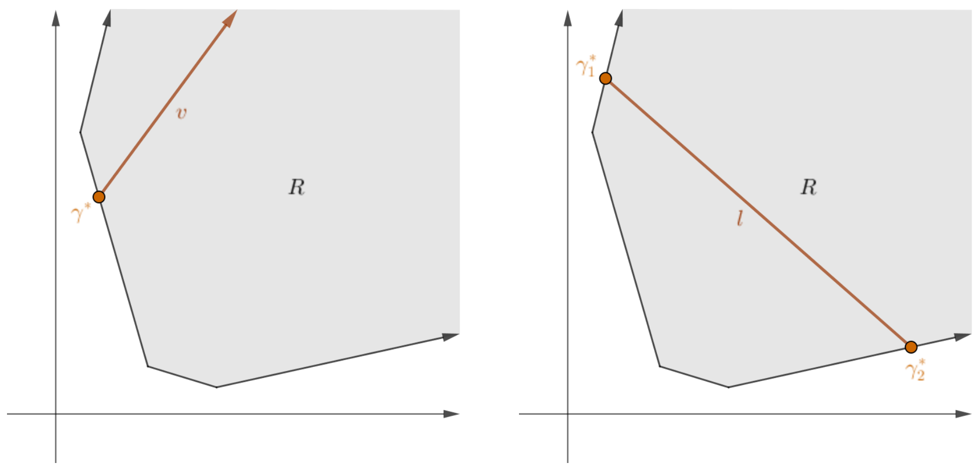

Fig. 27.: The left hand side figure demonstrates Case 4, and the right hand side figure demonstrates Case 5 when $k = 2$.

Fig. 28. shows the supporting plane $\mathcal{H}(\pi^2, 54)$ for $\mathrm{con}(\mathcal{A})$ in three views.

First, we define the new plane $\mathcal{H}(\pi^3, \beta^3)$ where $\pi^2 = (\gamma_1, 1, \gamma_2)^T$ and $\beta^3 = \langle \pi^3, a^3 \rangle$. We next have one equation

$$\langle \pi^3, a^4 \rangle = \langle \pi^3, a^3 \rangle \Longrightarrow 4\gamma_1 - \gamma_2 = -1. \tag{3.55}$$

There is a system with five inequalities and two variables:

$$\langle \pi^3, a^1 \rangle \leq \langle \pi^3, a^3 \rangle \Longrightarrow -3\gamma_1 - 5\gamma_2 \leq -4; \tag{3.56}$$

$$\langle \pi^3, a^2 \rangle \leq \langle \pi^3, a^3 \rangle \Longrightarrow -2\gamma_1 - 8\gamma_2 \leq -3; \tag{3.57}$$

$$\langle \pi^3, a^5 \rangle \leq \langle \pi^3, a^3 \rangle \Longrightarrow -4\gamma_1 - 6\gamma_2 \leq 2; \tag{3.58}$$

$$\langle \pi^3, a^6 \rangle \leq \langle \pi^3, a^3 \rangle \Longrightarrow 2\gamma_1 - 4\gamma_2 \leq 3; \tag{3.59}$$

$$\langle \pi^3, a^7 \rangle \leq \langle \pi^3, a^3 \rangle \Longrightarrow \gamma_1 - 7\gamma_2 \leq 1. \tag{3.60}$$

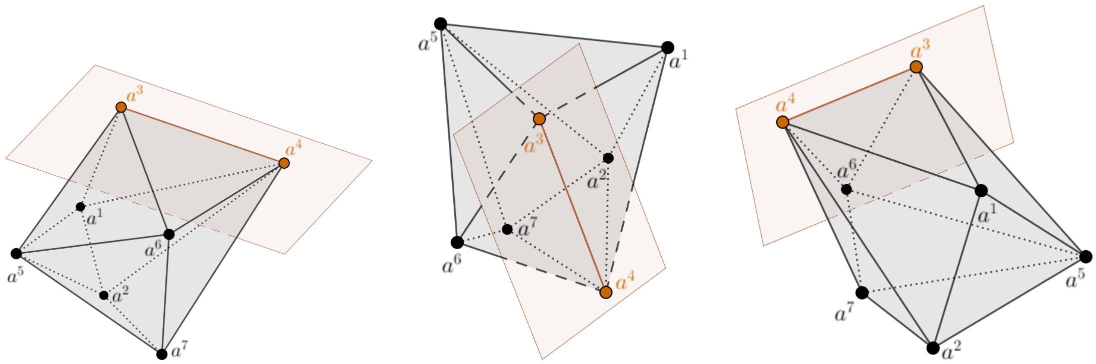

Fig. 28.: Three views of the supporting plane $\mathcal{H}(\pi^2, 54)$ where $\pi^2 = (1, 1, 5)^T$ for $\text{con}(\mathcal{A})$.

From System (3.55), we have $\gamma_1 = \frac{1}{4}\gamma_2 - \frac{1}{4}$. We therefore rewrite the system of five inequalities as follow based on just one variable γ_2: $\gamma_2 \geq \frac{19}{23}$; $\gamma_2 \geq \frac{7}{17}$; $\gamma_2 \geq -\frac{1}{7}$; $\gamma_2 \geq -1$; $\gamma_2 \geq -\frac{5}{27}$. This generates $\frac{19}{23}$ as an lower bound for γ_2. If we set $\gamma_2 = \frac{19}{23}$, then we get $\gamma_1 = -\frac{1}{23}$. The normal of the rotated plane is $\pi^3 = (-1, 23, 19)^T$, and $\beta^2 = 258$. The plane $\mathcal{H}(\pi^3, 258)$ is a supporting hyperplane for $\text{con}(\mathcal{A})$ with $\mathcal{S}_2 = \{a^1, a^3, a^4\}$ in the support set.

Fig. 29. shows the supporting plane $\mathcal{H}(\pi^3, 258)$ for $\text{con}(\mathcal{A})$ in three views.

Example 3.6 demonstrates Case 4. In Case 4, we can extract a second supporting hyperplane in a rotation that its support set contains two points a^3, a^4. When we reduce System (3.3 - 3.4) to one variable, wlog, γ_1, if there exists just one bound for γ_1, then the biggest and smallest values of the right hand side of the system of inequalities with one variable γ_1, when the coefficient of the variable γ_1 is one, are used to find the normal of two rotated hyperplanes.

Assume from System (3.3), we obtain all variables $\gamma_2, \ldots, \gamma_k$ based on the vari-

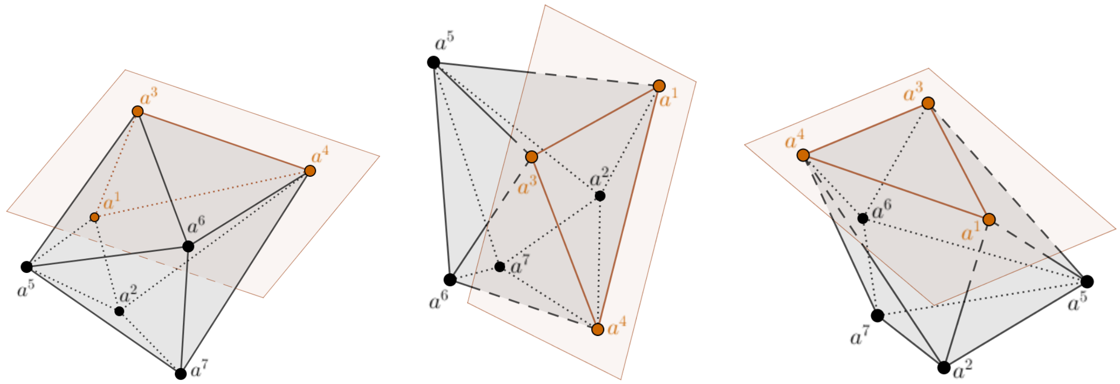

Fig. 29.: Three views of the supporting plane $\mathcal{H}(\pi^3, 258)$ where $\pi^3 = (-1, 23, 19)^T$ for $\text{con}(\mathcal{A})$.

able γ_1 and a constant:

$$\gamma_i = e_i \gamma_1 + f_i; \quad i = 2, \ldots, k. \tag{3.61}$$

If we rewrite System (3.4) based on just one variable γ_1, then we have

$$\left((a_1^j - a_1^1) + \sum_{i=2}^{k} e_i (a_i^j - a_i^1) \right) \gamma_1 \leq - \left(\sum_{i=2}^{k} f_i (a_i^j - a_i^1) + \sum_{i=k+1}^{m} \pi_i^k (a_i^j - a_i^1) \right); \ j = k+1, \ldots, n. \tag{3.62}$$

If all the coefficients of the variable γ_1 in System (3.62) be positive, then we have:

$$\gamma_1 \leq \bar{\gamma}_j; \ j = k+1, \ldots, n. \tag{3.63}$$

where

$$\bar{\gamma}_j = - \frac{\displaystyle\sum_{i=2}^{k} f_i (a_i^j - a_i^1) + \sum_{i=k+1}^{m} \pi_i^k (a_i^j - a_i^1)}{\displaystyle(a_1^j - a_1^1) + \sum_{i=2}^{k} e_i (a_i^j - a_i^1)}; \ j = k+1, \ldots, n. \tag{3.64}$$

Therefore, if we set:

$$\gamma_1 = \min \left\{ \bar{\gamma}_j;\ j = k+1, \ldots, n \right\}, \tag{3.65}$$

then we have a supporting hyperplane $\mathcal{H}(\pi^{k+1}, \beta^{k+1})$, which contains $a^1, \ldots, a^k, a^t$ where

$$t = \operatorname{argmin} \left\{ \bar{\gamma}_j;\ j = k+1, \ldots, n \right\}. \tag{3.66}$$

The following theorem presents a way to find a second supporting hyperplane.

Theorem 3.4. *If all the coefficients of the variable γ_1 in System (3.62) are positive, then the value for γ_1 in System (3.64) obtained from*

$$\gamma_1 = \ max \left\{ \bar{\gamma}_j;\ j = k+1, \ldots, n. \right\} \tag{3.67}$$

defines a supporting hyperplane $\mathcal{H}(-\pi^{k+1}, \beta^{k+1})$, which contains $a^1, \ldots, a^k, a^t$ where

$$t = \ argmax \left\{ \bar{\gamma}_j;\ j = k+1, \ldots, n. \right\} \tag{3.68}$$

Proof. We first show $\langle -\pi^{k+1}, a^t \rangle = \beta^{k+1}$. We have;

$$
\begin{aligned}
\langle -\pi^{k+1}, a^t\rangle - \langle -\pi^{k+1}, a^1\rangle &= -\sum_{i=1}^{m} \pi_i^{k+1}(a_i^t - a_i^1) \\
&= -\gamma_1(a_1^t - a_1^1) - \sum_{i=2}^{k} \gamma_i(a_i^t - a_i^1) - \sum_{i=k+1}^{m} \pi_i^k(a_i^t - a_i^1) \\
&= -\gamma_1(a_1^t - a_1^1) - \sum_{i=2}^{k} (e_i\gamma_1 + f_i)(a_i^t - a_i^1) - \sum_{i=k+1}^{m} \pi_i^k(a_i^t - a_i^1) \\
&= -\gamma_1 \left((a_1^t - a_1^1) + \sum_{i=2}^{k} e_i(a_i^t - a_i^1)\right) - \left(\sum_{i=2}^{k} f_i(a_i^t - a_i^1) + \sum_{i=k+1}^{m} \pi_i^k(a_i^t - a_i^1)\right) \\
&= -\left(-\frac{\sum_{i=2}^{k} f_i(a_i^t - a_i^1) + \sum_{i=k+1}^{m} \pi_i^k(a_i^t - a_i^1)}{(a_1^t - a_1^1) + \sum_{i=2}^{k} e_i(a_i^t - a_i^1)}\right)\left((a_1^t - a_1^1) + \sum_{i=2}^{k} e_i(a_i^t - a_i^1)\right) \\
&\quad - \left(\sum_{i=2}^{k} f_i(a_i^t - a_i^1) + \sum_{i=k+1}^{m} \pi_i^k(a_i^t - a_i^1)\right) \\
&= 0.
\end{aligned} \tag{3.69}
$$

We therefore have $\langle \pi^{k+1}, a^t\rangle = \langle \pi^{k+1}, a^1\rangle$, and then $\langle \pi^{k+1}, a^t\rangle = \beta^{k+1}$.

We further need to show all the points $a^{k+1}, \ldots, a^n$ are located in a half space defined by the hyperplane $\mathcal{H}(-\pi^{k+1}, \beta^{k+1})$. We will show $\langle -\pi^{k+1}, a^j\rangle \leq \beta^{k+1}$ for $j = k+1, \ldots, n$. Recall that we have:

$$
\gamma_1 \geq -\frac{\sum_{i=2}^{k} f_i(a_i^j - a_i^1) + \sum_{i=k+1}^{m} \pi_i^k(a_i^j - a_i^1)}{(a_1^j - a_1^1) + \sum_{i=2}^{k} e_i(a_i^j - a_i^1)} \Longrightarrow -\gamma_1 \leq \frac{\sum_{i=2}^{k} f_i(a_i^j - a_i^1) + \sum_{i=k+1}^{m} \pi_i^k(a_i^j - a_i^1)}{(a_1^j - a_1^1) + \sum_{i=2}^{k} e_i(a_i^j - a_i^1)}. \tag{3.70}
$$

Furthermore, we have $\left((a_1^j - a_1^1) + \displaystyle\sum_{i=2}^{k} e_i(a_i^j - a_i^1)\right) > 0$. We so have;

$$
\begin{aligned}
\langle -\pi^{k+1}, a^j\rangle - \langle -\pi^{k+1}, a^1\rangle &= -\gamma_1 \left((a_1^j - a_1^1) + \sum_{i=2}^{k} e_i(a_i^j - a_i^1) \right) - \left(\sum_{i=2}^{k} f_i(a_i^j - a_i^1) + \sum_{i=k+1}^{m} \pi_i^k(a_i^j - a_i^1) \right) \\
&\leq \left(\frac{\displaystyle\sum_{i=2}^{k} f_i(a_i^j - a_i^1) + \sum_{i=k+1}^{m} \pi_i^k(a_i^j - a_i^1)}{\displaystyle (a_1^j - a_1^1) + \sum_{i=2}^{k} e_i(a_i^j - a_i^1)} \right) \left((a_1^j - a_1^1) + \sum_{i=2}^{k} e_i(a_i^j - a_i^1) \right) \\
&\quad - \left(\sum_{i=2}^{k} f_i(a_i^j - a_i^1) + \sum_{i=k+1}^{m} \pi_i^k(a_i^j - a_i^1) \right) \\
&= 0.
\end{aligned}
\tag{3.71}
$$

Then we conclude, $\langle -\pi^{k+1}, a^j\rangle \leq \langle -\pi^{k+1}, a^1\rangle$. So, $\langle -\pi^{k+1}, a^j\rangle \leq \beta^{k+1}$, for $j = k+1, \ldots, n$. ■

In the following example, we find a second rotation for the hyperplane $\mathcal{H}(\pi^2, 54)$ in Example 3.6 by using the results of Theorem 3.4.

Example 3.7. According to Theorem 3.4, the other bound for γ_2 in Example 3.6 is -1. So the hyperplane $\mathcal{H}(\pi^3, 15)$ where $\pi^3 = (1, -2, 2)^T$, with $\mathcal{S}_3 = \{a^3, a^4, a^6\}$ in the support set, supports $\text{con}(\mathcal{A})$.

Fig. 30. shows $\text{con}(\mathcal{A})$ and the plane $\mathcal{H}(\pi^3, 15)$ in three views.

Another possibility in System (3.62) is that all the coefficients of the variable γ_1 are negative, then we have:

$$\gamma_1 \geq \bar{\gamma}_j;\ j = k+1, \ldots, n. \tag{3.72}$$

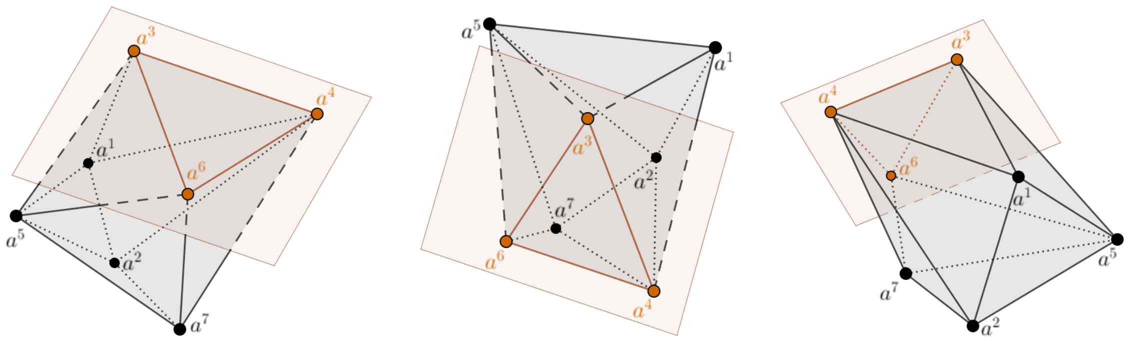

Fig. 30.: Three views of the supporting plane $\mathcal{H}(\pi^3, 15)$ where $\pi^3 = (1, -2, 2)^T$ for $\text{con}(\mathcal{A})$.

So, if we set:

$$\gamma_1 = \ \max \ \left\{\bar{\gamma}_j; \ j = k+1, \ldots, n.\right\} \tag{3.73}$$

defines a supporting hyperplane $\mathcal{H}(\pi^{k+1}, \beta^{k+1})$, which contains $a^1, \ldots, a^k, a^t$ where

$$t = \ \text{argmax} \ \left\{\bar{\gamma}_j; \ j = k+1, \ldots, n.\right\} \tag{3.74}$$

The following theorem presents a way to find a second supporting hyperplane in this scenario.

Theorem 3.5. *If all the coefficients of the variable γ_1 in System (3.62) are negative, then the value for γ_1 in System (3.64) obtained from*

$$\gamma_1 = \ \mathit{min} \ \left\{\bar{\gamma}_j; \ j = k+1, \ldots, n.\right\} \tag{3.75}$$

defines a supporting hyperplane $\mathcal{H}(-\pi^{k+1}, \beta^{k+1})$, *which contains* $a^1, \dots, a^k, a^t$ *where*

$$t = \ \textit{argmin} \ \left\{\bar{\gamma}_j; \ j = k+1, \dots, n.\right\} \tag{3.76}$$

Proof. The proof is similar to the proof of Theorem 3.4. ■

Example 3.8 demonstrates this scenario. In this example we find both possible of the rotated hyperplane by using Theorem 3.5. Example 3.8 is another possibility in Case 4.

Example 3.8. Consider the point set $\mathcal{A}$ from Example 3.1, $\mathcal{A} = \{a^1 = (2,8,4)^T, a^2 = (3,7,1)^T, a^3 = (5,4,9)^T, a^4 = (9,5,8)^T, a^5 = (1,2,3)^T, a^6 = (7,1,5)^T, a^7 = (6,3,2)^T\}$ in $\Re^3$, and the supporting plane $\mathcal{H}(\pi^2, 11)$ where $\pi^2 = (3,1,-5)^T$, with $\mathcal{S}_2 = \{a^2, a^7\}$ in the support set.

Fig. 31. shows the supporting plane $\mathcal{H}(\pi^2, 54)$ for con($\mathcal{A}$) in three views.

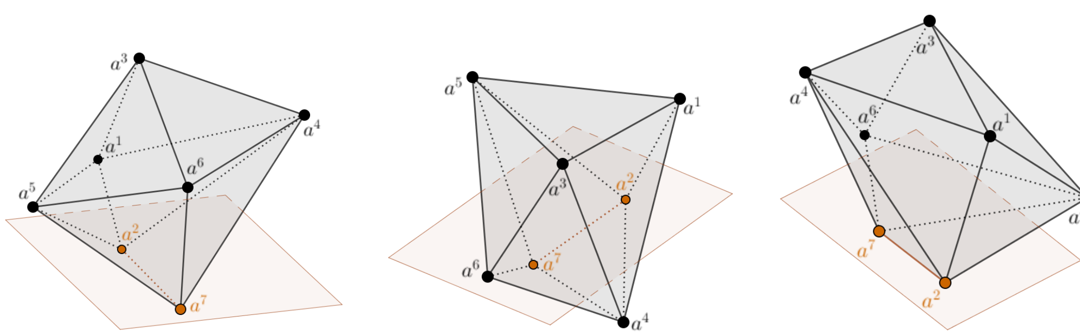

Fig. 31.: Three views of the supporting plane $\mathcal{H}(\pi^2, 54)$ where $\pi^2 = (3,1,-5)^T$ for con($\mathcal{A}$).

We define the new plane $\mathcal{H}(\pi^3, \beta^3)$ where $\pi^2 = (\gamma_1, 1, \gamma_2)^T$ and $\beta^3 = \langle \pi^3, a^2 \rangle$.

There are an equation and five inequalities and two variables. We solve this system similar to Example 3.7, then we find an upper bound $-\frac{6}{5}$ for γ_2.

If we set $\gamma_2 = -\frac{6}{5}$, then we get $\gamma_1 = \frac{26}{15}$. The normal of the rotated plane is $\pi^3 = (26, 15, -18)^T$, and $\beta^3 = 165$. The plane $\mathcal{H}(\pi^3, 165)$ is a supporting hyperplane for $\text{con}(\mathcal{A})$ with $\mathcal{S}_3 = \{a^2, a^4, a^7\}$ in the support set.

Fig. 32. shows the supporting plane $\mathcal{H}(\pi^3, 165)$ for $\text{con}(\mathcal{A})$ in three views.

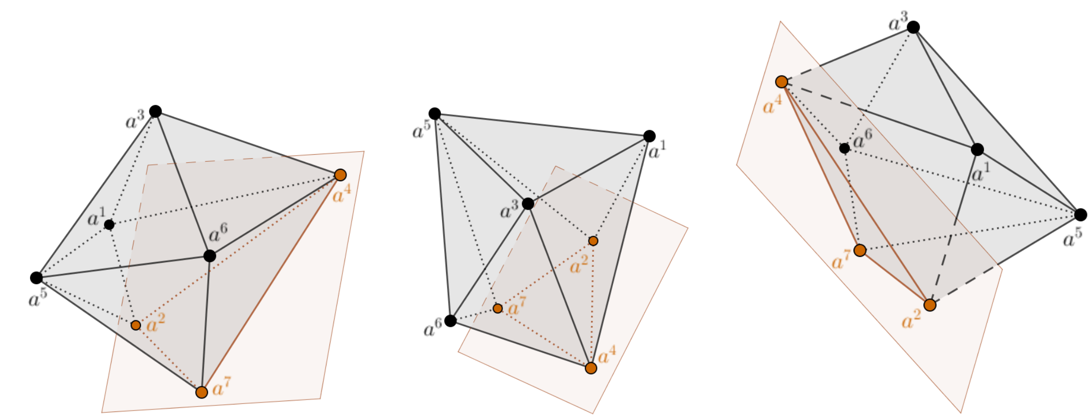

Fig. 32.: Three views of the supporting plane $\mathcal{H}(\pi^3, 165)$ where $\pi^3 = (26, 15, -18)^T$ for $\text{con}(\mathcal{A})$.

If we use Theorem 3.5, then the second rotation hyperplane is obtained by setting $\gamma_2 = \frac{23}{8}$, then we get $\gamma_1 = \frac{3}{8}$. So, the normal of the rotated plane is $\pi^3 = (-3, -8, -23)^T$, and $\beta^3 = -88$. The plane $\mathcal{H}(\pi^3, -88)$ is a supporting hyperplane for $\text{con}(\mathcal{A})$ with $\mathcal{S}_3 = \{a^2, a^5, a^7\}$ in the support set.

Fig. 33. shows the supporting plane $\mathcal{H}(\pi^3, -88)$ for $\text{con}(\mathcal{A})$ in three views.

Two Examples 3.6 and 3.8 demonstrate how Case 4 can be happened.

Finally we show how Case 5 is possible. This case happens when some of the coefficients of the variable γ_1 in System (3.62) be positive, and are negative for others.

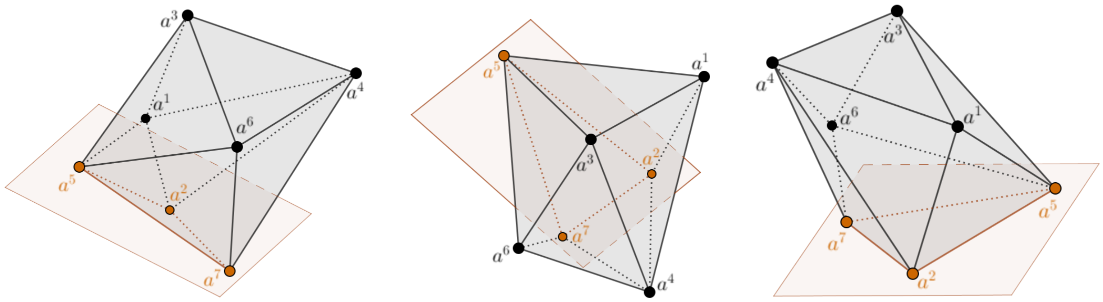

Fig. 33.: Three views of the supporting plane $\mathcal{H}(\pi^3, -88)$ where $\pi^3 = (-3, -8, -23)^T$ for $\text{con}(\mathcal{A})$.

Example 3.8 shows an example of this case.

Example 3.9. Consider the point set $\mathcal{A}$ from Example 3.1, $\mathcal{A} = \{a^1 = (2, 8, 4)^T, a^2 = (3, 7, 1)^T, a^3 = (5, 4, 9)^T, a^4 = (9, 5, 8)^T, a^5 = (1, 2, 3)^T, a^6 = (7, 1, 5)^T, a^7 = (6, 3, 2)^T\}$ in $\Re^3$, and the supporting plane $\mathcal{H}(\pi^2, 153)$ where $\pi^2 = (17, 16, -10)^T$, with $\mathcal{S}_2 = \{a^2, a^4\}$ in the support set.

Fig. 34. shows the supporting plane $\mathcal{H}(\pi^2, 54)$ for $\text{con}(\mathcal{A})$ in three views.

We define the new plane $\mathcal{H}(\pi^3, \beta^3)$ where $\pi^2 = (\gamma_1, 16, \gamma_2)^T$ and $\beta^3 = \langle \pi^3, a^2 \rangle$. There are an equation and five inequalities and two variables. We solve this system similar to the last example, then we find $-\frac{96}{5} \leq \gamma_2 \leq -\frac{64}{25}$.

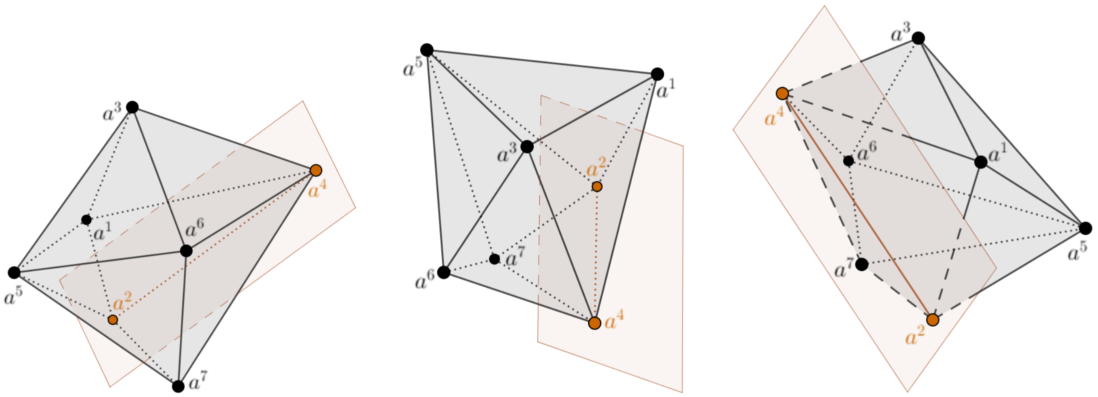

Fig. 34.: Three views of the supporting plane $\mathcal{H}(\pi^2, 54)$ where $\pi^2 = (17, 16, -10)^T$ for $\text{con}(\mathcal{A})$.

