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3-Maps And Their Generalizations

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by

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Abstract

A 3-map is a 3-region colorable map. They have been studied by Craft and White [4]. This thesis introduces topological graph theory and then investigates 3-maps in detail, including examples, special types of 3-maps, the use of 3-maps to find the genus of special graphs, and a generalization known as n-maps.

Chapter 1

Introduction

This thesis is primarily based on the paper *3-maps* [4] by Craft and White. The goal of this thesis is to explain Craft and White's research on 3-maps in full detail. The author has written this thesis in such a way that a reader with no prior knowledge of topological graph theory should be able to understand the proofs without too much difficulty. For this reason, there are many figures so that readers can quickly grasp the ideas used in the proofs. Also the proofs are written in a very detailed manner so that readers can see precisely why the theorem is true.

This thesis begins with an introduction to the basic concepts of graph theory and topology as well as miscellaneous ideas used later on. Next, common methods used to study topological graph theory: Euler's Polyhedron Formula, Edmonds Rotational Imbedding Scheme, and Voltage graphs are introduced and explained. 3-maps are then introduced with motivation, examples, and methods of generating new 3-maps from old. We will then study under what conditions 3-maps may exist on surfaces of given genus. The following chapter will describe 3-maps with all regions of the same length, called m-uniform maps. Next, we will study the genus of Cartesian products of graphs which have 3-maps. After this, we will investigate a generalization of 3-maps known as n-maps. We will then state and prove some historical theorems that relate to region coloring and cubic graphs, some of which relate to the 4-color theorem. We conclude with some open questions posed by Craft and White. [1]

[1] All figures are due to the author (mostly based on the figures in [4]), except for the 3-dimensional picture of the torus in Figure 3.2. That picture is a Wikimedia Commons picture made by the user Pokipsy76.

Chapter 2

Graphs Basic Concepts

2.1 Graph Theory Definitions

A **graph**, G is a set of **vertices**, denoted $V(G)$ and a set of **edges**, denoted $E(G)$. The edge set consists of unordered pairs of vertices of G. Two vertices are **adjacent** if the pair containing them occurs in the edge set, i.e., $\{v_1, v_2\} \in E(G)$. We denote an edge between v_1 and v_2 by v_1v_2. A graph is often represented by a picture:

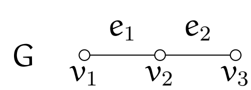

Figure 2.1: A graph with three vertices and two edges

In Figure 2.1, $V(G) = \{v_1, v_2, v_3\}$ and $E(G) = \{v_1v_2, v_2v_3\}$. Often, edges are denoted with their own labels, e.g. , $e_1 = v_1v_2$ and $e_2 = v_2v_3$.
A **loop** is an edge of the form vv and a **multiple edge** is one that appears more than once in $E(G)$. A **directed edge** is an ordered pair of distinct vertices. A **pseudograph** is a graph with loops and multiple edges allowed. A **digraph** (**directed graph**) on the other hand is a graph where every edge is a directed edge.

The **(closed) neighborhood** of v, denoted $N[v]$, is the set containing v and all vertices adjacent to v. The **open neighborhood** of v, denoted $N(v)$ is the set containing all vertices adjacent to v. In Figure 2.1, $N[v_2] = \{v_1, v_2, v_3\}$, while $N(v_2) = \{v_1, v_3\}$.

$|V(G)|$ is called the **order** of the graph, denoted by p while $|E(G)|$ is called the **size** of the graph, denoted by q. In Figure 2.1, G has order 3 and size 2.

A vertex, v, is **incident** with an edge, e, (or vice versa) if the vertex lies on either side of the edge, i.e. $e = vx = xv$, for some $x \in V(G)$. In Figure 2.1, v_2 is incident with e_1 and e_2, while v_1 is incident with e_1 and v_3 is incident with e_2.

The **degree** of a vertex is the number of edges incident with that vertex. A graph is **r-regular** if every vertex of the graph has degree **r**. In particular, a graph is **cubic** if every vertex of the graph has degree 3. The graph in Figure 2.2 is 3-regular, i.e. a cubic graph.

Theorem 1. *(First Theorem of Graph Theory) For a graph,* G*, the sum of the degrees of the vertices of* G *is twice the size of* G*, i.e.,*

$$\sum_{v \in V(G)} \deg(v) = 2q$$

.

Proof. The degree of a vertex is the number of edges incident with that vertex. Each edge is incident with exactly two vertices. If we sum the degrees of all vertices, each edge will be counted twice, so the sum must be twice the number of edges, i.e., $2q$. □

By Theorem 1, the degree sum of a graph must be even. Since an odd number of integers all of whom are odd cannot sum to an even number, *Theorem 1 implies that a graph with odd degree must have an even number of vertices*. In particular, for a cubic graph, p is even.

The **complete graph** on n vertices, denoted K_n is the graph with n vertices in which every vertex is adjacent to every other vertex. Figure 2.2 is a picture of K_4.

A $u-v$ **walk**, W, is a sequence of vertices which begins at v and ends at u such that adjacent vertices follow each other in the walk. In Figure 2.1, $W = (u = v_1v_2v_3v_2 = v)$ is a $v_1 - v_2$ walk. A **closed walk** is a walk where the beginning and ending vertices are the same. $W = (u = v_1v_2v_3v_2v_1 = v)$ is a closed $v_1 = v_1$ walk. A $u-v$ **path**, P, is a walk with no repeated vertices. $P = (u = v_1v_2v_3 = v)$ is a $v_1 - v_3$ walk. A **cycle** is a path which is also a closed walk. The graph of ordern n that consists entirely of a cycle on n vertices is denoted C_n. A graph is **connected** if for every two vertices $u, v \in V(G)$, there is a $u-v$ path in G.

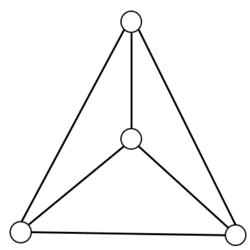

Figure 2.2: K_4

A **proper edge coloring** is a function $f : E(G) \to \mathbb{N}$, where an edge is labeled with a natural number (or a color) such that no two adjacent edges (edges who are incident with the same vertex) share the same color. The graph in Figure 2.4 has a proper edge coloring with 3 colors, called a **3-edge coloring**.

A graph is **bipartite** if $V(G)$ can be partitioned into two pieces, called partite sets, such that any edges in the graph run between partite sets. The filled vertices in Figure 2.4 form one partite set, while the hollow vertices form the other. The **complete bipartite graph** denoted $K_{n,m}$ is the graph with partite sets X and Y of size n and size m, respectively, where every vertex in X is adjacent to every vertex in Y and vice versa.

A **directed edge** or **arc** is an ordered pair of vertices. In Figure 2.3, $(1, 2)$ is a directed edge. A **digraph**, D consists of a set of vertices, $V(D)$ together with a set of arcs, $E(D)$. A **symmetric digraph** is a digraph in which for every arc $(u, v) \in E(D)$, there

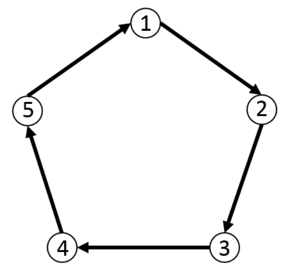

Figure 2.3: Directed Cycle on 5 vertices

is a corresponding arc $(v, u) \in E(D)$. Any graph can be transformed into a symmetric digraph by simply replacing each edge (u, v) as with the arcs (u, v) and (v, u). **Directed walks**, **closed directed walks**, and **directed paths**, are defined as walks, closed walks, and paths, respectively, which are sequences of directed edges. A **directed cycle** is a closed directed walk which is also a directed path. Figure 2.3 shows a directed cycle, which is also a digraph, though not a symmetric digraph.

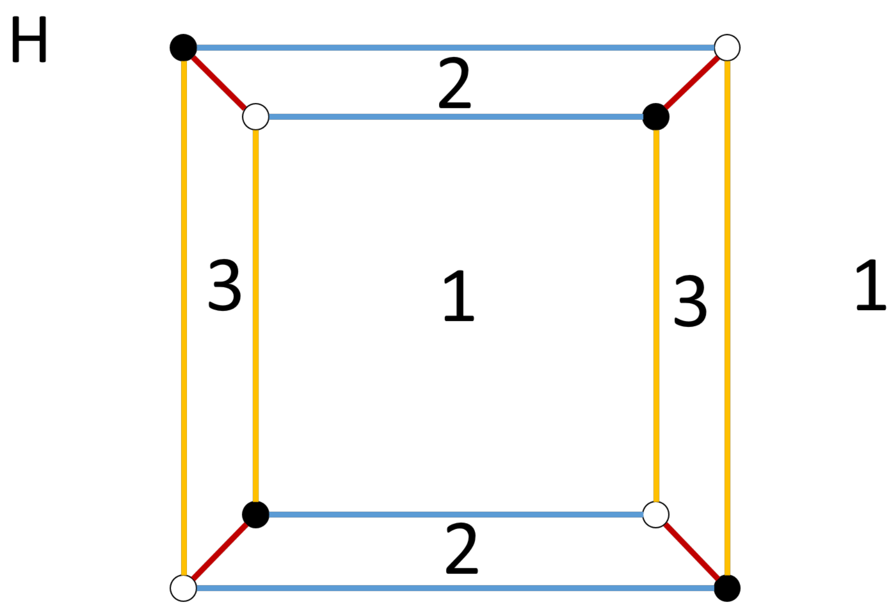

Figure 2.4: Q_3 on S_0

2.2 Topological Definitions

A **2 -manifold** is a connected topological space in which every point has a neighborhood homeomorphic to the open disk. This neighborhood is called a **2 - cell**. A 2-manifold is **orientable** if, for every simple closed curve C on M, a clockwise sense of rotation is preserved by traveling around C. Otherwise, M is **nonorientable**

M is **orientable** if it admits a 2-cell decomposition with **coherent** orientation, i.e. the boundary of each 2-cell is given an orientation so that a 1-cell portion of the boundary incident with two adjacent 2-cells is oppositely oriented within those two 2-cells.

A **surface** is a closed, orientable 2-manifold. Note that a point-set topologist calls this a **compact** 2-manifold. If M is a 2-manifold, then M is said to be **closed** if it is bounded and the boundary of M coincides with M.

A graph can be **imbedded** in a surface if it can be drawn on the surface with no edges crossing. Figure 2.4 is an imbedding of Q_3, the 3-cube on S_0, the sphere.

A **region** of an imbedding of a graph G in a surface M is said to be **2-cell** if it is homeomorphic to the open unit disk. If every region for an imbedding is a 2-cell, then the imbedding is said to be a **2-cell imbedding** The number of regions in a (2-cell) imbedding of a graph on a surface is denoted by r. In Figure 2.4, the regions of the imbedding are the areas inside each cycle as well as the outside (unbounded) region of the graph. For Figure 2.4, $r = 6$.

A **map** is an imbedding of a graph on a surface in which every edge separates two regions, i.e., there is no edge contained entirely within a single region. Figure 2.4 is an example of a map.

For a fixed imbedding, M, of a graph, G, a **proper region coloring** is a labeling of the regions of the imbedding with a natural number in which no two regions who share an edge have the same color.

An imbedding, M, of a graph, G, is called **k-region colorable** if M has a proper region coloring using exactly k colors.

A **3-map** is an imbedding of a cubic graph which is both a map and **3-region colorable**.

Figure 2.4 exhibits a proper region coloring with 3 colors and Q_3 is a cubic graph, so Figure 2.4 is, in fact, a 3-map.

The **genus** of an orientable surface is the number of holes or handles in the surface. The sphere S_0 is of genus 0, while the torus S_1 is of genus 1. Every orientable surface is homeomorphic to a sphere with n handles, denoted S_n.

A graph is **planar** if it can be imbedded in S_0 (which is homeomorphic to the plane (except for the north pole) using the stereographic projection). The graph, H, in Figure 2.4 is planar.

2.3 Miscellaneous Definitions

An **equivalence relation**, R, on a set S is a relation on S which is **reflexive**,i.e., $\forall x \in S, xRx$; **symmetric**, i.e., $\forall x, y \in S, xRy \rightarrow yRx$; and **transitive**, i.e. $\forall x, y, z \in S, xRy$ and $yRz \rightarrow xRz$. An equivalence relation of a set S forms a **partition** of S, i.e. a collection of subsets $T_i, i \in I$ (for some index set I) of S such that: 1. $\forall x \in S, x \in T_i$, 2. $x \in T_i$ and $x \in T_j \rightarrow T_i = T_j$, and 3. $\bigcup_{i \in I} T_i = S$. In a partition, P of S, each T_i is called an **equivalence class**.

The **Cartesian Product** of two sets A and B, denoted $A \times B$ is the set composed of all possible ordered pairs where the first element is from A and the second is from B. For example, if $A = \{a, b\}$ and $B = \{1, 2\}$, then $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$.

The **Cartesian Product** of graphs G and H, denoted $G \square H$ is the graph with vertex set $V(G \square H) = V(G) \times V(H)$, i.e., the vertex set is the Cartesian Product of the vertices of G with the vertices of H, and edge set $(uv, xy) \in E(G \square H)$ if $1.$ $u = x$ and $vy \in$

$E(H)$ or 2. $v = y$ and $ux \in E(G)$ [3]. In practice, the Cartesian Product of graphs can be viewed as replacing every vertex of H with a copy of G and for every edge of H, corresponding vertices in the copies of G representing the vertices are joined by an edge. Figure 4.6 is an example of a Cartesian product, specifically, $K_2 \square C_6$

Chapter 3

Methods of Topological Graph Theory

3.1 Euler's Polyhedron Formula

Theorem 2 (Euler's Polyhedron Formula)**.** *If* G *is a connected graph of order* p *and size* q *that is 2-cell imbedded in the sphere,* S_0*, with* r *regions, then*

$$p - q + r = 2$$

Proof. We proceed by induction on q, the size of G, a plane graph. If $q = 0$, there is only one connected graph of size 0, K_1. In this case, $p = 1$, $q = 0$ and $r = 1$. Since, $p - q + r = 2$, the anchor step of the induction holds. Assume that $q \in \mathbb{N}$ that if H is a connected plane graph of order p' and size q', where $q' < q$ such that there are r' regions, then $p' - q' + r' = 2$. Let G be a connected plane graph of order p and size q with r regions. We consider two cases;

- **Case 1**. If G is a tree, then $q = p - 1$ and $r = 1$ and $p - q + r = p - (p - 1) + 1 = 2$.
- **Case 2**. Let say that G is not a tree. Since, G is connected it contains a cycle. Let e be a cycle edge. In G, the edge e is on the boundaries of two regions. So, in $G - e$ these two regions merge to form a single region. Since, $G - e$ has order p, size

$q-1$ and $r-1$ regions and $q-1<q$, it follows by the induction hypothesis that $p-(q-1)+(r-1)=2$ and so, $p-q+r=2$.

□

For example Q_3, the 3-cube shown in Fig 2.4 is a connected plane graph (imbedded in S_0 the sphere or plane), $p=8, q=12$ and $r=6$, then $p-q+r=2 \Rightarrow 8-12+6=2$.

Theorem 3 (Generalized Euler's Polyhedron Formula). *If* G *is a connected graph of order* p *and size* q *that is 2-cell embedded in an orientable surface of genus* h*, with* r *regions, then*

$$p-q+r=2-2h$$

Corollary 1. *If* G *is a connected graph of order* $p \geqslant 3$*, size* q*, and girth* k *(i.e., the length of smallest cycle is* k*), then genus* $\gamma(G) \geqslant \frac{q}{2}\left(1-\frac{2}{k}\right)-\frac{p}{2}+1$*. In particular when the girth is 3, we have* $\gamma \geqslant \frac{q}{6}-\frac{p}{2}+1$ *and when* G *is bipartite (triangle free), then* $\gamma \geqslant \frac{q}{4}-\frac{p}{2}+1$*.*

Proof. If G is of girth k then $kr \leqslant 2q$ and $r \leqslant \frac{2}{k}q$. This inequality comes from going along the graph region by region and counting the number of edges that bound each region. The smallest number of edges a region can have is k and there are r regions, so the smallest this number can be is kr. Also, each edge will be counted at most twice because an edge can separate a maximum of two regions, so the largest this number can be is $2q$. Substituting for r in Euler's formula above we get $p+\left(\frac{2}{k}-1\right)q \geqslant 2-2\gamma$ giving the result, we get $\gamma \geqslant \frac{q}{2}\left(1-\frac{2}{k}\right)-\frac{p}{2}+1$. The special cases follow by taking $k=3$ and $k=4$ in that order.

□

3.2 Rotational Imbedding Scheme

The Rotational Imbedding Scheme is a method of describing imbeddings of graphs on surfaces in terms of permutations of vertices. In fact, for every imbedding of a graph on a surface, there is a unique n-tuple of cyclic permutations of the vertices of that graph and vice versa:

Theorem 4. *If* G *is a connected graph with* $V(G) = \{v_1, v_2, ...v_n\}$*, then for each 2-cell imbedding of* G *on a surface, there is a unique n-tuple* $(\pi_1, \pi_2, ...\pi_n)$ *such that* π_i *is a cyclic permutation of* $N(v_i)$ *in counterclockwise order about* v_i*. Conversely, if* $(\pi_1, \pi_2, ...\pi_n)$ *is an n-tuple of cyclic permutations, then there is a unique 2-cell imbedding of* G *on some surface such that for every vertex* $v_i \in V(G)$*,* π_i *denotes the vertices adjacent to* v_i *in counterclockwise order about* v_i*.*

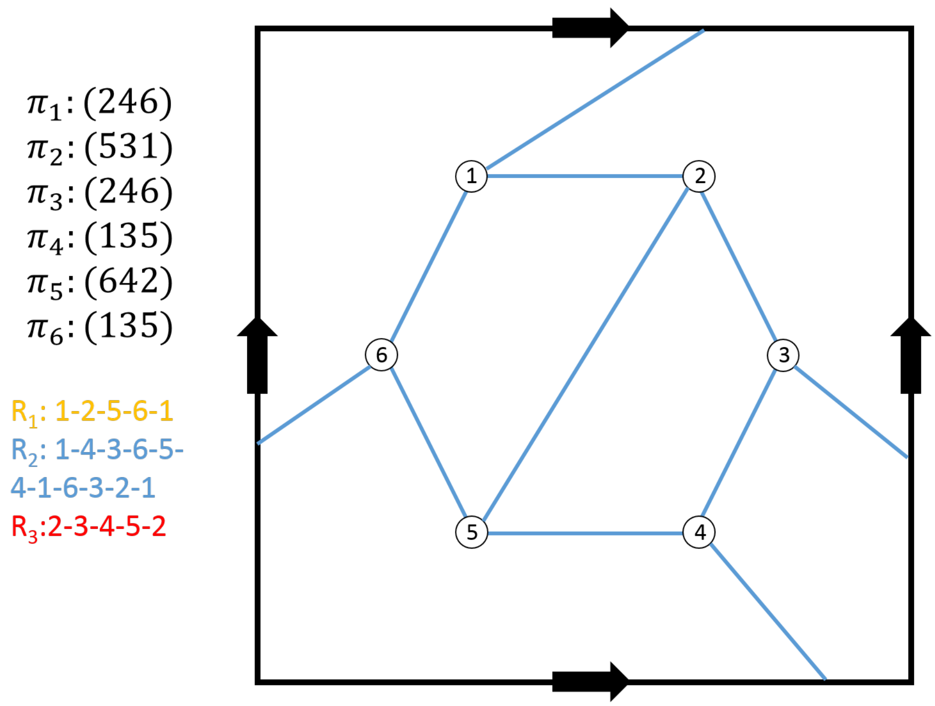

Figure 3.1: $K_{3,3}$ on S_1

Figure 3.1 gives an example of the Rotational Imbedding Scheme with $K_{3,3}$ imbedded on the torus. In Figure 3.1, the square with arrows is a representation of the torus or donut. In fact, S_n can be represented by a $4n-gon$ for all n. Figure 3.2 shows both the torus and its 2-dimensional representation. To see why the square is a representation of the torus, imagine folding two sides with arrows facing the same way to form a cylinder

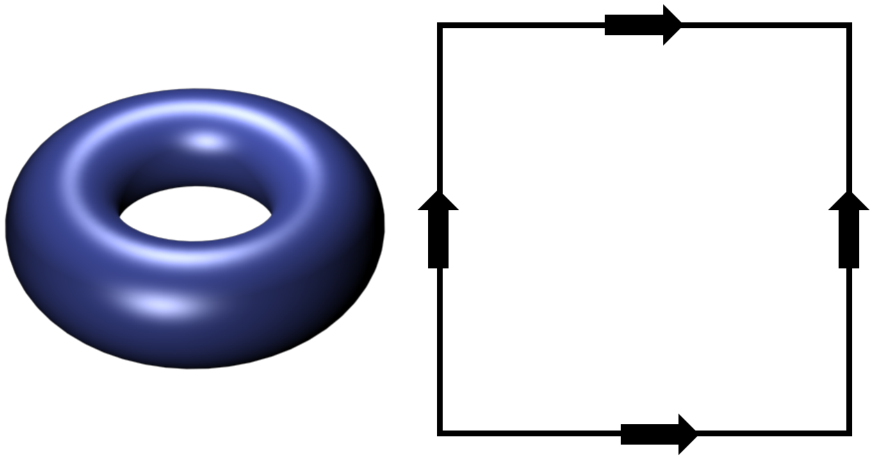

Figure 3.2: S_1 rolled and unrolled

and then joining the two ends of the cylinder to make a donut.

