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Elaborations on Multiattribute Utility Theory Dominance

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Elaborations on Multiattribute Utility Theory (MAUT) Dominance

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Virginia Commonwealth University

by

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List of Abbreviations

CPT – Cumulative Prospect Theory

EU – Expected Utility

MAUT – Multiattribute Utility Theory

OSPT – One-Switch Prospect Theory

RDU – Rank Dependent Utility

Abstract

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Multiattribute Utility Theory (MAUT) is used to structure decisions with more than one factor (attribute) in play. These decisions become complex when the attributes are dependent on one another. Where linear modeling is concerned with how factors are directly related or correlated with each other, MAUT is concerned with how a decision maker feels about the attributes. This means that direct elicitation of value or utility functions is required. This dissertation focuses on expanding the types of dominance forms used within MAUT. These forms reduce the direct elicitation needed to help structure decisions. Out of this work comes support for current criticisms of gain/loss separability that is assumed as part of Prospect Theory. As such, an alternative to Prospect Theory is presented, derived from within MAUT, by modeling the probability an event occurs as an attribute.

Summary

Multiattribute Utility Theory (MAUT) is used to structure decisions with more than one factor (attribute) in play. These decisions can become extremely complex when the attributes are dependent on one another. For example: consider a decision maker is attempting to decide on a new car to purchase and she wants to take into account a number of different attributes of the car that are important to her. Ultimately she is looking for a balance of features versus cost along with good gas mileage and enough room to fit her family. In this example, as features are added the price of the car will go up. Potentially, as gas mileage increased the price could also rise. Larger cars also carry a larger price tag and tend to have worse gas mileage than other cars.

Where linear modelling is concerned with how these are directly related/correlated with each other, MAUT is concerned about how a decision maker feels about these attributes. Are they willing to pay more for certain features? If will to spend some amount of money to increase gas mileage from 30 miles per gallon (mpg) to 35 mpg, are they willing to spend the same amount to go from 35 mpg to 40 mpg, or is there some diminished return? The assessment relies on figuring out which combination of attributes effect tradeoffs for the other attributes. For example: a decision maker would need to assess if the price of the car changes how they feel about tradeoffs on gas mileage or if types of features effect how they feel about tradeoffs on price.

Within MAUT there currently exists a concept of utility dominance (Abbas and Howard 2005). Said simply: utility dominance exists if total decision utility is forced to zero if one attribute is at its minimum. That is, nothing else matters. In the car example, if a person does not care what features exist or how cheap the car is, they will not take a car that seats less than five people, then when that attribute is at four or less, the entire utility is set to zero.

The dissertation expands dominance forms to include maximum and intermediate values; defines the relationship of these new dominance forms to known forms of independence among the

attributes; and shows the result of modelling the probability an event occurs as an attribute within this dominance form. This is largely in-line with the original proposal, which laid out four areas of related research:

1. Modeling chance as an attribute in decisions under risk
2. Exploration of max-dominance in MAUT
3. Modeling chance as an attribute in multi-attribute decisions under risk and modeling chance in decisions under uncertainty
4. Decision model selection – an application of the generalized linear model

The final dissertation is laid out into the follow areas:

1. Max-dominance
2. Mid-dominance
3. Application of mid-dominance, including:
    a. Modeling chance as an attribute
    b. Expansion to gambles with n-outcomes

I originally proposed to cover the new concept of max dominance and the final document does this. I also introduce mid-dominance as a new concept that was not originally proposed. This concept is complex and is usable in a wide variety of scenarios. I also show that modeling chance as an attribute is in fact a special case of this broader mid-dominance concept. This is expanded farther than originally thought with the introduction of restrictions on the representing function. As proposed, I was able to expand this concept to n-outcomes. Though the discussion is restricted to known probabilities, the underlying structure is decision making under uncertainty; so I do not take advantage of anything that decision making under risk would have provided. Additionally, with the implementation of cumulative probabilities (Quiggin 1982, 1991), I was able to develop a rival theory to prospect theory (by showing that the form does not support full gain/loss separability (Kahneman and Tversky 1979, Tversky and Kahneman 1992). Originally I was seeking to give further support to prospect theory; however, instead must now conclude that one of its underlying assumptions appears to be incorrect.

I did not include any material on the use of the generalized linear model. The original thought is that elicited values for a series of utility functions could be modelled to measure how close an individual is to a known form of independence. This was abandoned for a few reasons:

1. I believe the elicitation needed to perform this type of procedure would not lend itself to usability.
2. Other methods exist for assessing the relationship among attributes (e.g. swing weighting) and these methods are effective.
3. The most complex form of the utility function for two attributes requires the elicitation of: two marginal utility functions, a conditional utility function, and up to four corner points. This is not overly cumbersome and I feel there is a lack of need to move the research in this direction.