If we set $\gamma_2 = -\frac{96}{5}$, then we get $\gamma_1 = \frac{416}{15}$. The normal of the rotated plane is $\pi^3 = (26, 15, -18)^T$, and $\beta^2 = 165$. The plane $\mathcal{H}(\pi^3, 165)$ is a supporting hyperplane for $\text{con}(\mathcal{A})$ with $\mathcal{S}_2 = \{a^2, a^4, a^7\}$ in the support set. Fig. 35. shows the supporting plane $\mathcal{H}(\pi^3, 165)$ for $\text{con}(\mathcal{A})$ in three views.

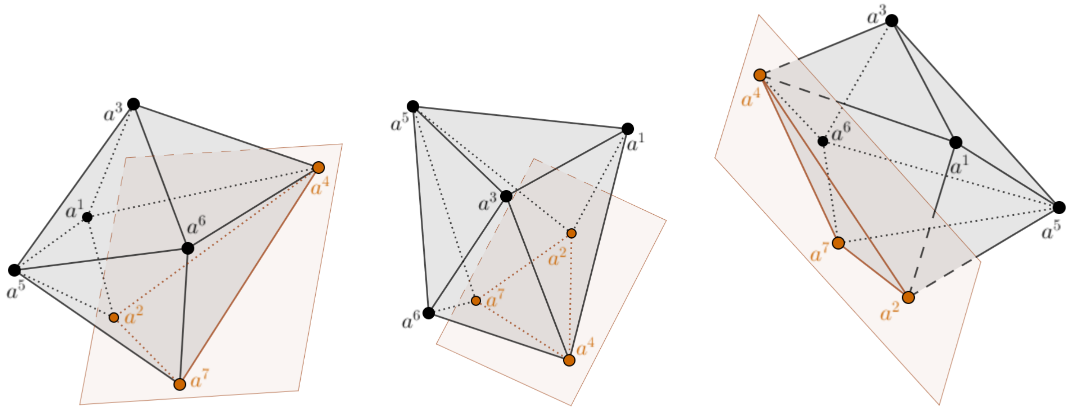

Fig. 35.: Three views of the supporting plane $\mathcal{H}(\pi^3, 165)$ where $\pi^3 = (26, 15, -18)^T$ for $\text{con}(\mathcal{A})$.

If we set $\gamma_2 = -\frac{64}{25}$, then we get $\gamma_1 = \frac{208}{25}$. So, the normal of the rotated plane is $\pi^3 = (13, 25, -4)^T$, and $\beta^2 = 210$. The plane $\mathcal{H}(\pi^3, 210)$ is a supporting hyperplane for $\text{con}(\mathcal{A})$ with $\mathcal{S}_2 = \{a^1, a^2, a^4\}$ in the support set. Fig. 36. shows the supporting plane $\mathcal{H}(\pi^3, 210)$ for $\text{con}(\mathcal{A})$ in three views.

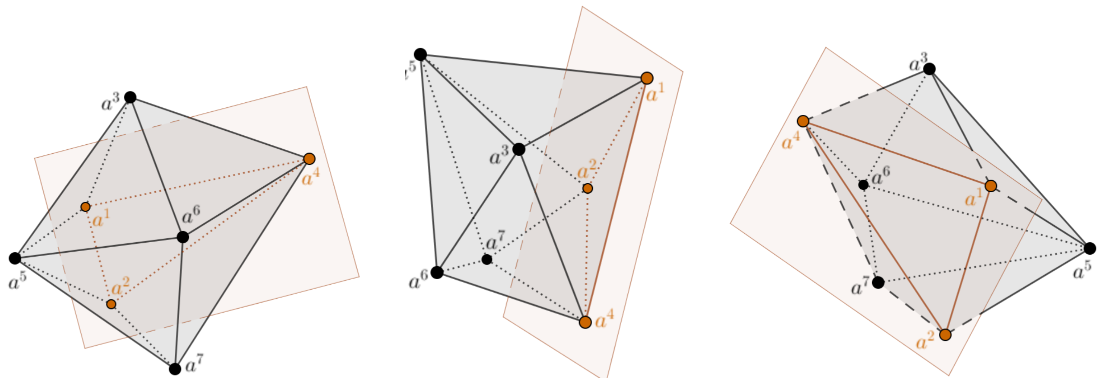

Fig. 36.: Three views of the supporting plane $\mathcal{H}(\pi^3, 210)$ where $\pi^3 = (13, 25, -4)^T$ for $\text{con}(\mathcal{A})$.

3.1.4 Procedure ARH pseudocode

Let a given point set $\mathcal{A} = \{a^1, \ldots, a^n\}$ and the supporting hyperplane $\mathcal{H}(\pi^k, \beta^k)$ in $\Re^m$. Assume, wlog, $\mathcal{S}_k = \{a^1, \ldots, a^k\}$ where $1 \leq k \leq m-1$. Procedure `ARH` to rotate the hyperplane $\mathcal{H}(\pi^k, \beta^k)$ is as follows.

Procedure `ARH`

Input : The point set $\mathcal{A} = \{a^1, \ldots, a^n\} \in \Re^m$, a supporting hyperplane $\mathcal{H}(\pi^k, \beta^k)$ with $\mathcal{S}_k = \{a^1, \ldots, a^k\}$ in the support set where $k < m$.

Output: The supporting hyperplane $\mathcal{H}(\pi^{k+1}, \beta^{k+1})$ with $\mathcal{S}_{k+1}$ in the support set that supports $\text{con}(\mathcal{A})$ at a facet.

while $(|\mathcal{S}_k| \; < \; m)$ **do**
- Define a hyperplane $\mathcal{H}(\pi^{k+1}, \beta^{k+1})$ where $\pi^{k+1} = (\gamma_1, \ldots, \gamma_k, \pi^k_{k+1}, \ldots, \pi^k_m)$. Note, wlog, all $\pi^k_{k+1}, \ldots, \pi^k_m$ cannot be zero together;
- Construct System (3.3), then find all the variables $\gamma_2, \ldots, \gamma_k$ in terms of γ_1;
- Construct System (3.4) in terms of just one variable γ_1;
- Identify a bound for γ_1 from System (3.4);
- Set the value of the variable γ_1 to the obtained bound, then find the values of all the variables $\gamma_2, \ldots, \gamma_k$;
- Set $\pi^k = \pi^{k+1}$;
- Update $\mathcal{S}_k$ by adding the new extreme point that is on the rotated hyperplane;

end

Calculate $\beta^{k+1} = \langle \pi^{k+1}, a^1 \rangle$;

The set $\mathcal{S}_{k+1}$ has at least m extreme points of $\text{con}(\mathcal{A})$. Therefore the dimension of the face that contains these m points is $m-1$, so this hyperplane contains a facet of $\text{con}(\mathcal{A})$.

Five observations about Procedure `ARH` are as follows:

Remark 1. There is no equation in System (3.3 - 3.3) if the support set of the hyperplane $\mathcal{H}(\pi^k, \beta^k)$ contains just a single point.

Remark 2. In all cases, the number of equalities and inequalities is always $n-1$, whereas the number of variables depends on the dimension of the point set $\mathcal{S}_k$, and is equal to k.

Remark 3. Procedures `ARH` and `FRH` can be initialized by a supporting hyperplane

$\mathcal{H}(\tilde{\pi}, \tilde{\beta})$ for $\text{con}(\mathcal{A})$ in $\Re^m$ where

$$\tilde{\pi} = \bar{1}, \text{ and } \tilde{\beta} = \max \left\{ \sum_{i=1}^{m} a_i^j \;\middle|\; a^j \in \mathcal{A} \right\}. \tag{3.77}$$

We have $\tilde{\pi} = (1, \ldots, 1)^T$, so the supporting hyperplane $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$ contains a point from $\mathcal{A}$ in which the sum of its elements is greater than the sum of the elements of any other point in $\mathcal{A}$.

Remark 4. Procedures `ARH` can be used to find a starting point (in the initial tableau) in dual simplex.

Remark 5. On the complexity of Procedures `ARH`, if the support set of the first supporting hyperplane has a single point from $\mathcal{A}$, then one overall needs to solve $\frac{(2n-m)(m-1)}{2}$ minimum and maximum ratio tests, and a system of $\frac{(m-1)(m-2)}{2}$ equations to find the final supporting hyperplane.

3.1.5 Example: Using Procedure `ARH` to rotate a plane in $\Re^3$

Consider the point set $\mathcal{A}$ from Example 3.1, $\mathcal{A} = \{a^1 = (2,8,4)^T, a^2 = (3,7,1)^T, a^3 = (5,4,9)^T, a^4 = (9,5,8)^T, a^5 = (1,2,3)^T, a^6 = (7,1,5)^T, a^7 = (6,3,2)^T\}$ in $\Re^3$, and the supporting plane $\mathcal{H}(\pi^1, 22)$ where $\pi^1 = (1,1,1)^T$, with the support set $\mathcal{S}_1 = \{a^4\}$.

This plane and $\text{con}(\mathcal{A})$ are shown in three views in Fig. 37.

We rotate this plane by applying Procedure `ARH`. To do that, we first define the plane $\mathcal{H}(\pi^2, \beta^2)$ where $\pi^2 = (\gamma_1, 1, 1)^T$, and $\beta^2 = \langle \pi^2, a^4 \rangle$. Here $k = 1$, so System (3.3 - 3.4) has no equation. We thus have a system with six inequalities and a single variable:

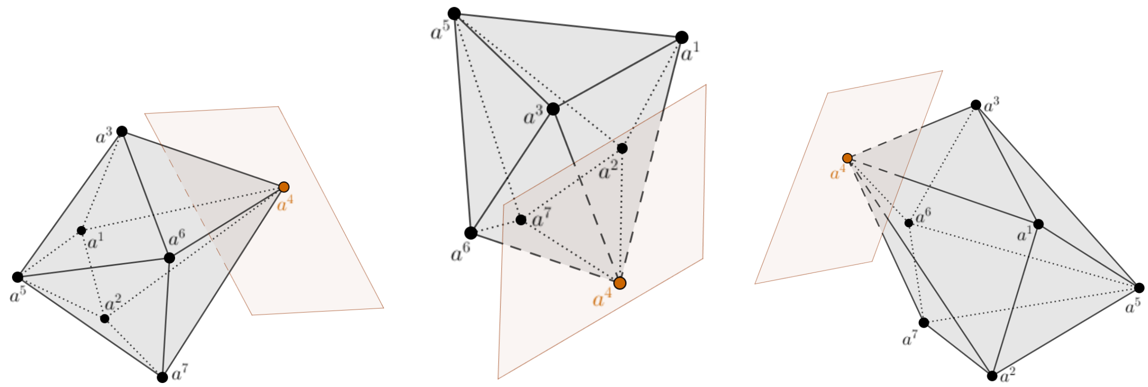

Fig. 37.: Three views of the supporting plane $\mathcal{H}(\pi^1, 22)$ where $\pi^1 = (1,1,1)^T$ for $\text{con}(\mathcal{A})$.

$$\gamma_1 \geq -\frac{\langle \hat{\pi}^2, \hat{a}^1 \rangle - \langle \hat{\pi}^2, \hat{a}^4 \rangle}{a_1^1 - a_1^4} = -\frac{12-13}{2-9} \implies \gamma_1 \geq -\frac{1}{7}; \tag{3.78}$$

$$\gamma_1 \geq -\frac{\langle \hat{\pi}^2, \hat{a}^2 \rangle - \langle \hat{\pi}^2, \hat{a}^4 \rangle}{a_1^2 - a_1^4} = -\frac{8-13}{3-9} \implies \gamma_1 \geq -\frac{5}{6}; \tag{3.79}$$

$$\gamma_1 \geq -\frac{\langle \hat{\pi}^2, \hat{a}^3 \rangle - \langle \hat{\pi}^2, \hat{a}^4 \rangle}{a_1^3 - a_1^4} = -\frac{13-13}{5-9} \implies \gamma_1 \geq \quad 0; \tag{3.80}$$

$$\gamma_1 \geq -\frac{\langle \hat{\pi}^2, \hat{a}^5 \rangle - \langle \hat{\pi}^2, \hat{a}^4 \rangle}{a_1^5 - a_1^4} = -\frac{5-13}{1-9} \implies \gamma_1 \geq \ -1; \tag{3.81}$$

$$\gamma_1 \geq -\frac{\langle \hat{\pi}^2, \hat{a}^6 \rangle - \langle \hat{\pi}^2, \hat{a}^4 \rangle}{a_1^6 - a_1^4} = -\frac{6-13}{7-9} \implies \gamma_1 \geq -\frac{7}{2}; \tag{3.82}$$

$$\gamma_1 \geq -\frac{\langle \hat{\pi}^2, \hat{a}^7 \rangle - \langle \hat{\pi}^2, \hat{a}^4 \rangle}{a_1^7 - a_1^4} = -\frac{5-13}{6-9} \implies \gamma_1 \geq -\frac{8}{3}. \tag{3.83}$$

This generates a bound 0 for γ_1: $\gamma_1 \geq 0$.

If we set $\gamma_1 = 0$, then we get the hyperplane $\mathcal{H}(\pi^2, \beta^2)$ where $\pi^2 = (0,1,1)^T$, and $\beta^2 = 13$. The support set for this hyperplane contains $\mathcal{S}_2 = \{a^3,\ a^4\}$. Fig. 38. shows $\text{con}(\mathcal{A})$ and the hyperplane $\mathcal{H}(\pi^2, 13)$ in three views.

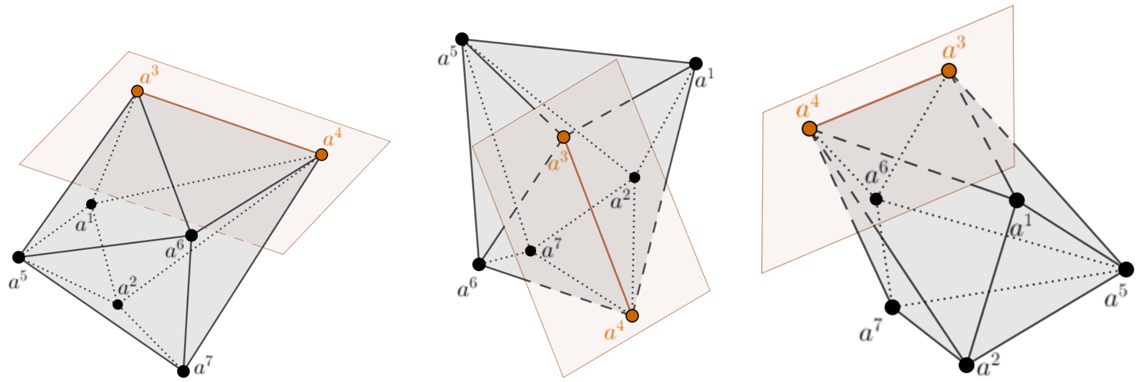

Fig. 38.: Three views of the supporting plane $\mathcal{H}(\pi^2, 13)$ where $\pi^2 = (0, 1, 1)^T$ for $\text{con}(\mathcal{A})$.

One more rotation is needed to get a facet of $\text{con}(\mathcal{A})$. To do so, we define the plane $\mathcal{H}(\pi^3, \beta^3)$ where $\pi^3 = (\gamma_1, \gamma_2, 1)^T$, and $\beta^3 = \langle \pi^3, a^4 \rangle$. There exists one equation:

$$\langle \pi^3, a^3 \rangle = \langle \pi^3, a^4 \rangle \Longrightarrow \gamma_2 = -4\gamma_1 + 1. \tag{3.84}$$

Here $k = 2$ and $n = 7$, so there are five inequalities:

$$\langle \pi^3, a^1 \rangle \leq \langle \pi^3, a^4 \rangle \Longrightarrow -7\gamma_1 + 3\gamma_2 \leq 4; \tag{3.85}$$

$$\langle \pi^3, a^2 \rangle \leq \langle \pi^3, a^4 \rangle \Longrightarrow -6\gamma_1 + 2\gamma_2 \leq 7; \tag{3.86}$$

$$\langle \pi^3, a^5 \rangle \leq \langle \pi^3, a^4 \rangle \Longrightarrow -8\gamma_1 - 3\gamma_2 \leq 5; \tag{3.87}$$

$$\langle \pi^3, a^6 \rangle \leq \langle \pi^3, a^4 \rangle \Longrightarrow -2\gamma_1 - 4\gamma_2 \leq 3; \tag{3.88}$$

$$\langle \pi^3, a^7 \rangle \leq \langle \pi^3, a^4 \rangle \Longrightarrow -3\gamma_1 - 2\gamma_2 \leq 6. \tag{3.89}$$

If we use the obtained γ_2 from (3.84) in these five inequalities, then we get five inequalities in terms of just the variable γ_1 and a constant as follows: $\gamma_1 \geq -\frac{1}{19}$, $\gamma_1 \geq -\frac{5}{14}$, $\gamma_1 \leq 2$, $\gamma_1 \leq \frac{1}{2}$, $\gamma_1 \leq \frac{8}{5}$. This generates the following bounds for γ_1;

$-\frac{1}{19} \leq \gamma_1 \leq \frac{1}{2}$.

If we set $\gamma_1 = -\frac{1}{19}$, then from (3.84), we get $\gamma_2 = \frac{23}{19}$. It yields the plane $\mathcal{H}(\pi^3, \beta^3)$ where $\pi^3 = (-1, 23, 19)^T$, and $\beta^3 = 258$. The support set for this hyperplane includes $\mathcal{S}_3 = \{a^1, a^3, a^4\}$.

The hyperplane $\mathcal{H}(\pi^3, 258)$ and con($\mathcal{A}$) in three views are shown in Fig. 39.

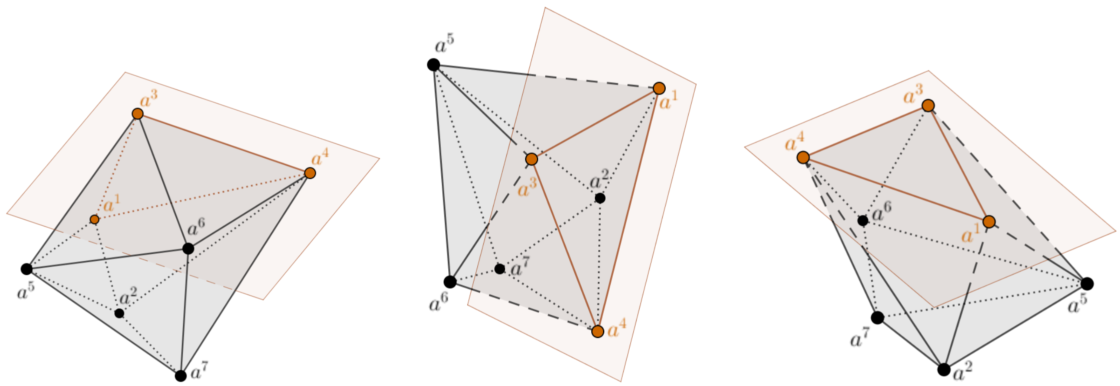

Fig. 39.: Three views of the supporting plane $\mathcal{H}(\pi^3, 258)$ where $\pi^3 = (-1, 23, 19)^T$ for con($\mathcal{A}$).

If we set $\gamma_1 = \frac{1}{2}$, then from (3.84), we get $\gamma_2 = -1$. It yields the plane $\mathcal{H}(\pi^3, \beta^3)$ where $\pi^3 = (1, 2, 2)^T$, and $\beta^3 = 15$. The support set for this hyperplane includes $\mathcal{S}_3 = \{a^3, a^4, a^6\}$.

The hyperplane $\mathcal{H}(\pi^3, 15)$ and con($\mathcal{A}$) in three views are shown in Fig. 40.

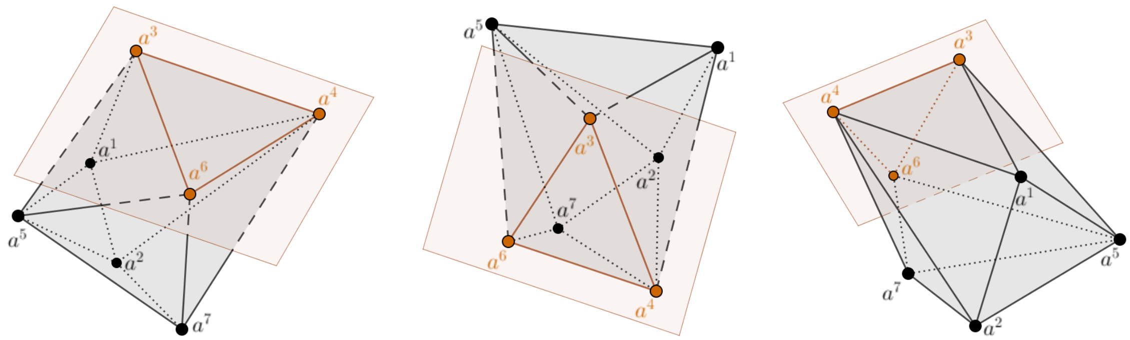

Fig. 40.: Three views of the supporting plane $\mathcal{H}(\pi^3, 15)$ where $\pi^3 = (1,2,2)^T$ for $\operatorname{con}(\mathcal{A})$.

3.2 Procedure Full Rotation Hyperplane (FRH)

In the previous section we introduced Procedure ARH to rotate a supporting hyperplane for the convex hull of a finite point set from a k-dimensional face to another face with one more dimension. Recall that both faces share k extreme points and the landing face has one additional extreme point. In this section we present a procedure to rotate a supporting hyperplane containing k extreme points that will land directly on a facet.

Recall that we assumed, the initial support set for the hyperplane $\mathcal{H}(\pi^k, \beta^k)$, wlog, includes $\mathcal{S}_k = \{a^1, \ldots, a^k\}$, and $\mathcal{S}_{k+1}$ of the rotated hyperplane $\mathcal{H}(\pi^{k+1}, \beta^{k+1})$ must contain $\mathcal{S}_k$. We rotate the supporting hyperplane $\mathcal{H}(\pi^k, \beta^k)$ such that its intersection with $\operatorname{con}(\mathcal{A})$ is a facet and it contains at least m extreme points of the set $\mathcal{A}$, regardless of the dimension of its initial support set. To do this, we define the hyperplane $\mathcal{H}(\pi^{k+1}, \beta^{k+1})$ where $\beta^{k+1} = \langle \pi^{k+1}, a^1 \rangle$. π^k and π^{k+1} will share one component. This component must be non-zero; the rest of the component will be variables. This component must be non-zero. Assume, wlog, π_m^k is this common element. Next, we define $\pi^{k+1} = (\gamma_1, \ldots, \gamma_{m-1}, \pi_m^k)$ with $m-1$ scalar variables $\gamma_1, \ldots, \gamma_{m-1}$. The k extreme points in the initial support set $\mathcal{S}_k$ will remain in the final rotation. This defines the following k equations:

$$\langle \pi^{k+1}, a^j \rangle = \langle \pi^{k+1}, a^1 \rangle; \quad j = 2, \ldots, k. \tag{3.90}$$

The remaining points $a^{k+1}, \ldots, a^n$ will be located in the halfspace defined by the hyperplane $\mathcal{H}(\pi^{k+1}, \beta^{k+1})$. This defines the following inequalities:

$$\langle \pi^{k+1}, a^j \rangle \leq \langle \pi^{k+1}, a^1 \rangle; \quad j = k+1, \ldots, n. \tag{3.91}$$

System (3.90 - 3.91) defines a polyhedral feasible region in $\Re^{m-1}$ with as many dimensions as the affine set defined by the solutions to System (3.90). Similar to System (3.3 - 3.4), it is the intersection of an affine set and a polyhedron defined by the intersection of halfspaces.

Similar arguments as those in the previous section can be used to show affine set has full dimension. So, the set of solutions to System (3.90 - 3.91) is not empty and must contain an infinity of points. The set of feasible solutions to system System (3.90 - 3.91) is a $m-k-1$-dimensional polyhedral set with, possibly, a multitude of extreme points. The next result establishes that an extreme point of this set defines a facet of $\text{con}(\mathcal{A})$.

Theorem 3.6. *An extreme point solution to System (3.90 - 3.91) defines the normal of a supporting hyperplane for a facet of* $\text{con}(\mathcal{A})$.

Proof. Recall that the current supporting hyperplane with $\mathcal{S}_k$ in the support set is a non-trivial solution to System (3.90 - 3.91). So, System (3.90 - 3.91) is non-empty.

We rewrite System (3.90) as follows:

$$\langle \pi^{k+1}, a^j \rangle - \langle \pi^{k+1}, a^1 \rangle = 0; \quad j = 2, \ldots, k. \tag{3.92}$$

This system has $k-1$ equations. We use similar arguments as those used in Theorem **??** to show that the $(k-1) \times (m-1)$ matrix of the coefficients of System (3.92) has full rank.

System (3.90 - 3.91) defines a polyhedral region in $\Re^{m-1}$. It is the intersection of an $(m-k)$-dimensional affine set defined by System (3.90) with the m-dimensional polyhedron defined by the feasible points in System (3.91). This itself is a polyhedron with $m-k$ dimensions. This polyhedron has extreme points which occur where the affine set intersects the boundary of the polyhedron defined by System (3.91). This intersection is a point where $m-k$ independent hyperplanes of the polyhedron defined by System (3.91) meet. There are many ways this can happen since there are $n-k$ inequalities System (3.91) and they are all independent by our assumptions. Such a point would be extreme to System (3.90 - 3.91). Its components are the normal of a supporting hyperplane where the number of extreme points of the convex hull in the support is at least m and hence a facet. ■

We use Theorem 3.6 to formulate the following LP for rotating a supporting hyperplane $\mathcal{H}(\pi^k, \beta^k)$ on the convex hull of the point set $\mathcal{A}$.

$$\begin{aligned}
&\max_{\gamma_1,\ldots,\gamma_{m-1}} && \sum_{j=k+1}^{n} \langle \bar{\pi}^{k+1}, \bar{a}^j - \bar{a}^1 \rangle && \textbf{FRH LP} \\
&\text{s.t.} && \langle \pi^{k+1}, a^j \rangle = \langle \pi^{k+1}, a^1 \rangle; && j = 2, \ldots, k, \\
& && \langle \pi^{k+1}, a^j \rangle \leq \langle \pi^{k+1}, a^1 \rangle; && j = k+1, \ldots, n,
\end{aligned} \tag{3.93}$$

where truncated normal $\bar{\pi}^{k+1} = (\gamma_1, \ldots, \gamma_{m-1})^T$, and truncated point $\bar{a}^j = (a_1^j, \ldots, a_{m-1}^j)^T$ for $j = 1, \ldots, n$. `FRH` LP has $m-1$ variables and $n-1$ constraints.

Recall that `FRH` LP is not empty. If the optimal value $\gamma_1^*, \ldots, \gamma_{m-1}^*$ of `FRH` LP is feasible and bounded, then the hyperplane $\mathcal{H}(\pi^{k+1}, \beta^{k+1})$ where $\pi^{k+1} = (\gamma_1^*, \ldots, \gamma_{m-1}^*, \pi_m^k)^T$ corresponds to a supporting hyperplane for con($\mathcal{A}$) with the $(m-1)$-dimensional support set. In the following theorem, we prove that the optimal value of `FRH` LP is bounded, so this procedure addresses the rotated hyperplane to a desired hyperplane.