In Figure 3.1, observe that using the imbedding of $K_{3,3}$ to write down the vertices in $N(v_i)$, for $1 \leqslant i \leqslant 6$, in counterclockwise order generates the six cyclic permutations. For example, v_1 is adjacent to $v_2, v_4,$ and v_6, so $\pi_1 = (246)$. Since π_1 is a cyclic permutation, any rewriting of the entries that does not change their cyclic order is permitted, i.e., π_1 could also be written (624) or (462), but not $(642), (264)$ or (426).

To generate the imbedding from the n-tuple of cyclic permutations of vertices, we will combine the six permutations π_1 through π_6 into a single permutation π. First, change $K_{3,3}$ into a symmetric digraph, D. π will be a permutation acting on the elements of $E(D)$, defined as follows: $\pi((v_i, v_j)) = (v_j, v_{\pi_j(i)})$. The **orbits** of this permutation, i.e., the sets $\{(v_s, v_t) \in E(D) : \pi((v_i, v_j)) = (v_s, v_t)\}$ are the equivalence classes of π for $E(D)$. Since an orbit is an equivalence class, the orbit of (v_i, v_j) can also be described as the orbit of (v_s, v_t), where (v_s, v_t) is any other arc in the orbit, so it makes sense to speak of the orbits of π, without reference to any particular arc. For any orbit, if we begin with (v_i, v_j) and repeatedly apply π, since there are finitely many arcs, we will eventually find an arc (v_k, v_l) such that $\pi((v_k, v_l)) = (v_l, v_i)$ (it is possible that there may be only one orbit). This traces the boundary of a 2-cell region. Once we have found every orbit of π, we can "glue" these regions together along corresponding arcs, as seen in Figure 3.3.

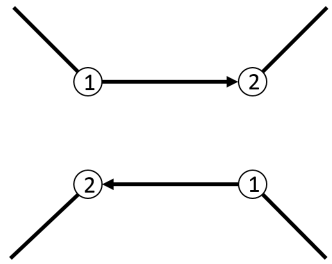

Figure 3.3: Two regions being pasted together

As an example of this process, in Figure 3.1 begin with the first entry of π_1, 2. Next, determine what number follows 1 in π_2, in this case, 5. Then, determine what number follows 2 in π_5, i.e., 6. This process will eventually repeat because 1 follows 5 in π_6, but 2 follows 6 in π_1. Since π acts on arcs, not vertices, a vertex may repeat in the same orbit, so it is necessary to check that an arc repeats to be sure the entire orbit is determined. In Figure 3.1, this is $R_1 = 1-2-5-6-1$. Observe that R_1 describes a region of this imbedding of $K_{3,3}$, the left cycle inside the hexagon.

3.3 Voltage Graphs

Consider K, a pseudograph and the set K^* associated with K as follows: with each edge $uv \in E(K)$ associate the oriented edge $e = (u,v)$ and $e^{-1} = (v,u)$ and let $K^* = \{(u,v)|uv \in E(K)\}$. A **voltage graph** is a group of (K, Γ, ϕ), where K is a connected pseudograph, Γ is a group, and ϕ a function such that $\phi : K^* \to \Gamma$ where $\phi\left(e^{-1}\right) = (\phi(e))^{-1}$ for all $e \in K^*$. Each value $\phi(e)$ is called a **voltage**. A covering graph $K \times_\phi \Gamma$ for (K, Γ, ϕ) has vertex set $V(K) \times \Gamma$ and each edge $e = (u,v)$ of K determines the edges $(v,g)(v, g\phi(e))$ of $K \times_\phi \Gamma$ for all $g \in \Gamma$. Voltage graphs work well in constructing embedding of regular graphs and graphs with lots of symmetry. Gross and Tucker proved that 2-cell embedding of G can be inferred from a 2-cell embedding of a voltage graph whose covering graph is G. Their result as summarized in [7] is stated below.

Theorem 5 (Gross and Alpert). *Let (K, Γ, ϕ), be a voltage graph with rotation scheme Π and Π^ϕ the lift of Π to $G \times_\phi \Gamma$.Let Π and Π^ϕ determine embeddings of G and $G \times_\phi \Gamma$ on S and S^ϕ in that order. The branched covering projection $p : S^\phi \to S$ such that*

1. $p^{-1}(G) = G \times_\phi \Gamma$
2. *If b is a branch point of multiplicity n, then b is in the interior of a region R such that $|R_\phi| = n$;*
3. *If R is a region of G which is a k-gon, then $p^{-1}(R)$ has $\frac{|\Gamma|}{|R_\phi|}$components, each which is a $k|R_\phi|$-gon.*

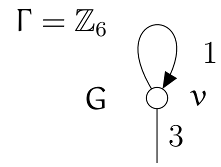

Figure 3.4: Voltage Graph

Figures 3.4 is an example of a voltage graph. In this case, we have a single vertex with a loop, and a half-edge. The group is $\mathbb{Z}_6$, the voltage assigned to the half-edge is 3 and the voltage assigned to the loop is 1. Figure 3.5 shows the covering graph. Since $V(K \times_\phi \Gamma) = V(G) \times \mathbb{Z}_6$ and G has only one vertex, the vertices of Figure 3.5 are labeled with a group element, for simplicity (rather than $(\nu, 0), (\nu, 1)$, etc). Since the vertex of G, is adjacent to itself because of the loop, every vertex in Figure 3.5 will be adjacent to the next group element in sequence, (adding $\bmod 6$). Furthermore, the half edge works by moving out along the half edge then moving back to the vertex, so we add for example 3 to 0, giving 3, then add 3 back to 3, giving 0 again, so we have two arcs from 0 to 3, each going in opposite directions. We can replace these with a single edge. Note that we actually have a *directed* edge from 0 to 1. However, for the purposes of determining an imbedding, we will treat this arc as the edge of a graph.

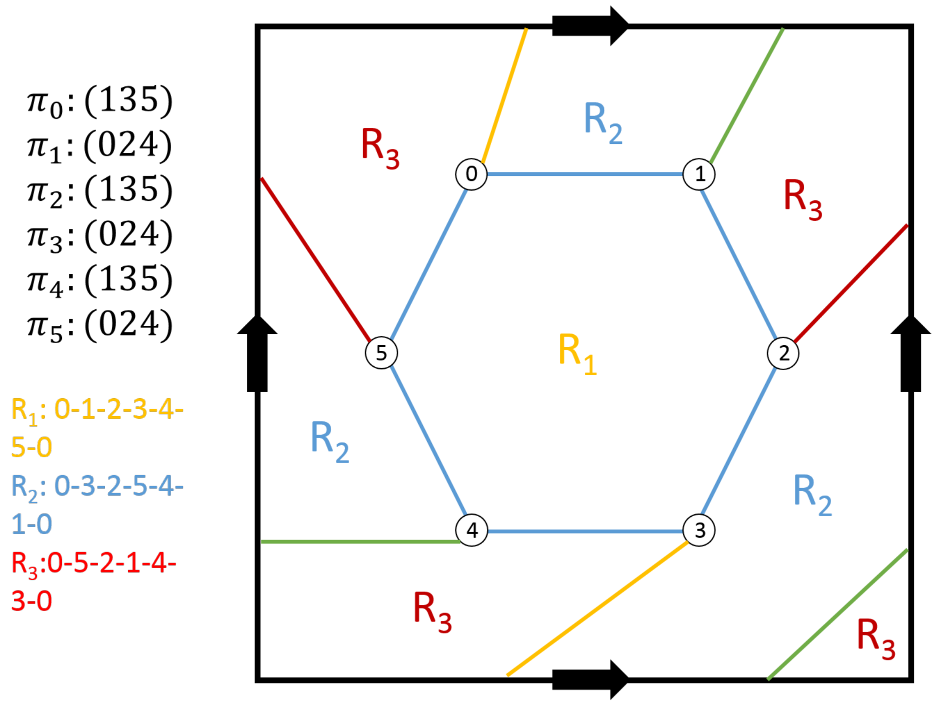

Figure 3.5: $K_{3,3}$ imbedded on S_0 as a 3-map

The regions are found by determining the lift of the rotation scheme of the voltage graph. If for (K, Γ, ϕ) we have $\pi = (\pi_1, \pi_2, ...\pi_p)$, then we define the lift $\widetilde{\pi}$ of π to $K \times_\phi \Gamma$ if $\pi_v(v, u) = (v, w)$, then $\widetilde{\pi}_{(v,g)}((v, g), (u, g\phi(v, u))) = ((v, g), (w, g\phi(v, w)))$ for every $g \in \Gamma$. In this case, $\widetilde{\pi}$ is just the collection of permutations $\widetilde{\pi}_{(v,g)}$ for every vertex in the covering graph [7]. It is very difficult to understand how this lifting works from reading the abstract definition, so let us see the lift in action by using Figures 3.4 and 3.5. For Figure 3.4, the imbedding scheme is $\pi_v = (5, 1, 3)$. This is because if we rotate clockwise around v, starting with the side of the loop with the arrow, we treat the loop as two edges, first an edge going backward, i.e., against the arrow (in which case the voltage is the inverse of 1 in $\mathbb{Z}_6$, i.e., -1 or 5), then as an edge going forward (in the same direction as the arrow), in which case the voltage is 1. The final entry in the cyclic permutation is 3, the voltage on the half edge.

For $\widetilde{\pi}_{(v,0)} = \widetilde{\pi}_0$, we have (by substituting into the definition) $\widetilde{\pi}_{(v,0)}((v, 0), (v, 5)) = ((v, 0), (v, 1))$,$\widetilde{\pi}_{(v,0)}((v, 0), (v, 1)) = ((v, 0), (v, 3))$, and $\widetilde{\pi}_{(v,0)}((v, 0), (v, 3)) = ((v, 0), (v, 5))$, which completes the cycle. For simplicity, we can write this as $\pi_0 = (135)$ as in Figure 3.5. The other permutations are computed similarly. Once this is accomplished, we have the full rotational imbedding scheme describing this imbedding and we can determine the regions following the process described above. Remark: the Rotational Imbedding Scheme only gives a description of imbeddings in terms of their region boundaries. We can use this to determine what surface the imbedding occurs on using the generalized Euler identity, but it is not always easy to move from a rotational imbedding scheme to a geometric description of an imbedding [3]. To aid understanding, in Figure 3.5, a geometric depiction of the imbedding is provided.

Chapter 4

3-maps

4.1 Motivation

The original motivation for this study was examining the properties of graphs of dice. There are five Platonic Solids: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron and five dice can be made, one for each of the solids. These dice can be transformed into graphs imbedded in S_0 by transforming vertices and edges of the die into vertices and edges of a graph and faces into regions of the imbedding. Figure 2.2 is an example of this process applied to the tetrahedron and Figure 2.4 provides an example for the cube. The numbers on the die then become labels for the regions. This leads us to White and Craft's first Theorem.

Theorem 6 (Craft and White)**.** *For a cubic map on S_n, with the r regions colored using all of 1,2, ...,r the number of clockwise vertices always equals the number of counterclockwise vertices. (A clockwise vertex is a vertex in which the region labels increase clockwise and a counterclockwise vertex is one in which the region labels increase counterclockwise.)*

Proof. Consider an arbitrary vertex and an arbitrary neighbor of that vertex. Since this imbedding is a map, every edge separates two regions, so every edge has two distinct labels on its opposite sides. Assign -1 or $-$ to the side with the smaller label and $+1$ or $+$ to the side with the larger label. Since S_n is orientable, we can impose a clockwise orientation on every vertex. Then, assign a cyclic triple of signs to every vertex.

For each vertex, start with one edge and the select $+$ or $-$ based on which sign we encounter first when moving we reach that edge. In Figure 4.2, the hollow vertex is assigned $(- + +)$ and the filled vertex is assigned $(- - +)$. Another way to view the assignment of triples to vertices is that we are making three comparisons. As we travel clockwise around a vertex, at each region we compare that region to the region that follows clockwise. If the region we are currently at has a larger color than the following region, assign $+$ for the edge separating the two regions. If less, assign $-$.
There are $2q$ signs in the entire graph, one $+$ and one $-$ for each edge. We will sum the $+$ and $-$ signs in the graph in two different ways. Since they come in pairs, one $+$ and one $-$ for each edge, the sum of all $+$ and $-$ labels in the entire graph must be 0. Furthermore, we know that if there are q edges, there must be $2q$ signs.
Next, group the signs into the triples each vertex is assigned by the above method. We can see by the Figure 4.2 that if one vertex makes the comparison $2 > 1$ and takes the plus, its neighbor must make the comparison $1 < 2$ and take the minus, simply because of the nature of the clockwise rotation. In other words, each triple contains a unique collection of three signs. We can now sum the signs in two stages. First, sum each triple, which must be either $+1$ or -1. The triple for a decreasing vertex will sum to $+1$, $+$ for short, and for an increasing vertex will sum to -1 or $-$. Then, sum all of these signs. But this sum must be 0 since each sign only appeared once. Thus, there are as many $+$'s as $-$'s and hence as many increasing as decreasing vertices. $\square$

In Figure 4.1, the filled vertices are clockwise and the hollow vertices are counterclockwise. The top left corner is adjacent to regions (2,5,6) and the bottom right corner is adjacent to regions (6,4,3). Observe that even though this graph, Q_3, is bipartite, the partition of $V(Q_3)$ into counterclockwise and clockwise vertices in Figure 4.1 does not

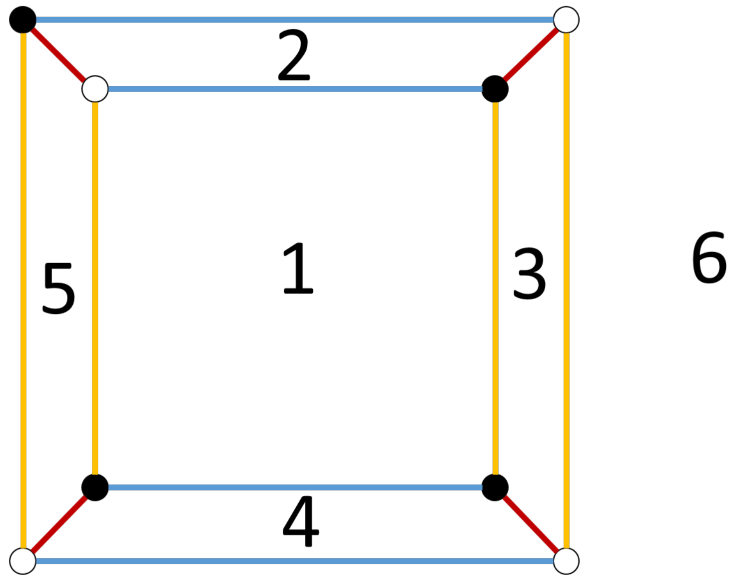

Figure 4.1: Q_3 with a non-canonical Bipartition

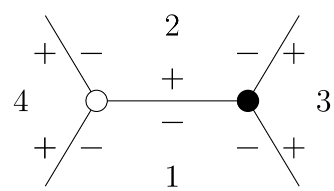

Figure 4.2: Clockwise and Counterclockwise Vertices

correspond to the bipartition of Q_3. In Figure 2.4, the two partitions do correspond and in this case, we say that the bipartition is **canonical**.

Notice the importance of assuming our graphs are cubic. In a cubic graph, each vertex is adjacent to at most 3 regions. And given any cyclic permutation of 3 distinct numbers, the permutation will either be increasing or decreasing. However, once we introduce four numbers we can have cyclic permutations that are neither increasing nor decreasing, such as (1324). Also note that since our imbedding is a map, every edge has two distinct regions on either side of the edge. In a 3-map, this ensures that every vertex is adjacent to exactly 3 regions. Even though the imbedding of Figure 3.1 is 3-region colorable, (since it has only 3 regions) this imbedding is not a 3-map because the edges $1-4$ and $3-6$ are entirely contained in R_2; they do not separate two regions. The imbedding shown in Figure 3.5 is a 3-map because it is both 3-region colorable and a map.

The reasoning of Theorem 5 also applies to maps colored with fewer than r colors; we only require that the map be proper region colored. In particular, Theorem 5 applies to maps using 3 colors, the aforementioned 3-maps. This second theorem by Craft and White characterizes 3-maps.

Theorem 7 (Craft and White)**.** *Let M be a map for a cubic graph G. Then the following statements are equivalent:*

(a) M is 3-region colorable (i.e. M is a 3-map).

(b) G is canonically 3-edge colorable, and every bi-colored cycle bounds a region. Moreover, every region is so described.

(c) $S(G) = (u, v) : uv \text{ is in } E(G)$ is partitioned into three sets, each inducing a collection of region-bounding directed cycles partitioning V(G).

Proof. (a) $\Rightarrow$ (b): As an intermediate step, we will first show that if a cubic map is 3-region colorable, the underlying graph must be bipartite and the bipartition is canonical. Consider an arbitrary edge of M, with colors x and y on opposite sides as shown in Figure 4.3. Since we have only 3 colors, it must be that the color for the remaining two regions is z. Since M is a map, each edge separates two regions, so we know that the situation shown in Figure 4.3 must obtain: all three colors will be used and one color will be used twice. Rotating clockwise around each vertex, the hollow vertex gives (x,y,z) and the filled (x,z,y). These are distinct cyclic permutations and, as noted above, a cyclic permutation of three different integers must be either increasing or decreasing.