Overall the contribution to the field can be broken into four specific areas:

1. Introduction of max-dominance as a new concept to better define the relationship among attributes and to cut the amount of work needed to assess utility values. This includes:
    a. Exploration and definition of the relationship between max-dominance and:
        i. Min-dominance (Abbas and Howard 2005)
        ii. Utility-independence (Keeney and Raiffa 1976)
        iii. One-switch independence (Abbas and Bell 2011, Tsetlin and Winkler 2012)
    b. Exploration and definition of the relationship between full-dominance and:
        i. Utility-independence
        ii. One-switch independence

2. Introduction of strong one-switch independence as a new concept that is free from utility-independence and an exploration of its impacts
3. Introduction of mid-dominance as a new concept
    a. Exploration and definition of the relationship between max-dominance and:
        i. Utility-independence
        ii. One-switch independence
    b. Direct counterexample to a proposition by Dr. Abbas that if an attribute x is utility independent of another attribute y and y is boundary independent of x, then y is utility independent of x. I show that while x is utility independent of y and y is boundary independent of x, y can be strong one-switch independent of x.
4. Derivation of a representing function on the standard gamble by modelling the probability as an attribute
    a. Includes a rescaling to have zero gain/loss be represented by a utility value of zero
    b. Includes a theorem that restricts the shape of the representing function on x because of strong one-switch independence
    c. Includes an expansion to gambles with n outcomes which allows us to:
        i. Establish a separate form of prospect theory derived from a multi-attribute framework
        ii. Directly refute the assumption of full gain/loss separability that exists in prospect theory.

The impacts can be simplified as follows. Multiple new concepts were introduced that change the way decision makers may look at the relationship among attributes. Included is a direct rebuttal of a current theorem in the literature around the concept of boundary independence. These concepts expand our understanding and offer new ready to use forms that cover families of decisions and make assessment easier. Additionally, this work formalizes the value of chance as a decision attribute. This was nodded at with the development of Prospect Theory and other decision models; however, the assessment of it within the MAUT is completely novel. This approach allowed us to refute a simplifying assumption in Prospect Theory (gain/loss separability) (Wu and Markle 2008, Bromiley 2010). This allowed us to develop a competing form of Prospect Theory without this problematic assumption.

Introduction

Expected Utility (EU) Theory is the leading prescriptive theory for choice under uncertainty. EU consists of a representing function on the probabilities and outcomes of events that is linear in the probabilities and applies a utility function to the outcomes to represent risk attitude. Multi-Attribute Utility Theory (MAUT) presents an expansion of EU by allowing for more than one attribute to be analyzed over the same series of events. Difficulty arises in defining the joint utility function over all attributes because the individual attributes are frequently dependent on each other. The following chapters cover multiple attempts to quantify the relationships among attributes and by extension reduce the work associated with eliciting a joint utility function. Additionally, I bring in concepts from other related modelling methods that have attempted to go beyond EU rationality to integrate inconsistences and paradoxes of human decision making.

Most theory will be focused on two attribute decisions, though some attention will be paid to decisions involving more than two attributes when discussing background material to show how expansion is possible. Where possible, this document covers examples and applications as a way to show how these formulas help decision analysts and decision makers in practice.

Chapter 1 covers some of the prior work in the field of MAUT. This include the concepts of utility independence, one-switch independence, attribute dominance, and briefly multi-attribute utility trees. The chapter paints a broad picture so that the work that follows can be viewed in the proper context. Chapter 2 focuses on redefining attribute dominance and challenging assumptions made about utility function bounds. Specifically, it introduces the new concepts of max-dominance and full-dominance. Chapter 3 introduces dominance at non-extreme values. Called mid-dominance, this work allows for a better modelling of gains and losses, by interpreting reference dependence as dominance.

Following this, Chapter 4 offers a new application of mid-dominance, by applying MAUT to single attribute decisions. In this way, the probability an event occurs is modeled as an attribute, with mid-dominance and one-switch independence forming that base of that work. The conclusion of Chapter 4 includes support for criticisms of full gain/loss seperability used in Cumulative Prospect Theory (CPT) and offers an alternative form of the theory. Finally, Chapter 5 offers some insights into potential future work.

Chapter 1: Literature Review

This chapter introduces key concepts in MAUT. Additional topics will be introduced as needed later; however, the definitions presented here form the backbone of all of the work to follow. The attempt is to be simultaneously as brief and as complete as possible for general understanding of utility independence, one-switch and n-switch independence, and attribute dominance.