Theorem 3.7. *The optimal value of* `FRH` *LP is bounded.*

Proof. We rewrite `FRH` LP as follows:

$$\begin{aligned}
&\max_{\gamma_1,\ldots,\gamma_{m-1}} && \sum_{i=1}^{m-1} \left(\sum_{j=k+1}^{n} (a_i^j - a_i^1) \right) \gamma_i && \\
&\text{s.t.} && \sum_{i=1}^{m-1} (a_i^j - a_i^1)\gamma_i = -(a_m^j - a_m^1); && j = 2, \ldots, k, \\
& && \sum_{i=1}^{m-1} (a_i^j - a_i^1)\gamma_i \leq -(a_m^j - a_m^1); && j = k+1, \ldots, n.
\end{aligned} \tag{3.94}$$

The dual of `FRH` LP is:

$$\min_{\lambda_2,\ldots,\lambda_n} \quad -\sum_{j=2}^{n}(a_m^j - a_m^1)\lambda_j \tag{3.95}$$
$$\text{s.t.} \quad \sum_{j=2}^{n}(a_i^j - a_i^1)\lambda_j = \sum_{j=k+1}^{n}(a_i^j - a_i^1); \qquad i = 1,\ldots,m-1,$$
$$\lambda_j \geq 0; \qquad j = k+1,\ldots,n.$$

LP formulation (3.95) is feasible if we set $\lambda_j = 0$ for $j = 2,\ldots,k$, and $\lambda_j = 1$ for $j = k+1,\ldots,n$. Therefore, based on weak duality theorem, `FRH` LP is feasible and bounded. ■

3.2.1 Procedure `FRH` pseudocode

Consider a given point set $\mathcal{A} = \{a^1,\ldots,a^n\}$ and the supporting hyperplane $\mathcal{H}(\pi^k,\beta^k)$ in $\Re^m$. Assume, wlog, $\mathcal{S}_k = \{a^1,\ldots,a^k\}$ where $1 \leq k \leq m-1$. Procedure `FRH` to rotate the hyperplane $\mathcal{H}(\pi^k,\beta^k)$ is as follows.

Procedure `FRH`

Input : The point set $\mathcal{A} = \{a^1,\ldots,a^n\} \in \Re^m$, a supporting hyperplane $\mathcal{H}(\pi^k,\beta^k)$ with $\mathcal{S}_k$ in the support set where $|\mathcal{S}_k| = k < m$.

Output: The supporting hyperplane $\mathcal{H}(\pi^{k+1},\beta^{k+1})$ with $\mathcal{S}_{k+1}$ in the support set that supports con($\mathcal{A}$) at a facet.

Define the hyperplane $\mathcal{H}(\pi^{k+1},\beta^{k+1})$ where $\pi^{k+1} = (\gamma_1,\ldots,\gamma_{m-1},\pi_m^k)^T$. Note, wlog, $\pi_m^k \neq 0$;
Find optimal basic feasible solution to LP (3.93), and suppose the optimum solution of the variables are $\gamma_1^*,\ldots,\gamma_{m-1}^*$;
Set $\pi^{k+1} = (\gamma_1^*,\ldots,\gamma_{m-1}^*,\pi_m^k)^T$;
Calculate $\beta^{k+1} = \langle\pi^{k+1},a^1\rangle$;
Find out $\mathcal{S}_{k+1}$;

The support set $\mathcal{S}_{k+1}$ includes at least m extreme points of con($\mathcal{A}$). Therefore the dimension of the face that contains these m points is $m-1$, so this hyperplane contains a facet of con($\mathcal{A}$).

Three observations about Procedure `FRH` are as follows:

Remark 1. There are no equations in System (3.90 - 3.91) if the support set of the hyperplane $\mathcal{H}(\pi^k, \beta^k)$ contains just a single point.

Remark 2. In LP (3.93), the different objective functions may define different facets. Individual inequalities or groups, can be the source for different objective functions.

Remark 3. On the complexity of Procedures `FRH`, one needs to solve an LP with $m-1$ variables and $n-1$ constraints for finding the final supporting hyperplane.

3.2.2 Example: Using Procedure `FRH` to rotate a plane in $\Re^3$

Use the point set $\mathcal{A}$ from Example 3.1, $\mathcal{A} = \{a^1 = (2,8,4)^T, a^2 = (3,7,1)^T, a^3 = (5,4,9)^T, a^4 = (9,5,8)^T, a^5 = (1,2,3)^T, a^6 = (7,1,5)^T, a^7 = (6,3,2)^T\}$ in $\Re^3$, and consider the supporting plane $\mathcal{H}(\pi^1, 22)$ where $\pi^1 = (1,1,1)^T$, with the support set $\mathcal{S}_1 = \{a^4\}$.

This plane and con($\mathcal{A}$) are shown in three views in Fig. 41.

We rotate this plane by applying Procedure `FRH` such that its intersection with con($\mathcal{A}$) contains at last three points of the set $\mathcal{A}$. To do so, we define the plane $\mathcal{H}(\pi^2, \beta^2)$ where $\pi^2 = (\gamma_1, \gamma_2, 1)^T$ and $\beta^2 = \langle \pi^2, a^4 \rangle$. We need to solve LP:

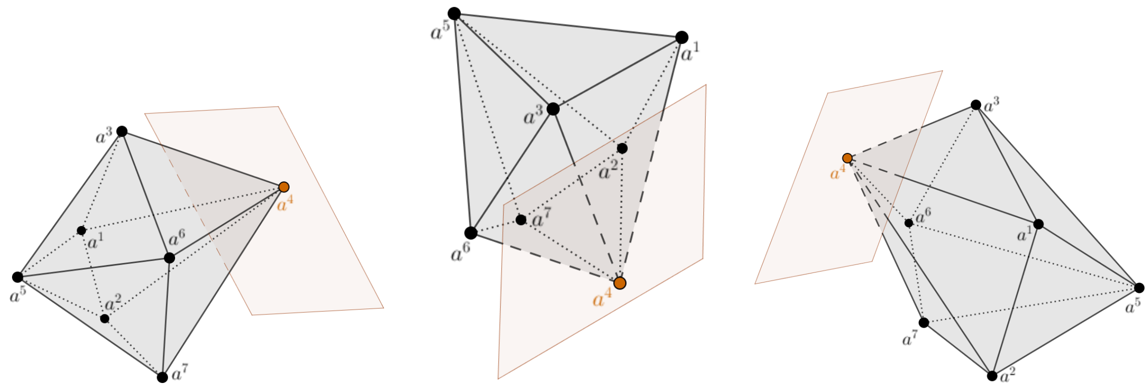

Fig. 41.: Three views of the supporting plane $\mathcal{H}(\pi^1, 22)$ where $\pi^1 = (1,1,1)^T$ for $\text{con}(\mathcal{A})$.

$$\begin{aligned}
\max_{\gamma_1,\gamma_2} \ & -30\gamma_1 - 5\gamma_2 && (3.96)\\
\text{s.t.} \ & -7\gamma_1 + 3\gamma_2 \leq 4,\\
& -6\gamma_1 + 2\gamma_2 \leq 7,\\
& -4\gamma_1 - \gamma_2 \leq -1,\\
& -8\gamma_1 - 3\gamma_2 \leq 5,\\
& -2\gamma_1 - 4\gamma_2 \leq 3,\\
& -3\gamma_1 - 2\gamma_2 \leq 6.
\end{aligned}$$

Notice that the objective function here is constructed by adding the six inequalities. By solving this LP, we get the optimal solution $\gamma_1^* = -\frac{1}{19}$ and $\gamma_2^* = \frac{23}{19}$. Hence we get the plane $\mathcal{H}(\pi^2, 258)$ where $\pi^2 = (-1, 23, 19)^T$. This plane contains the points a^1, a^2, a^4. Fig. 42. shows this rotation.

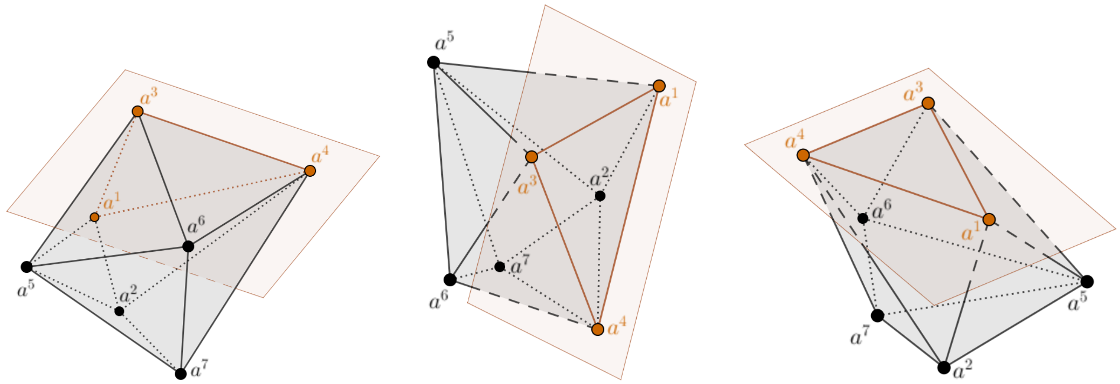

Fig. 42.: Three views of the supporting plane $\mathcal{H}(\pi^2, 258)$ where $\pi^2 = (-1, 23, 19)^T$ for con($\mathcal{A}$).

3.3 Procedure Facet To Facet (FTF)

We introduce a procedure to rotate a supporting hyperplane for con($\mathcal{A}$) to get another supporting hyperplane where the support sets of the initial and the rotated hyperplanes are two adjacent facets of con($\mathcal{A}$). We refer to this as facet to facet rotation.

If a hyperplane contains a facet of con($\mathcal{A}$), then it contains m points of the point set $\mathcal{A}$. Recall that we assumed there is no degenerate facet. Moreover, two supporting hyperplanes for con($\mathcal{A}$) that contain two adjacent facets, have $m-1$ points of $\mathcal{A}$ in common. Indeed, the intersection of these two supporting hyperplanes, contains a face of con($\mathcal{A}$). The dimension of this face for con($\mathcal{A}$) is $m-2$. Therefore, this face has $m-1$ points of $\mathcal{A}$. So, a hyperplane containing a facet of con($\mathcal{A}$) can be rotated in such a way that the new hyperplane will share $m-1$ extreme points of $\mathcal{A}$ and land in a different facet.

The first hyperplane can be calculated knowing the m extreme points of the support facet. The second hyperplane follows from deciding which extreme point will be replaced in the support set and identifying its replacement. The replacement is determined by a minimum ratio calculation. The algebraic derivation has been relegated to Appendix A.

3.3.1 Procedure `FTF` pseudocode

Procedure `FTF` for a given point set $\mathcal{A}$ and supporting hyperplane $\mathcal{H}(\pi^1, \beta^1)$ in $\Re^m$ is as follows.

Procedure `FTF`

Input : The point set $\mathcal{A} = \{a^1, \ldots, a^n\} \in \Re^m$, a supporting hyperplane $\mathcal{H}(\pi^1, \beta^1)$, and, wlog, $\mathcal{S}_1 = \{a^1, \ldots, a^{m-1}, a^m\}$ in the support set that supports con($\mathcal{A}$) at a facet.

Output: The supporting hyperplane $\mathcal{H}(\pi^2, \beta^2)$ with $\mathcal{S}_2 = \{a^1, \ldots, a^{m-1}, a^k\}$ in the support set where a^k is one of the points $a^{m+1}, \ldots, a^n$. This hyperplane supports con($\mathcal{A}$) at an adjacent facet with the facet that is contained by the hyperplane $\mathcal{H}(\pi^1, \beta^1)$.

1 Define System $\langle \pi, a^j \rangle = 1$; for $j = 1, \ldots, m-1$ where $\pi = (\gamma_1, \ldots, \gamma_m)^T$;
2 Find the variables $\gamma_2, \ldots, \gamma_m$ in terms of the variable γ_1 from this system of equations;
3 Define System $\langle \pi, a^j \rangle \leq 1$; $\quad j = m, \ldots, n$;
4 Rewrite this system of inequalities based on one variable γ_1 and a constant for each inequalities;
5 Solve this system of inequalities to find a lower bound γ_1^l and an upper bound γ_1^u for γ_1;
6 Set $\gamma_1 = \gamma_1^l$, find the values of the variables $\gamma_2, \ldots, \gamma_m$, and name them $\gamma_2^l, \ldots, \gamma_m^l$. Then define the hyperplane $\mathcal{H}(\pi^l, 1)$ where $\pi^l = (\gamma_1^l, \gamma_2^l, \ldots, \gamma_m^l)^T$;
7 Set $\gamma_1 = \gamma_1^u$, find the value of the variables $\gamma_2, \ldots, \gamma_m$, and name them $\gamma_2^u, \ldots, \gamma_m^u$. Then define the hyperplane $\mathcal{H}(\pi^u, 1)$ where $\pi^u = (\gamma_1^u, \gamma_2^u, \ldots, \gamma_m^u)^T$;
8 The intersection of each hyperplanes $\mathcal{H}(\pi^l, 1)$ and $\mathcal{H}(\pi^u, 1)$ with con($\mathcal{A}$) is a facet of con($\mathcal{A}$) such that these two facets are adjacent. One of these two hyperplanes $\mathcal{H}(\pi^l, 1)$ and $\mathcal{H}(\pi^u, 1)$ is exactly the hyperplane $\mathcal{H}(\pi^1, 1)$, and other one is the rotated of the hyperplane $\mathcal{H}(\pi^1, 1)$ that we name it $\mathcal{H}(\pi^2, 1)$;

Two observations about Procedure `FTF` are as follows:

Remark 1. In all cases, the number of equations is $m-1$, the number of inequalities is $n-m+1$, and the number of variables is m.

Remark 2. On the complexity of Procedures `FTF`, one needs to do one pivot that has $(n-m+1)$ minimum and maximum ratio tests, and a system of $(m-1)$ equations to find the final supporting hyperplane.

3.3.2 Example: Using Procedure FTF to rotate a plane in $\Re^3$

Let the point set $\mathcal{A} = \{a^1 = (2,8,4)^T, a^2 = (3,7,1)^T, a^3 = (5,4,9)^T, a^4 = (9,5,8)^T, a^5 = (1,2,3)^T, a^6 = (7,1,5)^T, a^7 = (6,3,2)^T\}$ in $\Re^m$ from Example 3.1, and the supporting plane $\mathcal{H}(\pi^1, 258)$ where $\pi^1 = (-1,23,19)^T$, with $\mathcal{S}_1 = \{a^1, a^3, a^4\}$ in the support set. This plane and con($\mathcal{A}$) are shown in three views in Fig. 43.

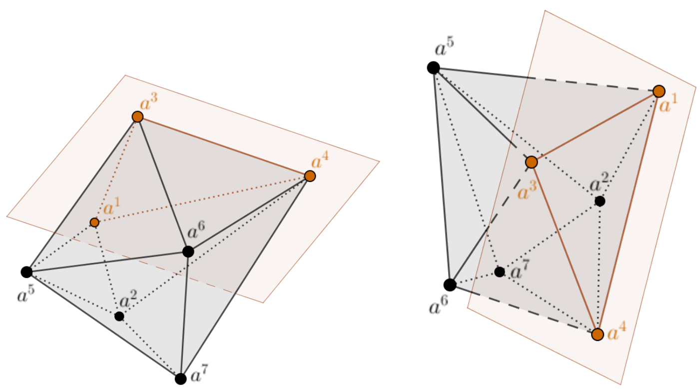

Fig. 43.: Three views of the supporting plane $\mathcal{H}(\pi^1, 258)$ where $\pi^1 = (-1,23,19)^T$ for con($\mathcal{A}$).

We define a family of a hyperplane $\mathcal{H}(\pi, 1)$ where $\pi = (\gamma_1, \gamma_2, \gamma_3)^T$ such that a^1 and a^3 are always in the its support set. Then, we have

$$2\gamma_1 + 8\gamma_2 + 4\gamma_3 = 1, \tag{3.97}$$

$$5\gamma_1 + 4\gamma_2 + 9\gamma_3 = 1. \tag{3.98}$$

We can solve this system of two equations to get γ_2 and γ_3 in terms of γ_1 and a constant value as follows.

$$\gamma_2 = \frac{1}{28}\gamma_1 + \frac{5}{56}; \tag{3.99}$$

$$\gamma_3 = -\frac{4}{7}\gamma_1 + \frac{1}{14}. \tag{3.100}$$

From (A.12), for five points a^2, a^4, a^5, a^6, and a^7 below the first hyperplane, we have:

$$3\gamma_1 + 7(\frac{1}{28}\gamma_1 + \frac{5}{56}) + (-\frac{4}{7}\gamma_1 + \frac{1}{14}) \leq 1 \Longrightarrow \gamma_1 \leq \frac{17}{150}; \tag{3.101}$$

$$9\gamma_1 + 5(\frac{1}{28}\gamma_1 + \frac{5}{56}) + 8(-\frac{4}{7}\gamma_1 + \frac{1}{14}) \leq 1 \Longrightarrow \gamma_1 \leq -\frac{1}{258}; \tag{3.102}$$

$$\gamma_1 + 2(\frac{1}{28}\gamma_1 + \frac{5}{56}) + 3(-\frac{4}{7}\gamma_1 + \frac{1}{14}) \leq 1 \Longrightarrow \gamma_1 \geq -\frac{17}{18}; \tag{3.103}$$

$$7\gamma_1 + (\frac{1}{28}\gamma_1 + \frac{5}{56}) + 5(-\frac{4}{7}\gamma_1 + \frac{1}{14}) \leq 1 \Longrightarrow \gamma_1 \leq \frac{31}{234}; \tag{3.104}$$

$$6\gamma_1 + 3(\frac{1}{28}\gamma_1 + \frac{5}{56}) + 2(-\frac{4}{7}\gamma_1 + \frac{1}{14}) \leq 1 \Longrightarrow \gamma_1 \leq \frac{33}{278}. \tag{3.105}$$

This generates the following bounds for γ_1:

$$-\frac{17}{18} \leq \gamma_1 \leq -\frac{1}{258}. \tag{3.106}$$

So we have this lower bound, a minimum ratio, $\gamma_1^l = -\frac{17}{18}$. This value γ_1^l is used to calculate the other two unknowns using System (3.99 - 3.100) to get $\gamma_2^l = \frac{1}{18}$, and $\gamma_3^l = \frac{11}{18}$. The new point is a^5 which along with the a^1 and a^3 define the landing facet. We can think of the two points common to both supporting hyperplanes as the "hinge" of this rotation. The level of this hyperplane is 1. This hyperplane is different from the starting supporting hyperplane $\mathcal{H}(\pi^1, \beta^1)$. We have the obtained hyperplane $\mathcal{H}(\pi^2, \beta^2)$ where $\pi^2 = (-17, 1, 11)^T$ and $\beta^2 = 18$, and its support set includes $\mathcal{S}_2 = \{a^1, a^3, a^5\}$. This plane and con($\mathcal{A}$) are shown in three views in Fig. 44.

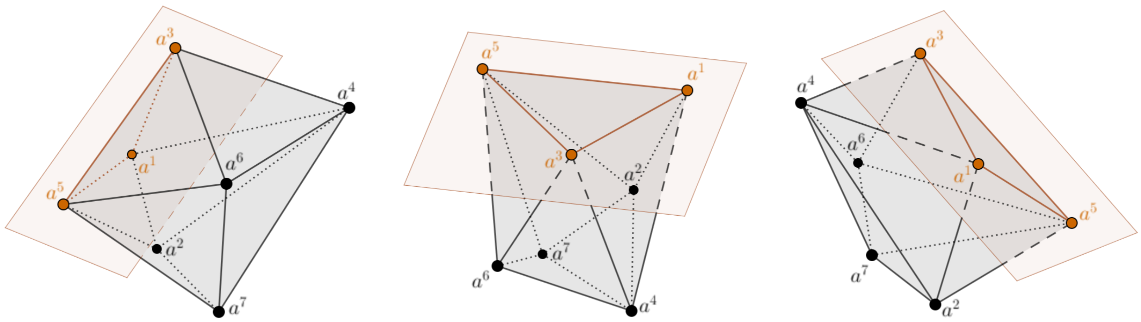

Fig. 44.: Three views of the supporting plane $\mathcal{H}(\pi^2, 18)$ where $\pi^2 = (-17, 1, 11)^T$ for con($\mathcal{A}$).

The upper limit of γ_1 from (3.106) is $\gamma_1^u = -\frac{1}{258}$, with this we get $\gamma_2^u = \frac{23}{258}$, and $\gamma_3^u = \frac{19}{258}$. The level of this hyperplane is also one. This is the the original starting hyperplane $\mathcal{H}(\pi^1, 258)$.

Theorem 3.8. *Procedure* `FTF` *is equivalent to a dual simplex pivot.*

Proof. The proof has been relegated to Appendix A. ■

3.4 Conclusion

In this chapter we presented three procedures to rotate a supporting hyperplane on the convex hull of a finite point set.

We presented Procedure `ARH` that relies on linear algebra operations to rotate a supporting hyperplane on the convex hull of a point set in $\Re^m$ such that the dimension of the support set for this initial hyperplane is fewer than $m - 1$. The support set for the rotated supporting hyperplane increases one at a time at each iteration. Therefore a facet will be reached in $m - k$ iterations if the dimension of the starting support face is k.

In second procedure Procedure `FRH`. Consider a supporting hyperplane for the convex hull of a point set $\mathcal{A}$. This hyperplane contains a face of con($\mathcal{A}$). The dimension of this face for con($\mathcal{A}$) is from 0 to $m - 1$. Similar to Procedure `ARH`, this procedure is applied to rotate a supporting hyperplane on the convex hull of a point set in $\Re^m$ when the dimension of the support set for this hyperplane is fewer than $m - 1$. To do so, we formulate an LP to find the normal and level of the rotated hyperplane. In just one iteration, the rotated hyperplane contains the convex hull's facet of the point set.

The last procedure was Procedure `FTF`. Unlike Procedures `ARH` and `FRH`, this procedure is applied to rotate a supporting hyperplane when the dimension of the support set is m that is a facet. This procedure uses just linear algebra operations. It was interesting to note how Procedure `FTF` is related to a dual simplex pivot.

Table 1 shows the complexity of three Procedures `ARH`, `FRH`, and `FTF`.

	System of equations	Min/Max ratio tests	Pivot	LP	
				Variables	Constraints
`ARH`	$\frac{(m-1)(m-2)}{2}$	$\frac{(2n-m)(m-1)}{2}$	-	-	-
`FRH`	-	-	-	m-1	n-1
`FTF`	-	-	1	-	-

Table 1.: On the complexity of three Procedures `ARH`, `FRH`, and `FTF`.

CHAPTER 4

SNUG CIRCUMSCRIBING SIMPLEXES

When the number of the extreme points is more than $m+1$ in m dimensions, finding the facets of the convex hull of a point set is sometimes necessary and always hard, certainly as the number of points and dimensions increases. There are many procedures to find a polytope that contains the point set, although it might not be exactly the convex hull of the point set. For a full dimensional polytope in m dimensions, the minimum number of facets that contain a point set is $m+1$. Such a polytope is a simplex. Finding the facets of a simplex given its $m+1$ extreme points is easy as is the reverse; i.e. finding the extreme points of a simplex defined by the intersection of $m+1$ halfspaces.

The goal of this part of the dissertation is to construct a special simplex that contains a given point set. These simplexes will contact a specified number of facets of the convex hull being enclosed. This is why they are referred to as "snug". To achieve this, we will use what we developed about rotating hyperplanes in the previous chapter.

We present three procedures to find a circumscribing simplex with special properties for a given point set. This simplex is named snug circumscribing simplex, because at least m facets of this simplex contain m facets of the convex hull of the given point set.

The first two procedures are Axis Rotation Snug `ARS` and Full Rotation Snug `FRS`, and these are based on the supporting hyperplane rotation ideas from Chapter 3. These two Procedures `ARS` and `FRS` are initialized by generating a simple large simplex that contains the point set $\mathcal{A}$. We refer to this initializing simplex as `PreSnug`. We explain how to find `PreSnug` in the next section. After generating `PreSnug`, we rotate its facets, one at a time, until they intersect with a facet of $\text{con}(\mathcal{A})$. We do this for m facets only. To do this, we use Procedures `ARH` and `FRH` from the previous chapter respectively for two Procedures `ARS` and `FRS`. Procedure `FTF` to rotate from one facet to another will be used in the event of duplication. One more step will be needed to complete the simplex by generating its m+1st facet. To do this we apply a procedure based on the result of Boundedness Theorem.

The third snug simplex procedure is BreakOut Simplex `BOS`. The first step of Procedure `BOS` finds m different facets of $\text{con}(\mathcal{A})$. Then, by using Boundedness Theorem, we find the last facet. These $m + 1$ facets correspond to our desired containment simplex.

4.1 Generating initializing simplex: `PreSnug`

The first two Procedures `ARS` and `FRS`, require an initializing circumscribing simplex. This is the purpose of `PreSnug`. To find a snug circumscribing simplex for a given point set $\mathcal{A}$, first we generate a large but simple simplex to contain $\mathcal{A}$. Then we translate its facets until they make contact the point set.

Consider the m hyperplanes each with the negative unit direction as its normal and with a large enough value for its level. We generate another hyperplane with normal $\bar{1}$, with a large enough value for its level such that all the points are in the halfspace defined by this hyperplane. These $m+1$ hyperplanes are translated until they make contact the point set. These $m+1$ hyperplanes correspond to $m+1$ facets of `PreSnug`. Any m combinations of these $m+1$ hyperplanes intersect at a single point. These $m+1$ points correspond to the `PreSnug` simplex.

More formally, the first m hyperplanes of `PreSnug` are $\mathcal{H}(\pi^i, \beta^i)$ for $i = 1, \ldots, m$, where

$$\pi^i_k = \begin{cases} -1; & \text{if } \ i = k \\ \\ 0; & \text{otherwise} \end{cases} \quad , \quad \beta^i = -\min \left\{ a^j_i \;\middle|\; a^j \in \mathcal{A} \right\}, \tag{4.1}$$

and the last hyperplane is $\mathcal{H}(\pi^{m+1}, \beta^{m+1})$ where

$$\pi^{m+1} = \bar{1}, \quad \beta^{m+1} = \max \left\{ \sum_{i=1}^{m} a^j_i \;\middle|\; a^j \in \mathcal{A} \right\}. \tag{4.2}$$

A point set in $\Re^2$ and its `Presnug` are shown in Fig. 45.

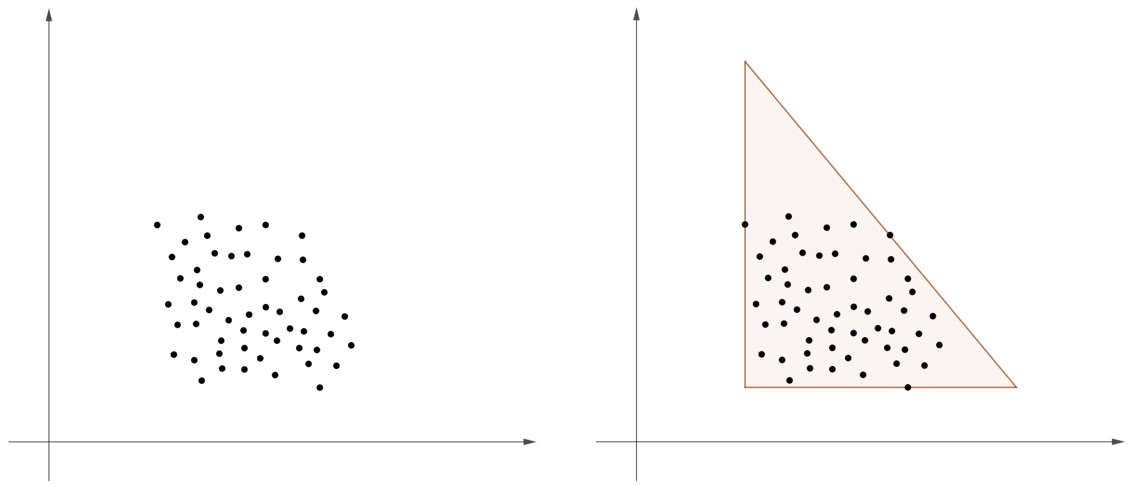

Fig. 45.: A point set in $\Re^2$ and its `Presnug`.