Next, color each edge with the unique color not used by either region it bounds. (In this thesis, we will let the color 1 correspond to red, 2 to blue, and 3 to yellow). This ensures that G is canonically 3-edge colorable, meaning that a decreasing vertex has edges of decreasing colors (when rotating clockwise around the vertex) and an increasing vertex has edges of increasing colors as shown in Figure 4.4. For a detailed proof of why every bi-colored cycle is a region and every region is a bi-colored cycle, see the $\Rightarrow$ part of the proof of Theorem 8.

(b) $\Rightarrow$ (c): Consider an arbitrary vertex, v. Since M is a map, every vertex is adjacent to exactly three regions. Each region is bounded by two edges which have colors with one of three color combinations: red and blue, blue and yellow, or yellow and red. By (b), for any such region, the two edges bounding it must form form part of a bi-colored cycle. Furthermore, (as shown in Figure 4.4) all color combinations are represented exactly once. So, v is part of three distinct bi-colored cycles distinguished by the colors that they use. Thus, we can form our three sets partitioning $V(G)$ by simply taking all bi-colored cycles of a given color combination or, equivalently, those not using a particular color. The above argument ensures that this is a partition. If we follow each region clockwise along the inside of the region-bounding bi-colored cycle for any region, R, considering each edge as a directed edge, we can see that these cycles can be changed to directed cycles (the edges in the other direction come from regions adjacent to R).

(c) $\Rightarrow$ (a) Paste the region-bounding directed cycles together as in the rotational imbedding scheme and color each region with the color not used by the bi-colored directed cycle bounding it.

$\square$

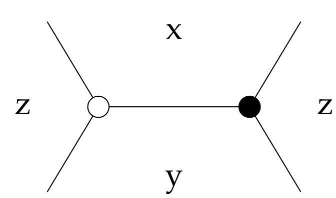

Figure 4.3: a 3-map induces a canonical bipartition

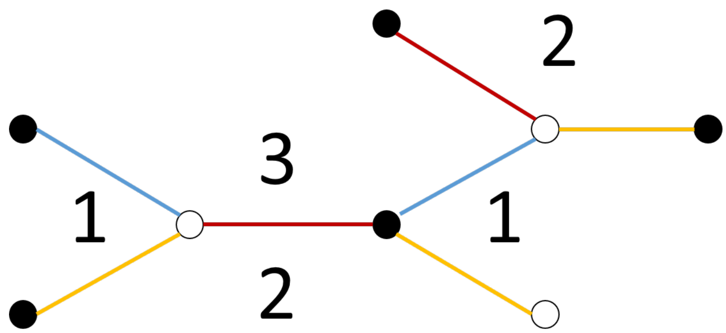

Figure 4.4: Property (b)

Observe that in Theorem 7 (c) each of the three sets is associated with all regions of particular color. These sets are called **color classes**.

Not only does a 3-map imply that the graph is bipartite, but, in fact, all cubic, bipartite graphs have a 3-map on some surface.

Theorem 8 (Craft and White)**.** *A connected cubic graph is the underlying graph for some 3-map if and only if it is bipartite.*

Proof. $\Rightarrow$ This was proved in the intermediate step of (a) $\Rightarrow$ (b).

$\Leftarrow$ Suppose G is a cubic, connected, and bipartite graph. By König's edge coloring theorem, G is 3-edge colorable. Fix some edge coloring of G using colors 1,2, and 3. This induces a rotation scheme for an imbedding of G. For some vertex v, let $N(v) = \{u_1, u_2, u_3\}$ such that the color of edge vu_1, denoted $c\{v, u_1\}$ is 1, $c\{v, u_2\} = 2$, and $c\{v, u_3\} = 3$. In other words, we will refer to the vertex incident with the edge colored 1 as u_1 and so on. Let V_1, V_2 be the partite sets of $V(G)$. Fix a (counterclockwise) rotation scheme on $V(G)$ so that if $\pi(v) = (u_1, u_2, u_3)$, $v \in V_1$ but if $\pi(v) = (u_3, u_2, u_1)$, then $v \in V_2$. Ignoring the region colors, Figure 4.5 shows that the three edge coloring will naturally lead to such a bipartition. In this case, filled vertices will have $\pi(v) =$ (yellow, blue, red) or $(3, 2, 1)$, while hollow vertices will have $\pi(v) =$ (red, blue, yellow) or $(1, 2, 3)$.

In this rotation scheme we need to keep track of not only vertices but also the colors of their respective edges. There are 3 possibilities. If we start with a vertex in V_1 and an edge colored 1, the next edge must be colored 3. If we start with a vertex in V_2 and an edge colored 3, the next edge must be colored 1. (This is because we jump from V_1 to V_2 with every new edge) A vertex in V_1 and an edge colored 2 forces the next edge to be colored 1. A vertex in V_2 and an edge colored 1 ensures that the following edge will be colored 2. Lastly, a vertex in V_1 and an edge colored 3 means the next edge will be colored 2, while a vertex in V_2 and an edge colored 2 means the next edge will be colored 3.

To see why every bi-colored cycle bounds a region consider an arbitrary bi-colored cycle of G. Starting with an arbitrary vertex, trace out the cycle clockwise. Because of the rotation scheme induced by the edge coloring, the path traced along the cycle will correspond exactly to an orbit of the rotation scheme. Thus, the cycle is a region. Such a process is shown in Figure 4.5 for a six-sided region colored red and blue. The reasoning in the above paragraph shows that every region must be bi-colored. In addition, every region is a cycle because once a vertex is entered by an edge of one color, it must be left by the edge of the second color of the region, so no vertex will be revisited except for the first vertex of the cycle. We have now shown that all of the conditions of Theorem 6 (b) hold, except for one, that our imbedding is a map. This is proved as follows: If every region is a directed cycle, such as in Figure 2.3, then given any arc, such as $(1, 2)$, $(2, 1)$, the opposite arc that will be pasted together to form an edge must come from another region. Therfore, every edge divides two regions and we have a map. By Theorem 6 (b), our imbedding is a 3-map and the proof is complete.

We have now shown that a canonical 3-edge coloring of a bipartite cubic graph ensures that every bi-colored cycle bounds a region and every region is formed from a bi-colored cycle, so this also provides a proof of the second part of (a) $\Rightarrow$ (c). $\square$

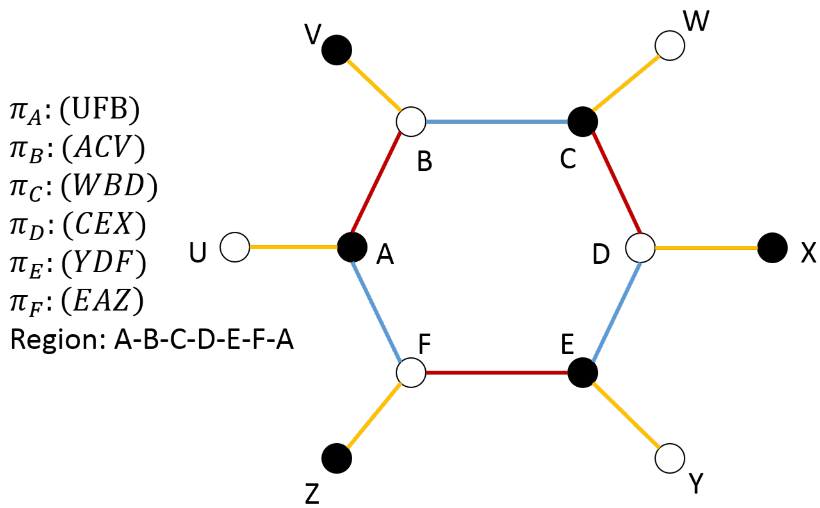

Figure 4.5: Every bi-colored cycle bounds a region

4.2 Examples of 3-maps

Every even **prism**, i.e., every graph of the form $K_2 \square C_{2n}$ is a 3-map on the plane. Color the two $2n - gons$ which form the central region and the outer region with 1 and then alternate colors 2 and 3 for the $4 - gons$ that are placed between these two regions. This induces a 3-edge coloring as in Theorem 2 b, so we have a 3-map. As an illustration, observe that Figure 4.6 is a prism, $K_2 \square C_6$ and is also a 3-map. In Figure 2.4, $Q_3 = K_2 \square C_4$ is also a prism.

4.2.1 New 3-maps from old

The following method is called internal extension or $C1$. Given a 3-map of order p and size q with r regions on a surface S_n, select two edges of the same color, c_1, from the same region boundary. By Theorem 2c, this region is bounded by a bi-colored cycle, so call the other color c_2. Subdivide each edge twice, (place two new vertices on each edge but do not add any additional edges to these vertices) and then join the new vertices

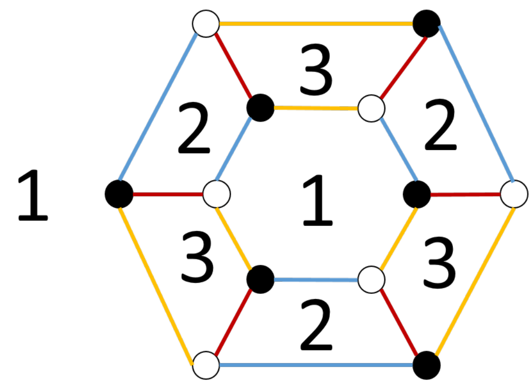

Figure 4.6: $K_2 \square C_6$ as a 3-map on S_0

with two new edges as shown in Figure 4.7 for Q_3 on S_0 [4]. The two internal edges formed by the subdivisions will be colored c_3, while the four external edges will be colored c_1. The edges joining the new vertices will be colored c_2. These colors for the edges form bi-colored cycles which force colors for the two new regions. Figure 4.7 illustrates this process. We will be using red as color 1, blue as color 2, and yellow as color 3. This process adds four new vertices, six new edges, and two new regions, so we have $2 - 2h = p + 4 - (q + 6) + (r + 2) = p - q + r = 2 - 2n$, so $h = n$ and C1 forms a new 3-map on the same surface.

This method is called external extension or C2. Given two copies of the same 3-map both imbedded on their respective surfaces, S_n, select two edges of the same color and subdivide and join as before [4]. In order to form the new edges, it is necessary to join the surfaces by the following process. Extend a cylinder from region of the first surface containing the subdivided edge to the region containing the subdivided edge of the second, connect this cylinder to the second surface and run the new edges along this cylinder. This produces a new 3-map with 4 more vertices, 4, more edges and 0 more regions, but because of the joining of the two surfaces, this new 3-map map is imbedded on S_{2n}. Figure 4.8 illustrates this process for two copies of Q_3 imbedded on S_0.

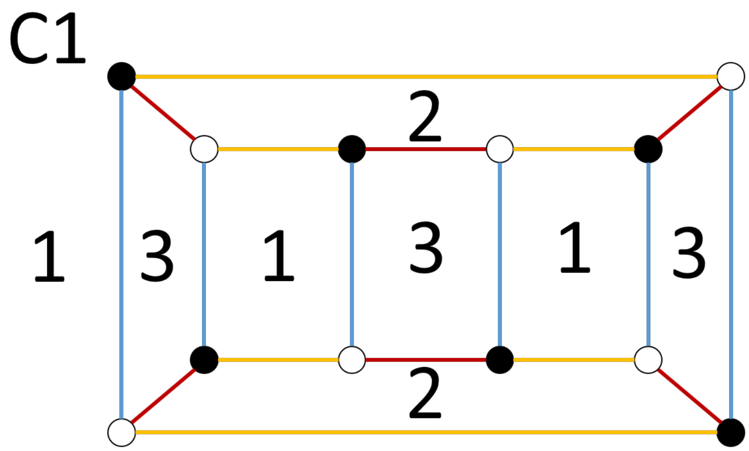

Figure 4.7: C1 internal extension

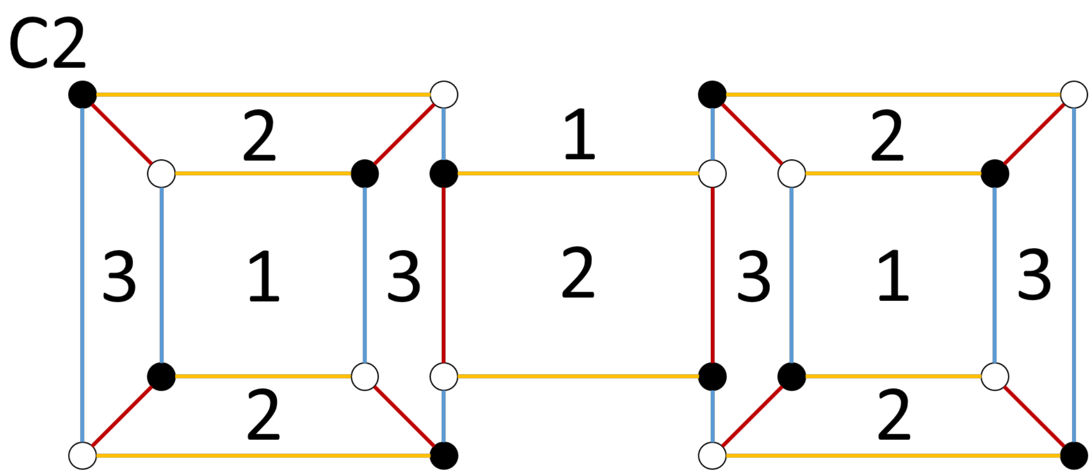

Figure 4.8: C2 external extension

This method can also be used to join two different 3-maps (on surfaces S_n and S_m, respectively) as long as the cylinder combines two regions of the same color and connects, by the edges running along the cylinder, two edges of the same color. Figure 4.8 illustrates this process and also illustrates why C2 does, in fact, produce a new 3-map.

In Figure 4.8, the topological cylinder connecting the two regions is colored green. The 3-edge coloring and three region coloring force one of the new edges created by the subdivision to be the same color as the original edges (blue) and the other to be yellow. This edge cannot be blue because it is adjacent to a blue edge and cannot be red because red is color 1. But then, the two edges running along the cylinder cannot be yellow or blue, so they must be red and therefore, the region enclosed by the red and yellow edges must have color 2. The rest of the respective regions are not affected by C2, so the two original 3-maps are still 3-maps and we can see that we now have a single 3-map on S_{n+m}.

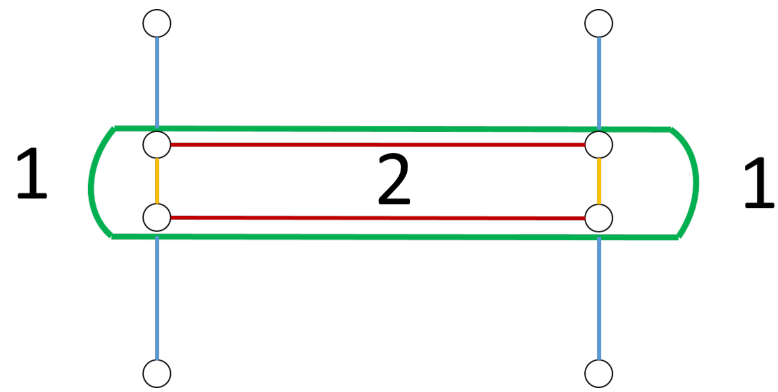

Figure 4.9: C2 with two different 3-maps

4.2.2 New 3-maps from bipartite maps

Not only can we produce new 3-maps from existing 3-maps, we can in fact produce a 3-map from an arbitrary map of a bipartite graph, imbedded on a surface S_n. We will break this result down into 5 propositions. Before proving these propositions, we will give some notation and definitions. Given an arbitrary map, to **stellate** the map means to add a vertex to the interior of each region (including the outer region), and join each new vertex by an edge to each old vertex in the boundary of the region [4]. Denote the stellation of a map, M, by M^*. Figure 4.10 shows C_4^*, the stellation of the cycle on 4 vertices where the black and white filled vertices represent the bipartition of C_4 and the red vertices are the new vertices that result from stellating C_4.

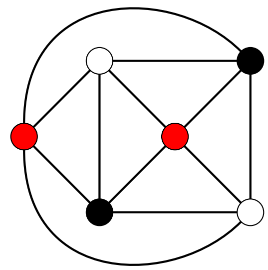

Figure 4.10: C_4^*

Truncate a map by adding a cycle around every vertex, introducing new vertices at the intersection with each edge and deleting all vertices and edges within each new cycle [4]. The operation of truncating a map, M, is denoted $T[M]$. Figure 4.11 illustrates $T[C_4]$, the result of truncating C_4.

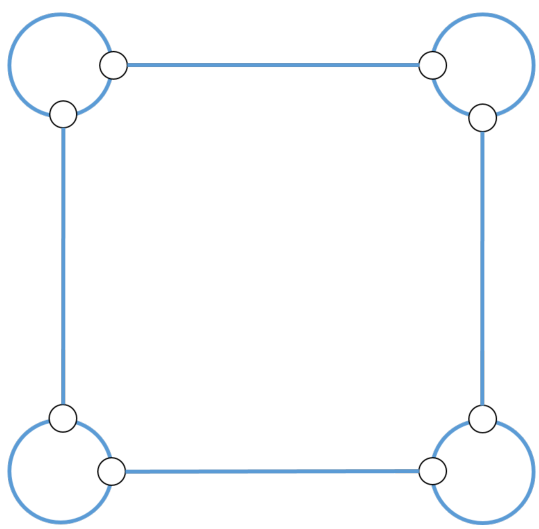

Figure 4.11: $T[C_4]$

The **dual** of a map, M, denoted M', is the imbedding produced by first replacing

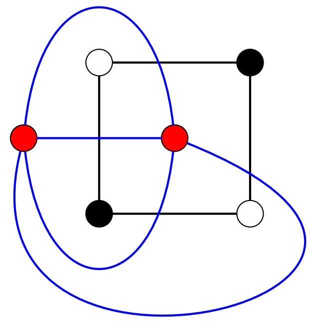

Figure 4.12: C_4'

each region (including the outside region) with a vertex and then joining each vertex with every vertex that shared an edge with that vertex when it was a region. Vertices of M will become regions in M' The dual of a graph can be a multigraph or pseudograph. Figure 4.12 shows the process of forming the dual of C_4. The red vertices connected by blue edges represent C_4', which is a multigraph.

A graph, G, is **tripartite** if $V(G)$ can be partitioned into three sets, X, Y, and Z, called partite sets, such that edges run between vertices in different partite sets, but not between vertices in the same partite set. A **triangulation** of a surface is an imbedding of a graph on a surface where every region, including the outside region is a triangle, i.e., a 3-cycle. A **tri-map** is a triangulation of an orientable surface by a tripartite graph.