To start a utility function over two attributes, x and y is defined as $u(x,y)$. This function forms a mapping $\mathbb{R}^2 \to [0,1]$, though in some future cases I will argue for a more relaxed mapping. When discussing two attributes, I will refer to them as above; however, when adding a third attribute, I will introduce it as z. Introducing z here is for illistrutive purposes to show the complexity of adding a third attribute. Note that only two attributes are used throughout Chapters 2 and 3, though the theory behind the use of more than three attributes is used in Chapter 4. For those instances where I show generalizations to more than three attributes, I switch to an slightly different syntax, instead using $x_1,\ldots,x_n$. I choose not to use this for three attributes and fewer due to concerns about readability and the introduction of multiple indices. The values for any attribute x and y exist such that $x \in \left[x^0, x^*\right]$ and $y \in \left[y^0, y^*\right]$. Unless otherwise stated, $u\left(x^0, y^0\right) = 0$ and $u\left(x^*, y^*\right) = 1$ without loss of generality, reflecting a 'more is better' approach. Two additional corner points are defined as $u\left(x^0, y^*\right) = k_x$ and $u\left(x^*, y^0\right) = k_y$.

A conditional utility function over two attributes is defined as follows:

(1.1) $$u(x \mid y) = \frac{u(x,y) - u\left(x^0, y\right)}{u\left(x^*, y\right) - u\left(x^0, y\right)}$$

This conditional function acts very much like a conditional probability in that it shifts the reference point for the function and standardizes it to be between zero and one.

The first independence topic is essential to MAUT and was covered in depth by Keeney and Raiffa in their seminal work Decisions with Multiple Objectives in 1976: Utility independence offers a benchmark against which some other forms of independence can be measured and significantly reduces the work required to define a joint utility.

Definition: As updated in Abbas and Bell (2011) an attribute x is utility independent of an attribute y if preferences for gambles on x, for a fixed level of y, do not depend on the level to which y is set. Said another way, x is utility independent of y if when y is fixed at any y', there exist functions $f(y)$ and $g(y) > 0$, such that:

(1.2)
$$u(x,y) = f(y) + g(y)u(x,y')$$

As the above shows, if x is utility independent of y, the joint utility function can be written as a linear combination of functions of y. This means that the attribute y scales the attribute x; however, preferences across x are not dependent of y.

What's more, by fixing $y = y^0$ in (1.1), combining with (1.2), and solving for $u(x,y)$: if x is utility independent of y then:

(1.3)
$$u(x,y) = u(x^0,y) + \left[u(x^*,y) - u(x^0,y)\right]u(x \mid y^0)$$

If both x and y are utility independent of each other, elicitation becomes far easier. Keeney and Raiffa (1976) covered this topic in depth and defined:

Definition: If both x is utility independent of y and y is utility independent of x, the attributes are said to be <u>mutually utility independent</u> and the joint utility takes the <u>multilinear</u> form, which is as follows:

$$u(x,y) = k_x u(x \mid y') + k_y u(y \mid x') + k_{xy} u(x \mid y') u(y \mid x') \tag{1.4}$$

for all $x' \in \left[x^0, x^*\right]$, all $y' \in \left[y^0, y^*\right]$, and where $k_{xy} = 1 - k_x - k_y$. For elicitation purposes, x' and y' are traditionally set at extreme values; however, any value that is convenient will work. Additionally, the marginal functions used in the multilinear form are by far the simplest to elicit. It is easy to see that two special forms come out of the multilinear form. If $k_{xy} = 0$, the utility function is additive. Additionally, if $k_x = k_y = 0$, the utility function is multiplicative. Note that in the multilinear form k_{xy} can be negative.

1.1: One-Switch Independence:

This section in its entirety, unless otherwise stated, is taken from Abbas and Bell (2011). To start, it is useful to supply a definition for zero-switch independence:

Definition: An attribute x is <u>zero-switch independent</u> from an attribute y if preferences for gambles on x does not change as the fixed level of y within the gambles is changed.

It should be clear that utility independence and zero-switch independence each imply the other. Despite this, the use of zero-switch independence as a concept has not replaced utility independence in the literature; however, it provides a useful insight into the actual relationship between the two attributes. For example, the addition of a monotonicity condition to each of the marginal preferences indicates that zero-switch independence of x from y implies one-switch independence of y from x. This means if preferences on x do not switch as the fixed level of y increases, monotonicity enforces that preferences on y will switch no more than once as the fixed level of x increases. A proof for this provided in Abbas and Bell (2011). A formal definition for one-switch follows:

Definition: An attribute x is <u>one-switch independent</u> from an attribute y if preference between any pair of gambles on x can switch at most once as y increases in level.

The following theorem provides a functional form for a one-switch independent attribute.