4.2 Generating a snug circumscribing simplex with Procedures `ARS` or `FRS`

After either of the two Procedures `ARH` and `FRH` are initialized with `PreSnug`. The m `PreSnug`'s facets get rotated by applying Procedures `ARH` and `FRH` respectively. As needed, we will apply Procedure `FTF` to make sure different facets of con($\mathcal{A}$) are identified. We demonstrate the details of these two procedures in the next two sections.

Recall that, we assumed no two facets of con($\mathcal{A}$) are parallel. After generating m snug's facets, we can solve the following system of equations to find their common extreme point. Assume m different hyperplanes $\mathcal{H}(\pi^i, \beta^i)$ for $i = 1, \ldots, m$ contain the first m snug's facets. So we have:

$$\langle \pi^i, y \rangle = \beta^i; \quad i = 1, \ldots, m. \tag{4.3}$$

To find the last facet of the snug circumscribing simplex, we solve Boundedness LP. We use the following two theorems to define Boundedness LP.

Theorem 4.1 (**Boundedness Theorem**). *Consider m linearly independent vectors $d^1, \ldots, d^m$ in $\Re^m$. If $d^{m+1} = -\sum_{i=1}^{m} d^i \mu_i$, where $\mu_i > 0$ for $i = 1, \ldots, m$, then the positive hull of these $m+1$ vectors defines $\Re^m$.*

Proof. Let $A = [d^1 \ d^2 \ \ldots \ d^{m+1}]$, and assume there exists some b in $\Re$ such that $\sum_{i=1}^{m+1} d^i \mu_i = b$ has no solution where $\mu_i \geq 0$ for $i = 1, 2, \ldots, m+1$, therefore the system $\{A\mu = b,\ \mu \geq 0\}$ has no solution. According to Farkas' Lemma, the system $\{\pi A \geq 0,\ \pi b < 0\}$ must have a solution. So we have

$$\langle \pi, d^{m+1} \rangle \geq 0 \Longrightarrow \left\langle \pi, -\sum_{i=1}^{m} d^i \mu_i \right\rangle \geq 0 \tag{4.4}$$

$$\Longrightarrow \sum_{i=1}^{m} \mu_i \langle \pi, d^i \rangle \leq 0. \tag{4.5}$$

On the other hand, since $d^1, d^2, \ldots,$ and d^m are linearly independent vectors, at least for some $k \in \{1, 2, \ldots, m\}$, $\langle \pi, d^k \rangle > 0$, therefore $\sum_{i=1}^{m} \langle \pi, d^i \rangle > 0$ and we know $\mu > 0$, so $\sum_{i=1}^{m} \mu_i \langle \pi, d^i \rangle > 0$. This is a contradiction. ■

Theorem 4.2 (**Full Body Theorem**). *Consider a point set $\mathcal{G} = \{g^1, \ldots, g^\ell\}$ in $\Re^m$, and a polyhedron $\mathcal{P} = \{y \mid \langle g^j, y \rangle \leq b^j;\ j = 1, \ldots, \ell\}$. The polyhedron $\mathcal{P}$ is bounded if and only if the positive hull of the point set $\mathcal{G}$ spans $\Re^m$.*

Proof. First we prove that if the positive hull of the point set $\mathcal{G}$ defines $\Re^m$ ($\text{pos}(g^1, \ldots, g^\ell) = \Re^m$), then the polyhedron $\mathcal{P}$ is bounded. When $\text{pos}(g^1, \ldots, g^\ell) = \Re^m$, it is equivalent to the system of equations $\{\forall b \in \Re^m : \sum_{j=1}^{\ell} g^j \mu_j = b \mid \mu_j \geq 0;\ j = 1, \ldots, \ell\}$, which is equivalent to $\{\forall b \in \Re^m : A\mu = b \mid \mu \geq 0\}$, where $A = [g^1 \ldots g^\ell]$ and $\mu = [\mu_1 \ldots \mu_\ell]^T$. This system has a solution, so based on Farkas' Lemma, the system $\{\forall y \in \Re^m :\ A^T y \geq 0,\ b^T y < 0\}$ has no solution. Recall that, the polyhedron $\mathcal{P}$ is bounded if and only if there does not exist a solution for the system $\{A^T y \leq 0,\ y \neq 0\}$. In the system $\{\forall y \in \Re^m :\ A^T y \geq 0,\ b^T y < 0\}$, $y = 0$ is the only solution to $A^T y \geq 0$. Since, otherwise there exists some b such that $b^t y < 0$. It means System $\{\langle g^j, y\rangle \leq b^j;\ j = 1, \ldots, \ell\}$ is bounded.

To prove the reverse direction, consider when obtained polyhedron by system $\{\langle g^j, y\rangle \leq b^j;\ j = 1, \ldots, \ell\}$ is bounded, then the homogeneous system $\{A^T y \leq 0,\ y \neq 0\}$ has no solution. According to Farkas' Lemma, the system $\{\forall b \in \Re^m :\ A\mu = b \mid \mu \geq 0\}$ has a solution. Therefore, $\text{pos}(g^1, \ldots, g^\ell) = \Re^m$. ■

To get the last facet of the snug circumscribing simplex, we define a hyperplane $\mathcal{H}(\pi, \beta)$. The points $a^1, \ldots, a^n$ should be located in a halfspace defined by the hyperplane $\mathcal{H}(\pi, \beta)$. So, wlog, we have the inequalities

$$\langle \pi, a^j\rangle \leq \beta; \quad j = 1, \ldots, n. \tag{4.6}$$

We need to assure all $m + 1$ hyperplanes correspond to $m + 1$ facets of a full dimension polytope (simplex) in $\Re^m$. Thus, according to Boundedness Theorem, we need to have

$$\pi = -\sum_{i=1}^{m} \pi^i \mu_i, \tag{4.7}$$

where $\mu_i \geq \epsilon$ for $i = 1, \ldots, m$, and small positive value ϵ.

The optimal solution of the following LP corresponds to a new hyperplane $\mathcal{H}(\pi^{m+1}, \beta^{m+1})$ that contains the last facet of the final simplex.

$$\begin{array}{lll}
\max\limits_{\substack{\pi \in \Re^m, \\ \mu \in \Re^m, \beta}} & \displaystyle\sum_{j=1}^{n} \Big(\langle \pi, a^j \rangle - \beta \Big) & \textbf{Boundedness LP} \\
\text{s.t.} & \langle \pi, a^j \rangle \leq \beta; & j = 1, \ldots, n, \\
& \pi = -\displaystyle\sum_{i=1}^{m} \pi^i \mu_i, & \\
& \mu_i \geq \epsilon; & i = 1, \ldots, m.
\end{array}$$

There are overall $2m + 1$ variables $\pi_1, \ldots, \pi_m, \beta, \mu_1, \ldots, \mu_m$, and $2m + n$ constraints. In the next theorem, we prove that how this LP derives the desired results.

Theorem 4.3 (**Boundedness LP Theorem**). *The optimal solution π and β of Boundedness LP corresponds to the normal and the level of a supporting hyperplane for* $\mathrm{con}(\mathcal{A})$ *being the last facet of the final simplex.*

Proof. Boundedness LP is feasible if we set $\pi = -\frac{1}{m} \sum_{i=1}^{m} \pi^i$, $\beta = \max \left\{ \langle -\frac{1}{m} \sum_{i=1}^{m} \pi^i, a^j \rangle \mid j = 1, \ldots, n \right\}$, and $\mu_i = \frac{1}{m}$ for $i = 1, \ldots, m$. Notice that the normal of the obtained hyperplane in this feasible solution is the negative barycenter of the obtained cone from the normals of the m hyperplanes $\mathcal{H}(\pi^i, \beta^i)$ for $i = 1, \ldots, m$.

Zero is an upper bound for Boundedness LP. To indicate this, it is enough to sum all inequalities of the first set of the constraints. ■

Boundedness LP has $2m+1$ variables. So, its optimal solution has at least $2m+1$ binding constraints. Boundedness LP has three sets of the constraints. The second and third sets of the constraints have $2m$ constraints together. Therefore at least one constraint from the first set of the constraints is binding. We conclude that the obtained hyperplane makes contact with $\text{con}(\mathcal{A})$. This hyperplane definitely contains a face of $\text{con}(\mathcal{A})$. This contained face may be a facet of $\text{con}(\mathcal{A})$.

There is an important observation in Boundedness LP that we are able to find the last facet of snug circumscribing simplex without using an LP as follows:

Remark. If we set $\pi = -\frac{1}{m}\sum_{i=1}^{m}\pi^i$, $\beta = \max\,\{\langle -\frac{1}{m}\sum_{i=1}^{m}\pi^i, a^j\rangle \mid j = 1,\ldots,n\}$, and $\mu_i = \frac{1}{m}$ for $i = 1,\ldots,m$, then we have a feasible solution the Boundedness LP that yields a bounded simplex.

We prefer to use the optimal solution to the Boundedness LP, because of the hope that the last supporting hyperplane will support the convex hull at a higher dimensional face.

4.3 Procedure Axis Rotation Snug (`ARS`)

The first step of this procedure is to generate `PreSnug`. Assume the $m+1$ hyperplanes $\mathcal{H}(\pi^i, \beta^i)$, $i = 1,\ldots,m+1$, contain the $m+1$ facets of `PreSnug`.

The next step is to apply Procedure `ARH` to rotate m hyperplanes of these $m+1$ hyperplanes until they make contact m facets of $\text{con}(\mathcal{A})$. We check for duplicate facets after each rotation. If there exists a duplicate facet, we perform Procedure `FTF` and rotate to an adjacent facet. We do this until the duplication is gone. The

m rotated hyperplanes that contain m different facets of $\text{con}(\mathcal{A})$, correspond to m facets of a snug circumscribing simplex.

The last step is to construct Boundedness LP to find the last facet of the snug circumscribing simplex.

4.3.1 Procedure ARS pseudocode

Finding a snug circumscribing simplex by using Procedure `ARS` for a given point set is as follows.

Procedure `ARS`

Input : A point set $\mathcal{A} = \{a^1, \ldots, a^n\} \in \Re^m$.
Output: A snug circumscribing simplex for $\mathcal{A}$.

Generate `PreSnug`: Assume `PreSnug`'s facets are contained by the hyperplanes $\mathcal{H}(\pi^i, \beta^i)$ with $\mathcal{S}_i$ in the support set for $i = 1, \ldots, m+1$;
for $i \leftarrow 1$ **to** m **do**
    Use Procedure `ARH` to rotate $\mathcal{H}(\pi^i, \beta^i)$ on a facet of $\text{con}(\mathcal{A})$;
    Update $\mathcal{S}_i$;
    Set $j = 1$;
    while $(j < i)$ **do**
        if $\left(\mathcal{H}(\pi^i, \beta^i) \equiv \mathcal{H}(\pi^j, \beta^j) \right)$ **then**
            Use Procedure `FTF` to rotate $\mathcal{H}(\pi^i, \beta^i)$ on another facet of $\text{con}(\mathcal{A})$;
            Set $j = 0$;
        end
        Set $j = j + 1$;
    end
end
Solve Boundedness LP for the variables π, β, and μ;
The optimal solution of variables π^{m+1} and β^{m+1} correspond to the normal and the level of the last facet respectively;

These $m+1$ obtained hyperplanes $\mathcal{H}(\pi^i, \beta^i)$ for $i = 1, \ldots, m+1$ contain $m+1$ facets of a snug circumscribing simplex for the point set $\mathcal{A}$.

Four observations about Procedure `ARS` are as follows:

Remark 1. The m facets of the snug circumscribing simplex contain m different facets of con($\mathcal{A}$), and the last facet contains a face (may be a facet that is different from the previous m found facets) of con($\mathcal{A}$).

Remark 2. Procedure `ARS` just uses an LP formulation to find the last facet. Other facets are found by linear algebra operations.

Remark 3. According to Remark at the end of the previous section, Procedure `ARS` can find a snug circumscribing simplex using only algebra operations (without using an LP).

Remark 4. Assume to find a snug circumscribing simplex for a certain point set by applying Procedures `ARS`, it does not need to use Procedures `FTF`. Then on the complexity of Procedures `ARS`, one overall needs to solve $\frac{m(2n-m)(m-1)}{2}$ minimum and maximum ratio tests, a system of $\frac{m(m-1)(m-2)}{2}$ equations, and an LP with $2m+1$ variables and $2m+n$ constraints to find the final snug circumscribing simplex.

4.3.2 Example: Using Procedure `ARS` to find a snug circumscribing simplex for a finite point set in $\Re^3$

Recall that we assumed the convex hull of a point set has no degenerate facet. All examples in Chapter 3, we considered this assumption. Here in this example, we demonstrate how to use `ARH` in the case of having degenerate facet.

Consider the point set $\mathcal{A} = \{a^1 = (1,4,2)^T,\ a^2 = (3,1,5)^T,\ a^3 = (2,2,3)^T,\ a^4 = (2,5,6)^T,\ a^5 = (4,3,1)^T,\ a^6 = (3,5,4)^T,\ a^7 = (9,3,2)^T\}$ in $\Re^3$.

The initializing `PreSnug` simplex is characterized by the following four hyper-

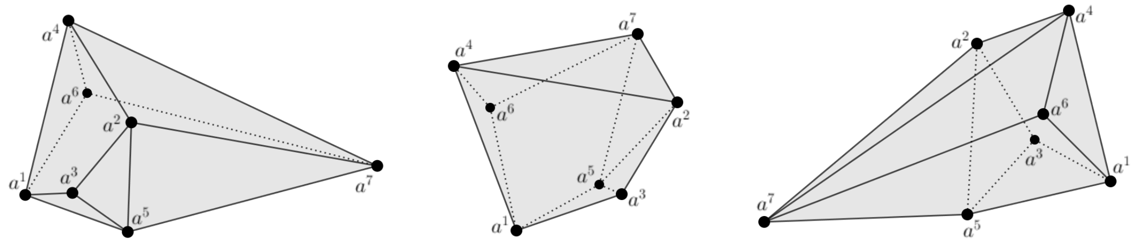

Fig. 46.: Three views of a point set and its convex hull in $\Re^3$ with 7 extreme points.

planes:

$$\mathcal{H}(\pi^1, \beta^1): \; -e^1 \qquad\qquad = -1; \tag{4.8}$$

$$\mathcal{H}(\pi^2, \beta^2): \qquad -e^2 \qquad = -1; \tag{4.9}$$

$$\mathcal{H}(\pi^3, \beta^3): \qquad\qquad -e^3 = -1; \tag{4.10}$$

$$\mathcal{H}(\pi^4, \beta^4): \quad e^1 + e^2 + e^3 = \;\; 14; \tag{4.11}$$

with four support sets $\mathcal{S}_1 = \{a^1\}$, $\mathcal{S}_2 = \{a^2\}$, $\mathcal{S}_3 = \{a^5\}$, and $\mathcal{S}_4 = \{a^7\}$. The following figure shows three views of `PreSnug` that contains $\text{con}(\mathcal{A})$.

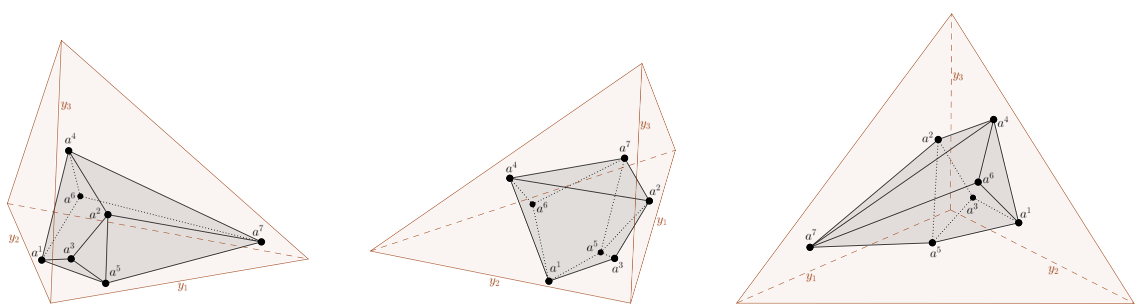

Fig. 47.: Three views of the same `PreSnug` for the convex hull of eight points in $\Re^3$.

Procedure `ARS` begins by rotating one of `PreSnug`'s hyperplanes. Just recall that to rotate a supporting hyperplane $\mathcal{H}(\pi, \beta)$ on the convex hull of a point set

$\mathcal{A} = \{a^1, \ldots, a^n\} \in \Re^m$, wlog, with $\mathcal{S} = \{a^1\}$ in the support set by using Procedure `ARH`, we first define $\tilde{\pi} = (\gamma_1, \pi_2, \ldots, \pi_m)^T$. When $a_1^j - a_1^1 \neq 0$ for $j = 1, \ldots, n$, we have:

$$\gamma_1 \begin{cases} \leq -\frac{\langle \overline{\pi}, \overline{a}^j \rangle - \langle \overline{\pi}, \overline{a}^1 \rangle}{a_1^j - a_1^1}; & \text{if } \; a_1^j - a_1^1 < 0; \\ \\ \geq -\frac{\langle \overline{\pi}, \overline{a}^j \rangle - \langle \overline{\pi}, \overline{a}^1 \rangle}{a_1^j - a_1^1}; & \text{if } \; a_1^j - a_1^1 > 0; \end{cases} \tag{4.12}$$

where $\overline{\pi} = (\pi_2, \ldots, \pi_m)^T$, and $\overline{a}^j = (a_2^j, \ldots, a_m^j)^T$ for $j = 1, \ldots, n$. This system of inequalities generates at least a bound for γ_1. If we set γ_1 to this bound, the obtained hyperplane with the normal $\tilde{\pi}$ and the level $\tilde{\beta} = \langle \tilde{\pi}, a^1 \rangle$ corresponds to a supporting hyperplane for $\text{con}(\mathcal{A})$ such that contains a^1 and another point of $\mathcal{A}$.

To rotate the first hyperplane $\mathcal{H}(\pi^1, \beta^1)$, we define $\tilde{\pi}^1 = (-1, \gamma_1, 0)^T$. So we have:

$$\langle \tilde{\pi}^1, a^2 \rangle \leq \langle \tilde{\pi}^1, a^1 \rangle \Longrightarrow \gamma_1 \geq -\frac{(-1)(1) + (0)(2) - (-1)(3) - (0)(5)}{(4) - (1)} = -\frac{2}{3}; \tag{4.13}$$

$$\langle \tilde{\pi}^1, a^3 \rangle \leq \langle \tilde{\pi}^1, a^1 \rangle \Longrightarrow \gamma_1 \geq -\frac{(-1)(1) + (0)(2) - (-1)(2) - (0)(3)}{(4) - (2)} = -\frac{1}{2}; \tag{4.14}$$

$$\langle \tilde{\pi}^1, a^4 \rangle \leq \langle \tilde{\pi}^1, a^1 \rangle \Longrightarrow \gamma_1 \leq -\frac{(-1)(1) + (0)(2) - (-1)(2) - (0)(6)}{(4) - (5)} = \quad 1; \tag{4.15}$$

$$\langle \tilde{\pi}^1, a^5 \rangle \leq \langle \tilde{\pi}^1, a^1 \rangle \Longrightarrow \gamma_1 \geq -\frac{(-1)(1) + (0)(2) - (-1)(4) - (0)(1)}{(4) - (3)} = -3; \tag{4.16}$$

$$\langle \tilde{\pi}^1, a^6 \rangle \leq \langle \tilde{\pi}^1, a^1 \rangle \Longrightarrow \gamma_1 \leq -\frac{(-1)(1) + (0)(2) - (-1)(3) - (0)(4)}{(4) - (5)} = \quad 2; \tag{4.17}$$

$$\langle \tilde{\pi}^1, a^7 \rangle \leq \langle \tilde{\pi}^1, a^1 \rangle \Longrightarrow \gamma_1 \geq -\frac{(-1)(1) + (0)(2) - (-1)(9) - (0)(2)}{(4) - (3)} = -8. \tag{4.18}$$

This generates the following bounds for γ_1; $-1/2 \leq \gamma_1 \leq 1$.

So we have the lower bound, a minimum ratio, is $-1/2$ for γ_1. If we set $\gamma_1 =$

$-1/2$, we get the hyperplane $\mathcal{H}(\pi^{1L}, -6)$ where $\pi^{1L} = (-2, -1, 0)^T$ with the support set $\mathcal{S}_{1L} = \{a^1, a^3\}$.

The upper bound, a minimum ratio, is 1 for γ_1. By setting $\gamma_1 = 1$, we get the hyperplane $\mathcal{H}(\pi^{1U}, 3)$ where $\pi^{1U} = (1, -1, 0)^T$ with $\mathcal{S}_{1U} = \{a^1, a^4\}$ in the support set.

The dimension of the support set of the hyperplane $\mathcal{H}(\pi^{1L}, -6)$ for con($\mathcal{A}$) is 2. To rotate this hyperplane using ARH, we define $\tilde{\pi}^{1L} = (-2, \gamma_1, \gamma_2)^T$. Thus, we have:

$$\langle \tilde{\pi}^{1L}, a^2 \rangle \leq \langle \tilde{\pi}^{1L}, a^1 \rangle \Longrightarrow -3\gamma_1 + 3\gamma_2 \leq 4; \tag{4.19}$$

$$\langle \tilde{\pi}^{1L}, a^3 \rangle \leq \langle \tilde{\pi}^{1L}, a^1 \rangle \Longrightarrow -2\gamma_1 + \gamma_2 = 2; \tag{4.20}$$

$$\langle \tilde{\pi}^{1L}, a^4 \rangle \leq \langle \tilde{\pi}^{1L}, a^1 \rangle \Longrightarrow \gamma_1 + 4\gamma_2 \leq 2; \tag{4.21}$$

$$\langle \tilde{\pi}^{1L}, a^5 \rangle \leq \langle \tilde{\pi}^{1L}, a^1 \rangle \Longrightarrow -\gamma_1 - \gamma_2 \leq 6; \tag{4.22}$$

$$\langle \tilde{\pi}^{1L}, a^6 \rangle \leq \langle \tilde{\pi}^{1L}, a^1 \rangle \Longrightarrow \gamma_1 + 2\gamma_2 \leq 4; \tag{4.23}$$

$$\langle \tilde{\pi}^{1L}, a^7 \rangle \leq \langle \tilde{\pi}^{1L}, a^1 \rangle \Longrightarrow -\gamma_1 \leq 16. \tag{4.24}$$

From(4.20), we get $\gamma_2 = 2\gamma_1 + 2$. If in all the inequalities we set $\gamma_2 = 2\gamma_1 + 2$, then it generates the bounds; $-8/3 \leq \gamma_1 \leq -2/3$.

For a lower bound $\gamma_1 = -8/3$, we get $\gamma_2 = -10/3$. It yields the hyperplane $\mathcal{H}(\pi^{1LL}, -29)$ where $\pi^{1LL} = (-3, -4, -5)^T$ with the support set $\mathcal{S}_{1LL} = \{a^1, a^3, a^5\}$. This hyperplane contains a facet of con($\mathcal{A}$) that contains three points a^1, a^3 and a^5.

When we set $\gamma_1 = -2/3$, an upper bound for γ_1, then we get $\gamma_2 = 2/3$. This corresponds to the hyperplane $\mathcal{H}(\pi^{1LU}, -5)$ where $\pi^{1LU} = (-3, -1, 1)^T$ with $\mathcal{S}_{1LU} = \{a^1, a^2, a^3, a^4\}$ in the support set. This hyperplane contains a facet of con($\mathcal{A}$) that contains four points a^1, a^2, a^3 and a^4. By rotating the hyperplane $\mathcal{H}(\pi^{1U}, 3)$, we get two supporting hyperplanes $\mathcal{H}(\pi^{1UL}, 20)$ and $\mathcal{H}(\pi^{1UU}, -5)$ where $\pi^{1UL} = (-2, 6, -1)^T$ and $\pi^{1UU} = (-3, -1, 1)^T$ with $\mathcal{S}_{1UL} = \{a^1, a^4, a^6\}$ and $\mathcal{S}_{1UU} = \{a^1, a^2, a^3, a^4\}$ in their support sets. Among these four rotated hyperplanes, we select the hyperplane $\mathcal{H}(\pi^{1UU}, -5)$ to be a hyperplane containing a facet of the final snug circumscribing simplex.

By using Procedure ARH to rotate the hyperplane $\mathcal{H}(\pi^2, -1)$, we get four hyperplanes $\mathcal{H}(\pi^{2LL}, -5)$, $\mathcal{H}(\pi^{2LU}, -7)$, $\mathcal{H}(\pi^{2UL}, -65)$, and $\mathcal{H}(\pi^{2UU}, 169)$ where $\pi^{LL} = (-3, -1, 1)^T$, $\pi^{2LU} = (0, -2, -1)^T$, $\pi^{2UL} = (2, -21, -10)^T$, and $\pi^{2UU} = (14, -3, 26)^T$ with $\mathcal{S}_{2LL} = \{a^1, a^2, a^3, a^4\}$, $\mathcal{S}_{2LU} = \{a^2, a^3, a^5\}$, $\mathcal{S}_{2UL} = \{a^2, a^5, a^7\}$, and $\mathcal{S}_{2UU} = \{a^2, a^4, a^7\}$ in their support sets. Among all these hyperplanes, we pick up the hyperplane $\mathcal{H}(\pi^{2LL}, -5)$ as the hyperplane that contains a facet of the final snug circumscribing simplex.

Finally, rotating the hyperplane $\mathcal{H}(\pi^3, -1)$ by using Procedure ARH, can yield four different hyperplanes $\mathcal{H}(\pi^{3LL}, -29)$, $\mathcal{H}(\pi^{3LU}, 23)$, $\mathcal{H}(\pi^{3UL}, -65)$, and $\mathcal{H}(\pi^{3UU}, 23)$ where $\pi^{3LL} = (-3, -4, -5)^T$, $\pi^{3LU} = (1, 8, -5)^T$, $\pi^{3UL} = (2, -21, -10)^T$, and $\pi^{3UU} = (1, 8, -5)^T$ with $\mathcal{S}_{3LL} = \{a^1, a^3, a^5\}$, $\mathcal{S}_{3LU} = \{a^1, a^5, a^6, a^7\}$, $\mathcal{S}_{3UL} = \{a^2, a^5, a^7\}$, and $\mathcal{S}_{3UU} = \{a^1, a^5, a^6, a^7\}$ in their support sets. We keep the hyperplane $\mathcal{H}(\pi^{3UU}, 9)$ as the hyperplane that contains a facet of the final snug circumscribing simplex.