Proposition 1 (Craft and White)**.** *A map, M is a 3-map if and only if its dual M' is a tri-map.*

Proof. $\Rightarrow$ Suppose M is a 3-map. Place a vertex in every region of M. Consider an arbitrary vertex, v of M. v is adjacent to three regions, each of which will each be joined to the other two by a single edge. This forms a 3-cycle or triangle. Since v will become a region of M' and v was arbitrary, we see that M' is a triangulation. Furthermore, since no two regions of the same color are adjacent in M, but every region will be adjacent to

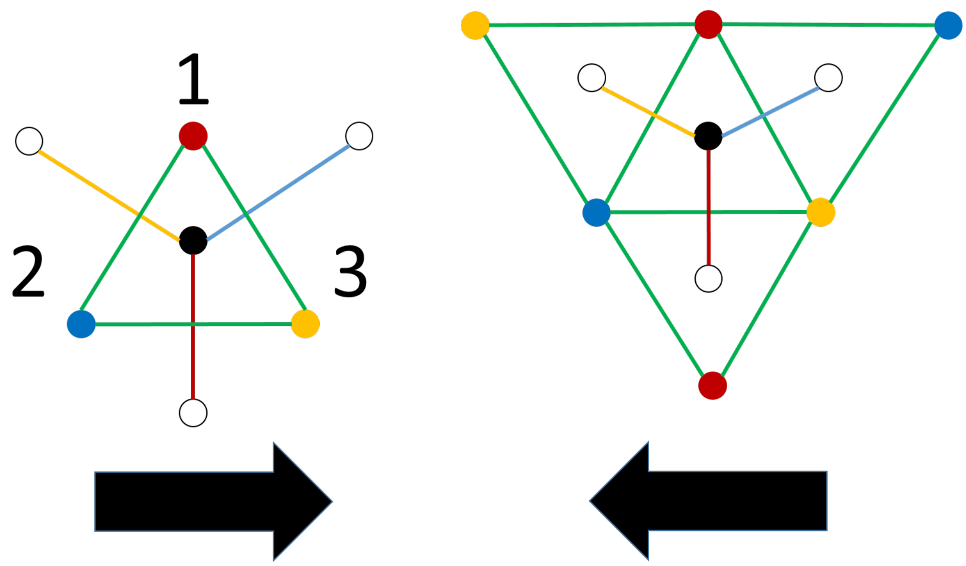

Figure 4.13: Proposition 1

regions of different colors, we can form three the three partite sets of M' by using the region colors of M.

$\Leftarrow$ Now, Suppose M' is a tri-map. M' is a triangulation and a map, so every region of M' is adjacent to exactly three other regions of M'. Thus, M is cubic. In addition, since M' is tripartite, we can color the vertices of the first partite set 1 (or red), the second 2 (or blue) and the third 3 (or yellow). The tripartiteness of M' ensures that this will form a proper region coloring in M because no vertices of the same color are adjacent in M'. We also know that M will be a map because if M was not a map, some region would share an edge with itself, so M' would have a loop, but M' is a triangulation, so this does not occur. Figure 4.13 illustrates both directions of this proof.

□

Proposition 2 (Craft and White)**.** *If* M *is a map for a bipartite graph, then* M^* *is a tri-map.*

Proof. Figure 4.10 illustrates the ideas in this proof. Let X and Y be the two partite sets in M. M^*, does not add any edges between X and Y; edges are only added between new vertices and vertices in M. So, X and Y still form two partite sets of M^*. Furthermore, none of these new vertices are adjacent to any other new vertex, so they form a third partite set, Z and thus, M^* is tripartite.

For any k-sided region, R, in M and any two adjacent vertices in the boundary of R, each is adjacent to the new vertex formed by M^*, which forms a triangle. Thus, M^* converts a k-sided region in M into k triangles, so M^* is a triangulation. M is a map, so all of its edges separate two regions. All of the new edges formed by M^* also separate new regions, namely the triangles formed inside each region. The original edges from M now separate two triangles formed from the original two regions they separated, so M^* is a map and thus, M^* is a 3-map. □

Proposition 3 (Craft and White)**.** *If M is a map for a bipartite graph, then $(M^*)'$ is a 3-map.*

Proof. By Proposition 2, M^* is a tri-map and by Proposition 1, $(M^*)'$ is a 3-map. □

Proposition 4 (Craft and White)**.** *For any map, M, $(T[M'])' = M^*$, or equivalently $T[M'] = (M^*)'$*

Proof. For the first form of Proposition 4, observe that regions of M become vertices of M' and vertices of M become regions of M'. Truncating preserves the adjacencies and nonadjacencies of the original regions of M' because the edges between these regions do not change and no new edges are added between regions. Furthermore, truncating M' transforms the vertices of M' into cycles, so these vertices all become regions of $T[M']$. Any new region formed by such a cycle is adjacent to the same regions as the vertex of M' which it replaced. When we then move to $(T[M'])'$, the old regions become vertices again with no change in their adjacencies and nonadjacencies: we retain the original graph as a subgraph of $(T[M'])'$. Also, for each cycle of $T[M']$ that replaced a truncated vertex of M' (which was originally a region of M), we now have an additional vertex adjacent to every vertex which surrounded the original region in M.

Finally, every region of M will be replaced by k triangles in $(T[M'])'$, where k is the number of sides of the region. Observe that if M is a map, M' has no loops. Also, since a region of M must be at least 3-sided, every vertex of M' has degree at least 3. Thus,

$T[M']$ is cubic because each new vertex is part of a cycle with at least 3 vertices, so each new vertex is adjacent to the two on either side in the cycle and the vertex it is adjacent to by the original edge from before the truncation. Since $T[M']$ is a cubic map, $(T[M'])'$ is a triangulation. So, we can see that, in fact, $(T[M'])' = M^*$.

Figure 4.14 demonstrates the process of moving from $T[C_4']$ to $(T[C_4'])' = C_4^*$. The gray vertices connected by black edges represent $T[C_4']$, while the red, black, and white vertices connected by blue edges represent $(T[C_4'])'$ colored so that it can be observed that $(T[C_4'])'$ is in fact the same as C_4^* by comparison to Figure 4.10.

For the second form of Proposition 4, note that taking the dual of a graph returns the original graph, so thus $(M^*)' = ((T[M'])')' = T[M']$.
This proposition is very difficult to follow simply by reading the proof. The main purpose of the above proof is guide understanding by presenting the logic in full detail and to supplement Figures 4.10,4.11,4.12, and 4.14 which should be used to understand the concepts. □

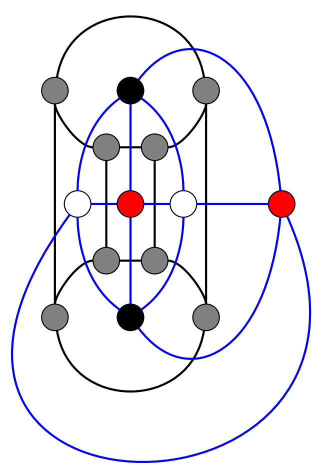

Figure 4.14: $(T[C_4'])' = C_4^*$

Proposition 5 (Craft and White)**.** *If* M *is a map for a bipartite graph, then* $T[M^*]$ *is a 3-map.*

Proof. By Proposition 3, $(M^*)'$ is a 3-map. By Theorem 2, a 3-map is bipartite, so Proposition 3 guarantees that $(((M^*)')^*)'$ is also a 3-map. By the second form of Proposition 4, this is $T[((M^*)')']$, which is $T[M^*]$ because taking the dual is self-inverse. $\square$

Chapter 5

Realizable 3-maps

A pair (p, n) is **realizable** if there is a 3-map of order p on S_n. In this chapter, we will prove two theorems characterizing which pairs are realizable. First, we will produce two base 3-maps and two infinite classes of 3-maps. Denote the 3-map created by imbedding Q_3 on S_0 and shown in Figure 2.4 by $\mathbf{M_1}$.

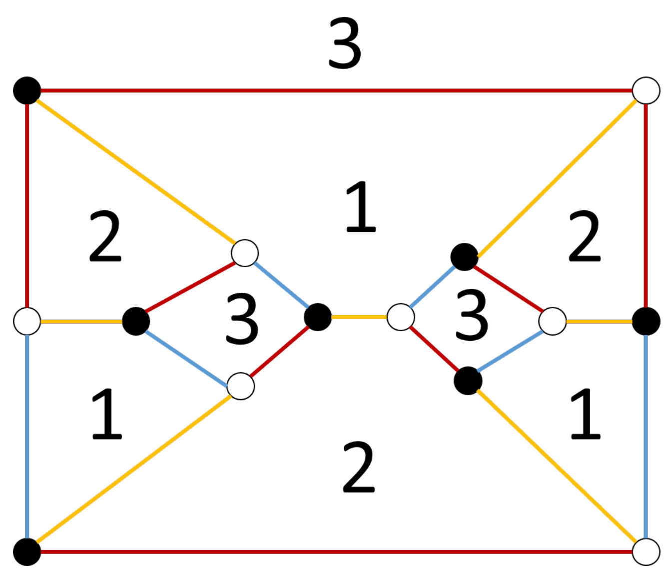

Figure 5.1: $\mathbf{M_2}$

Figure 5.1 shows $\mathbf{M_2}$, a planar 3-map of order 14 with 21 edges, 9 regions, and where each color class consists of 2 triangles and 1 hexagon.

$\Gamma = \mathbb{Z}_{4n+2}$

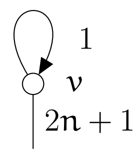

Figure 5.2: $\mathbf{M_{4n+2}}$

The first of the infinite classes of 3-maps is the set of 3-maps generated by the voltage graph shown in Figure 5.2, denoted $\mathbf{M_{4n+2}}$. This was the same voltage graph shown in Figure 3.4 whose 3-map imbedding on S_1 was depicted in Figure 3.5. $\mathbf{M_{4n+2}}$ generates a 3-map for all n. Again, since there is only one vertex in the voltage graph, we will refer to vertices in the covering graph by their group element. If a is some element of $\mathbb{Z}_{4n+4}$, then $\pi_a = (a-1, a+1, a+2n+1)$. So, for example, $\pi_0 = (4n+1, 1, 2n+1), \pi_{4n+1} = (4n, 0, 2n)$, and $\pi_{2n} = (2n, 2n+1, 4n+1)$ and so on. Using the rotation scheme and starting with 0, we can determine the regions: $R_1 : 0-(4n+1)-(2n)-(2n-1)-(4n)-(4n-1)-...$, $R_2 : 0-1-2-3-4-5-...$, and $R_3 : 0-(2n+1)-(2n)-(4n+1)-(4n)-....$

These 3 regions are all Hamiltonian cycles. Clearly R_2 is Hamiltonian. To see why the others are Hamiltonian, observe that R_1 and R_3 alternate between even and odd elements of $\mathbb{Z}_{4n+2}$. Starting with 0 for the even elements and 4n+1 for the odd elements, we can use the rotation scheme to see the following pattern for R_1: $R_1(even) : 0-(2n)-(4n)-(2n-2)-(4n-2)-(2n-4)-...$ and $R_1(odd) : (4n+1)-(2n-1)-(4n-1)-(2n-3)-(4n-2)-....$ We add $2n$ to the previous element to reach the next element. In the even, we have two parallel countdowns one moving from (4n) to (2n) and the other from (2n) to (0) counting down by 2. We can see that these countdowns (and those for the odd elements) will pass each number exactly once. R_3 is similar. The three Hamiltonian cycles fulfill the conditions of Theorem 7(c), so the imbedding is a 3-map.

The covering graphs generated by the voltage graph in Figure 5.3 form the sec-

$\Gamma = \mathbb{Z}_{4n+4}$

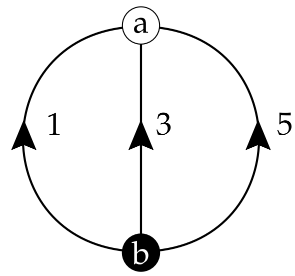

Figure 5.3: $\mathbf{M_{4n+4}}$

ond infinite class of 3-maps, denoted $\mathbf{M_{4n+4}}$. For the covering graph, we will denote (a, g) or $(b, g), g \in \Gamma$ by a_g or b_g The rotation scheme for the covering graph is $\pi_{a_g} = (b_{g-5}, b_{g-3}, b_{g-1})$ and $\pi_{b_g} = (a_{g+5}, a_{g+3}, a_{g+1})$. We can use this rotation scheme to find the regions of the covering graph. Another method is to simply use the regions on the voltage graph. Observe that the voltage graph has 3 regions, the left digon, the right digon and the outer digon. For the left digon, we would start with a_g, move to b_{g-3}, then a_{g-3+1} and so on.

The regions for the covering graph are:

$R_1 : a_0 - b_{4n+3} - a_2 - b_1 - a_4 - b_3 - a_6 - b_5 - a_8 - b_7 - ....$ In other words, a vertices with even subscripts and b vertices with odd subscripts.

$R_2 : a_1 - b_0 - a_3 - b_2 - a_5 - b_4 - a_7 - b_6 - a_9 - ...$, i.e., a odd and b even.

$R_3 : a_0 - b_{4n+1} - a_2 - b_{4n+3} - a_4 - b_1 - a_6 - b_3 - a_8 - ...$ a even and b odd but arranged differently.

$R_4 : a_1 - b_{4n+2} - a_3 - b_0 - a_5 - b_2 - a_7 - b_4 - a_9 - ...$ a odd and b even but staggered.

$R_5 : a_0 - b_{4n+3} - a_4 - b_3 - a_8 - b_7 - a_{12} - ...$ $a \equiv 0 (\mathrm{mod} 4), b \equiv 3 (\mathrm{mod} 4)$

$R_6 : a_1 - b_0 - a_5 - b_4 - a_9 - b_8 - a_{13} - b_{12} - a_{17} - ...$ $a \equiv 1 (\mathrm{mod} 4), b \equiv 0 (\mathrm{mod} 4)$

$R_7 : a_2 - b_1 - a_6 - b_5 - a_{10} - b_9 - a_{14} - b_{13} - a_{18} - ...$ $a \equiv 2 (\mathrm{mod} 4), b \equiv 1 (\mathrm{mod} 4)$

$R_8 : a_3 - b_2 - a_7 - b_6 - a_{11} - b_1 - a_{15} - ...$ $a \equiv 3 (\mathrm{mod} 4), b \equiv 2 (\mathrm{mod} 4)$

Note that a even vertices are only adjacent to b odd and b odd vertices are only adjacent to a even. This is because we add or subtract an odd number from each and even $-$ odd is odd while odd $+$ odd is even. Similarly, a odd and b even are only adjacent to each other. In fact, this voltage graph produces two disconnected copies of the same graph each 2-cell imbedded on two disconnected copies of the same surface. One way to see this is that if the covering graph is connected, there must be a path from an a even to an a odd vertex. Yet, a even goes to b odd and then back to a even bouncing back and forth and will never reach an a odd vertex. Furthermore, we can see from the rotation scheme that the vertices are all cubic, each a even is adjacent to exactly 3 b odd each b odd is adjacent to exactly 3 a even vertices. Likewise for the other two types of vertices. So, both form a full graph of order 4n + 4. We only need one graph, so consider the graph with a even and b odd vertices. Each of the two regions with 4n+4 sides forms a color class. The two regions with 2n+2 sides have no vertices in common, so they will not be adjacent and thus they form the third color class. By Theorem 7 (c), the covering graph is indeed a 3-map.

Theorem 9 (Craft and White)**.** *Let p be even. The ordered pair (p,0) is realizable if and only if p = 8 or p $\geqslant$ 12.*

Proof. There is no cubic graph with p = 2. The only cubic graph with p = 4 is K_4, which is not bipartite. In addition, since K_4 is self-dual and every vertex is adjacent to every other vertex, K_4 has no 3-region coloring. The only cubic bipartite graph with p = 6 is $K_{3,3}$, which is nonplanar. For p = 8, Q_3 forms a 3-map on S_0 (Figure 2.4).

There is no 3-map of order 10 on S_0. Suppose there was such a 3-map. Then, since our graph is cubic, $q = \frac{3}{2}(10) = 15$. By Theorem 2, $10 - 15 + r = 2$, so $r = 7$. By Theorem 7 (c), the sum of the sides of all the regions for a color class must be 10. Any region must have at least 3 sides and since a bipartite graph has no odd cycles, a region must have at least 4 sides. There can be no 8 or 10 sided region because that would force the other region to be 2 or 0, which is impossible. So, the only possible region sizes for a color class are one 4 and 6. Neither of these can be divided up further so each color class must have two regions: a 4-gon and a 6-gon. But, this is only six regions, not seven, a contradiction.

Now, we will show that if $p \geqslant 12$, then there is a 3-map of order p on S_0. First, take $\mathbf{M_1}$ and apply C1 $m-2$ times, (in sequence, not simultaneously). Each operation of C1 creates 4 more vertices, so our final map will have $8+4(m-2) = 8+4m-8 = 4m$, $m \geqslant 2$ vertices. Next, apply C1 $m-3$ times to $\mathbf{M_2}$. This will make a 3-map with $14+4(m-3) = 14+4m-12 = 4m+2$, $m \geqslant 3$. We now have 3-maps of order $p = 4m, m \geqslant 2$ and $p = 4m+2, m \geqslant 3$, which covers all even values of p equal to 8 or greater than or equal to 12. $\square$

Theorem 10 (Craft and White). *Let p be even. For* $n \geqslant 1$, *(p,n) is realizable if and only if* $p \geqslant 4n+2$

Proof. Fix n.

First, we will show that if $p < 4n+2$ then (p,n) is not realizable.

Suppose $p < 4n+2$, i.e., $p = 4n-2k$, k a nonnegative integer. Then, because our graph is cubic, $q = 6n-3k$, so $2-2n = (4n-2k)+(6n-3k)+r$. After some algebra, we find that $r = 2-k$, i.e. $r \leqslant 2$, but this cannot be a 3-map because there are less than 3 regions.

Second, we will show that if $p \geqslant 4n+2$, (p,n) is realizable. If $p \geqslant 4n+2$ and p is even, there are two possibilities: $p = 4n+2+4m$ or $p = 4n+4+4m$, $m \geqslant 0$. For the first possibility, apply C1 to $\mathbf{M_{4n+2}}$ m times, which will create a 3 map with $4m$ new vertices and so $4n+2+4m$ vertices total. For the second, apply C1 to $\mathbf{M_{4n+4}}$ $4m$.

Note that $\mathbf{M_{4n+2}}$ generates 3-maps with order $\equiv 2$ (mod 4) and $\mathbf{M_{4n+4}}$ generates 3-maps with order $\equiv 0$ (mod 4). The reason we say $\mathbf{M_{4n+4}}$ instead of simply $\mathbf{M_{4n}}$ because if $n = 1$, we would have $\mathbf{M_4}$ but there is no 3-map of order 4, so we skip 4 and begin at 8. $\square$

Observe that in proving there was no 3-map of order 10 on S_0 we investigated what

the regions would look like. A **multiset**, i.e., a set which can have multiple copies of an element, is called **feasible** for (p,n) if the multiset could possibly describe the regions for a 3-map of order p on S_n. A 3-map on S_n has r, regions, so by Theorem 2, $p - \frac{3}{2}p + r = 2 - 2n$. After some algebra, we find that $r = \frac{p}{2} + 2 - 2n$. Thus, a feasible multiset must have $r = \frac{p}{2} + 2 - 2n$ elements, each element must be an even natural number larger than 3, and, furthermore, the multiset must be able to be partitioned into three sub-multisets (one for each color class) each of whose elements sum to p. As with a pair (p,n), a multiset is **realizable** if it does in fact describe the regions of a 3-map.
$\{6 \times 4, 3 \times 6\}$ is realizable for $(14, 0)$ because it describes the regions of $\mathbf{M_2}$. The multiset is feasible because it has $\frac{14}{2} + 2 + 2(0) = 9$ elements and the three submultisets $\{2 \times 4, 1 \times 6\}$ sum to p.