Therefore, we have three hyperplanes $\mathcal{H}(\pi^{1UU}, -5)$, $\mathcal{H}(\pi^{2LL}, -5)$, and $\mathcal{H}(\pi^{3UU}, 9)$ that contain two facets of the final snug circumscribing simplex. The hyperplanes $\mathcal{H}(\pi^{1UU}, -5)$ and $\mathcal{H}(\pi^{2LL}, -5)$ correspond to the same facet of $\text{con}(\mathcal{A})$. In this case, we use Procedure FTF to rotate one of these two hyperplanes $\mathcal{H}(\pi^{1UU}, -5)$ and $\mathcal{H}(\pi^{2LL}, -5)$. We select $\mathcal{H}(\pi^{2LL}, -5)$. The support set of the hyperplane $\mathcal{H}(\pi^{2LL}, -5)$

contains four extreme points. This is a degenerate facet. We need to select two extreme points from the support set $\mathcal{S}_{\in\mathcal{LL}}$ to locate in the next rotated hyperplane using Procedure `FTF`. This is 4-choose-2. There are 6 different possibilities. Not all these possibilities can be conducted to rotate the hyperplane $\mathcal{H}(\pi^{2LL}, -5)$ using Procedure `FTF`. If two selected points to locate in the next rotated hyperplane are adjacent, then using Procedure `FTF` yields a new hyperplane containing a facet of $\mathrm{con}(\mathcal{A})$. We select two adjacent points a^2 and a^4 to be in the support set of the next rotated hyperplane. We define a family of a hyperplane $\mathcal{H}(\pi, 1)$ where $\pi = (\gamma_1, \gamma_2, \gamma_3)^T$. So we have the system of equations:

$$\langle \pi, a^2 \rangle = 1 \Longrightarrow 3\gamma_1 + \gamma_2 + 5\gamma_3 = 1; \tag{4.25}$$

$$\langle \pi, a^4 \rangle = 1 \Longrightarrow 2\gamma_1 + 5\gamma_2 + 6\gamma_3 = 1. \tag{4.26}$$

We can solve this system of equations to get γ_2 and γ_3 in terms of γ_1 and a constant value as follows:

$$\gamma_2 = \frac{8}{19}\gamma_1 - \frac{1}{19}; \tag{4.27}$$

$$\gamma_3 = -\frac{13}{19}\gamma_1 + \frac{4}{19}. \tag{4.28}$$

Next, for other points a^1, a^3, a^5, a^6, a^7 below the hyperplane $\mathcal{H}(\pi^{2LL}, -5)$, and considering System (4.27 - 4.28), we have the system of inequalities:

$$\langle \pi, a^1 \rangle \leq 1 \Longrightarrow \quad \gamma_1 + 4\gamma_2 + 2\gamma_3 \leq 1 \Longrightarrow \gamma_1 \leq \frac{3}{25}; \tag{4.29}$$

$$\langle \pi, a^3 \rangle \leq 1 \Longrightarrow 2\gamma_1 + 2\gamma_2 + 3\gamma_3 \leq 1 \Longrightarrow \gamma_1 \leq \frac{3}{5}; \tag{4.30}$$

$$\langle \pi, a^5 \rangle \leq 1 \Longrightarrow 4\gamma_1 + 3\gamma_2 + \quad \gamma_3 \leq 1 \Longrightarrow \gamma_1 \leq \frac{9}{44}; \tag{4.31}$$

$$\langle \pi, a^6 \rangle \leq 1 \Longrightarrow 3\gamma_1 + 5\gamma_2 + 4\gamma_3 \leq 1 \Longrightarrow \gamma_1 \leq \frac{8}{45}; \tag{4.32}$$

$$\langle \pi, a^7 \rangle \leq 1 \Longrightarrow 9\gamma_1 + 3\gamma_2 + 2\gamma_3 \leq 1 \Longrightarrow \gamma_1 \leq \frac{14}{169}. \tag{4.33}$$

This generates an upper bound for γ_1; $\gamma_1 \leq 14/169$.

Therefore we use this value $\gamma_1 = 14/169$ to calculate the other two unknowns using System (4.27 - 4.28) to get $\gamma_2 = -3/169$ and $\gamma_3 = 26/169$. This is a new hyperplane $\mathcal{H}(\hat{\pi}, 169)$ where $\hat{\pi} = (14, -3, 26)^T$, and its support set includes $\hat{\mathcal{S}} = \{a^2, a^4, a^7\}$.

So three hyperplanes $\mathcal{H}(\pi^1, -5)$, $\mathcal{H}(\pi^2, 169)$, and $\mathcal{H}(\pi^3, 9)$ where $\pi^1 = (-3, -1, 1)^T$, $\pi^2 = (14, -3, 26)^T$, and $\pi^3 = (1, 8, -5)^T$ contain three facets of the final snug circumscribing simplex.

To find the last facet of the final snug, we use Boundedness LP for $\epsilon = 0.01$ as follows:

$$
\begin{aligned}
\max_{\substack{\pi\in\Re^3,\\ \mu\in\Re^3,\beta}} \quad & 24\pi_1 + 23\pi_2 + 23\pi_3 - 7\beta \\
\text{s.t.} \quad & \pi_1 + 4\pi_2 + 2\pi_3 \le \beta, \\
& 3\pi_1 + \pi_2 + 5\pi_3 \le \beta, \\
& 2\pi_1 + 2\pi_2 + 3\pi_3 \le \beta, \\
& 2\pi_1 + 5\pi_2 + 6\pi_3 \le \beta, \\
& 4\pi_1 + 3\pi_2 + \pi_3 \le \beta, \\
& 3\pi_1 + 5\pi_2 + 4\pi_3 \le \beta, \\
& 9\pi_1 + 3\pi_2 + 2\pi_3 \le \beta, \\
& \pi_1 = 3\mu_1 - 14\mu_2 - \mu_3, \\
& \pi_2 = \mu_1 + 3\mu_2 - 8\mu_3, \\
& \pi_3 = -\mu_1 - 26\mu_2 + 5\mu_3, \\
& \mu_i \ge 0.01; \qquad i = 1, 2, 3.
\end{aligned} \tag{4.34}
$$

By solving this LP, we get the optimal solution $\pi_1^* = 0, \pi_2^* = -0.23451, \pi_3^* = -0.117255, \beta^* = -0.820784, \mu_1^* = 0.060196, \mu_2^* = 0.01, \mu_3^* = 0.040588$. So we get the hyperplane $\mathcal{H}(\pi^4, -7)$ where $\pi^4 = (0, -2, -1)^T$ with $\mathcal{S}_4 = \{a^2, a^3, a^5\}$ in the support set. Fig. 48. shows how final snug circumscribing simplex contains con($\mathcal{A}$).

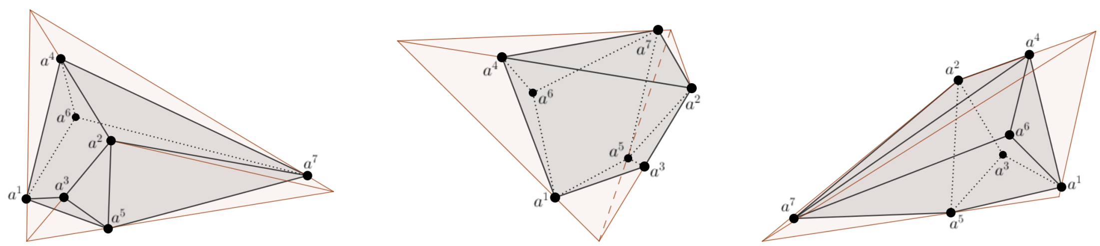

Fig. 48.: Three views of the final snug circumscribing simplex obtained by applying Procedure `ARS`.

4.4 Procedure Full Rotation Snug (`FRS`)

In Procedure `FRH`, similar to Procedure `ARH`, first we generate `PreSnug` for the given point set. Assume all $m+1$ hyperplanes $\mathcal{H}(\pi^j, \beta^j)$ for $j = 1, \ldots, m+1$ contain $m+1$ facets of `PreSnug`. Then, select m hyperplanes among all these $m+1$ hyperplanes and perform Procedure `FRH` to rotate them such that each of them contains a facet of $\text{con}(\mathcal{A})$. The procedures to avoid having duplicate hyperplanes, and finding last facet of the final snug circumscribing simplex are the same as Procedure `ARS`.

4.4.1 Procedure `FRS` pseudocode

Recall that to rotate a hyperplane $\mathcal{H}(\pi^k, \beta^k)$ with, wlog, $\mathcal{S}_k = \{a^1, \ldots, a^k\}$ in the support set, we define a hyperplane $\mathcal{H}(\pi^{k+1}, \beta^{k+1})$ where $\pi^{k+1} = (\gamma_1, \ldots, \gamma_k, \pi^k_{k+1}, \ldots, \pi^k_m)$ where all the constant values are not zero together. Then, the optimal solution of the following LP formulation yields the normal of the rotated hyperplane

$\mathcal{H}(\pi^{k+1}, \beta^{k+1})$.

$$
\begin{array}{lll}
\displaystyle\max_{\gamma_1,\ldots,\gamma_k} & \displaystyle\sum_{j=k+1}^{n} \langle \pi^{k+1}, a^j - a^1 \rangle & \textbf{FRH LP} \\
\text{s.t.} & \langle \pi^{k+1}, a^j \rangle = \langle \pi^{k+1}, a^1 \rangle; & j = 2, \ldots, k, \\
 & \langle \pi^{k+1}, a^j \rangle \leq \langle \pi^{k+1}, a^1 \rangle; & j = k+1, \ldots, n.
\end{array}
$$

Finding a snug circumscribing simplex by using Procedure `FRS` for a given point set is as follows.

Procedure `FRS`

Input : A point set $\mathcal{A} = \{a^1, \ldots, a^n\} \in \Re^m$.
Output: A snug circumscribing simplex for $\mathcal{A}$.

Generate `PreSnug`: Assume `PreSnug`'s facets are contained by the hyperplanes $\mathcal{H}(\pi^i, \beta^i)$ with $\mathcal{S}_i$ containing the extreme points of $\mathcal{A}$ from its support set for $i = 1, \ldots, m+1$;
for $i \leftarrow 1$ **to** m **do**
  Use Procedure `FRH` to rotate $\mathcal{H}(\pi^i, \beta^i)$ on a facet of con($\mathcal{A}$);
  Update $\mathcal{S}_i$;
  Set $j = 1$;
  while $(j < i)$ **do**
    if $\big(\mathcal{H}(\pi^i, \beta^i) \equiv \mathcal{H}(\pi^j, \beta^j) \big)$ **then**
      Use Procedure `FTF` to rotate $\mathcal{H}(\pi^i, \beta^i)$ on another facet of con($\mathcal{A}$);
      Set $j = 0$;
    end
    Set $j = j + 1$;
  end
end
Solve Boundedness LP for the variables π, β, and μ;
The optimal solution of variables π^{m+1} and β^{m+1} correspond to the normal and the level of the last facet respectively;

Three observations about Procedure `FRS` are as follows:

Remark 1. The m facets of the snug circumscribing simplex contain m different facets of con($\mathcal{A}$), and the last facet contains a face (may be a facet that is different from the previous m found facets) of con($\mathcal{A}$).

Remark 2. There is needed solving an LP to find each facet of the snug circumscribing simplex, and overall the $m+1$ LPs.

Remark 3. Assume to find a snug circumscribing simplex for a certain point set by applying Procedures `FRS`, it does not need to use Procedures `FTF`. Then on the complexity of Procedures `FRS`, one overall needs to solve $m+1$ LPs that each of the first m LPs has $m-1$ variables and $n-1$ constraints, and the last LP has $2m+1$ variables and $2m+n$ constraints to find the final snug circumscribing simplex.

4.4.2 Example: Using Procedure `FRS` to find a snug circumscribing simplex for a finite point set in $\Re^3$

Consider the point set $\mathcal{A} = \{a^1 = (1,4,2)^T,\ a^2 = (3,1,5)^T,\ a^3 = (2,2,3)^T,\ a^4 = (2,5,6)^T,\ a^5 = (4,3,1)^T,\ a^6 = (3,5,4)^T,\ a^7 = (9,3,2)^T\}$ in $\Re^3$ and its `PreSnug` from last example. To rotate the first hyperplane $\mathcal{H}(\pi^1, -1)$ of `PreSnug` where $\pi^1 = (-1,0,0)^T$ with the support set $\mathcal{S}_1 = \{a^1\}$ by using Procedure `FRH`, we define the hyperplane $\mathcal{H}(\tilde{\pi}^1, \tilde{\beta}^1)$, and parametrize $\tilde{\pi}^1 = (-1, \gamma_1, \gamma_2)^T$. We thus have the `FRH`

LP:

$$\begin{aligned}
\max_{\gamma_1,\gamma_2} \quad & -5\gamma_1 + 9\gamma_2 \\
\text{s.t.} \quad & -3\gamma_1 + 3\gamma_2 \le 2, \\
& -2\gamma_1 + \gamma_2 \le 1, \\
& \gamma_1 + 4\gamma_2 \le 1, \\
& -\gamma_1 - \gamma_2 \le 3, \\
& \gamma_1 + 2\gamma_2 \le 2, \\
& -\gamma_1 \le 8.
\end{aligned} \tag{4.35}$$

We get the optimal solution $\gamma_1^* = -\frac{1}{3}$ and $\gamma_2^* = \frac{1}{3}$. These values drive the hyperplane $\mathcal{H}(\tilde{\pi}^1, \tilde{\beta}^1)$ where $\tilde{\pi}^1 = (-3, -1, 1)^T$ and $\tilde{\beta}^1 = -3$, with $\tilde{\mathcal{S}}_1 = \{a^1, a^2, a^3, a^4\}$ in the support set.

We use the same way to rotate the second hyperplane $\mathcal{H}(\pi^2, -1)$ of `PreSnug` where $\pi^2 = (0, -1, 0)^T$ with the support set $\mathcal{S}_2 = \{a^2\}$, and third hyperplane $\mathcal{H}(\pi^3, -1)$ of `PreSnug` where $\pi^3 = (0, 0, -1)^T$ with $\mathcal{S}_2 = \{a^5\}$ in the support set. We then get the supporting hyperplanes $\mathcal{H}(\tilde{\pi}^2, \tilde{\beta}^2)$ and $\mathcal{H}(\tilde{\pi}^3, \tilde{\beta}^3)$ where $\tilde{\pi}^2 = (2, -21, -10)^T$, $\tilde{\beta}^2 = -65$, $\tilde{\pi}^3 = (1, 8, -5)^T$, and $\tilde{\beta}^3 = 23$, with $\tilde{\mathcal{S}}_2 = \{a^2, a^5, a^7\}$ and $\tilde{\mathcal{S}}_3 = \{a^1, a^5, a^6, a^7\}$ in their support sets.

Finally, we use Boundedness LP to find the last hyperplane $\mathcal{H}(\tilde{\pi}^4, \tilde{\beta}^4)$ where $\tilde{\pi}^4 = (14, -3, 26)^T$ and $\tilde{\beta}^4 = 169$, with the support set $\tilde{\mathcal{S}}_4 = \{a^2, a^4, a^7\}$.

The final snug circumscribing simplex and $\text{con}(\mathcal{A})$ is shown in Fig. 49.

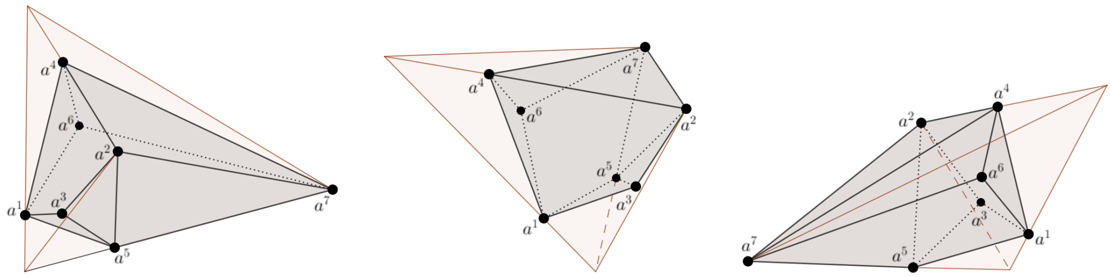

Fig. 49.: Three views of the final snug circumscribing simplex obtained by applying Procedure `FRS`.

4.5 Procedure Breakout Snug (`BOS`)

In this procedure we identify faces of the convex hull of a point set $\mathcal{A} = \{a^1, \ldots, a^n\}$ in $\Re^m$ to find a circumscribing simplex. An assumption for the procedure is that con($\mathcal{A}$) contains origin. The points in $\mathcal{A}$ can be translated such that the convex hull contains origin. We will assume that is already the case.

Recall the dual of the Gauge LP from (2.21) placed here again:

$$\begin{aligned} &\max_{\pi \in \Re^m} \langle \pi, b \rangle && \textbf{Dual Gauge LP} \\ &\text{s.t. } \langle \pi, a^j \rangle \leq 1; && j = 1, \ldots, n; \end{aligned} \tag{4.36}$$

for some $0 \neq b \in \Re^m$. The Gauge LP is feasible and bounded (Theorem 2.1). An optimal solution, π^* to this LP, is the normal of a hyperplane supporting con($\mathcal{A}$). If the optimal solution is basic, the supporting hyperplane contains a facet of the convex hull.

The following two results will be used in the procedure. Let $\mathcal{P}$ be a polyhedron with r facets the normals of which are $\pi^1, \ldots, \pi^r$ in $\Re^m$.

Theorem 4.4. *The normals of the facets of $\mathcal{P}$ positively span $\Re^m$ if and only if $\mathcal{P}$ is a full-dimensional bounded polyhedron.*

Proof. Assume r hyperplanes $\mathcal{H}(\pi^j, \beta^j)$ for $j = 1, \dots, r$ contains r facets of $\mathcal{P}$. The polyhedron $\mathcal{P}$ is:

$$\mathcal{P} = \left\{ y \in \Re^m | \langle \pi^j, y \rangle \leq \beta^j; \quad j = 1, \dots, r \right\}. \tag{4.37}$$

We first prove that if $\mathrm{pos}(\pi^1, \dots, \pi^r) = \Re^m$, then $\mathcal{P}$ is a full-dimensional bounded polyhedron. When $\mathrm{pos}(\pi^1, \dots, \pi^r) = \Re^m$, the system

$$\sum_{j=1}^{r} \pi^j \lambda_j = b, \;\; \lambda_j \geq 0; \quad j = 1, \dots, r, \tag{4.38}$$

has a solution for any $b \in \Re^m$. We rewrite this system as

$$\text{System I} \equiv \begin{cases} \Pi\lambda = b; \\ \quad \lambda \geq 0, \end{cases} \tag{4.39}$$

where $\Pi = [\pi^1 \dots \pi^r]$ and $\lambda = [\lambda_1 \dots \lambda_n]^T$. By Farkas' Lemma,

$$\text{System II} \equiv \begin{cases} y^T\Pi \geq 0; \\ b^T y \;\; < 0, \end{cases} \tag{4.40}$$

cannot have a solution. No solution to System II means there is no non-trivial solution to the homogeneous system in (4.37) which means no unbounded ray is possible in the polyhedron must be bounded.

To show the converse we note that if the positive hull of the vectors $\mathrm{pos}(\pi^1, \dots, \pi^r)$ do not span $\Re^m$ then for some nonzero b, System I has no solution and System II can be solved for that b. This solution provides a direction of recession for $\mathcal{P}$. ■

An immediate corollary of Theorem 4.4 is this:

Corollary 4.1. *The normals of the $m+1$ facets of a full-dimensional bounded simplex positively span $\Re^m$.*

Recall that the normals of the $m+1$ facets of a full-dimensional bounded simplex form an affinely independent set.

The third and last procedure for circumscribing simplexes, `BOS` relies on these and other geometrical properties. The idea is to generate a sequence of $m+1$ hyperplanes each supporting a face of $\text{con}(\mathcal{A})$ in a part of the polytope that is far away from the previous faces. All this is an attempt to capture as much of the geometry of the convex hull. To this end, new supporting hyperplanes are generated one at a time so that each normal is in the polar of the cone of normals of all previous supporting hyperplanes until there are $m+1$ of them. The polar cone is, in a sense, an extreme geometric counterpart to the original cone since it is in the region of the space were vectors are beyond their orthogonals. The more encompassing the generating cone, the narrower its polar; and vice-versa. Each iteration of `BOS` will generate a hyperplane by adding one new "polarity" constraint to the Dual Gauge LP in (4.36) in addition to the original support constraints. This requires modifying the Dual Gauge LP in (4.36) at every one of m iterations by adding a "polarity" constraint.

The following result establishes the validity of Procedure `BOS`.

Theorem 4.5. *Consider $m+1$ vectors in $\Re^m : \pi^1, \ldots, \pi^{m+1}$, such that the first m are linearly independent and $\pi^k \in \text{pos}^*(\pi^1, \ldots, \pi^{k-1})$; $k = 2, \ldots, m+1$, where pos^* is the polar cone. Then, $\text{pos}(\pi^1, \ldots, \pi^{m+1}) = \Re^m$.*

Proof. Suppose $\text{pos}(\pi^1, \ldots, \pi^{m+1}) \neq \Re^m$. Then by Farkas' Lemma the following system has a non-trivial solution.

$$\langle \pi^k, y \rangle \leq 0;\ k = 1, \ldots, m+1.$$

This means there exists a hyperplane through the origin that supports the cone generated by the $m+1$ vectors. One such solution defines a supporting hyperplane which contains $m-1$ of these vectors. Assume, wlog, this supporting hyperplane contains the first $m-1$ of the vectors. Since the first m vectors are linearly independent, π^m is left off the hyperplane. Consider the two-dimensional cone $\text{pos}(\pi^m, \pi^{m+1})$. The angle between these two vectors is greater than $90°$ by construction. Then, necessarily π^m must have an angle less than $90°$ with the supporting hyperplane and hence with all the other vectors. This contradicts that π^m is in the polar of the previous $m-1$ vectors and therefore rejects the possibility that $\text{pos}(\pi^1, \ldots, \pi^{m+1}) \neq \Re^m$ under the premises of the theorem. ■

To find the k^{th} normal π^k of a hyperplane $\mathcal{H}(\pi^k, 1)$ that contains a facet of the snug circumscribing simplex where $2 \leq k \leq m+1$, we form the LP:

$$\begin{aligned}
\max_{\pi \in \Re^m} \quad & \sum_{i=1}^{k-1} \left\langle -\frac{\pi^i}{\|\pi^i\|}, \pi \right\rangle & & \textbf{Polar Cone LP} \qquad (4.41)\\
\text{s.t.} \quad & \langle a^j, \pi \rangle \leq 1; & & j = 1, \ldots, n,\\
& \langle \pi^i, \pi \rangle \leq 0; & & i = 1, \ldots, k-1.
\end{aligned}$$

The optimal solution of this LP π^k yields the normal of a new facet that we have not met before. Therefore we get $m+1$ hyperplanes $\mathcal{H}(\pi^k, 1)$ for $k = 1, \ldots, m+1$

that contain $m+1$ facets of the final circumscribing simplex.

4.5.1 Procedure BOS pseudocode

The pseudo-code of Procedure BOS for a given point set $\mathcal{A}$ that contains origin is as follows.

Procedure BOS

Input : A point set $\mathcal{A} = \{a^1, \ldots, a^n\} \in \Re^m$.
Output: A circumscribing simplex for $\mathcal{A}$.

Choose an arbitrary point $b \in \Re^m$;
The optimal solution π^1 of Dual Gauge LP in (4.36) is the normal of a hyperplane $\mathcal{H}(\pi^1, \beta^1)$ that contains a facet con($\mathcal{A}$);
for $k \leftarrow 2$ **to** $m+1$ **do**
    The optimal solution π^k of Polar Cone LP, (4.41), is the normal of a new supporting hyperplane $\mathcal{H}(\pi^k, \beta^k)$ that contains a facet of the final circumscribing simplex;
end

The $m+1$ hyperplanes $\mathcal{H}(\pi^k, \beta^k)$ for $k = 1, \ldots, m+1$ contains $m+1$ facets of the final circumscribing simplex.

Four observations about Procedure BOS are as follows:

Remark 1. All the facets of a final circumscribing simplex support con($\mathcal{A}$), and at least one facet (the first founded facet) of the final circumscribing simplex contains a facet of con($\mathcal{A}$).

Remark 2. There is needed solving an LP to find each facet of the circumscribing simplex, and overall the $m+1$ LPs.

Remark 3. It is possible that one of the polar hyperplane's normal is generated by Polar Cone LP is not linear independent from the previous ones.

Remark 4. On the complexity of Procedures BOS, one overall needs to solve $m+1$

LPs that each of them has m variables and the i^{th} LP has $n+i-1$ constraints for $i=1,\ldots,n+1$ to find the final circumscribing simplex.

4.5.2 Example: Using Procedure BOS to find a circumscribing simplex for a finite point set in $\Re^3$

Consider the point set $\mathcal{A}=\{a^1=(1,4,2)^T,\ a^2=(3,1,5)^T,\ a^3=(2,2,3)^T,\ a^4=(2,5,6)^T,\ a^5=(4,3,1)^T,\ a^6=(3,5,4)^T,\ a^7=(9,3,2)^T\}$ in $\Re^3$ from the first example. The polytope con($\mathcal{A}$) does not contain the origin. To use Procedure `BOS`, we translate $\mathcal{A}$ such that the origin is contained by con($\mathcal{A}$). Consider translated point set $\tilde{\mathcal{A}}=\{\tilde{a}^1=(-4,1,-1)^T,\tilde{a}^2=(-2,-2,2)^T,\tilde{a}^3=(-3,-1,0)^T,\tilde{a}^4=(-3,2,3)^T,\tilde{a}^5=(-1,0,-2)^T,\tilde{a}^6=(-2,2,1)^T,\tilde{a}^7=(4,0,-1)^T\}$.

To find the first facet of a circumscribing simplex, we select a point $b=(-5,1,-2)^T$, and construct Dual Gauge LP in (4.36). So, we have:

$$\begin{aligned}
\max_{\pi\in\Re^3}\ & -5\pi_1+\pi_2-2\pi_3 \\
\text{s.t.}\ & -4\pi_1+\pi_2-\pi_3\le 1, \\
& -2\pi_1-2\pi_2+2\pi_3\le 1, \\
& -3\pi_1-\pi_2\le 1, \\
& -3\pi_1+2\pi_2+3\pi_3\le 1, \\
& -\pi_1-2\pi_3\le 1, \\
& -2\pi_1+2\pi_2+\pi_3\le 1, \\
& 4\pi_1-\pi_3\le 1.
\end{aligned}\tag{4.42}$$

The optimal solution of this LP yields the hyperplane $\mathcal{H}(\pi^1,13)$ where $\pi^1=(-3,-4,-5)^T$, with $\tilde{\mathcal{S}}_1=\{\tilde{a}^1,\tilde{a}^3,\tilde{a}^5\}$ in the support set. This hyperplane contains a facet of the final circumscribing simplex.

To find the second facets of the final circumscribing simplex, we construct Polar Cone LP in (4.41).

$$
\begin{aligned}
\max_{\pi \in \Re^3} \quad & \frac{3}{\sqrt{50}}\pi_1 + \frac{4}{\sqrt{50}}\pi_2 + \frac{5}{\sqrt{50}}\pi_3 \\
\text{s.t.} \quad & -4\pi_1 + \pi_2 - \pi_3 \le 1, \\
& -2\pi_1 - 2\pi_2 + 2\pi_3 \le 1, \\
& -3\pi_1 - \pi_2 \le 1, \\
& -3\pi_1 + 2\pi_2 + 3\pi_3 \le 1, \\
& -\pi_1 - 2\pi_3 \le 1, \\
& -2\pi_1 + 2\pi_2 + \pi_3 \le 1, \\
& 4\pi_1 - \pi_3 \le 1, \\
& -3\pi_1 - 4\pi_2 - 5\pi_3 \le 0.
\end{aligned}
\tag{4.43}
$$

The optimal solution yields the hyperplane $\mathcal{H}(\pi^2, 30)$ where $\pi^2 = (14, -3, 26)^T$, with $\tilde{\mathcal{S}}_2 = \{\tilde{a}^2, \tilde{a}^4, \tilde{a}^7\}$ in the support set.

We found two facets of the circumscribing simplex so far. We need to find two more facets. To find the third facet, we construct Polar Cone LP in (4.41), by having two hyperplanes $\mathcal{H}(\pi^1, 13)$ and $\mathcal{H}(\pi^2, 30)$ as follows:

$$
\begin{aligned}
\max_{\pi\in\Re^3} \quad & (\frac{3}{\sqrt{50}} - \frac{14}{\sqrt{881}})\pi_1 + (\frac{4}{\sqrt{50}} + \frac{3}{\sqrt{881}})\pi_2 + (\frac{5}{\sqrt{50}} - \frac{26}{\sqrt{881}})\pi_3 \\
\text{s.t.} \quad & -4\pi_1 + \pi_2 - \pi_3 \leq 1, \\
& -2\pi_1 - 2\pi_2 + 2\pi_3 \leq 1, \\
& -3\pi_1 - \pi_2 \leq 1, \\
& -3\pi_1 + 2\pi_2 + 3\pi_3 \leq 1, \\
& -\pi_1 - 2\pi_3 \leq 1, \\
& -2\pi_1 + 2\pi_2 + \pi_3 \leq 1, \\
& 4\pi_1 - \pi_3 \leq 1, \\
& -3\pi_1 - 4\pi_2 - 5\pi_3 \leq 0, \\
& 14\pi_1 - 3\pi_2 + 26\pi_3 \leq 0.
\end{aligned}
\tag{4.44}
$$

The optimal solution of this LP yields the hyperplane $\mathcal{H}(\pi^3, 9)$ where $\pi^3 = (1, 8, -5)^T$, with $\tilde{\mathcal{S}}_3 = \{\tilde{a}^1, \tilde{a}^5, \tilde{a}^6, \tilde{a}^7\}$ in the support set.

Finally, to find the last facet, we construct Polar Cone LP in (4.41) as follows:

$$
\begin{aligned}
\max_{\pi\in\Re^3} \quad & (\frac{3}{\sqrt{50}} - \frac{14}{\sqrt{881}} - \frac{1}{\sqrt{90}})\pi_1 + (\frac{4}{\sqrt{50}} + \frac{3}{\sqrt{881}} - \frac{8}{\sqrt{90}})\pi_2 + (\frac{5}{\sqrt{50}} - \frac{26}{\sqrt{881}} + \frac{5}{\sqrt{90}})\pi_3 \\
\text{s.t.} \quad & -4\pi_1 + \pi_2 - \pi_3 \leq 1, \\
& -2\pi_1 - 2\pi_2 + 2\pi_3 \leq 1, \\
& -3\pi_1 - \pi_2 \leq 1, \\
& -3\pi_1 + 2\pi_2 + 3\pi_3 \leq 1, \\
& -\pi_1 - 2\pi_3 \leq 1, \\
& -2\pi_1 + 2\pi_2 + \pi_3 \leq 1, \\
& 4\pi_1 - \pi_3 \leq 1, \\
& -3\pi_1 - 4\pi_2 - 5\pi_3 \leq 0, \\
& 14\pi_1 - 3\pi_2 + 26\pi_3 \leq 0, \\
& \pi_1 + 8\pi_2 - 5\pi_3 \leq 0.
\end{aligned}
\tag{4.45}
$$

This LP yields the last hyperplane $\mathcal{H}(\pi^4, 10)$ where $\pi^4 = (-3, -1, 1)^T$, with $\tilde{\mathcal{S}}_4 = \{\tilde{a}^1, \tilde{a}^2, \tilde{a}^3, \tilde{a}^4\}$ in the support set.

We found the four facets of a circumscribing simplex that this simplex contains $\mathrm{con}(\tilde{\mathcal{A}})$. If translate back the point set to the original place, the four hyperplanes $\mathcal{H}(\pi^1, 29)$, $\mathcal{H}(\pi^2, 169)$, $\mathcal{H}(\pi^3, 23)$, and $\mathcal{H}(\pi^4, -5)$, where $\pi^1 = (-3, -4, -5)^T$, $\pi^2 = (14, -3, 26)^T$, $\pi^3 = (1, 8, -5)^T$, and $\pi^4 = (-3, -1, 1)^T$, contain four facets of the final circumscribing simplex that contains $\mathrm{con}(\mathcal{A})$. The support sets of these four hyperplanes include $\mathcal{S}_1 = \{a^1, a^3, a^5\}$, $\mathcal{S}_2 = \{a^2, a^4, a^7\}$, $\mathcal{S}_3 = \{a^1, a^5, a^6, a^7\}$, and $\mathcal{S}_4 = \{a^1, a^2, a^3, a^4\}$ respectively.

In Fig 50, the obtained circumscribing simplex by this procedure is illustrated in three views.

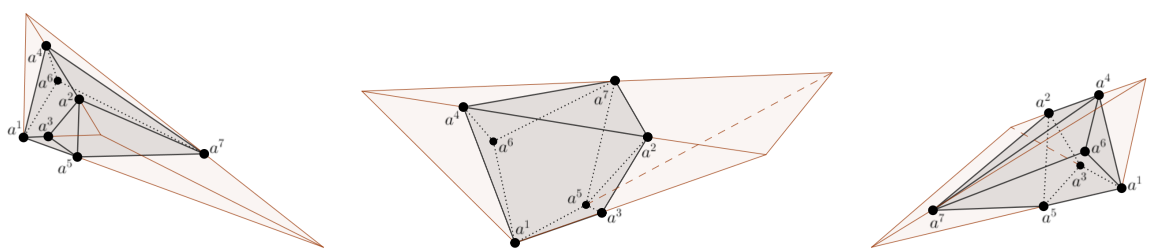

Fig. 50.: Three views of the final circumscribing simplex obtained by applying Procedure `BOS`.

In this example, the four facets of the final circumscribing simplex contain the four facets of $\mathrm{con}(\mathcal{A})$. As we mentioned earlier in this section, it does not happen always.

4.6 Conclusion

In this chapter we present three Procedures ARS, FRS, and BOS to find a circumscribing simplex for a finite point set.

To apply two Procedures ARS and FRS, we first initialize it with PreSnug. To do so, we use linear algebraic operations to find PreSnug. Then, by applying Procedures ARH and FRH in Procedures ARS and FRS respectively, we rotate m hyperplanes containing m facets of PreSnug such that rotated hyperplanes contain m facets of the convex hull of the point set. If there exists duplicate facets, then we use Procedure FTF to rotate it. Finally, we apply *Boundedness LP* to find the last facet of a snug circumscribing simplex.

Procedure BOS uses polar cone's properties and *Gauge LP* to find a circumscribing simplex.

Table 2 shows the complexity of three Procedures ARS, FRS, and BOS when they do not need to use Procedure FTF.

	System of equations	Min/Max ratio tests	LP (rows) $\times$ (columns)
ARS	$m * \left(\frac{(m-1)(m-2)}{2}\right)$	$m * \left(\frac{(m-1)(2n-m)}{2}\right)$	$(2m+n) \times (2m+1)$
FRS	-	-	$\big((2m+n) \times (2m+1)\big) + \sum_{i=1}^{m} \big((n-1) \times (m-1)\big)$
BOS	-	-	$\sum_{i=1}^{m+1} (n+i-1) \times (m)$

Table 2.: On the complexity of three Procedures ARS, FRS, and BOS.

CHAPTER 5

COMPUTATIONAL EXPERIMENTS

In this chapter we present comprehensive numerical experiments for all the procedures developed in Chapters 3 and 4. To reveal how rotating hyperplanes and circumscribing simplexes procedures perform, we implement, test, and report the results for the procedures a large and varied suite of test data in the form of point sets. In this chapter, we discuss about different procedures for rotating a hyperplane or finding a circumscribing simplex, regarding the computational time (seconds).

We test and report the results for the procedures with the 24 different point sets in dimensions $m = 5,\ 10,\ \ 15,\ 20$, with cardinalities $n = 100,\ 1000,\ \ 2500,\ 5000,$ $10000,\ 25000$. Note that all the points in each point set are extreme points for their convex hull.

We report the results in two sections for two separate topics. There are two collections of figures in each section. The first has four figures where each one demonstrates a comparison between time versus dimension ($m = 5,\ 10\ ,\ 15,\ 20$) for the procedures. There are seven figures in the second collection where each figure presents a comparison between time versus cardinality ($n = 100,\ 1000,\ 2500,\ 5000,\ 7500,\ 10000,\ 25000$) for the procedures.

The experiments were conducted on an OS X Yosemite machine with 3 GHz Intel Core i7 and 16 GB 1600 MHz DDR3 RAM. All the procedures were implemented in C and we used GNU - GSL version 1.2 to perform all the matrix operations. All LPs were solved with Gurobi 6.5.0.

5.1 Results and analysis for hyperplane rotation

We presented three procedures in this topic of the dissertation. Let the point set $\mathcal{A} = \{a^1, \dots, a^n\}$ in $\Re^m$. Recall that from Chapter 3, Procedures `ARH` and `FRH` can be initialized by a supporting hyperplane $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$ for $\operatorname{con}(\mathcal{A})$ in $\Re^m$ where

$$\tilde{\pi} = \bar{1}, \text{ and } \tilde{\beta} = \max \left\{ \sum_{i=1}^{m} a_i^j \;\middle|\; a^j \in \mathcal{A} \right\}. \tag{5.1}$$

We have $\tilde{\pi} = (1, \dots, 1)^T$, so the supporting hyperplane $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$ contains a point from $\mathcal{A}$ in which the sum of its elements is greater than the sum of the elements of any other point in $\mathcal{A}$.

We rotated the hyperplane $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$ on the convex hull of 24 point sets by applying the two Procedures `ARH` and `FRH`.

Recall that in Procedure `ARH`, we do not solve LPs and only apply algebra operations and that the number of the extreme points from $\mathcal{A}$ in the support set for the hyperplanes increases at least one at a time (in all our experiments it increased exactly one). Consider a supporting hyperplane containing k points of $\mathcal{A}$. To rotate this hyperplane such that it contains a facet of $\operatorname{con}(\mathcal{A})$, there are needed $m - k$

iterations at most.

If we apply Procedure `FRH` to rotate a supporting hyperplane on the convex hull of a point set, the rotated hyperplane contains a facet of the convex hull of this point set after only one iteration.

We use the same point sets to evaluate Procedure `FTF`. To this end, we rotate the obtained hyperplanes that contain a facet of the convex hull from Procedure `FRH` in the 24 point sets. Recall that similar to Procedure `ARH`, Procedure `FTF` does not rely on LP.

Table 3 presents the computational times for the three procedures `ARH`, `FRH`, and `FTF`. An immediate observation from this table is that Procedure `ARH` is faster than Procedure `FRH` to arrive at a facet of a convex hull in all our experiments. This comparison implemented when the cardinality of a point set is fewer then 5000, then there is not huge different to use either Procedures `ARH` and `FRH` for rotating a supporting hyperplane on the convex hull of the point set.

It takes 0.375 seconds to rotate a supporting hyperplane $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$ on con(20by25000) when Procedure `FRH` is applied. This is the largest time in all our experiments.

Procedure `FTF` is fast to rotate the supporting hyperplane $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$ on the convex hull of the tested point sets, even at the highest dimension and largest cardinality point set (20by25000).

Fig. 51. illustrates the different CPU times for the 24 point sets for $m = 5,\ 10$ $,\ 15,\ 20$ separately. They indicate that by increasing cardinality, the difference of the obtained times between the two Procedures `ARH` and `FRH` becomes larger. When the cardinality n is fewer than 2500, there is not huge different in times by applying

either Procedure ARH or Procedure FRH to rotate the supporting hyperplane $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$ on the convex hull of the tested point sets. Increasing the cardinality, this difference becomes more tangible. It becomes more clear if there is a choice of Procedures ARH or FRH: it is faster to rotate a supporting hyperplane on the convex hull of a point set in large scale cardinality, regardless of the dimension, to use Procedure ARH. Moreover, recall that another advantage of Procedure ARH is that it relies on linear algebra operations, whereas Procedure FRH requires formulating and solving an LP. Therefore, it saves time to apply Procedure ARH for rotating a supporting hyperplane on the convex hull of a point set for large cardinality ($n \geq 2500$).

Although Procedure ARH is faster than Procedure FRH in all tested point sets to rotate a supporting hyperplane $\mathcal{H}(\tilde{\pi}, \tilde{\beta})$, but Procedure FRH has a simpler structure and pseudocode. So, we advise when the time to rotate a supporting hyperplane is not critical, or the cardinality n is fewer than 2500, analysts apply Procedure FRH.

Procedure FTF should not be compared directly to the two Procedures ARH and FRH. The reason refers to the dimension of the support set for the supporting hyperplane that get rotated on the convex hull of the point sets. We apply Procedure FTF to rotate a supporting hyperplane on the convex hull of a point set when the dimension of this supporting hyperplane for the convex hull of the point set is $m-1$ in $\Re^m$, whereas Procedures ARH and FRH are applied for rotating a supporting hyperplane on the convex hull of a point set when the dimension of the support set for this hyperplane is fewer than $m-1$. Thus, the requirements for applying Procedure FTF with two other Procedures ARH and FRH are different.

In all our experiments, the obtained time to rotate a supporting hyperplane on the convex hull of a point set by applying Procedure `FTF` is extremely fast, even when the largest point set (20by25000) is involved. The largest value of the obtained time in Procedure `FTF` is just 0.008 seconds.

Another indication is that by increasing the cardinality of the point set, the obtained time by applying Procedures `ARH` and `FRH` are growing much faster than applying Procedure `FTF`.

Fig. 52. illustrates of CPU times for all 24 tested point sets are depicted based on different cardinalities $n = 100,\ 1000, 2500,\ 5000,\ 7500, 10000,\ 25000$.

Figures in 51 and 52 suggest that the three Procedures `ARH`, `FRH`, and `FTF` are polynomial on their dimensions, and linear on their cardinalities.

File name	ARH	FRH	FTF
05by00100	0.00E+0	1.00E-3	0.00E+0
05by01000	0.00E+0	5.00E-3	0.00E+0
05by02500	1.00E-3	1.10E-2	0.00E+0
05by05000	3.00E-3	1.70E-2	0.00E+0
05by07500	4.00E-3	2.60E-2	1.00E-3
05by10000	5.00E-3	3.50E-2	1.00E-3
05by25000	1.20E-2	8.30E-2	2.00E-3
10by00100	0.00E+0	2.00E-3	0.00E+0
10by01000	2.00E-3	8.00E-3	0.00E+0
10by02500	5.00E-3	1.90E-2	0.00E+0
10by05000	9.00E-3	3.20E-2	1.00E-3
10by07500	1.30E-2	5.00E-2	1.00E-3
10by10000	1.80E-2	6.90E-2	2.00E-3
10by25000	3.90E-2	1.67E-1	4.00E-3
15by00100	1.00E-3	3.00E-3	0.00E+0
15by01000	5.00E-3	1.20E-2	0.00E+0
15by02500	9.00E-3	2.50E-2	1.00E-3
15by05000	1.80E-2	5.10E-2	1.00E-3
15by07500	2.90E-2	7.60E-2	2.00E-3
15by10000	3.40E-2	1.04E-1	3.00E-3
15by25000	8.90E-2	2.49E-1	6.00E-3
20by00100	1.00E-3	4.00E-3	0.00E+0
20by01000	9.00E-3	1.70E-2	0.00E+0
20by02500	1.60E-2	3.90E-2	1.00E-3
20by05000	3.10E-2	7.30E-2	2.00E-3
20by07500	4.60E-2	1.06E-1	3.00E-3
20by10000	6.40E-2	1.43E-1	3.00E-3
20by25000	1.48E-1	3.75E-1	8.00E-3

Table 3.: Comporison of CPU times (seconds) for rotating a supporting hyperplanes for Procedures ARH, FRH, and FTF.

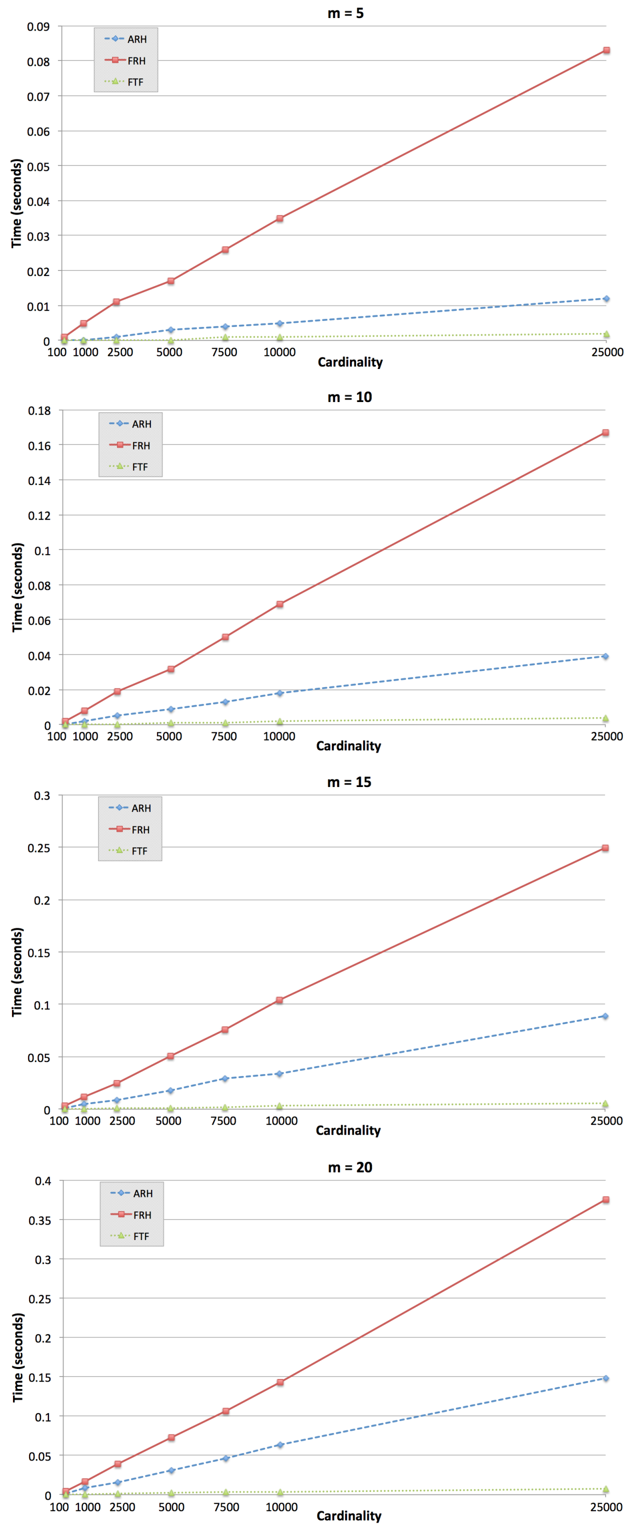

Fig. 51.: CPU times for rotating hyperplanes for the 24 point sets in dimensions $m = 5,\ 10,\ 15,\ 20$.

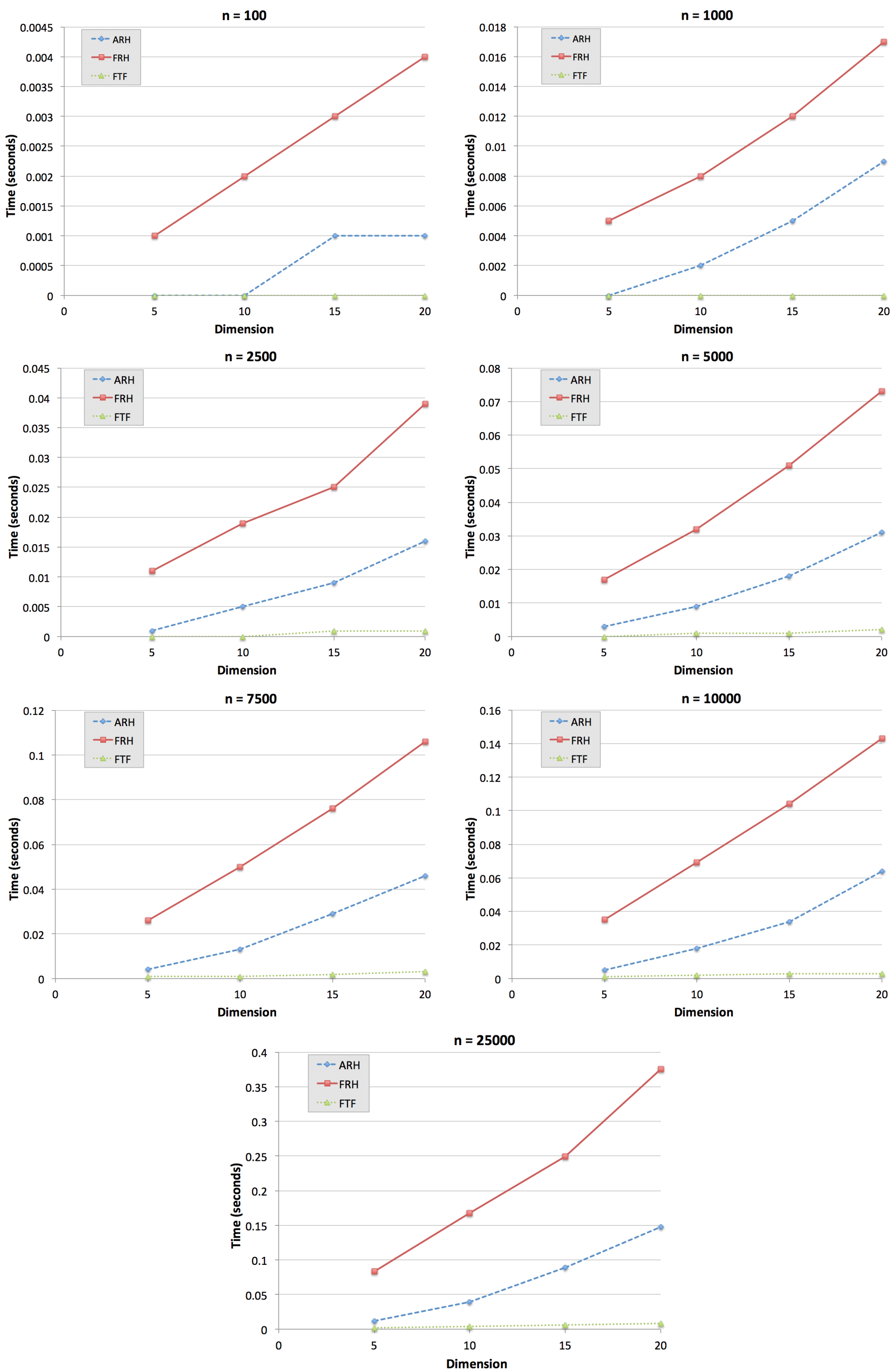

Fig. 52.: CPU times for rotating hyperplanes for the 24 point sets with cardinalities $n = 100, 1000, 2500, 5000, 7500, 10000, 25000$.

5.2 Results and analyses in circumscribing simplexes

In this section of the dissertation we compare the computational times of applying the three Procedures ARS, FRS, and BOS to find a circumscribing simplex for the 24 point sets.

Recall that a circumscribing simplex is snug (at least the m facets of the circumscribing simplex contains the m different facets of the convex hull of a given point set) when we apply either ARS or FRS. In both procedures ARS or FRS, we initialize using linear algebra operations to find PreSnug. Then, by applying Procedures ARH and FRH in Procedures ARS and FRS respectively, we rotate to find m facets of PreSnug such that the m rotated facets contain the m different facets of the convex hull for each point set. Finally, we apply Boundedness LP to find the last facet. Note that to find the first m facets of a snug circumscribing simplex using ARS, we just use linear algebra operations. To find each facet of a circumscribing simplex by using either ARS orBOS, we need to solve an LP.

The computational times for the three Procedures ARS, FRS, and BOS are presented in Table 4. The obtained times reveal that the times taken by applying Procedures FRS and BOS are almost double and sometimes triple the time taken to find a circumscribing simplex by applying Procedure ARS respectively for the same point set and dimension. The longest time for finding a circumscribing simplex is 11.911 seconds. This value refers to Procedure BOS in the largest point set (20by25000). Procedure ARS is fastest among all procedures for finding a circumscribing simplex for a given point set with any cardinality and dimension.

In Fig. 53., the different CPU times when we apply the three Procedures `ARS`, `FRS`, and `BOS` to find a circumscribing simplex for the 24 point sets for $m = 5,\ 10$ $,\ 15,\ 20$ are depicted separately. There are four charts in Fig. 53. These four charts are indicated that by increasing cardinality, the difference of the obtained times between two Procedures `FRS` and `BOS` with Procedure `ARS` is becoming larger. Another immediate indication is that the diagram of all these three procedures appear to be linear. This is an important result to predict the time taken by applying Procedure `ARS` to find a circumscribing simplex in a larger point set with certain dimension and any cardinality.

In Fig. 54., the illustrations of obtained CPU times when we apply Procedures `ARS`, `FRS`, and `BOS` to find a circumscribing simplexes for all the 24 point sets are depicted using cardinalities $n = 100,\ 1000\ ,\ 2500,\ 5000,\ 7500\ ,\ 10000,\ 25000$. The diagram of either Procedures `ARS`, `FRS`, or `BOS` are appear to be polynomial, perhaps a quadratic in all seven charts. An observation about these seven figures is that Procedure `ARS` is faster than other two procedures for any of the cardinalities. When $n = 100$, Procedure `BOS` is faster than Procedure `FRS`. This is the only case in all our experiments that Procedure `BOS` is faster than Procedure `FRS`.

File name	ARS	FRS	BOS
05by00100	3.00E-3	4.00E-3	3.00E-3
05by01000	9.00E-3	2.10E-2	2.20E-2
05by02500	2.20E-2	4.00E-2	4.80E-2
05by05000	3.50E-2	8.10E-2	9.20E-2
05by07500	4.90E-2	1.22E-1	1.47E-1
05by10000	6.30E-2	1.68E-1	1.99E-1
05by25000	1.55E-1	4.57E-1	5.93E-1
10by00100	5.00E-3	1.20E-2	9.00E-3
10by01000	2.60E-2	5.80E-2	6.60E-2
10by02500	5.70E-2	1.37E-1	1.60E-1
10by05000	1.13E-1	2.79E-1	3.16E-1
10by07500	1.59E-1	4.31E-1	5.27E-1
10by10000	2.14E-1	6.21E-1	7.82E-1
10by25000	5.23E-1	1.73E+0	1.92E+0
15by00100	1.40E-2	2.30E-2	1.80E-2
15by01000	6.90E-2	1.36E-1	1.38E-1
15by02500	1.50E-1	3.22E-1	3.63E-1
15by05000	2.92E-1	6.19E-1	7.08E-1
15by07500	4.36E-1	1.00E+0	1.05E+0
15by10000	5.85E-1	1.44E+0	1.58E+0
15by25000	1.47E+0	3.64E+0	4.81E+0
20by00100	2.70E-2	4.30E-2	3.60E-2
20by01000	1.37E-1	2.50E-1	3.35E-1
20by02500	3.39E-1	6.36E-1	1.02E+0
20by05000	6.36E-1	1.29E+0	2.08E+0
20by07500	9.34E-1	1.91E+0	3.26E+0
20by10000	1.27E+0	2.77E+0	4.69E+0
20by25000	3.09E+0	7.36E+0	1.19E+1

Table 4.: Comparison of CPU times (seconds) for finding a circumscribing simplex for Procedures ARS, FRS, and BOS.