$\{7 \times 4, 1 \times 8\}$ is feasible for (12,0) but is not realizable: One color class of the map must consist of a non-adjacent octagon and square, as shown in Figure 5.4. A cubic graph of order 12 has 18 edges. The octagon and the square by themselves (the black edges in Figure 5.4) have only 12 edges, so 6 more edges must be added to make a 3-map. However, none of these edges can be formed by joining vertices in the square because this would make a 3-sided region. Thus, each of the four vertices in the square must be joined to a vertex in the octagon. The remaining two edges will come from joining vertices of the octagon with each other. Vertices cannot be joined inside the octagon because that would destroy the 8-sided region. Pick an arbitrary vertex of the octagon, say vertex 1. To form a 4-sided region, 1 can only be joined to 6 or 4. But if this is done, for example by the blue edge in Figure 5.4, then two vertices are now inaccessible to vertices from the square. Our map is planar so the edges from the square cannot cross through the blue edge or the edges of the octagon. But then, even if we join every edge of the square to a distinct vertex of the octagon, the map will only have 17 edges, so $\{7 \times 4, 1 \times 8\}$ is not realizable for (12,0).
Thus, feasibility is a necessary but not a sufficient condition for realizability.

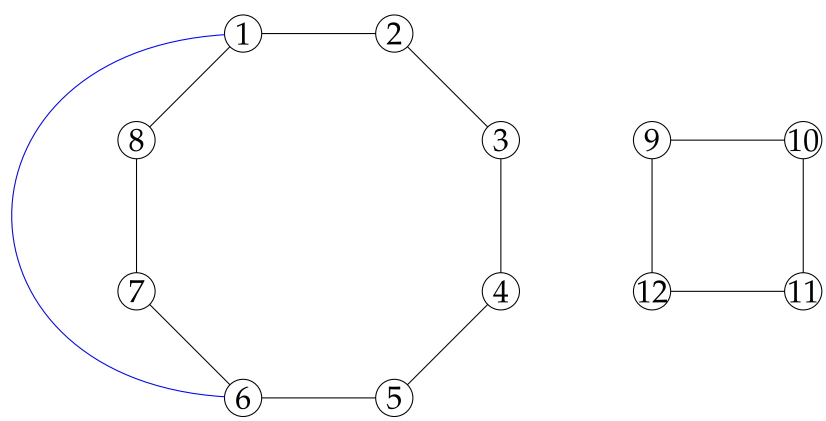

Figure 5.4: $\{7 \times 4, 1 \times 8\}$ not realizable for (12,0)

Chapter 6

m-uniform maps

In this chapter we will investigate 3-maps in which every region is an m-gon, called **m-uniform maps**. For a fixed m, we can refer to the map by the size of the region. For example, an imbedding where every region is 4-sided is called a **quadrilateral imbedding**.

4 sides is the smallest possible region size for a bipartite graph, so let us investigate quadrilateral imbeddings for 3-maps. By Theorem 7 (c) we know that the size of the regions of each color class must sum to p, so for a quadrilateral imbedding p must be a multiple of 4, i.e., $p = 4k$, for some positive integer k. Furthermore, since each color class is a partition of the vertices of the graph, we can divide p by 4 to see that k is the number of regions in each color class. Thus, $r = 3k$ and because we have a cubic graph $q = 6k$. $p - q + r = k = 2 - 2n$. Solving for n shows that $n = \frac{2-k}{2}$. k cannot be 1 because then we would have $n = \frac{1}{2}$ and k also cannot be greater than or equal to 3 because then we would have $n < 0$. So, $k = 2, n = 0, p = 8, q = 12,$ and $r = 6$. Q_3 imbedded on S_0 clearly fits these parameters, but is there any other 3-map with a quadrilateral imbedding?

The answer is no. To see why, first observe that any color class consists of two non-adjacent squares, as shown by the vertices in Figure 6.1 connected by black edges. We need 4 more edges to form a cubic graph but no vertex in a square can connect to any other in that same square because this would create a 3-sided region. Thus, the remaining edges are formed by joining vertices of the two different squares. Pick an arbitrary vertex on the left square, a, and join it to an arbitrary vertex on the right, c as shown by the blue edge in Figure 6.1. Now, consider another vertex, b on the left square. If b is joined to any other vertex except d, a region of more than 4 sides will be formed. So, b must be joined to d as shown by the red edge. Next, consider the vertex u as shown in the bottom part of Figure 6.1. If u is joined to x instead of w, v be be cut off from forming any more edges. So, uw and vx, as shown by the violet edges in Figure 6.1, are the last two edges of the graph, which is Q_3.

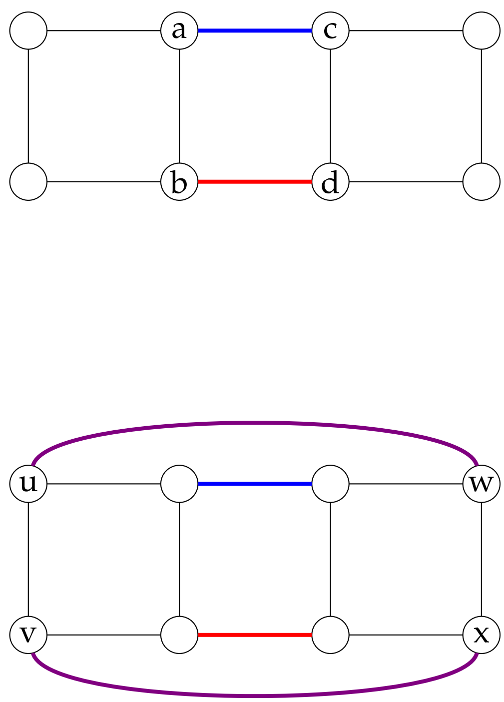

Figure 6.1: Q_3 is the only 4-uniform 3-map

Next, let us examine the imbeddings with the second-smallest size, i.e., hexagonal imbeddings. p is a multiple of 6, i.e., so by similar reasoning as above, $p = 6k, q = 9k, r = 3k$. Then, $p - q + r = 0 = 2 - 2n$ and $n = 1$. Any hexagonal imbedding for a 3-map must be on S_1, the torus. Furthermore, there is a 3-map on the torus for every positive integer k. These are generated by the voltage graph shown in Figure 6.2, which is the same voltage graph as in Figure 5.3 but imbedded on the torus instead of the sphere.

By starting with either a or b and following the edges, we can see that the voltage graph in Figure 6.2 has only one region. The region is 6-sided and follows this form: $a \overset{-1}{—} b \overset{+3}{—} a \overset{-5}{—} b \overset{+1}{—} a \overset{-3}{—} b \overset{+5}{—} a$ (where the number above each line is the voltage associated with a particular edge and direction). Notice that we only add or subtract odd numbers. Thus, for any choice of k, a even vertices will only be adjacent to b odd and vice versa, while b even vertices will only be adjacent to a odd and vice versa. Thus, as with $\mathbf{M_{4n+4}}$, this voltage graph will generate two identical 6-uniform 3-maps. We will choose the map with a even and b odd vertices.

A description of the regions generated by the voltage graph in Figure 6.2 will be more complicated than the descriptions of the regions of previous voltage graphs. Partition $\mathbb{Z}_k$ into k sets, each consisting of six consecutive elements, each associated with a different congruence class mod 6. Call these sets **bands** and number them. We will start counting with 0, so $\{0, 1, 2, 3, 4, 5\}$ is band 1, $\{6, 7, 8, 9, 10, 11\}$ is band 2, $\{6k-6, 6k-5, 6k-4, 6k-3, 6k-2, 6k-1\}$ is band k etc. Since we are using a even b odd vertices, we will start each region with an a even vertex and the color classes will be decided from the congruence class of the group element associated with a. For $g \in \mathbb{Z}_k$, color regions 1 that begin with $a_g, g \equiv 0 \pmod 6$, color regions 2 that begin with $a_g, g \equiv 2 \pmod 6$, and color regions 3 that begin with $a_g, g \equiv 4$ (mod) 6.

Some sample regions:

0 (mod 6) $a_0 - b_{6k-1} - a_2 - b_{6k-3} - a_{6k-2} - b_{6k-5} - a_0$

2 (mod 6) $a_2 - b_1 - a_4 - b_{6k-1} - a_0 - b_{6k-3} - a_2$

4 (mod 6) $a_4 - b_3 - a_6 - b_1 - a_2 - b_{6k-1} - a_4$

We can see from these samples that if i is the band of g for a_g, then 0 (mod 6) regions have 2 elements from a band i, 1 from a band $i-1$ (mod k), and 3 from b band $i-1$. 2 (mod 6) regions have 3 elements from a band i, 2 from b band $i-1$, and 1 from b band

i.

4 (mod 6) regions have 2 elements from a band i, 1 from a band $i+1$, 2 from b band i, and 1 from b band $i-1$.

Starting from the first region in any color class, simply add 6 to the subscript of each element (mod $6k$) to generate the next region in that color class. The six subscripts for each region are in different congruence classes, so adding 6 will preserve these same congruence classes. Thus, no vertex can repeat in a region and so each color class will, in fact form directed cycles that partition the vertex set and thus by Theorem (c) we have a 3-map.

$\Gamma = \mathbb{Z}_{6k}$

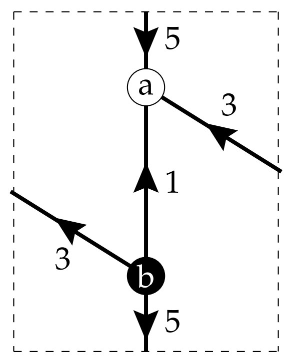

Figure 6.2: a voltage graph for 6-uniform 3-maps on S_1

By similar reasoning as with quadrilateral and hexagonal imbeddings, we can see that a 3-map with an m-uniform imbedding has k m-gons in each color class. This integer k is called the **partition size** of the map. Here is what has been shown so far regarding partition size of m-uniform maps:

(1) For every n, $\mathbf{M_{4n+2}}$ is (4n+2)-uniform with partition size $k = 1$ (because each region is a Hamiltonian cycle).

(2) The only 4-uniform 3-map is $\mathbf{M_1}$, i.e., Q_3 imbedded on S_0, which has $k = 2$.

(3) For every positive integer k, there is a 6-uniform 3-map on the torus with partition size k.

There is no (4n)-uniform 3-map with $k = 1$ because this would force $p = 4n$, $q = 6n$, $r = 3$

and then we would have $4n - 6n + 3 = 2 - 2h$, which simplifies to $h = n - \frac{1}{2}$. But this is impossible because a surface cannot have a fractional genus. In addition, there is no m-uniform map of order (4n+2) with $k = 2$ because then we would have $\frac{4n+2}{2} = 2n+1 = m$, i.e., odd cycles in a bipartite graph.

(4) For every $s \in \mathbb{N}$, there is a (4s)-uniform 3-map of order 8s on S_{2s-2}. Since $\frac{8s}{4s} = 2$, this has partition size $k = 2$.

$$\Gamma = \mathbb{Z}_{2s}$$

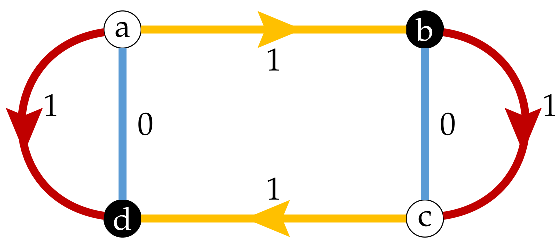

Figure 6.3: A voltage graph for (4s)-uniform 3-maps of order 8s

These maps are generated by the voltage graph in Figure 6.3. The group is $\mathbb{Z}_{2s}$. As before, the regions for the covering graph can be found using the rotation scheme or by tracing around the regions of the voltage graph. For example choose the left digon and start with a_0, add 1 to form d_1, then add 0 to form a_1 and so on, which leads to:

$R_1 : a_0 - d_1 - a_1 - d_2 - a_2 - d_3 - a_3 - d_4 - a_4 - d_5 - \ldots$ all a and d vertices.

Even if a region in the voltage graph generates multiple regions in the covering graph continue choosing vertices from the covering graph not used yet until all vertices have been used in a region. For the outer region we start with a_0 add 1 to make b_1, add 1 to form c_2, then d_3, and since we are moving backward along the red edges, subtract 1 to make a_2 and so on:

$R_2 : a_0 - b_1 - c_2 - d_3 - a_2 - b_3 - c_4 - d_5 - a_4 - \ldots$ a even c even, b odd d odd

Next, start with an odd a vertex:

$R_3 : a_1 - b_2 - c_3 - d_4 - a_3 - b_4 - c_5 - d_6 - a_5 - ...$ a odd c odd, b even d even

The outer region of the voltage graph can only generate these two in the covering graph because between both of them, all vertices are accounted for. We could start with vertices other than a_0 or a_1, but this would only trace out one of the above two regions while starting in a different place. The remaining regions are:

From the right digon $R_4 : b_0 - c_1 - b_1 - c_2 - b_2 - c_3 - ...$ all b and c vertices

From the middle square $R_5 : a_0 - b_1 - c_1 - d_2 - a_2 - b_3 - c_3 - d_4 - a_4 - b_5 - ...$ a even d even, b odd c odd

$R_6 : a_1 - b_2 - c_2 - d_3 - a_3 - b_4 - c_4 - d_5 - a_5 - ...$ a odd d odd, b even c even.

The first two color classes are formed from the two regions generated by each square and the two regions generated by the two digons form the third color class. By Theorem 7 (c), the covering graph is a 3-map. Recall that in a 3-map every bi-colored cycle bounds a region, just as shown in the voltage graph in Figure 6.3. The color classes correspond to the bi-colored cycles of the voltage graph. Furthermore, since each color class does in fact have two regions, k is equal to 2, as promised. Since, $|\mathbb{Z}_{2s}|$ = $2s$ and we have 4 vertices in the voltage graph, $p = 8s$ and $q = 12s$. We know $r = 6$, so we have $8s - 12s + 6 = 2 - 2n$ and thus $n = 2s - 2$.

(5) For each $s \in \mathbb{N}$, there is a (4s+2)-uniform 3-map of order (8s+4), and thus partition number $k = 2$ on S_{2s-1}. To form this map, we will start with a (4s)-uniform 3-map of order (8s), as constructed in (4) and then modify the graph and the surface. This sort of topological modification is called surgery. Each of the regions is either has a even or b even vertices and there are 6 regions total,so a_0 is adjacent to 3 regions and b_0 is adjacent to the other three regions. If we delete both of these vertices and their edges, the three regions they are adjacent to will run together and only two regions will remain in the imbedding, as shown in Figure 6.4.

Join these two regions together by a topological cylinder also called a handle, as shown in Figure 6.5. The handle (as seen from above) is represented by the light gray ellipse. Imagine the handle coming out of the bottom region at the dark gray circle, lifting up into the air in an arch and touching down again at the dark gray circle on the top region. Next, place six new vertices around the center of the handle. Since our original (4s)-uniform map was a 3-map, c_0, c_1 and c_{2s-1} were attached to b_0 with one edge of each of the three colors. We can now attach them to the filled vertices in the center of the handle with the same color edges with which they were attached to b_0 in the previous map and likewise for b_1, d_0, and d_1. Observe that this operation extends the original 3 regions by 2 vertices since we have deleted one vertex and added 3. Each region is now a (4s + 2)-gon. Also note that, as shown in Figure 6.5 this new map is still a 3-map. The new vertices and edges simply extend the old bi-colored cycles rather than disrupting them. Since we have added a new handle, the surface is now S_{2s-1}.

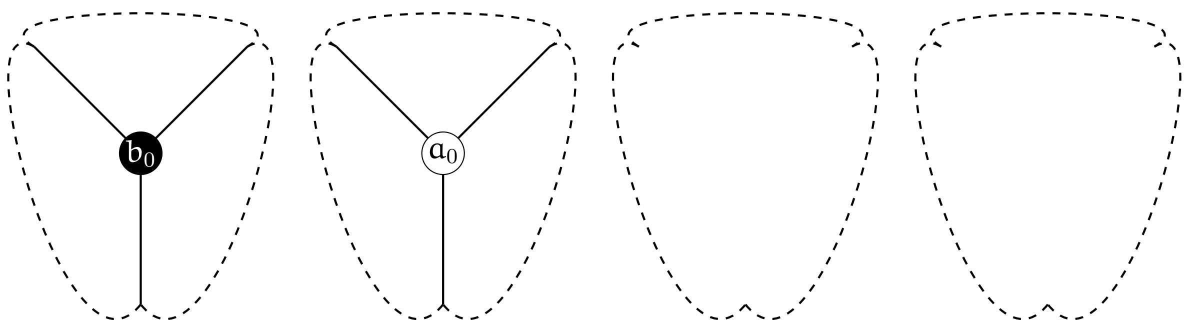

Figure 6.4: Deleting b_0 and a_0 causes the six region to blend together into two regions

(6) For each $k \in \mathbb{N}$, there is a (2k)-uniform 3-map of partition size k and order $2k^2$ on $S_{(k-1)(k-2)/2}$. We will construct this 3-map in three stages. First, we will construct an imbedding of the complete bipartite graph $K_{k,k}$ on $S_{(k-1)(k-2)/2}$. Using the dipole voltage graph shown in figure 8.4, choose $\Gamma = \mathbb{Z}_k$ and replace the voltages $\{1, 3, ..., 2k-1\}$ with $\{0, 1, ..., k-1\}$, respectively. The regions will be the digons between successive edges and the outer region. For example, the region between edge 0 and edge 1 will

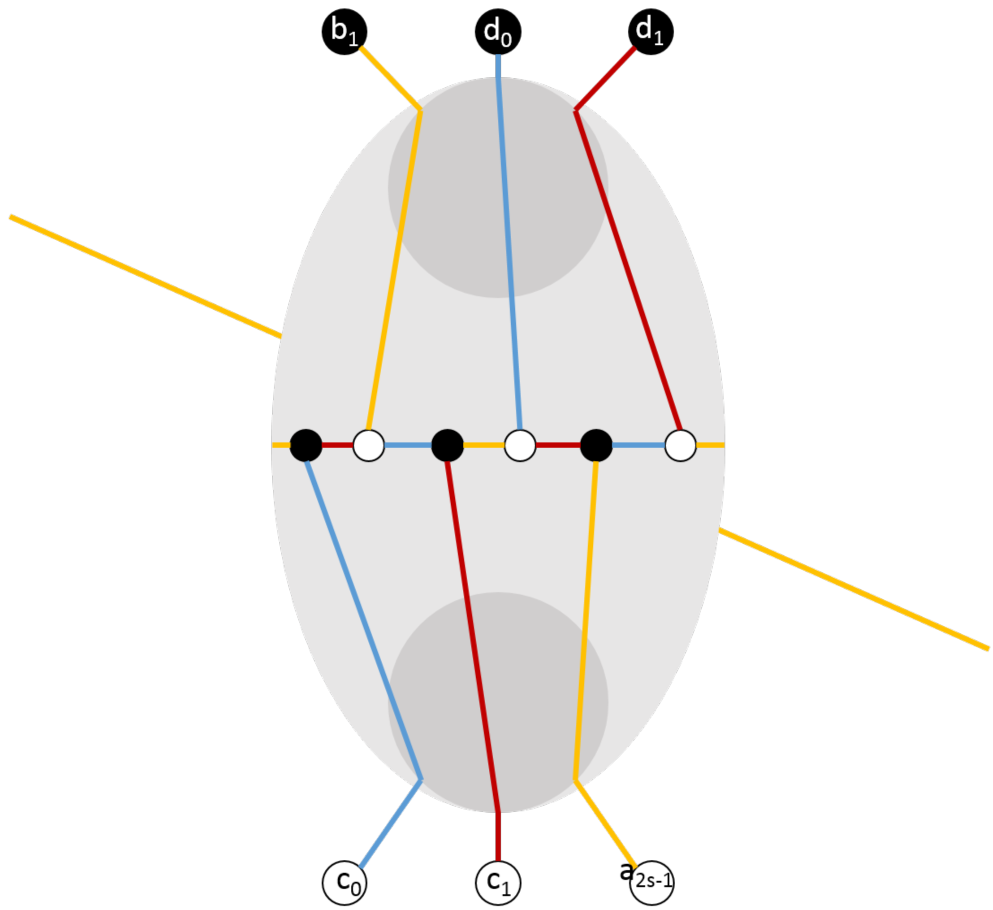

Figure 6.5: Surgery to produce (4s + 2)-uniform 3-maps of order 8s + 4

be $a_0 - b_0 - a_1 - b_1 - a_2 - b_2 - \ldots$ and the region between edge 1 and edge 2 will be $a_0 - b_{k-1} - a_1 - b_0 - a_2 - b_1 - a_3 - b_2 - \ldots$ Consider an arbitrary region, with voltages $i, i+1$ (adding mod k so that the outside region is included in this description). Start with vertex b_0. The next vertices will be a_{i+1}, $b_{i+1-i} = b_1$, a_{i+2}, $b_{i+2-i} = b_2$, etc. We can see that the vertices alternate between a and b and increase by one each time. Thus, every region is a Hamiltonian cycle and our imbedding is a map. Furthermore, since there is an edge for each element of the group, each b vertex is adjacent to every a vertex and vice versa. However, no a vertex is adjacent to any other a vertex, so the covering graph is indeed $K_{k,k}$ Furthermore, $p = 2k, q = k^2, r = k$, so $2-2n = 2k-k^2+k$ and after simplifying, we find that $n = (k-1)(k-2)/2$. Call this map of $K_{k,k}$ M

For the next step, we will stellate M. Recall that this means placing a new vertex in the center of each region and joining the new vertex to all the vertices on the boundary. None of the new vertices will be adjacent any of the other new vertices and, because the regions are Hamiltonian cycles each new vertex is adjacent to all of the original vertices. Thus we have formed a map of the complete tripartite graph, $K_{k,k,k}$. We will denote this M^*.