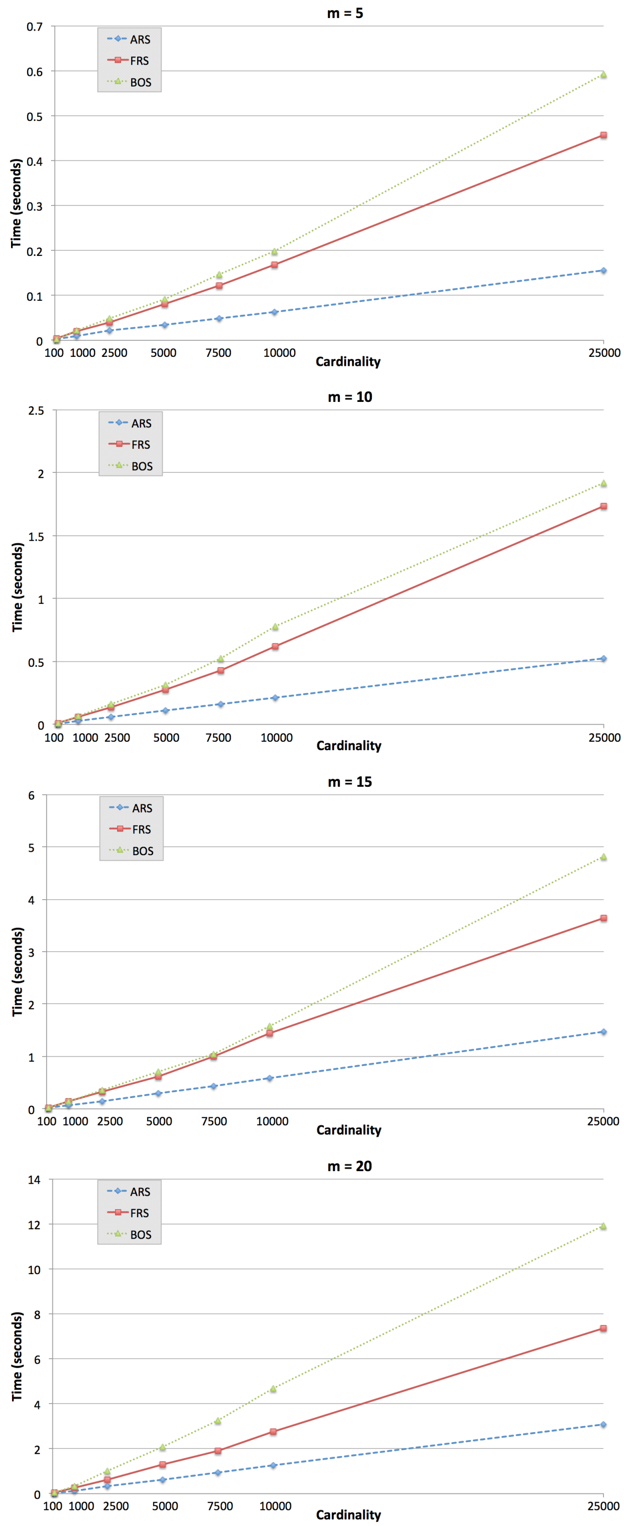

Fig. 53.: CPU times for finding a snug circumscribing simplex for the 24 point sets in dimensions $m = 5,\ 10,\ 15,\ 20$.

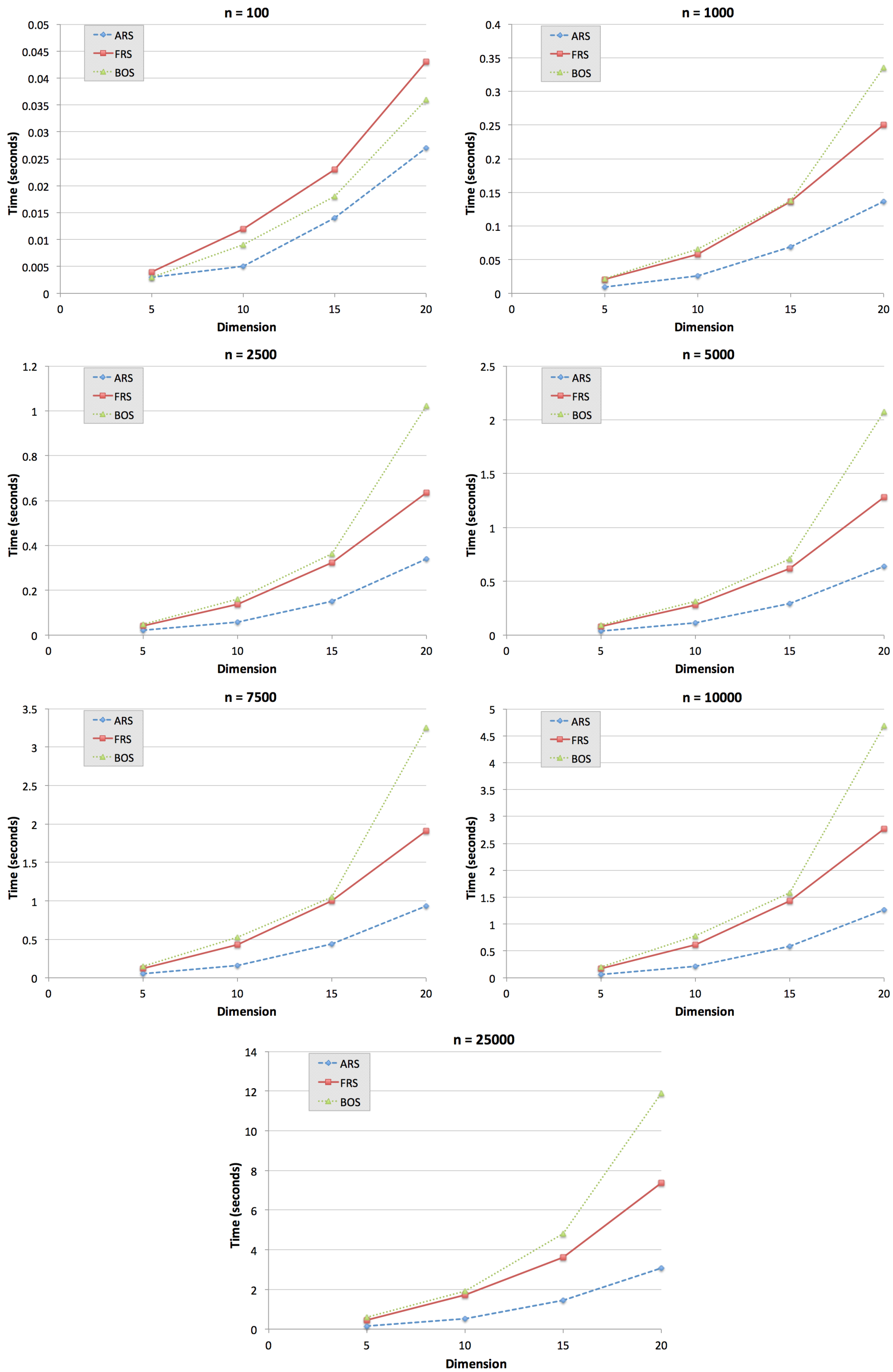

Fig. 54.: CPU times for finding a snug circumscribing simplex for the 24 point sets with cardinalities $n = 100,\ 1000,\ 2500,\ 5000,\ 7500,\ 10000,\ 25000$.

5.3 Conclusion

This chapter presents the results of applying the procedures for rotating hyperplanes and snug circumscribing simplex for the convex hull of a finite point set in multiple dimensions using the 24 point sets. The point sets are in dimensions $m = 5,\ 10,\ 15,\ 20$. In each dimension we test seven point sets in cardinalities $n = 100,\ 1000,\ 2500,\ 5000,\ 7500,\ 10000,\ 25000$.

The results of rotating hyperplanes show that Procedure `ARH` is faster than Procedure `FRH` to rotate a supporting hyperplane on the convex hull of a given point set. Procedure `FTF` performs a different task and is much faster in rotating a supporting hyperplane from one facet to an adjacent one for the largest point set (20by25000).

In the second section we presented the results of applying Procedures `ARS`, `FRS`, and `BOS` to find a circumscribing simplex in the 24 point sets. The results showed that, to find a circumscribing simplex in the 24 point sets, the time taken by applying Procedure `FRS` and `BOS` are double and triple of applying Procedure `ARS` respectively.

CHAPTER 6

APPLICATION: HYPERSPECTRAL UNMIXING

In hyperspectral unmixing, the extreme points are known as "endmembers". Theses endmembers are unknown usually and calculated by the existence pixels [41]. Having a mixture of pixels is a typical problem in satellite images [41]. One goal in hyperspectral unmixing is to find (or estimate) endmembers based on the existence mixed pixels [42] This procedure is so called "mixed pixel decomposition", because endmembers are based on a decomposition of abundance of the mixed pixels [42]. The procedures to identify the endmembers in hyperspectral unmixing are based on computational geometric and statistical [43]. The way how to deal with mixed pixels to find a containment polytope plays the central role in hyperspectral unmixing [43]. One suite polytope to contain mixed pixels is a simplex [43].

To make this problem clearer, and interpret its relation with computational geometry, consider a small example using concrete. Imagine concrete as a blend of three ingredients cement, sand, and gravel. Depending on how the concrete will be used, a specified blend of these ingredients will be required. Two measurements can be made about a given blend: volume and weight. These are depicted in Fig. 55. The object in the figure is a simplex, because there are three ingredients and two measurements. Note that m measurements and $m+1$ ingredients will produce a simplex in m dimensions. From a geometrical point of view, consider Fig. 55.

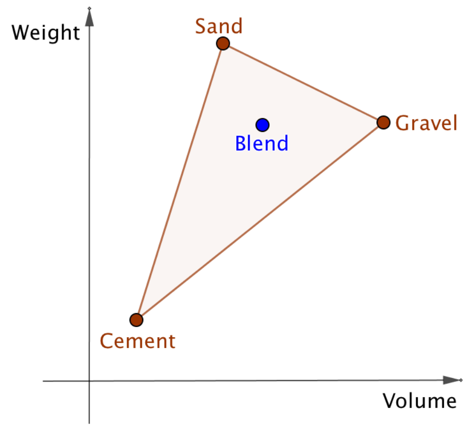

Fig. 55.: Different concretes with two measurements and three ingredients.

There is a unique amount of each of the three ingredients, cement, sand, and grave, for any blend. These three ingredients correspond to what are referred to as "endmembers" in this simple analogy to hyperspectral unmixing. Consider a concrete blend $\hat{x}$, and assume $w_{\hat{x}}$, $v_{\hat{x}}$ show the weight and the volume of blend $\hat{x}$ respectively. So to find the proportions of the three ingredients in this blend we need to solve the following system of equations.

$$w_{cement}\ \lambda_1 + w_{sand}\ \lambda_2 + w_{gravel}\ \lambda_3 = w_{\hat{x}}, \tag{6.1}$$

$$v_{cement}\ \lambda_1 + v_{sand}\ \lambda_2 + v_{gravel}\ \lambda_3 = v_{\hat{x}}, \tag{6.2}$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1, \tag{6.3}$$

where $\lambda_1,\ \lambda_2,\ \lambda_3$ are non-negative variables. Every point inside of the convex hull of a point set is a linear combination of the extreme points, also referred to as "endmembers" in hyperspectral unmixing, of the set with non-negative coefficients. Because the convex hull is a simplex, System (6.1-6.3) has a unique solution.

This system of equations in the special case when $\hat{x}$ contains a single ingredient has two zero variables, and the value of the another variable is one. In hyperspectral unmixing, this case happens when there exists pure pixels.

In this example the ingredients of the tested substance was three, so we defined two measurements. In general, if the number of ingredients is m, there needs to be $m-1$ measurements. If there are more than $m-1$ measurements, the $m-1$ most important measurements can be used, or the data can be projected to lower dimensions.

Li et al., 2015 presented Procedure `MVSA`, Minimum Volume Simplex Analysis, to find a simplex for hyperspectral unmixing with minimum volume [44]. The authors distributed their procedure in MATLAB. There is a parameter p as a number of endmembers. When this parameter is defined, then this code selects p affinely independent points, True, in $\Re^{P-1}$, and generates N points inside the convex hull of these p points. The left hand side figure in Fig. 56, is a simulation of their code's output when $p = 4$, and the right hand side figure is this point set with their convex hull (a simplex). To depict these two figures we use their code's output.

We used their code to generate the point sets in dimensions $m = 2, 3,\ 5, 10$ with cardinalities $n = 1000, 15000$,$30000, 45000$. The point sets are generated inside a simplex. It selects p affinely independent points, and generates data inside of their

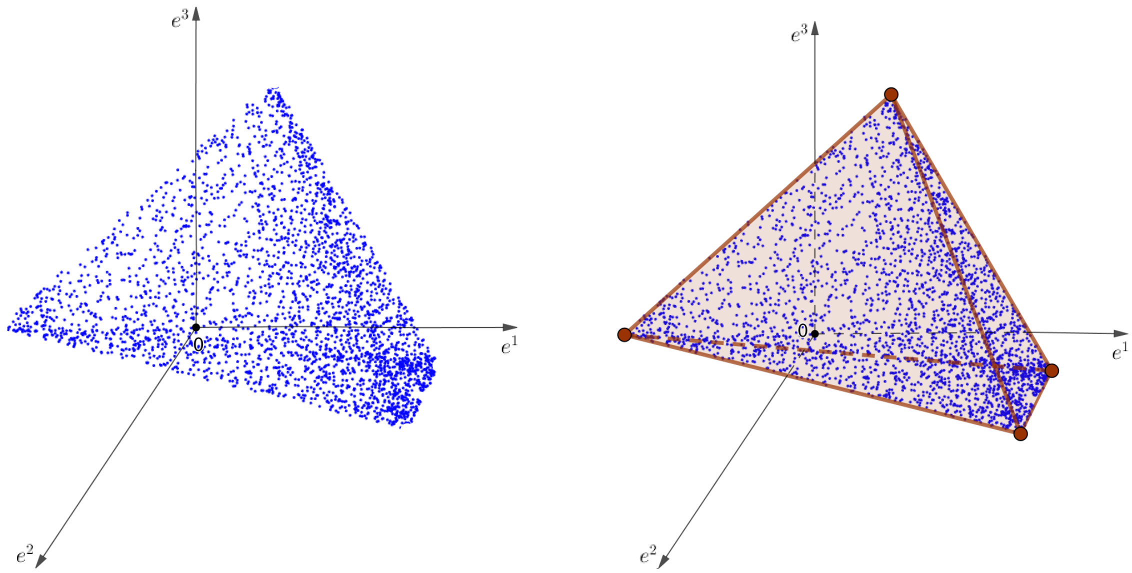

Fig. 56.: The left hand-side is a generated point set in three dimensions, and the right hand side shows a simplex that contains this point set.

convex hull. We added these p points to the point sets for finding a simplex by three procedures `ARS`, `FRS`, and `BOS`. The experiments were conducted on an OS X Yosemite machine with 3 GHz Intel Core i7 and 16 GB 1600 MHz DDR3 RAM. All the procedures were implemented in C and we used GNU - GSL version 1.2 to perform all the matrix operations. All LPs were solved with Gurobi 6.5.0.

In the generated point sets, we set *SIGNATURES-TYPE = 3*. The point sets are generated uniformly in $[0, 1]$. We multiplied point sets to 100 to manage them better, specialty in large cardinalities. We set signal-to-noise ratio and number of bands to 100 and 200 respectively: *SNR = 100, L =200*. To generate data uniformly over the simplex, we set *SHAPE-PARAMETER = 1*. Moreover, to avoid having outliers in the point sets, we set *OUTLIERS = 0*.

We consider the case of having pure pixels. In this case, we set *MAX-PURIRY = 1.00*.

Li et al., implemented their results by average 30 independent runs [44]. We thus report the obtained times, and the volumes of the obtained simplexes by applying four procedures `MVSA`, `ARS`, `FRS`, and `BOS`. In this dissertation, we did same to report the obtained times for Procedure `MVSA`. To report the volume of the obtained simplexes, we selected a random point set in each dimension and cardinality.

Table 5 presents the obtained volume in Case 1, when we have pure pixels.

File name	`MVSA`	`ARS`	`FRS`	`BOS`
02by01000	1.50191E+05	1.46233E+05	1.46233E+05	1.46233E+05
02by15000	1.43447E+05	1.43504E+05	1.43504E+05	1.43504E+05
02by30000	1.38154E+05	1.38170E+05	1.38170E+05	1.38170E+05
02by45000	1.45507E+05	1.43623E+05	1.43623E+05	1.43623E+05
03by01000	2.40477E+07	2.29582E+07	2.29582E+07	2.29582E+07
03by15000	2.27274E+07	2.27406E+07	2.27406E+07	2.27408E+07
03by30000	2.32552E+07	2.32646E+07	2.32646E+07	2.32647E+07
03by45000	2.46771E+07	2.46809E+07	2.46809E+07	2.46809E+07
05by01000	2.60052E+11	2.69471E+11	2.69469E+11	2.69469E+11
05by15000	1.97712E+11	1.98074E+11	1.98042E+11	1.98045E+11
05by30000	2.20555E+11	2.04353E+11	2.04353E+11	2.04353E+11
05by45000	2.08174E+11	2.08269E+11	2.08258E+11	2.08252E+11
10by01000	9.70105E+19	1.07413E+20	1.07367E+20	2.09818E+21
10by15000	1.19588E+20	1.20312E+20	1.20312E+20	1.81531E+21
10by30000	1.22849E+20	1.00914E+20	1.00910E+20	3.88743E+21
10by45000	9.97874E+19	1.01093E+20	1.01092E+20	8.50383E+21

Table 5.: The volumes of the obtained simplexes for the point sets using Procedures `MVSA`, `ARS`, `FRS`, and `BOS`.

In dimensions $m = 2, 3, 5$, the volumes of the obtained simplexes by either of the four procedures are very close to each other.

In dimension $m = 10$, the obtained volumes for Procedure BOS is not as good as other three procedures. In this dimension three Procedures MVSA, ARS, and FRS yields some simplexes that their volumes are kind of equal.

Table 6 presents the obtained times. We must emphasize that we implemented Procedures ARS, FRS, and BOS in C, and Procedures MVSA is implemented in MATLAB. Each value is obtained by the average of 30 independent runs.

File name	MVSA	ARS	FRS	BOS
02by01000	1.386E+00	1.000E-03	1.000E-02	8.000E-03
02by15000	2.201E+00	4.600E-02	8.200E-02	5.600E-02
02by30000	4.214E+00	7.900E-02	1.580E-01	1.070E-01
02by45000	3.778E+00	1.270E-01	2.410E-01	1.790E-01
03by01000	1.379E+00	6.000E-03	1.200E-02	9.000E-03
03by15000	2.653E+00	5.800E-02	1.340E-01	8.300E-02
03by30000	2.616E+00	1.110E-01	2.710E-01	1.690E-01
03by45000	4.793E+00	1.620E-01	4.800E-01	2.480E-01
05by01000	1.276E+00	1.200E-02	2.200E-02	1.700E-02
05by15000	2.544E+00	9.500E-02	2.650E-01	1.480E-01
05by30000	5.155E+00	1.930E-01	5.600E-01	2.850E-01
05by45000	7.370E+00	2.930E-01	8.940E-01	4.540E-01
10by01000	1.294E+00	3.000E-02	5.400E-02	2.900E-02
10by15000	8.326E+00	3.700E-01	9.310E-01	4.510E-01
10by30000	1.479E+01	7.160E-01	2.058E+00	8.470E-01
10by45000	2.006E+01	1.075E+00	3.075E+00	1.580E+00

Table 6.: Comparison of the real times (seconds) to find a simplex using Procedures MVSA, ARS, FRS, and BOS.

In all experiments, Procedure ARS is fastest. Procedure BOS is faster than Procedure FRS. Although there is not huge different between these two procedures.

Fig. 57 shows the different real time (seconds) for the point sets in dimensions $m = 2, 3, 5, 10$. Procedure `MVSA` increases faster than other procedures, although the four procedure looks behaving linear or polynomial. Procedure `MVSA` does not behave consistently in dimensions $m = 2, 3$. For example in dimension $m = 2$, the average time of finding a simplex with cardinality $n = 30000$ is larger than the average time of finding a simplex with cardinality $n = 45000$. Moreover, in dimension $m = 3$, the average time of finding a simplex with cardinality $n = 15000$ is larger than the average time of finding a simplex with cardinality $n = 30000$.

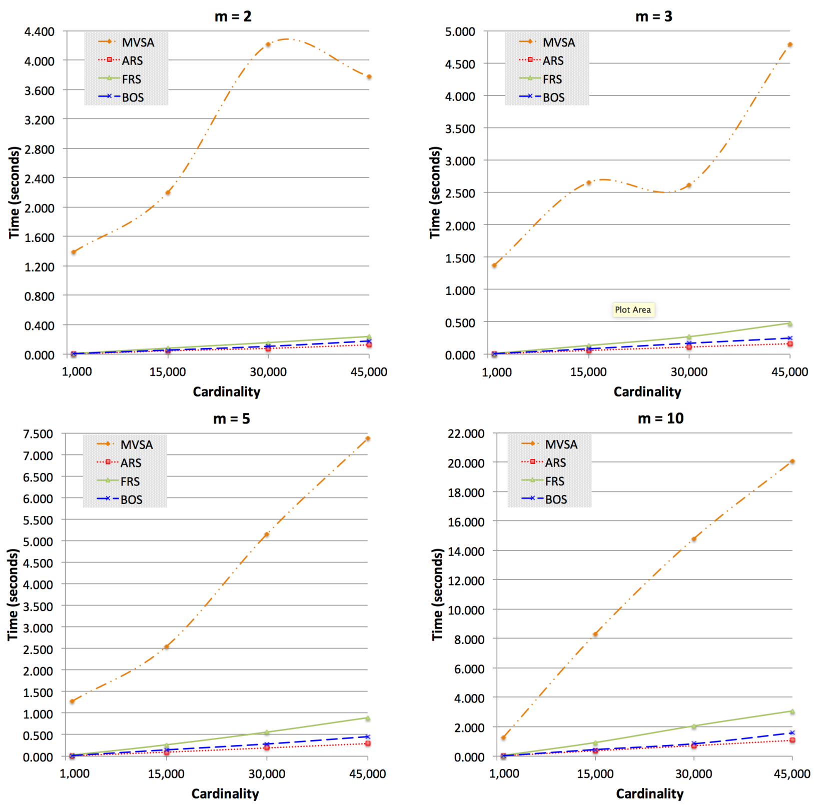

Fig. 57.: The real times for finding a simplex for the point sets using Procedures `MVSA`, `ARS`, `FRS`, and `BOS` in dimensions $m = 5,\ 10,\ 15,\ 20$.

Fig. 58 shows the different real times (seconds) for the point sets in cardinalities $n = 1000,\ 15000,\ \ 30000,\ 45000$. These figures illustrate Procedures `ARS` and `BOS` are linear. They also indicate that Procedures `FRS` is quadratic. In cardinalities $n = 15000,\ \ 30000,\ \ 45000$, Procedure `MVSA` is exponential. Notice that, when we use Procedure `MVSA` with cardinality $n = 1000$, the average time of finding a simplex decreases by increasing the dimension.

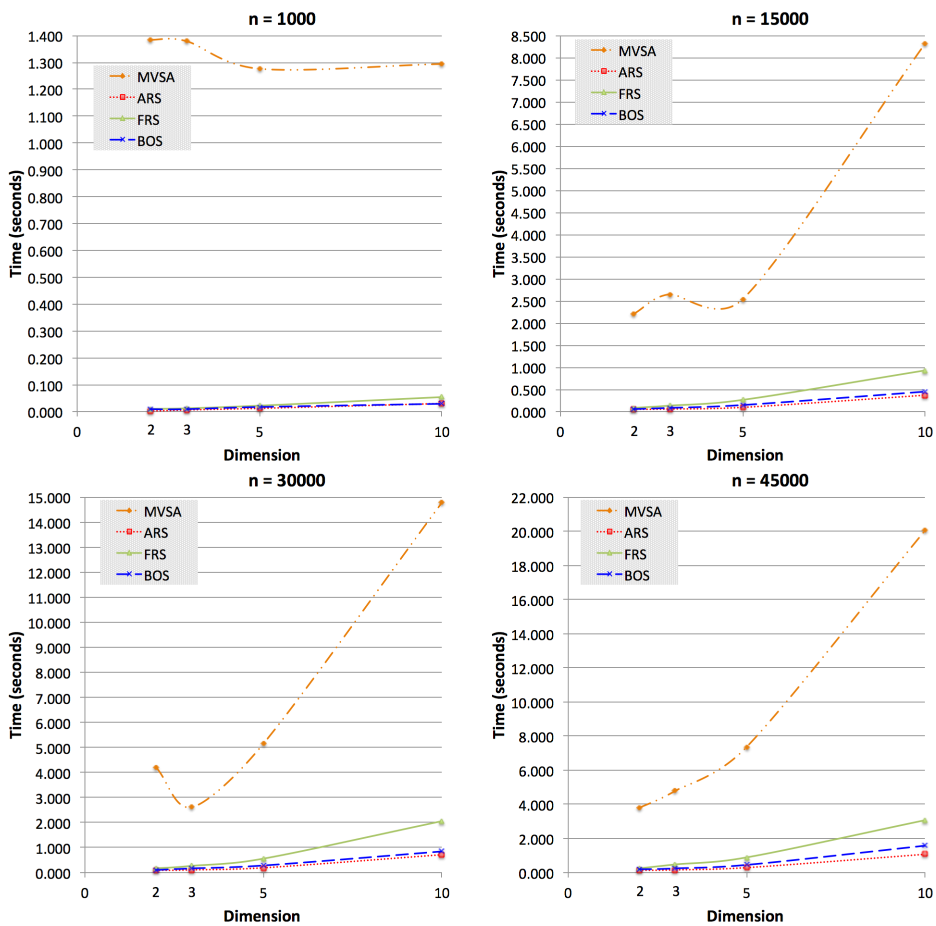

Fig. 58.: The real times for finding a simplex for the point sets using Procedures `MVSA`, `ARS`, `FRS`, and `BOS` in cardinalities $n = 1000,\ 15000,\ 30000,\ 45000$.

6.1 Conclusion

In this chapter we demonstrate an application of finding a snug circumscribing simplex for a finite point set is in hyperspectral unmixing. We compare our three procedures with one existed procedure, `MVSA`. Based on obtained times and volumes, we suggest to the users that to use Procedures `ARS`, `FRA`, and `BOS` to find a circumscribing simplex in low dimensions e.g. in dimensions $m = 2, 3, 5$ when there exists pure pixels and we can add them to the point set.

CHAPTER 7

CONCLUSION

In this dissertation we have two research topics: Rotating Supporting Hyperplanes, and Snug Circumscribing Simplexes.

We present three Procedures `ARH`, `FRH`, and `FTF` to rotate a supporting hyperplane on the convex hull of a finite point set. Procedures `ARH` and `FTF` relies on linear algebraic operations, and Procedure `FRH` uses the solution of an LP to rotate a hyperplane on the polytope. Procedures `ARH` and `FTF` are used when the dimension of the support set for the initial hyperplane in $\Re^m$ is fewer than $m-1$. When this dimension is m, Procedures `FTF` is applied. The results of this research topic has applications in DEA, integer programming, finding a circumscribing for a finite point set simplex.

In the second topic, we present three Procedures `ARS`, `FRS`, and `BOS` to find a circumscribing simplex for a finite point set. To apply two Procedures `ARS` and `FRS`, we first initialize it with `PreSnug`. To do so, we use linear algebraic operations to find `PreSnug`. Then, by applying Procedures `ARH` and `FRH` in Procedures `ARS` and `FRS` respectively, we rotate m hyperplanes containing m facets of `PreSnug` such that rotated hyperplanes contain m facets of the convex hull of the point set. If there exists duplicate facets, then we use Procedure `FTF` to rotate it. Finally, we apply *Boundedness LP* to find the last facet of a snug circumscribing simplex. Procedure `BOS` uses polar cone's properties and *Gauge LP* to find a circumscribing simplex. The results of this research topic has applications in NMF and hyperspectral unmixing. We applied the procedures for snug simplexes to this last topic.

7.1 Future works

We here present five possible future works that can extend the two presented research topics in this dissertation as follows:

1. Adopted Procedure `ARH` for using in dual simplex: implementation and test as an efficient for dual simplex.
2. Improve Procedure `ARS` to find a snug circumscribing simplex using only linear algebraic operations.
3. Improve Procedures `ARS`, `FRS`, and `BOS` to find a simplex with less volumes for some applications.
4. Present a new procedure to find a snug circumscribing simplex that it first finds the m facets containing a certain vertex of a polytope. Then, it finds the last facet (cap).
5. Using the obtained circumscribing simplex to construct a robust set.

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Appendix A

A.1 Procedure FTF:

Consider two hyperplanes $\mathcal{H}(\pi^1, 1)$ and $\mathcal{H}(\pi^2, 1)$ containing two adjacent facets of $\text{con}(\mathcal{A})$. Assume, wlog, the hyperplane $\mathcal{H}(\pi^1, 1)$ contains the points $a^1, \ldots, a^{m-1}, a^m$, and the hyperplane $\mathcal{H}(\pi^2, 1)$ in the adjacent facet will contain the points $a^1, \ldots, a^{m-1}, a^k$ where a^k is one of the point $a^{m+1}, \ldots, a^n$.

One way to find the normals of the hyperplane $\mathcal{H}(\pi^1, 1)$ is to solve the system of equations $B\pi^1 = \bar{1}_{m\times 1}$ where

$$B = \begin{bmatrix} a_1^1 & \ldots & a_m^1 \\ \vdots & \ddots & \vdots \\ a_1^m & \ldots & a_m^m \end{bmatrix}_{m\times m} . \tag{A.1}$$

We define the matrix

$$B_{ij} = \begin{bmatrix} a_1^1 & \dots & a_{j-1}^1 & a_{j+1}^1 & \dots & a_m^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_1^{i-1} & \dots & a_{j-1}^{i-1} & a_{j+1}^{i-1} & \dots & a_m^{i-1} \\ \\ a_1^{i+1} & \dots & a_{j-1}^{i+1} & a_{j+1}^{i+1} & \dots & a_m^{i+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_1^m & \dots & a_{j-1}^m & a_{j+1}^m & \dots & a_m^m \end{bmatrix}_{(m-1)\times(m-1)} \tag{A.2}$$

for $i = 1, \dots, m$ and $j = 1, \dots, m$. This is the matrix B without i^{th} row and j^{th} column.

The matrix B is non-singular, so we have

$$B^{-1} = \begin{bmatrix} b_{11} & \dots & b_{m1} \\ \vdots & \ddots & \vdots \\ b_{1m} & \dots & b_{mm} \end{bmatrix}_{m\times m}, \tag{A.3}$$

where

$$b_{ij} = (-1)^{i+j} \frac{\mid B_{ij} \mid}{\mid B \mid}, \quad i = 1, \dots, m, \text{ and } j = 1, \dots, m. \tag{A.4}$$

The matrix B is non-singular, hence the determinant of B is not zero. We have

$$\mid B \mid = \sum_{j=1}^{m} a_j^m \mid B_{mj} \mid, \tag{A.5}$$

so at least determinant of one B_{mj} for $j = 1, \dots, m$ is not zero. Assume, wlog, $\mid B_{m1} \mid \neq 0$, then remove the last row of the matrixes B (related to m^{th} point) and

$\bar{1}$. Hence, we get

$$\begin{bmatrix} a_1^1 & \dots & a_m^1 \\ \vdots & \ddots & \vdots \\ a_1^{m-1} & \dots & a_m^{m-1} \end{bmatrix} \begin{bmatrix} \pi_1^1 \\ \vdots \\ \pi_m^1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} \pi_2^1 \\ \vdots \\ \pi_m^1 \end{bmatrix} = \begin{bmatrix} a_2^1 & \dots & a_m^1 \\ \vdots & \ddots & \vdots \\ a_2^{m-1} & \dots & a_m^{m-1} \end{bmatrix}^{-1} \begin{bmatrix} 1 - a_1^1\pi_1^1 \\ \vdots \\ 1 - a_1^{m-1}\pi_1^1 \end{bmatrix}. \tag{A.6}$$

We have the matrix

$$B_{m1} = \begin{bmatrix} a_2^1 & \dots & a_m^1 \\ \vdots & \ddots & \vdots \\ a_2^{m-1} & \dots & a_m^{m-1} \end{bmatrix}_{(m-1)\times(m-1)}, \tag{A.7}$$

and we define the matrix

$$B_{m1}^{ij} = \begin{bmatrix} a_2^1 & \dots & a_{j-1}^1 & a_{j+1}^1 & \dots & a_m^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_2^{i-1} & \dots & a_{j-1}^{i-1} & a_{j+1}^{i-1} & \dots & a_m^{i-1} \\ a_2^{i+1} & \dots & a_{j-1}^{i+1} & a_{j+1}^{i+1} & \dots & a_m^{i+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_2^{m-1} & \dots & a_{j-1}^{m-1} & a_{j+1}^{m-1} & \dots & a_m^{m-1} \end{bmatrix}_{(m-2)\times(m-2)}, \tag{A.8}$$

for $i = 1, \dots, m-1$ and $j = 2, \dots, m$. This is the matrix B_{m1} without i^{th} row and j^{th} column. So, we get

$$B_{m1}^{-1} = \begin{bmatrix} g_{12} & \cdots & g_{(m-1)2} \\ \vdots & \ddots & \vdots \\ g_{1m} & \cdots & g_{(m-1)m} \end{bmatrix}_{(m-1)\times(m-1)}, \tag{A.9}$$

where

$$g_{ij} = (-1)^{i+j+1} \frac{\mid B_{m1}^{ij} \mid}{\mid B_{m1} \mid}; \quad i = 1, \ldots, m-1, \text{ and } j = 2, \ldots, m. \tag{A.10}$$

From (A.6) and (A.10), we get

$$\pi_k^1 = \sum_{i=1}^{m-1} g_{ik} \left(1 - a_1^i \pi_1^1\right); \quad k = 2, \ldots, m. \tag{A.11}$$

The points $a^1, \ldots, a^{m-1}$ are common in both hyperplanes, hence for the points $a^m, \ldots, a^n$, we construct the system of inequalities

$$\sum_{j=1}^{m} a_j^k \pi_j^1 \leq 1; \quad k = m, \ldots, n. \tag{A.12}$$

From (A.11) and (A.12), we get

$$\begin{aligned} a_1^k \pi_1^1 + \sum_{j=2}^{m} a_j^k \pi_j^1 \leq 1 &\Longrightarrow a_1^k \pi_1^1 + \sum_{j=2}^{m} \sum_{i=1}^{m-1} a_j^k g_{ij} \left(1 - a_1^i \pi_1^1\right) \leq 1 \\ &\Longrightarrow \left(a_1^k - \sum_{j=2}^{m} \sum_{i=1}^{m-1} a_j^k a_1^i g_{ij}\right) \pi_1^1 \leq 1 - \sum_{j=2}^{m} \sum_{i=1}^{m-1} a_j^k g_{ij}. \end{aligned} \tag{A.13}$$

By applying (A.10), we get

$$\left(a_1^k - \frac{1}{\mid B_{m1} \mid}\sum_{j=2}^{m}\sum_{i=1}^{m-1}(-1)^{i+j+1}a_j^k a_1^i \mid B_{m1}^{ij} \mid\right)\pi_1^1 \leq 1 - \frac{1}{\mid B_{m1} \mid}\sum_{j=2}^{m}\sum_{i=1}^{m-1}(-1)^{i+j+1}a_j^k \mid B_{m1}^{ij} \mid. \tag{A.14}$$

We define the matrix

$$E_i^k = \begin{bmatrix} a_2^1 & \dots & a_m^1 \\ \vdots & \ddots & \vdots \\ a_2^{i-1} & \dots & a_m^{i-1} \\ \\ a_2^{i+1} & \dots & a_m^{i+1} \\ \vdots & \ddots & \vdots \\ a_2^{m-1} & \dots & a_m^{m-1} \\ \\ a_2^k & \dots & a_m^k \end{bmatrix}_{(m-1)\times(m-1)} \tag{A.15}$$

for $i = 1, \dots, m-1$. In fact, the matrix E_i^k is the matrix B_{m1} by removing i^{th} row, shifting up the rows of $(i+1)^{th}, \dots, (m-1)^{th}$, and adding the row with the values $a_2^k, \dots, a_m^k$ as a last row. The determinant of the matrix E_i^k is

$$\mid E_i^k \mid = \sum_{j=2}^{m}(-1)^{j+m}a_j^k \mid B_{m1}^{ij} \mid. \tag{A.16}$$

Then, from (A.14) and (A.16), we get

$$\left(a_1^k - \frac{1}{\mid B_{m1} \mid}\sum_{i=1}^{m-1}(-1)^{i+m+1}a_1^i \mid E_i^k \mid\right)\pi_1^1 \leq \; 1 - \frac{1}{\mid B_{m1} \mid}\sum_{i=1}^{m-1}(-1)^{i+m+1} \mid E_i^k \mid. \tag{A.17}$$

We define two matrixes

$$F^k = \begin{bmatrix} a_1^1 & \dots & a_m^1 \\ \vdots & \ddots & \vdots \\ a_1^{m-1} & \dots & a_m^{m-1} \\ \\ a_1^k & \dots & a_m^k \end{bmatrix}_{m\times m}, \text{ and } \; E^k = \begin{bmatrix} 1 & a_2^1 & \dots & a_m^1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_2^{m-1} & \dots & a_m^{m-1} \\ \\ 1 & a_2^k & \dots & a_m^k \end{bmatrix}_{m\times m}. \tag{A.18}$$

Matrix F^k is the matrix B that by removing its last row (related to the point a^m), and using the element of the point a^k instead of it. The matrix E^k is the matrix F^k that we set the first element of each row to 1 value.

Based on Assumption 4, the determinant of F_m^k is non-zero. The determinant of the matrix F^k is

$$\mid F^k \mid = (-1)^{m+1}a_1^k \mid B_{m1} \mid + \sum_{i=1}^{m-1}(-1)^{i+1}a_1^i \mid E_i^k \mid$$

$$= (-1)^{m+1} \mid B_{m1} \mid \left(a_1^k - \frac{1}{\mid B_{m1} \mid}\sum_{i=1}^{m-1}(-1)^{i+m+1}a_1^i \mid E_i^k \mid\right). \tag{A.19}$$

So, we get

$$a_1^k - \frac{1}{\mid B_{m1} \mid} \sum_{i=1}^{m-1} (-1)^{i+m+1} a_1^i \mid E_i^k \mid = \frac{\mid F^k \mid}{(-1)^{m+1} \mid B_{m1} \mid}. \tag{A.20}$$

The determinant of the matrix E^k is

$$\mid E^k \mid = (-1)^{m+1} \mid B_{m1} \mid + \sum_{i=1}^{m-1} (-1)^{i+1} \mid E_i^k \mid$$

$$= (-1)^{m+1} \mid B_{m1} \mid \left(1 - \frac{1}{\mid B_{m1} \mid} \sum_{i=1}^{m-1} (-1)^{i+m+1} \mid E_i^k \mid \right). \tag{A.21}$$

So, we have

$$1 - \frac{1}{\mid B_{m1} \mid} \sum_{i=1}^{m-1} (-1)^{i+m+1} \mid E_i^k \mid = \frac{\mid E^k \mid}{(-1)^{m+1} \mid B_{m1} \mid}. \tag{A.22}$$

Next, from (A.17), (A.20), and (A.22), we get

$$\left(\frac{\mid F^k \mid}{(-1)^{m+1} \mid B_{m1} \mid} \right) \pi_1^1 \leq \frac{\mid E^k \mid}{(-1)^{m+1} \mid B_{m1} \mid} \Longrightarrow \pi_1^1 \leq \frac{\mid E^k \mid}{\mid F^k \mid}\ , \text{ or } \ \pi_1^1 \geq \frac{\mid E^k \mid}{\mid F^k \mid}. \tag{A.23}$$

From [A.23], we get the biggest closed interval $[l,\ u]$ for two real numbers l and u such that π_1^1 is feasible for all $k = m, \ldots, n$. The last step is to set π_1^1 to the end points l and u separately, and by finding the values of $\pi_2^1, \ldots, \pi_m^1$ from (A.11), we get two different supporting hyperplanes. These two hyperplanes coincide with two adjacent facets of $\text{con}(\mathcal{A})$. One of them contains the points $a^1, \ldots, a^m$, and another one contains the points $a^1, \ldots, a^{m-1}, a^k$, where a^k is one of the point $a^{m+1}, \ldots, a^n$.

A.2 The proof of Theorem 3.8

For the given point set $\mathcal{A}$, consider the following LP.

$$\begin{aligned} \min_{\lambda} \quad & \sum_{i=1}^{n} \lambda_i \\ \text{s.t.} \quad & A^T\lambda = a^o, \\ & \lambda_i \geq 0, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{A.24}$$

where a^o is an arbitrary point in $\Re^m$, $\lambda^T = [\lambda_1 \ldots \lambda_n]_{1\times n}$, and

$$A^T = \begin{bmatrix} a_1^1 & \ldots & a_1^n \\ \vdots & \ddots & \vdots \\ a_m^1 & \ldots & a_m^n \end{bmatrix}_{m\times n} . \tag{A.25}$$

Assume, $\lambda_1, \ldots, \lambda_m$ are optimal basic feasible solutions. We denote the optimal basis as B^T. Therefore the updated tableau at optimality is as follows.

$$\begin{array}{c|cccccccc|c} & \lambda_1 & \lambda_2 & \ldots & \lambda_m & \lambda_{m+1} & \ldots & \lambda_n & & \text{updated rhs} \\ \hline \lambda_1 & 1 & 0 & \ldots & 0 & \sum_{j=1}^m a_j^{m+1} b_{1j} & \ldots & \sum_{j=1}^m a_j^n b_{1j} & & \sum_{j=1}^m a_j^o b_{1j} \\ \lambda_2 & 0 & 1 & \ldots & 0 & \sum_{j=1}^m a_j^{m+1} b_{2j} & \ldots & \sum_{j=1}^m a_j^n b_{2j} & & \sum_{j=1}^m a_j^o b_{2j} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \vdots \\ \lambda_m & 0 & 0 & \ldots & 1 & \sum_{j=1}^m a_j^{m+1} b_{mj} & \ldots & \sum_{j=1}^m a_j^n b_{mj} & & \sum_{j=1}^m a_j^o b_{mj} \\ \hline & 0 & 0 & \ldots & 0 & 1 - \sum_{i=1}^m \sum_{j=1}^m a_j^{m+1} b_{ij} & \ldots & 1 - \sum_{i=1}^m \sum_{j=1}^m a_j^n b_{ij} & & \end{array} \tag{A.26}$$

The hyperplane $\mathcal{H}(\pi, 1)$ that contains the points $a^1, \ldots, a^m$, is a supporting hyperplane for con($\mathcal{A}$). Some subsets of $m - 1$ points on the hyperplane $\mathcal{H}(\pi, 1)$

define a supporting hyperplane with $m-2$ dimensionally for con($\mathcal{A}$). If one variable leaves from basis, the entering variable to basis is found by applying the minimum ratio test for DSP. Assume λ_l leaves the basis for $l = 1, \ldots, m$, then the entering variable is found from following test.

$$\operatorname{argmin} \left\{ \; - \; \frac{1 - \sum_{i=1}^{m} \sum_{j=1}^{m} a_j^k b_{ij}}{\sum_{j=1}^{m} a_j^k b_{lj}}; \; k = m+1, \ldots, n \;\middle|\; \sum_{j=1}^{m} a_j^k b_{lj} < 0 \right\}. \tag{A.27}$$

Suppose λ_m leaves the basis. From (A.27), we have $n-m$ values

$$MRT^k = -\frac{1 - \sum_{i=1}^{m} \sum_{j=1}^{m} a_j^k b_{ij}}{\sum_{j=1}^{m} a_j^k b_{mj}}, \quad k = m, \ldots, n. \tag{A.28}$$

Clearly, MRT^m is zero. In addition, from (A.23), for $k = m, \ldots, n$, we get $n-m$ values for $\frac{|E^k|}{|F^k|}$. We just need to show that there is a linear relation between MRT^k and $\frac{|E^k|}{|F^k|}$ for $k = m, \ldots, n$.

In (A.27), assume λ_m leaves from the basis. Regardless of its condition, $\sum_{j=1}^{m} a_j^k b_{lj} < 0$, we have $n-m$ values

$$MRT^k = -\frac{1 - \sum_{i=1}^{m} \sum_{j=1}^{m} a_j^k b_{ij}}{\sum_{j=1}^{m} a_j^k b_{mj}}, \quad k = m, \ldots, n. \tag{A.29}$$

Clearly, MRT^m is zero. In addition, from (A.23), for $k = m, \ldots, n$, we get $n-m$ values for $\frac{|E^k|}{|F^k|}$. Prove that there is a linear relation between MRT^k and $\frac{|E^k|}{|F^k|}$ for $k = m, \ldots, n$ as follows.

$$MRT^k \;=\; \frac{|\, B \,|}{|\, B_{m1} \,|} \cdot \frac{|\, E^k \,|}{|\, F^k \,|} - \frac{|\, H \,|}{|\, B_{m1} \,|}, \tag{A.30}$$

where the matrix $\mathcal{H}$ is

$$H = \begin{bmatrix} 1 & a_2^1 & \dots & a_m^1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_2^m & \dots & a_m^m \end{bmatrix}_{m\times m} . \tag{A.31}$$

The matrix $\mathcal{H}$ is the matrix $\mathcal{B}$ that the first element of the each row of it is replaced by one value.

Proof. Assume λ_m leaves from the basis. So, λ_k enters to the basis, where $k = m+1, \dots, n$. Then, from (A.4) and (A.29), we get

$$MRT^k = -\frac{\mid B \mid - \sum_{i=1}^{m} \sum_{j=1}^{m} (-1)^{i+j} a_j^k \mid B_{ij} \mid}{\sum_{j=1}^{m} (-1)^{m+j} a_j^k \mid B_{mj} \mid}. \tag{A.32}$$

We define the matrix

$$F_i^k = \begin{bmatrix} a_1^1 & \dots & a_m^1 \\ \vdots & \ddots & \vdots \\ a_1^{i-1} & \dots & a_m^{i-1} \\ & & \\ a_1^{i+1} & \dots & a_m^{i+1} \\ \vdots & \ddots & \vdots \\ a_1^m & \dots & a_m^m \\ & & \\ a_1^k & \dots & a_m^k \end{bmatrix}_{m\times m} \tag{A.33}$$

for $i = 1, \dots, m$, and $k = m+1, \dots, n$. This is the matrix B without i^{th} row, and adding the elements of the point a^k as the last row for $k = m+1, \dots, n$.

The determinant of the matrix F_i^k is

$$| F_i^k |= \sum_{j=1}^{m} (-1)^{m+j} a_j^k \mid B_{ij} \mid . \tag{A.34}$$

In addition, we have

$$| F^k |= \sum_{j=1}^{m} (-1)^{m+j} a_j^k \mid B_{mj} \mid . \tag{A.35}$$

Therefore, From (A.32), (A.34) and (A.35), we get

$$MRT^k = - \frac{| B | - \sum_{i=1}^{m} (-1)^{i+m} \mid F_i^k \mid}{| F^k |}. \tag{A.36}$$

We define the matrix

$$F = \begin{bmatrix} 1 & a_1^1 & \dots & a_m^1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_1^m & \dots & a_m^m \\ \\ 1 & a_1^k & \dots & a_m^k \end{bmatrix}_{(m+1)\times(m+1)} . \tag{A.37}$$

The determinant matrix F is

$$| F |= (-1)^m \left(| B | - \sum_{i=1}^{m} (-1)^{i+m} \mid F_i^k \mid \right). \tag{A.38}$$

Finally, from (A.36) and (A.38), we get

$$MRT^k = - (-1)^m \frac{| F |}{| F^k |}. \tag{A.39}$$

If λ_m leaves from the basis, and λ_k enters to the basis, then the optimal solution of variables correspond to the new hyperplane with the level of one. This hyperplane, is a supporting hyperplane for con($\mathcal{A}$) that coincides with a facet of con($\mathcal{A}$), and contains the points $a^1, \ldots, a^{m-1}, a^k$. This is a way for rotating a supporting hyperplane from a facet to another facet of the convex hull of a point set. According to the defined matrixes, (A.30), and (A.39), we need to prove the equation

$$-\frac{|\ F\ |}{|\ F^k\ |} \ = \ (-1)^m \ \frac{|\ B\ |}{|\ B_{m1}\ |} \left(\frac{|\ E^k\ |}{|\ F^k\ |} - \frac{|\ H\ |}{|\ B\ |} \right). \tag{A.40}$$

To continue the proof, we use the following theorem.

Blocks matrixes Theorem [45]. Consider the matrix $M_{m \times m}$ with four blocks matrixes A_1, A_2, A_3, and A_4

$$M = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}_{m \times m} \tag{A.41}$$

such that A_4 is invertible. Then we have $|\ M\ | = |\ A_4\ |\ \ .\ det\left(A_1 - A_2 A_4^{-1} A_3\right)$. four submatrixes of matrixes F, B, F^k, E^k, and H are as follows.

$$A_1^F = \begin{bmatrix} 1 & a_1^1 \\ \vdots & \vdots \\ 1 & a_1^{m-1} \end{bmatrix}_{(m-1) \times 2}, \quad A_2^F = B_{m1}, \quad A_3^F = \begin{bmatrix} 1 & a_1^m \\ 1 & a_1^k \end{bmatrix}_{2 \times 2}, \quad A_4^F = \begin{bmatrix} a_2^m & \ldots & a_m^m \\ a_2^k & \ldots & a_m^k \end{bmatrix}_{2 \times (m-1)}, \tag{A.42}$$

$$A_1^B = \begin{bmatrix} a_1^1 \\ \vdots \\ a_1^{m-1} \end{bmatrix}_{(m-1) \times 1}, \qquad A_2^B = B_{m1}, \quad A_3^B = \begin{bmatrix} a_1^m \end{bmatrix}_{1 \times 1}, \qquad A_4^B = \begin{bmatrix} a_2^m & \ldots & a_m^m \end{bmatrix}_{1 \times (m-1)}, \tag{A.43}$$

$$A_1^{F^k} = \begin{bmatrix} a_1^1 \\ \vdots \\ a_1^{m-1} \end{bmatrix}_{(m-1)\times 1}, \quad A_2^{F^k} = B_{m1}, \quad A_3^{F^k} = \begin{bmatrix} a_1^k \end{bmatrix}_{1\times 1}, \quad A_4^{F^k} = \begin{bmatrix} a_2^k & \dots & a_m^k \end{bmatrix}_{1\times(m-1)}, \tag{A.44}$$

$$A_1^{E^k} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{(m-1)\times 1}, \qquad A_2^{E^k} = B_{m1}, \quad A_3^{E^k} = \begin{bmatrix} 1 \end{bmatrix}_{1\times 1}, \quad A_4^{E^k} = \begin{bmatrix} a_2^k & \dots & a_m^k \end{bmatrix}_{1\times(m-1)}, \tag{A.45}$$

$$A_1^{H} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{(m-1)\times 1}, \qquad A_2^{H} = B_{m1}, \quad A_3^{H} = \begin{bmatrix} 1 \end{bmatrix}_{1\times 1}, \quad A_4^{H} = \begin{bmatrix} a_2^m & \dots & a_m^m \end{bmatrix}_{1\times(m-1)}. \tag{A.46}$$

To simplify the proof, we define $P_1 = \sum_{i=1}^{m-1}\sum_{j=2}^{m} a_j^m g_{ij}$, $P_2 = \sum_{i=1}^{m-1}\sum_{j=2}^{m} a_j^k g_{ij}$, $P_3 = \sum_{i=1}^{m-1}\sum_{j=2}^{m} a_1^i a_j^m g_{ij}$, and $P_4 = \sum_{i=1}^{m-1}\sum_{j=2}^{m} a_1^i a_j^k g_{ij}$. Then, by applying the Blocks matrixes theorem, the determinant of the matrix F, B, F^k, E^k, and H are as follows.

$$\mid F \mid \;= \begin{vmatrix} A_1^F & A_2^F \\ A_3^F & A_4^F \end{vmatrix} = \mid A_2^F \mid \;.\; det\left(A_3^F - A_4^F \left(A_2^F\right)^{-1} A_1^F\right) = \mid B_{m1} \mid \; \begin{vmatrix} 1-P_1 & a_1^m - P_3 \\ 1-P_2 & a_1^k - P_4 \end{vmatrix}. \tag{A.47}$$

$$\mid B \mid = \begin{vmatrix} A_1^B & A_2^B \\ A_3^B & A_4^B \end{vmatrix} = (-1)^{m-1} \mid A_2^B \mid \;.\; det\left(A_3^B - A_4^B \left(A_2^B\right)^{-1} A_1^B\right) = (-1)^{m-1} \mid B_{m1} \mid (a_1^m - P_3). \tag{A.48}$$

$$| F^k | = \begin{vmatrix} A_1^{F^k} & A_2^{F^k} \\ A_3^{F^k} & A_4^{F^k} \end{vmatrix} = (-1)^{m-1} \mid A_2^{F^k} \mid \ . \ det \left(A_3^{F^k} - A_4^{F^k} \left(A_2^{F^k} \right)^{-1} A_1^{F^k} \right) = (-1)^{m-1} \mid B_{m1} \mid (a_1^k - P_4). \tag{A.49}$$

$$| E^K | = \begin{vmatrix} A_1^{E^k} & A_2^{E^k} \\ A_3^{E^k} & A_4^{E^k} \end{vmatrix} = (-1)^{m-1} \mid A_2^{E^k} \mid \ . \ det \left(A_3^{E^k} - A_4^{E^k} \left(A_2^{E^k} \right)^{-1} A_1^{E^k} \right) = (-1)^{m-1} \mid B_{m1} \mid (1 - P_2). \tag{A.50}$$

$$| H | = \begin{vmatrix} A_1^{H} & A_2^{H} \\ A_3^{H} & A_4^{H} \end{vmatrix} = (-1)^{m-1} \mid A_2^{H} \mid \ . \ det \left(A_3^{H} - A_4^{H} \left(A_2^{H} \right)^{-1} A_1^{H} \right) = (-1)^{m-1} \mid B_{m1} \mid (1 - P_1). \tag{A.51}$$

From (A.40), (A.47), (A.48), (A.49), (A.50), and (A.51), we get

$$(-1)^m \frac{| B |}{| B_{m1} |} \left(\frac{| E^k |}{| F^k |} - \frac{| H |}{| B |} \right)$$

$$= \frac{(-1)^m}{| B_{m1} | \, | F^k |} \Big(| E^k | \, | B | - | H | \, | F^k | \Big)$$

$$= \frac{(-1)^m}{| B_{m1} | \, | F^k |} \Big((-1)^{2m-2} \, | B_{m1} |^2 \, (1 - P_2)(a_1^m - P_3) - (-1)^{2m-2} \, | B_{m1} |^2 \, (1 - P_1)(a_1^k - P_4) \Big)$$

$$= (-1)^m \, \frac{| B_{m1} |}{| F^k |} \Big((1 - P_2)(a_1^m - P_3) - (1 - P_1)(a_1^k - P_4) \Big)$$

$$= - \, (-1)^m \, \frac{| F |}{| F^k |} \, . \blacksquare \tag{A.52}$$