For the final step, observe that M^* is, in fact, a tri-map, i.e., a triangulation of an orientable surface by a tripartite graph. By Proposition 1, the dual of M^*, $(M^*)'$ is a 3-map. M^* has order $3k$. M^* replaces each of the k regions of M with $2k$ triangles so M^* has $2k^2$ regions. Taking the dual interchanges regions and vertices and does not change the surface, so our 3-map $(M^*)'$ has order $2k^2$ and $3k$ regions total, with k regions for each color class (corresponding to the k vertices in each of the three partite sets in M^*). Lastly, since each vertex in M^* has degree $2k$, each region in $(M^*)'$ has $2k$ sides and we have a uniform 3-map.

Chapter 7

Genus Contructions with Product Graphs

Lemma 1. *If* G *and* H *are both bipartite, then* $G \square H$ *is bipartite.*

Proof. Based on proof in [1]. Suppose G and H are both bipartite with partite sets G_1, G_2 and H_1, H_2, respectively. Then, the partite sets of $G \square H$ are $G_1 \times H_1 \cup G_2 \times H_2$ and $G_1 \times H_2 \cup G_2 \times H_1$. To check that this partition of $V(G \square H)$ does in fact form a bipartition, it is sufficient to show that there are no edges within partite sets. If (g, h) and (k, l) are in $G_i \times H_j, i, j \in \{1, 2\}$, then there can be no edges between the two vertices because, if $g = k$, then $hl \notin E(H)$ (because h and l are from the same partite set) and similarly if $h = l$. If $(h_1, g_1) \in H_1 \times G_1$ and $(h_2, g_2) \in H_2 \times G_2$, then can also be no edges between the two vertices because $h_1 \neq h_2$ and $g_1 \neq g_2$. For the same reason, there can be no edges if $(g_1, h_2) \in G_1 \times H_2$ and $(g_2, h_1) \in G_2 \times H_1$. □

Theorem 11 (Craft and White)**.** *Let G be an order p(G) graph for a 3-map M and let H be a connected, bipartite, cubic graph with order p(H). Then,* $\gamma(G \square H) = 1 + p(G)p(H)/4$.

Proof. By lemma 1, $G \square H$ is bipartite. Therefore, a quadrilateral imbedding will be a minimal imbedding. We will use surgery to construct a quadrilateral imbedding of

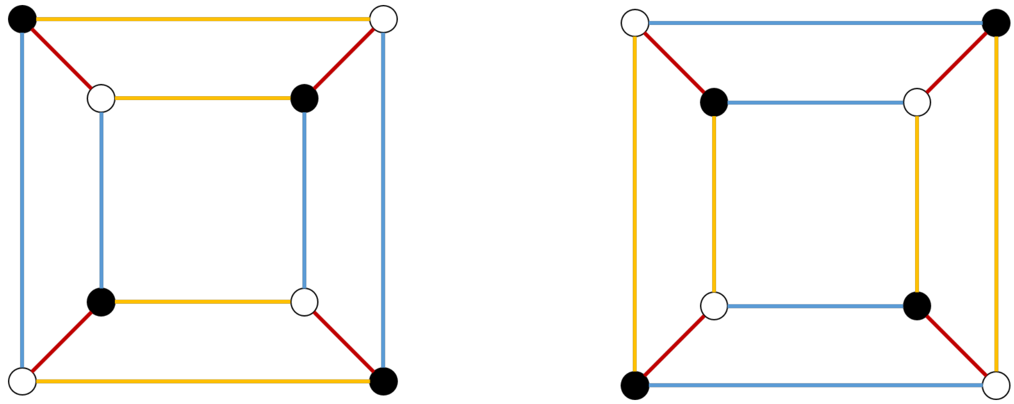

Figure 7.1: Two mirror image copies of Q_3 with opposite orientations

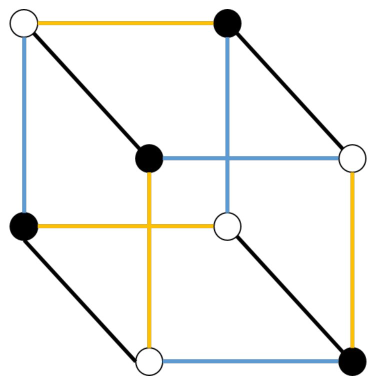

Figure 7.2: Joining the central regions of the copies of Q_3 with a tube

$G \square H$. Start with $p(H)$ copies of M and associate each copy with a vertex of H. Denote the copy associated with vertex v by $M(v)$. Since H is bipartite, by König's Theorem, H has as 3-edge coloring. As in Theorem 8, associate a clockwise or counterclockwise orientation with each vertex of H, corresponding to the 3-edge coloring. For every $v \in V(H)$, give $M(v)$ an orientation that corresponds to the orientation of v. The effect of this will be to flip the bipartitions of oppositely oriented copies of M, as shown in Figure 7.1. So, when we join vertices to form $G \square H$, there is no need to find the bipartition of $G \square H$; it will still be given by filled and hollow vertices as in M.

If $uv \in E(H)$ and uv is colored i, then join the regions colored i in $M(u)$ and $M(v)$ by a tube. Since each color class partitions the vertices of M, every vertex of $M(u)$ will be joined to the corresponding vertex of $M(v)$. This process is shown in Figure 7.2, where the inner regions of the copies of Q_3 are joined by a tube. As we can see from Figure 7.2, the old regions are inside the tube, so they are eliminated and the new regions formed along the sides of the tubes are all quadrilaterals. By Corollary 1 (Corollary 6-15 in [7]), a quadrilateral imbedding of a graph is on a surface of genus $q/4 - p/2 + 1$. $G \square H$ has $p = p(G)p(H)$. The degree of a vertex (x, y) for a Cartesian product of graphs is the sum of the degrees of x and y. G and H are cubic, so $G \square H$ is 6-regular. Thus, $6p(G)p(H) = 2q$. $\gamma(G \square H) = 3p(G)p(H)/4 - p(G)p(H)/2 + 1$, which simplifies to $1 + p(G)p(H)/4$. □

Note that in Figure 7.1, if the tubes for the inner and outer regions (both colored 1) both emerged from the front of S_0, the tube for the inner region would be *inside* the tube for the outer region. However, if the tubes for the outer region come from the back of the sphere, there is no such problem.

For a concrete example of this construction, consider $Q_3 \square Q_3$, with Q_3 imbedded on S_0. $p = 64$ and $q = 192$, so by Theorem 11, $\gamma(Q_3 \square Q_3)$ should be $1 + 64/4 = 17$. We will have 8 copies of Q_3 each on S_0. There are 2 regions in each color class and 12 edges in Q_0, so for each edge uv we will add two tubes between $M(u)$ and $M(v)$ for a total of 24. Seven of the tubes will join the spheres into a single surface and the remaining 17 will form handles so that $\gamma(Q_3 \square Q_3) = S_{17}$, as expected.

Theorem 12 (Craft and White)**.** *Let G be a cubic graph of order p having a 3-map M. Then* $\gamma(G \square G) = 1 + p^2/4$.

Proof. This proof is directly from Theorem 11 with $p(G) = p(H) = p$. □

Theorem 13 (Craft and White)**.** *Let G be a cubic graph of order p having a 3-map M. Then,* $\gamma(G^m) = 1 + ((3m - 4)/8)p^m$, *for* $m \geqslant 2$.

Proof. We will prove this by induction on m. Theorem 11 provides the base case. Now, assume that $\gamma(G^m) = 1 + ((3m - 4)/8)p^m$ up to m. By Lemma 1 and a simple induction,

G^m is bipartite. Thus, a quadrilateral imbedding will be minimal. Denote G^m with such an imbedding by M^m. Just as in Theorem 11, associate a copy of M^m with each vertex of G, oriented according to the bipartition of G. M^m is not a 3-map since G^m is not cubic but $3m$-regular. However, recall that in Theorem 11 each new tube joined two regions of the same color. Thus, even though the original regions were covered by a tube, each tube in M^2, the mimimal imbedding of $G \square G$, is associated with a region color. Each tube also contains four quadrilateral regions. We can choose either the top and bottom or left and right regions to partition the vertices on the tube. For M^2, pick the left and right regions from each tube. Color them the same color as the regions covered by the tube. Now, we can form G^3 by associating a copy of M^2 with each vertex of G, oriented according to the bipartition of G. For each edge uv of G colored i, join each quadrilateral region of $M^2(u)$ and $M^2(v)$ colored i. Just as before, this will produce a quadrilateral imbedding of G^3. For each tube with two quadrilateral regions colored i in G^2, there will now be two tubes and four quadrilateral regions colored i in G^3. Assume that we have continued to follow this process of coloring quadrilateral regions up to m. We can now form G^{m+1} in the same way and the induction is complete.

Since G is a cubic graph, G^m will be $3m$ regular with order p^m and size $3mp^m/2$, so by Corollary 1, we have $1 + 3mp^m/8 - p^m/2$, which simplifies to $1 + ((3m-4)/8)p^m$. $\square$

Chapter 8

n-maps

Now, we will examine a generalization of 3-maps. A map is **n-chromatic** if the regions of the map can be colored with a minimum of n colors such that regions who share a vertex are not colored the same color. If a map is n-chromatic, it must be n-region colorable because if two regions share an edge, they must share a vertex. However, n-chromatic is more stringent than being n-region colorable because a map may have regions that do not share a boundary edge but share vertices. The two regions of the pentagon marked 1 in Figure 8.1 share vertex v but have no boundary edges in common. The two notions do coincide for 3-maps, however, because every region shares an edge with any region with which it shares a vertex and vice versa, as shown in Figure 8.1 where the regions marked 1, 2, and 3 share the filled vertex and also each region shares an edge with the other two. It is clear from the definition of n-chromaticity that for an n-chromatic map, n must be greater than or equal to the maximum degree of the underlying graph.

An n-regular, n-chromatic map for which there are only two mirror image ordering of region colors around all vertices (1,2, ..., n) and (n, n-1, n-2, ...) is called **canonically n-chromatic** or an **n-map**. Figure 8.2 shows a 4-map with $K_{4,4}$ imbedded on S_1. As can be seen from Figure 8.2, adjacent vertices will have opposite region-coloring orders, which forms, as before, a canonical bipartition of the map. We can use the region coloring to

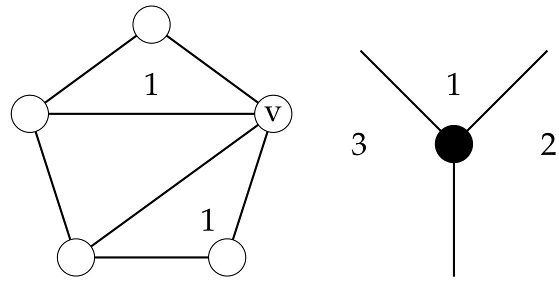

Figure 8.1: Two regions sharing a vertex but not an edge

give an n-edge coloring by coloring an edge color i if it separates regions colored $i+1$ and $i+2$ where we add mod n. For example, on a 6-map, an edge separating regions colored 6 and 1 would be colored 5. As can be seen in Figure 8.2 (where green is used for color 4), the edge colors also increase clockwise or counterclockwise in accordance with the bipartition, so we have a canonical edge coloring.

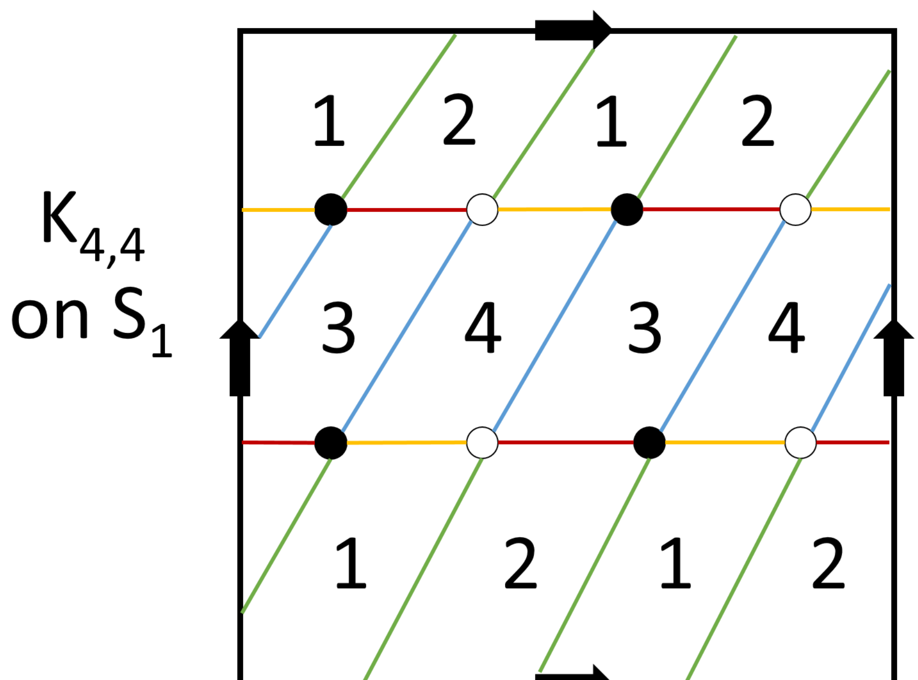

Figure 8.2: $K_{4,4}$ on S_1 as a 4-map

We have the following analogous theorems:

Theorem 14 (Craft and White)**.** *Let M be a map for an n–regular graph G. Then, the following statements are equivalent:*
(a) M is canonically n-chromatic (i.e. M is an n-map)
(b) G is canonically n-edge colorable and every cycle colored with two consecutive colors bounds a region. Moreover, every region is so described.
In addition, each of these statements implies:
$S(G) = \{(u, v) : uv \text{ is in } E(G)\}$ *is partitioned into n sets, each inducting a collection of region - bounding directed cycles partitioning V(G).*

Proof. (b) ⇒ (a) Every region is a bi-colored cycle with two consecutive colors. If a region is bounded by edges colored i and $i+1$, color the region $i+2$ (working mod n). For example, if the edge colors are $n-1$ and n, color the region 1 and if the colors are n and 1, color the region 2. Observe that the regions are colored this way in Figure 8.2 and that this naturally leads to a canonical rotation of region colors corresponding to the canonical edge rotation. Because of the way regions are colored, every vertex is adjacent to exactly one region of each color, so M is n-chromatic as well as canonical and the proof is complete.

□

Theorem 15 (Craft and White)**.** *A connected n-edge colorable, n-regular graph is the underlying graph for some n-map if and only if it is bipartite.*

We will not prove the rest of these theorems because their proofs are very similar to the two above. For example, (a) ⇒ (b) in Theorem 14 will make use of an intermediate step of first showing that G is bipartite and the bipartition is canonical. Theorem 15 will use that intermediate step for the first direction. Next, we would use König's edge coloring theorem to guarantee that G has an n-edge coloring and then use the rotation scheme induced by the edge coloring to form a 3-map. One difference is that Theorem 14 (c) does imply that M is n-chromatic, but not *canonically* n-chromatic. Figure 8.3 shows the octahedral map on the plane, which is 4-regular and 4-chromatic, but not bipartite. In addition, as shown in Figure 8.3 the vertices show all possible 6 cyclic permutations of the four region colors.

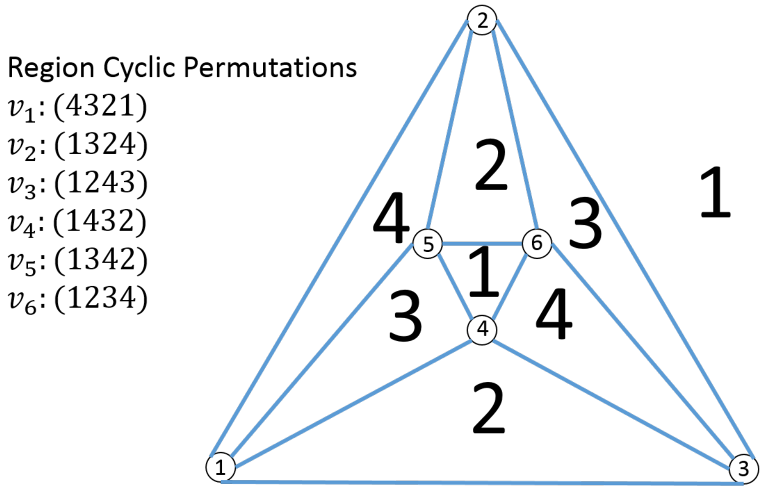

Figure 8.3: The octahedral map on the plane is 4-chromatic but not canonically 4-chromatic

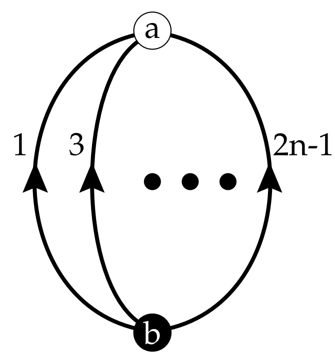

Figure 8.4: The dipole voltage graph

The voltage graph in Figure 8.4, called a dipole because of its shape, provides an example of an n-map for all $n \geqslant 3$. Choose $\Gamma = \mathbb{Z}_{4n-2}$. For this choice of Γ, the voltage graph has n edges. This is because there are as many edges as odd numbers from 1 to 2n-1. If we start with 1 and repeatedly add 2 n times, we will reach 2n + 1, so if we only add 2 (n - 1) times, we will reach 2(n-1) + 1 = 2n - 2 + 1 = 2n - 1. So, the total number is n-1 and 1 more for 1, n total. There are also n regions. Each edge is to the left of exactly one region (including the outside region for the 2n-1 edge) and since there are n edges, there are exactly n regions. As with the voltage graphs in Figures 5.3 and 6.2, the dipole will have b even vertices only adjacent to a odd and vice versa , while a even will only be adjacent to b even and vice versa. Therefore, this voltage graph will also make two identical cubic graphs of order 4n - 2, imbedded on disconnected surfaces. Each graph will be n-regular, so by Theorem 1, we will have $2q = (4n-2)(n) = 4n^2 - 2n$, so $q = 2n^2 - n$. Thus, we have $2 - 2h = (4n-2) - (2n^2 - n) + n$. After some algebra, we conclude that $h = n^2 - 3n + 2 = (n-1)(n-2)$.

Furthermore, the regions form Hamiltonian cycles. Consider the graph with b even a odd vertices. Start with b_0 and follow the left edge of an region, so the next vertex is a_i, then we will follow the right edge backwards, so the next vertex is $b_{i-(i+2)} = b_{-2}$, a_{i-2}, $b_{(i-2)-(i+2)} = b_{-4}$. Both the a and b vertices are alternating and then counting down by 2, so we can see eventually all the vertices will be passed exactly once. Since the regions are all cycles, we have a map. Next, color the edges in the covering graph with color i if they were generated by the edge with voltage $2i-1$. This will ensure that each region is a bi-colored cycle of successive colors. Moreover, as we can see from the rotation of the dipole, this coloring will also induce a canonical n-edge coloring. As in Theorem 15, the canonical edge coloring will ensure that every bi-colored cycle bounds a region. All the conditions of Theorem 14 (b) are fulfilled, so we have an n-map.

Theorem 16 (Craft and White)**.** *Let M be an n-map on S_k with underlying graph G having r_i regions of color i.*

(a) There is an (n + 1)-map on S_{2k+r_i-1} with underlying graph $G \square K_2$

(b) There is a 4-uniform 2n-map on $S_{1+p^2(n-2)/4}$ with underlying graph $G \square G$.

Proof. (a) Suppose M is an n-map as described in the statement of the theorem. Let M_1 be an identical M. Fix some color i, $1 \leqslant i \leqslant n$. Join the interior of all i-colored regions in M to that of all i-colored regions in M_1 with a separate handle for each region. The first handle will fuse the surfaces, producing a surface S_{2k}. The remaining tubes will add a handle for each of the remaining $r_i - 1$ regions, so that the resulting graph will be imbedded on S_{2k-r_i-1}.

Recall that the regions of M and M_1 are colored such that every region colored k is bounded by a bi-colored cycle with edges colored alternately $k-1$ and $k-2$. Thus, the cylinders have added a new edge between two edges colored $i-1$ and $i-2$. Color this new edge $i-1$, retain the color of the edges colored 1 to $i-2$ and add one to the color of the edges whose original colors were $i-1$ and above. Thus, the canonical ordering of edges from M and M_1 remains but with an extra edge color added. Color the regions according to the previous rule and we have an (n + 1)-map.

(b) As in Theorem 11, associate p copies of M each with a vertex of G, where p is the order of G. Also as in Theorem 11, if v is a clockwise vertex of G, then give $M(v)$ a clockwise orientation and likewise if v is counterclockwise. If uv is an edge colored i in G, then join the regions colored i in $M(u)$ and $M(v)$ for all $i = 1, 2, ..., n$. Since each color class partitions the vertices of G, all the vertices of $M(u)$ will be joined with all the vertices of $M(v)$ as is necessary for the Cartesian product. This process of joining the interiors of regions by handles will eliminate the old regions and produce new quadrilateral regions, as shown in Figure 7.2. As in part (a), each of these new edges will be placed between an edge colored $i-1$ and $i-2$. To verify that this new construction produces an n-map, begin with the old regions that were bi-colored cycles colored 1 and 2. Retain the color of edges colored 1, color the new edges along the tube 2 and color 3 the edges previously colored 2. Then, the cycles that were originally colored 2 and 3 will now be colored 3 and 3. Color the new edges 4 and color 5 the edges previously colored 3. Continue this process, shifting the colors along, never violating the bipartition and remembering that the new edges are added between the original edges, so we must color them a color between the new colors of the old edges. This somewhat more complicated process produces, just as in (a), a canonical ordering of the vertices, which induces an n-map.

Since G has order p, then $G \square G$ has order p^2. G is also n-regular, so $G \square G$ is 2n-regular and $2q = 2np^2$. Since we have a quadrilateral imbedding, $4r = 2q = 2np^2$. This yields $p^2 - np^2 + np^2/2 = 2 - 2h$. After simplifying, we find that $h = 1 + p^2(n-2)/4$. $\square$

Theorem 17 (Craft and White)**.** *If* (n, m_1) *and* (n, m_2) *are realizable, then* $(n, m_1 + m_2)$ *is realizable.*

Proof. Suppose M_1 is an n-map on S_{m_1}, M_2 is an n-map on S_{m_2} and u and v are oppositely oriented vertices in M_1 and M_2, respectively. Denote the vertices clockwise ordering of the vertices adjacent to u by $x_1, x_2, ..., x_n$ and the counterclockwise ordering of the vertices adjacent to v by $y_1, y_2, ..., y_n$. Remove u and v. Since M_1 and M_2 are n-maps, u and v are each adjacent to n regions, so removing these two vertices will lead to a similar situation to that shown in Figure 6.4. Now, join these two newly created regions by a handle, joining vertices with the same subscripts with an edge of the same color to which they were joined to u or v. This will create a situation very similar to that pictured in Figure 6.5. The reason for the opposite orientations of the vertices is so that when the two regions are joined, each pair of vertices with the same subscript can be joined with no worry about edges crossing. Also, the new edges are the exact same color as the previous edges and in the same position relative to the other edges of the vertices so that we do in fact have an n-map on the surface $S_{m_1+m_2}$. $\square$

Lemma 2 (Craft and White)**.** *An n-map of order p on* S_m *satisfies* $m \geqslant \lceil \frac{1}{4}(n-2)^2 \rceil$.

Proof. Since an n-map is bipartite, $4r \leqslant 2q$, so $r \leqslant q/2$. We then have $2 - 2m \leqslant p - q/2$. Since an n-map is n-regular, $2q = pn$, hence $q = pn/2$. So, now we have $2 - 2m \leqslant p - pn/4$. After simplifying, this yields $m \geqslant 1 + \frac{p(n-4)}{8}$. Above, we used the facts that an n-map is n-regular and bipartite separately, but, taken together, these properties imply that each partite set must have at least n elements, so $p \geqslant 2n$.

This gives $m \geqslant 1 + \frac{p(n-4)}{8} \geqslant 1 + \frac{2n(n-4)}{8} = \frac{n^2-4n}{4} + \frac{4}{4} = \frac{(n-2)^2}{4}$. The genus of a surface must be a nonnegative integer so we can round up, hence the ceiling function. $\square$

Note that Lemma 2 is actually false for $n = 3$. $\lceil \frac{1}{4}(3-2)^2 \rceil = \lceil \frac{1}{4} \rceil = 1$, which would imply that there are no 3-maps on S_0. The problem is that for $n = 3, 1 + \frac{p(n-4)}{8} = 1 - \frac{p}{8}$, which is negative or zero for $p \geqslant 8$, so we would have $m \geqslant 1 - \frac{p}{8} \leqslant \frac{(3-2)^2}{4} = \frac{1}{4}$, which is not very helpful. However, for $n \geqslant 4$, $1 - \frac{p(n-4)}{8}$ is either positive or zero and then both inequalities agree. Thus, a consequence of the above lemma is that there are no n-maps on the plane for $n \geqslant 4$.

Theorem 18 (Craft and White)**.** *(4,m) is only realizable if and only if* $m \geqslant 1$.

Proof. Suppose (4,m) is realizable. Then, by Lemma 2, it must be that $m \geqslant 1$. Conversely, suppose that $m \geqslant 1$. We will show that $(4, m)$ is realizable by induction. Figure 8.2 shows that $m = 1$ is realizable. Now, suppose that $(4, k)$ is realizable up to k. By the base case and Theorem 17, $(4, k+1)$ is also realizable. $\square$

Theorem 19 (Craft and White)**.** *(5,m) is realizable if and only if* $m \geqslant 3$

Proof. Suppose (5,m) is realizable. Then, by Lemma 2, m must be $\geqslant 3$.
Now, sppose $m \geqslant 3$. We will construct a (5,m) map to a similar process as with the Cartesian product constructions. Begin with $m-1$ identical copies of the 4-map of $K_{4,4}$ imbedded on S_1 shown in Figure 8.2. Denote these copies $M(1), M(2), ..., M(m-1)$. Denote the two regions colored 2, with edges 4 and 1 as A and B. Join A in $M(i)$ to B in $M(i{+}1)$ (adding mod m-1 so that A in $M(m{-}1)$ is joined to to B in $M(1)$). In this case, we do not need to worry about regions of any particular size, so when joining the interiors of these regions, it is sufficient to join vertices so long as they are in different partite sets so that our final imbedding is also bipartite. $m-2$ of these handles join the copies of S_1 to produce S_{m-1} and the final tube adds one handle to produce an imbedding on S_m, as desired. Since the new edges occur between edges colored 1 and edges colored 4,

we may color these new edges 5. This produces a canonical edge coloring and so our imbedding is a 5-map. □

Theorem 20 (Craft and White)**.** *(6,m) is realizable if and only if* $m \geqslant 4$

Proof. Suppose (6,m) is realizable. Then, by Lemma 2, $m \geqslant 4$. Next, suppose $m \geqslant 4$. We will use the dipole graph of Figure 8.4 to construct imbeddings of $K_{6,6}$ on S_4, S_5, S_6, and S_7. With these surfaces, we will have surfaces with genus numbers of the form $4, 4+1, 4+2, 4+3$, so we can add them using Theorem 17 to produce surfaces of any higher genus.

For S_4, we will use $\Gamma = D_3$ and alternate reflections and rotations along the edges of the voltage graph so that the edges read: $id = r_0, s, r_1, sr_1, r_2, sr_2$. This will produce a quadrilateral imbedding (for more detail on the calculation of these regions, see the proof of Theorem 21 below). This particular voltage graph will also produce a single copy of our graph (for an explanation, see Theorem 20). $p = 12, q = 36$. We can use the formula for quadrilateral imbeddings to find that the imbedding is on a surface of genus $36/4 - 12/2 + 1 = 4$.

For S_5 we will still use $\Gamma = D_3$ but the edges of the voltage graph will be labeled $r_0, s, r_1, sr_1, sr_2, r_2$. Using part 3. of Theorem 5, we will calculate the number of regions. Recall that the dipole voltage graph with 6 edges will have 6 regions (the region between any two edges and the outer region). Starting with the region between r_0 and s (R_1) and ending with the outer region (R_6), the following list shows the voltages around each region (denoted R_ϕ) and their orders (denoted $|R_\phi|$):

$R_{1,\phi} = r_0 s, |R_{1,\phi}| = 2$

$R_{2,\phi} = sr_2, |R_{2,\phi}| = 2$

$R_{3,\phi} = r_1 sr_1, |R_{3,\phi}| = 2$

$R_{4,\phi} = sr_1 sr_2 = r_2 ssr_2 = r_2 r_2 = r_1, |R_{4,\phi}| = 3$

$R_{5,\phi} = sr_2 r_1 = s|R_{5,\phi}| = 2$

$R_{6,\phi} = r_0 r_1 = r_1, |R_{6,\phi}| = 3$

Following the formula from Theorem 5, this gives us a total of 16 regions. $2 - 2h = p - q + r = 12 - 36 + 16$ and $h = 5$.

For S_6, we will still use $\Gamma = D_3$ but label the edges $r_2, r_0, r_1, s, sr_1, sr_2$. Starting with the region between r_2 and r_0 and ending with the outer region, the voltages around each region and their orders are: $R_{1,\phi} = r_2 r_0 = r_2, |R_{1,\phi}| = 3$

$R_{2,\phi} = r_0 r_2 = r_2, |R_{2,\phi}| = 3$

$R_{3,\phi} = r_1 s, |R_{3,\phi}| = 2$

$R_{4,\phi} = ssr_1 = r_1, |R_{4,\phi}| = 3$

$R_{5,\phi} = sr_1 sr_2 = r_2 ssr_2 = r_2 r_2 = r_1, |R_{5,\phi}| = 3$

$R_{6,\phi} = r_2 sr_2 = sr_1 r_2 = s, |R_{6,\phi}| = 2$

Theorem 5 tells us that this imbedding has 14 regions, so $2-2h = p-q+r = 12-36+14$ and $h = 6$.

Finally, for S_7, we will use $\Gamma = \mathbb{Z}_6$ and label the edges with $0, 2, 4, 1, 3, 5$. Starting with the region between 0 and 2 and ending with the outer region, the voltages around each region and their orders are:

$R_{1,\phi} = 0 + 4 = 4, |R_{1,\phi}| = 3$

$R_{2,\phi} = 2 + 2 = 4, |R_{2,\phi}| = 3$

$R_{3,\phi} = 4 + 5 = 9, |R_{3,\phi}| = 2$

$R_{4,\phi} = 1 + 3 = 4, |R_{4,\phi}| = 3$

$R_{5,\phi} = 3 + 1 = 4, |R_{5,\phi}| = 3$

$R_{6,\phi} = 0 + 1 = 1, |R_{6,\phi}| = 6$

Using Theorem 5, we see that we have 12 regions, so $2 - 2h = 12 - 36 + 12$ and $h = 7$.

We have not yet shown that each of these imbeddings is an n-map. First, all of the voltage graph labellings used in this proof will create a single graph. This is because for any element g of D_3 either or $\mathbb{Z}_6$, there exists a second element h such that $gh = l$, where l is any element of the group Γ. In other words, it is possible to move from any element to any other by a single group operation. Since the edges of the voltage graph contain all possible group elements of either D_3 or $\mathbb{Z}_6$, no matter which subscript an a vertex has, that a vertex is adjacent to every b vertex in the covering graph (and vice versa for b vertices).

Next, number the labels of the edges of the voltage graph from left to right. For any fixed labelling, color the edges in the covering graph generated by that edge by the number of the label. By looking at the dipole voltage graph, we can see that b vertices will be clockwise and a counter clockwise. In the covering graph, this combined with the edge colors will induce a canonical edge coloring. The only regions in the covering graph are those produced by the digons, which due to the coloring process above will all be bi-colored. The regions are all cycles because they are digons in the voltage graph. Thus, once we return to the original vertex there is no where else to go but to start the cycle over again. Thus, by Theorem 14 (b), we have an n-map.

□

Theorem 21 (Craft and White)**.** *For* $n \geqslant 1$, $K_{2n,2n}$ *has a quadrilateral imbedding as a 2n-map on* $S_{(n-1)^2}$, *so* $(2n,(n-1)^2)$ *is realizable.*

Proof. For n = 1, $K_{2,2}$ is C_4, which is planar and forms a 2-map with quadrilateral imbedding. For n = 2, Figure 8.2 shows $K_{4,4}$ as a 4 map on S_1. Now, suppose $n \geqslant 3$. Consider the dipole voltage graph in Figure 8.4. Let $\Gamma = D_n$ so that the voltage graph has $2n$ edges. Alternate rotations and reflections so each digon region in the voltage graph will be between an edge of the form r_i and sr_i (if we think of the identity as r_0) except the outer region which will have as edges r_0 and sr_n.

Suppose (b, r_j) is some vertex in the covering graph. Consider an arbitrary region in the voltage graph (except the outside region). Then, the region generated in the covering graph will be $(b, r_j) - (a, r_{j+i}) - (b, r_{j+i}sr_i) = (b, sr_{n-j-i+i}) = (b, sr_{n-j}) - (a, sr_{n-j+i}) - (b, sr_{n-j+i}sr_i) = (b, r_{n-n+j-i}ssr_i) = (b, r_j)$.

The outer region would generate $(b, r_j) - (a, r_j) - (b, r_jsr_n) = (b, sr_{2n-j}) - (a, sr_{2n-j}) - (b, sr_{2n-j}sr_n) = (b, r_{j-n}ssr_n) = (b, r_j)$.

Using $(b, sr_j), (a, r_j)$, or (a, sr_j) would produce similar regions. In all cases, we can see that the regions are quadrilateral. In addition, the voltage graph only generates one component. To see this, recall that the elements of D_n can be viewed as reflections and rotations of an n-gon. Starting from any position, we can reach any other by either rotating the appropriate number of times or reflecting and then rotating the appropriate number of times. In either case, starting with any element of D_n, we can reach any other element by adding a single group element. Since each a or b vertex is adjacent to an edge with every voltage, we can see that every a vertex is adjacent to every b vertex and vice versa.

Number the elements of D_n so that $r_0 = 1, s = 2, r_1 = 3, ...s_n = 2n$ and color the edges in the covering graph accordingly. As can be seen from the voltage graph, this will produce a canonical rotation, all regions will be bi-colored, and the only bi-colored cycles will be regions. Thus, we will have a 2n map. $p = 4n, q = 4n^2$ Using the formula for quadrilateral imbeddings, we find that the graph is imbedded on a surface of genus $4n^2/4 - 4n/2 + 1 = n^2 - 2n + 1 = (n-1)^2$. □

Chapter 9

Historical Theorems and Connections to 4-color Theorem

Theorem 22 (Vizing)**.** *For every nonempty graph* G*,* G *can be edge colored with either* $\Delta(G)$ *or* $\Delta(G)+1$ *many colors, where* $\Delta(G)$ *is the maximum degree of* G*.*

Denote the mimimum number of colors required to edge-color a graph, G by $\chi'(G)$. $\chi'(G)$ must be at least $\Delta(G)$ because if we use fewer than $\Delta(G)$ colors, a vertex with degree Δ would be incident with two edges of the same color.

Theorem 23 (König, also see Chartrand, Lesniak, Zhang [3] pgs. 456 - 457)**.** *A cubic bipartite graph is 3-edge colorable. More generally, if* G *is a nonempty bipartite graph, then* G *can be colored with* Δ*-edge colored.*

Proof. We will prove the more general version of this theorem because it is also used for n-maps, $n \geqslant 3$. Suppose, to the contrary, that the theorem is false. Choose a bipartite graph G such that among all bipartite graphs which cannot be edge-colored with Δ colors, G has the mimimum size. Then, by Vizing's Theorem, G can be colored with $\Delta(G)+1$ many colors. Choose an arbitrary edge of $G, e = uv$. Delete this edge and call the resulting graph $(G-e)$. Since G was of mimimum size among counterexamples, $\chi'(G-e) = \Delta(G-e)$. Also, $\Delta(G-e) = \Delta(G)$. If this was not the case, that would mean that u and v were the only vertices of degree $\Delta(G)$ in G, so every vertex of $(G-e)$ must have degree $< \Delta(G)$. But then, since $\chi'(G-e) = \Delta(G-e) = \Delta(G) - 1$, we can color $(G-e)$ with $\Delta(G-e)$ colors and then color e with the remaining color to form a $\Delta(G)$-edge coloring of G, contradicting our assumption.

Since $\chi'(G-e) = \Delta(G-e)$ and $\Delta(G-e) = \Delta(G)$, we can color $(G-e)$ with $\Delta(G)$ many colors. Each of the $\Delta(G)$ colors must be assigned to an edge incident with either u or v. If not, we could color e with the missing color and obtain a $\Delta(G)$-edge coloring of G. Also, deg(u) and deg(v) are both $< \Delta(G)$, so it must be that one of the edges incident with u is assigned a color used by no edge incident with v and vice versa. Call these colors α and β, respectively.

Let P be a path starting at v which alternates edges of color β and α. Furthermore, let P be a path of maximum length with this property. This ensures that every vertex of P, including the final vertex are not adjacent to any edge colored α or β outside the path. Thus, we can change every α-edge to a β-edge and vice versa without disrupting the proper region coloring. Since v and u are now adjacent to no edge colored β we could color e β, giving a $\Delta(G)$-edge coloring of G towards a contradiction. There is only one problem. P might terminate at u, in which case, the color swapping would cause u to be adjacent to an edge colored β and the contradiction would fail. However, this is impossible. Suppose P did terminate at u. Then, since the final edge of P would have a different color than the initial edge of P, P must have an even number of edges and hence an odd number of vertices. But then, in G, $P + e$ would form an odd cycle in a bipartite graph, a contradiction. Thus, we can go ahead with our originally planned contradiction and the proof is complete. □

Theorem 24 (Grötzsch)**.** *Every plane graph with no triangles is 3-vertex colorable.*

A simplified proof of this theorem plus preliminaries given in [2] is roughly 16 pages, so we will not prove this theorem here.

Theorem 25. *(see Wilson [9] pg. 92) If every cubic plane map is 4-region colorable, then every* *plane map is 4-region colorable.*

Proof. Suppose that every cubic plane map is 4-region colorable and that G is a plane map (not necessarily cubic). Since G is a map, G has no vertices of degree 1 (because then the edge incident with that vertex would not separate two regions). For any vertex of degree larger than 3, truncate those vertices. As described in the proof of Proposition 4, truncation will replace a vertex of degree d with d vertices of degree 3. Moreover, truncation will not change the adjacencies and non-adjacencies of the original regions of G. If G has any vertices of degree 2, these lie at the intersection of only two edges, so we can remove the vertices and leave the edges unchanged. This may create unusual structures like edges with no vertices or multiple edges as shown in Figure 9.1. Nevertheless, even though our map may no longer be a graph, it will still have the same structure as a map. Now, we have replaced G with a cubic plane map and, by assumption, this new map is 4-region colorable. Since transforming G into a cubic map did not change the structure of the regions, we now undo the truncation and removal of vertices and return to G, keeping the region colors the same. This provides a 4-region coloring of G. □

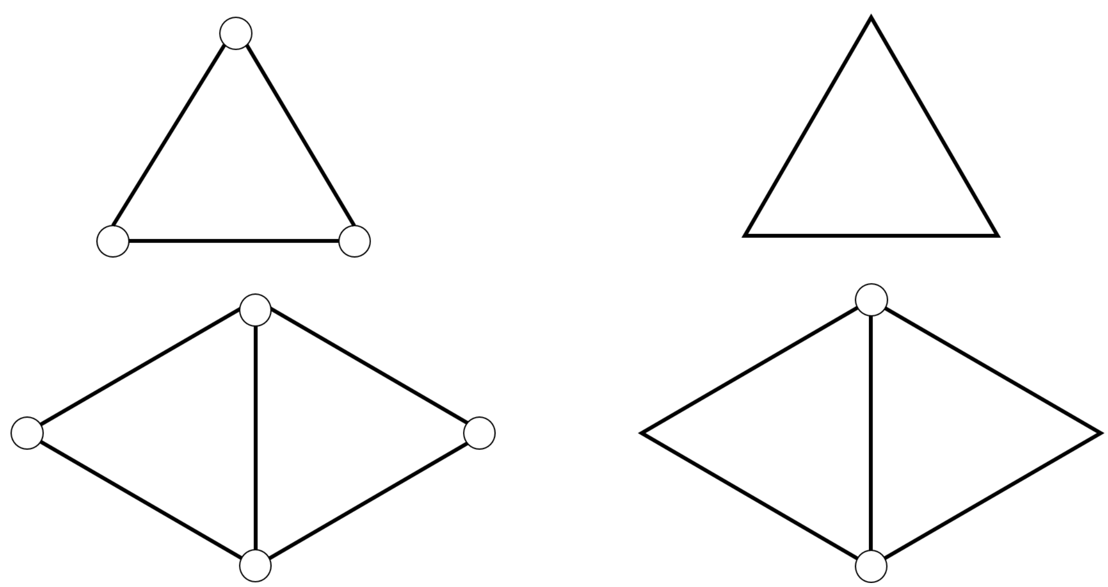

Figure 9.1: Removing vertices of degree 2 from a plane map

Theorem 26 (Tait [6])**.** *Let G be a bridgeless cubic plane graph. Then G can be 3-edge colored if and only if G can be 4-region colored.*

Theorem 27 (Tait)**.** *Let* G *be a cubic graph (not necessarily planar). Then* G *can be 3-edge colored if and only if* G *is spanned by a collection of disjoint cycles of even length.*

The following theorem comes from combining the previous two theorems of Tait.

Theorem 28. *Let* G *be a bridgeless cubic plane graph. Then* G *can be 4-region colored if and only if* G *is spanned by a collection of disjoint cycles of even length.*

Theorem 29 (Heawood [5]; Also Chartrand, Lesniak, Zhang [3] pg. 377 - 378)**.** *A plane triangulation can be 3-vertex colored if and only if all vertices have even degree.*

Proof. $\Rightarrow$ Contrapositive. Suppose that G is the underlying graph of a plane triangulation but G has some vertex, v of odd degree. Since all regions are triangles, v and its neighbors appear as in Figure 9.2. The neighbors of v form an odd cycle, which has chromatic number 3. Thus, v must be colored by a fourth color and the chromatic number of $G, \chi(G)$ is at least 4.

$\Leftarrow$ We will use induction on the order of plane triangulations with all vertices even. Base Case: The smallest such graph is K_3, with $p = 3$ and $\chi(K_3) = 3$. Assume that all plane triangulations of order less than or equal to k with vertices of even degree are 3 vertex colorable. Suppose G is a plane triangulation with even vertices of order $k + 1$. Consider an arbitrary edge in G, uw. Since G is a triangulation, uw is on the border of two triangular regions. Suppose x and y are the remaining vertices in the regions to the right and left of uw, respectively, as shown in the top left corner of Figure 9.3. Since the graph is a triangulation, the neighbors of x form a cycle $C = (u = x_1, x_2, ..., x_s = w, x_1)$ and likewise the neighbors of y: $C' = (u = y_1, y_2, ..., y_t = w, y_1)$. Since all vertices have even degree, both s and t are even.

Now, delete both x and y as well as the edge uw, leaving a region with $s+t-2$ sides (since we only count u and w once in the new region) as shown in the top right corner of Figure 9.3. Now, add a new vertex z to the center of this region and join z to every vertex on the boundary of the region, as shown in the bottom right of Figure 9.2. This will triangulate the region. We deleted one edge from every vertex on the boundary of the region and then, with z, added one edge, so all these vertices have even degree. Furthermore $s+t-2$ is an even number so z also has even degree. This new graph, G' is a plane triangulation with even vertices of order k, so, by our induction hypothesis, G' is 3-vertex colorable. Consider an arbitrary 3-vertex coloring of G'. The vertices adjacent to z are part of a cycle, so they will alternate two colors. z itself must be a third color, as shown in Figure 9.3. Leaving the colors of all vertices of G' unchanged, remove z and reattach x and y as before. We may color x and y the same color as G and G now has a 3-vertex coloring.

□

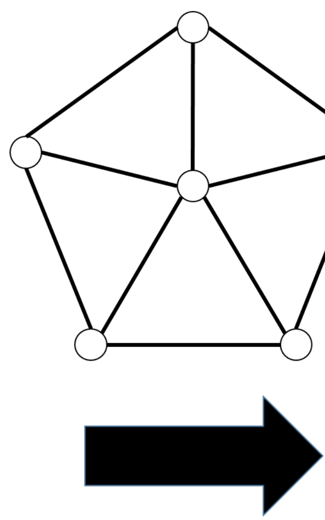

Figure 9.2: Heawood's theorem on plane triangulations

Theorem 30. *In dual form, Heawood's theorem states that a plane cubic map can be 3-region colored if and only if all regions lengths are even.*

Proof. Consider a plane cubic map G. The dual of G, G', which will be a plane triangulation. $\Rightarrow$ Suppose G can be 3-region colored. Using the same colors, G' has a 3-vertex

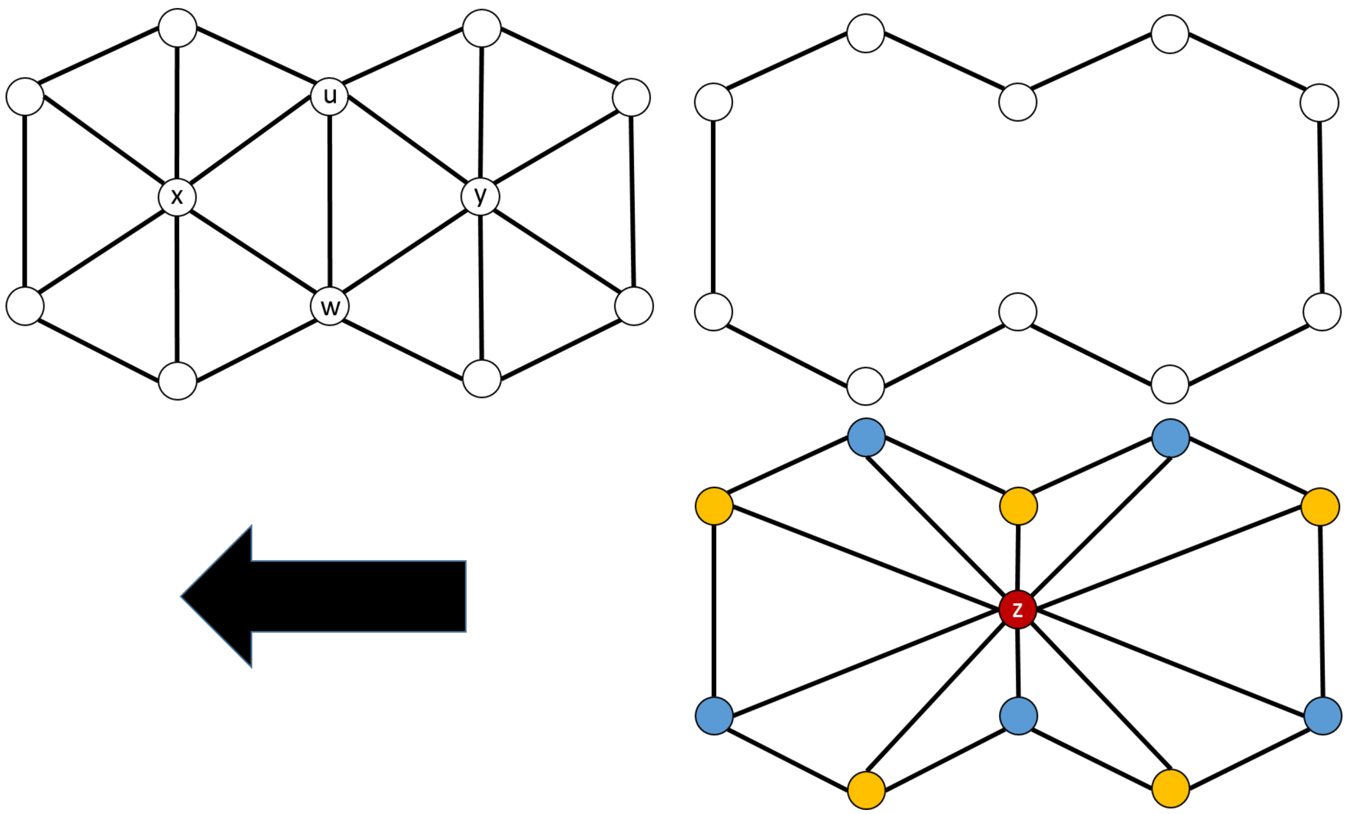

Figure 9.3: Heawood's theorem on plane triangulations

coloring By Heawood's theorem, all vertices of G' have even degree. Therefore, G must have had all regions of even length.

$\Leftarrow$ Suppose all region lengths of G are even. Then, G' has all vertices of even degree. By Heawood's theorem, G' has a 3-vertex coloring. Color the regions of G with the same colors for a 3-region coloring of G. $\square$

Lemma 3. *A plane graph is bipartite if and only if all region lengths are even.*

This lemma is false for surfaces of higher genus. For example, $C_3 \square C_3$ is not bipartite, yet has a quadrilateral imbedding on S_1 as shown in Figure 9.3 [4].

Theorem 31. *A plane cubic map can be 3-region colored if and only if it is bipartite.*

Proof. This theorem comes from combining the dual form of Heawood's theorem with Lemma 3. $\square$

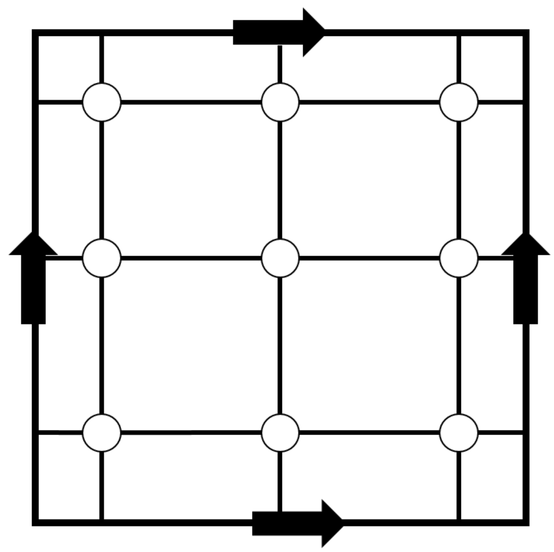

Figure 9.4: $C_3 \square C_3$ on S_1

Chapter 10

Open Questions

All open questions taken from [4]

(1) In reference to Theorem 6, characterize those partitions of the vertex set of a cubic map into two, equal-sized classes A and B for which A is the set of clockwise vertices B is the set of counterclockwise vertices for some region coloring.

(2) For a given (p,n) characterize those feasible multisets which are realizable, for a 3-map of order p on S_n.

(3) Is there an intermediate value theorem for 3-maps; that is, if G has a 3-map on both S_n and S_m, with $n < k < m$, does G have a 3-map on S_k?

(4) For a given cubic bipartite graph G, find all n for which G has a 3-map on S_n. For example, if $G = K_2 \square C_6$, then G has a 3-map precisely when $0 \leqslant n \leqslant 2$. G has no 3-map on $S_n, n \geqslant 3$ because then the map would have less than 3 regions.

(5) Define the 3-map genus of a cubic bipartite graph G as $\gamma_{3m}(G)$ is the minimum n such that G has a 3-map on S_n. For example, $\gamma_{3m}(G)(K_2 \square C_6) = 0$. Study this parameter. Since $q = \frac{3p}{2}$ and $2q \geqslant 4r$ (this inequality comes from Corollary 1 of Theorem 3), $\gamma_{3m} = p - q + r \leqslant p - \frac{3p}{2} + \frac{3p}{4}$. Simplifying shows that $\gamma_{3m} \geqslant 1 - \frac{p}{8}$. This is not very helpful except to check that $2q = 4r$ if and only if $p = 8$.

(6) Likewise, define the 3-map maximum genus as: $\gamma_{3M}(G)$ is the maximum n such that

G has a 3-map on S_n. For example, $\gamma_{3M}(K_2 \,\Box\, C_6) = 2$. Study this parameter too. A 3-map must have at least 3 regions, so $r \geqslant 3$ and $2 - 2\gamma_{3M} = p - q + r \geqslant p - q + 3$. After some algebra, we find that $\gamma_{3M} \leqslant \frac{p-2}{4}$. Note that using this inequality for $(K_2 \,\Box\, C_6)$, we find that $\gamma_{3M}(K_2 \,\Box\, C_6) \leqslant 2\frac{1}{2}$. Since γ_{3M} must be an integer, we round down to 2 and see that the upper bound is attained.

(7) Continue the study of n-maps.

Chapter 11

A Note on Edge Coloring

In Theorem 7, we assigned each 3-map a *canonical* edge coloring. Is this the only possible edge coloring for a 3-map? No. In fact, it is possible for a 3-map to have a 3-edge coloring that is exactly opposite to the canonical bipartition where clockwise vertices have counterclockwise edge colorings and vice versa. Figure 11.1 shows such an edge coloring (remember that red is color 1, blue is color 2, and yellow is color 3).

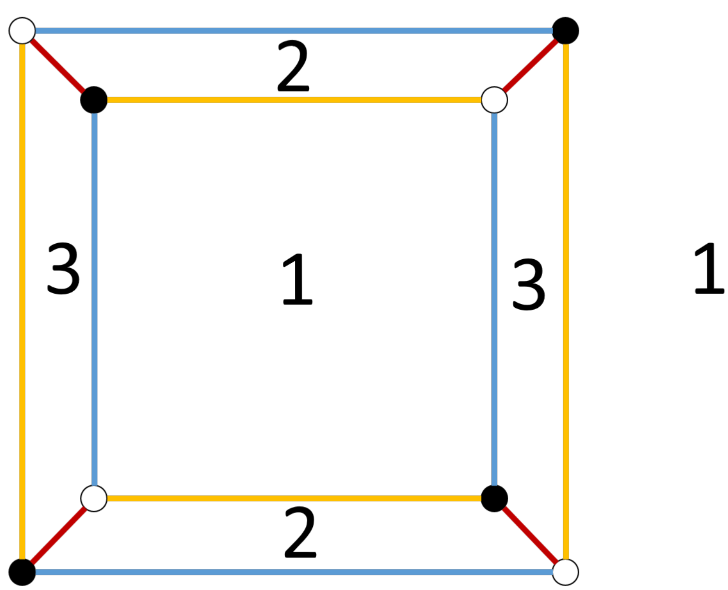

Figure 11.1: An edge coloring that is opposite to the canonical bipartition

Bibliography

[1] G. Abay-Asmerom, Graph Products and Covering Graph Imbeddings, Ph.D. Dissertation, Western Michigan University, 1990.

[2] N. Asghar, Grötzsch's Theorem, Master's Thesis, University of Waterloo, 2012.

[3] G. Chartrand, L. Lesniak, and P. Zhang, Graphs and Digraphs, CRC Press, Florida, 5th edition, 2011.

[4] D. L. Craft and A. T. White, 3-maps, *Discrete Math.* **309** (2009) 5857-5869.

[5] P.J. Heawood, On the four-colour map theorem, *Quart. J. Pure Appl. Math.* **29** (1898) 270-285.

[6] P.G. Tait, On the colouring of maps, *Proc. Roy. Soc. Edinburgh Sect. A 10* (1880) 501-503. 729 (1878-80)

[7] A. T. White, Graphs of Groups on Surfaces: Interactions and Models, Elsevier, Amsterdam, 2001.

[8] A.T. White, The Genus of Repeated Products of Bipartite Graphs, *Trans. Amer. Math. Soc.* **151** (1970) 393 - 404.

[9] R.J. Wilson, Introduction to Graph Theory, Longman, New York, 1985.

Vita

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