Theorem: x is one-switch independent from an attribute y if and only if:

(1.5) $$u(x,y)=g_0(y)+f_1(x)g_1(y)+f_2(x)g_2(y)$$

with $g_1(y)\neq 0$, $\frac{d}{dy}g_1(y)>0$ or $\frac{d}{dy}g_1(y)<0$, and $g_2(y)=g_1(y)\phi(y)$ where $\phi(y)$ is monotonic (Abbas and Bell 2011).

They took the above a step further and were able to define each of its components:

Theorem: x is one-switch independent from an attribute y if and only if:

(1.6) $$u(x,y)=g_0(y)+g_1(y)\left[f_1(x)+f_2(x)\phi(y)\right]$$

with $g_0(y)=u(x^0,y)$, $g_1(y)=\left[u(x^*,y)-u(x^0,y)\right]>0$, $f_1(x)=u(x\,|\,y^0)$, $f_2(x)=u(x\,|\,y^*)-u(x\,|\,y^0)$, and:

(1.7) $$\phi(y)\triangleq\frac{u(x\,|\,y)-u(x\,|\,y^0)}{u(x\,|\,y^*)-u(x\,|\,y^0)},$$

which is independent of x and monotonic (Abbas and Bell 2011).

The above theorem can be expanded to incorporate more than two attributes as follows:

Theorem: Attributes x and y are one-switch independent from an attribute z if preference between any pair of joint gambles on x, y can switch at most once as z increases in level giving us the following equation:

$$u(x, y, z) = g_0(z) + g_1(z)\left[f_1(x, y) + f_2(x, y)\phi(z) \right] \tag{1.8}$$

This equation makes no assumptions about the relationship between x and y (Abbas and Bell 2011).

Similar to one-switch, for n-switch independence allows preference to switch at most n time. Therefore, if x is <u>*n*-switch independent</u> of an attribute y, it is also $n+h$ -switch independent of y for $h = 1, 2, 3, \ldots$ What is further, (1.5) expands for n switches as follows:

Theorem: If x is *n*-switch independent from an attribute y, it implies that two functions, $f_i(x)$ and $g_i(y)$ exist such that:

$$u(x, y) = \sum_{i=0}^{n+1} f_i(x) g_i(y) \tag{1.9}$$

with the following permanent condition $f_0(x) = 1$ (Abbas and Bell 2011).

Assessing if one-switch independence exists consists of having a decision maker think on the following: For any two gambles on x, one of the following three must be true:

1. They always prefer one over the other no matter the value of y,
2. They are always indifferent no matter the value of y, or
3. They switch their preference exactly one time at a point y' so that one preference exists for all $y < y'$ and the other preference exists for all $y > y'$ (Abbas and Bell 2011).[1]

Assessment of a one-switch utility function requires two tests for verification, two corner points, and the direct assessment of five single-attribute utilities. The two verification tests are:

- Verify that either $\frac{d}{dy} g_1(y) > 0$ or $\frac{d}{dy} g_1(y) < 0$.
- Verify that $\phi(y)$ is monotonic (Abbas and Bell 2011).[2]

The two corner points are k_x and k_y and the five single-attribute utilities are:

- $u(x \mid y^0)$ and $u(x \mid y^*)$ for use in calculating $f_1(x)$ and $f_2(x)$,
- $u(y \mid x^0)$ and $u(y \mid x^*)$ for use in calculating $g_0(x)$ and $g_1(x)$, and
- $u(x' \mid y)$, which gives us both $u(x', y^0)$ and $u(x', y^*)$ and allows for the calculation of $\phi(y)$ (Abbas and Bell 2011).

[1] A diligent reader may ask what happens at the point y'. This point can be included in the range for either of the two preferences; it is only important that a point exists where the switch occurs.

[2] This is clearly true only if $u(x \mid y)$ is for a selected value of x, call it x', due to the fact that the other terms in the equation are all constants.

Note that x' should be chosen to be a value that is convenient for the decision maker. Also, of important note, if the decision maker is not able to give a value for $u(x' \mid y)$ but is for $u(y \mid x')$, the following two equations can be used for a "Bayes' Rule" type conversion:

(1.10) $$u(x', y) = u(x^0, y) + \left[u(x^*, y) - u(x^0, y)\right] u(x' \mid y) \text{ and}$$

(1.11) $$u(x', y) = u(x', y^0) + \left[u(x', y^*) - u(x', y^0)\right] u(y \mid x') .$$

If this conversion is needed, the further assessment of two constants is required: $u(x', y^0)$ and $u(x', y^*)$ (Abbas and Bell 2011).

Mutual One-Switch Independence

As with other types of independence, one-switch independence can be mutual.

Definition: x and y are said to be <u>mutually one switch independent</u> if x is one-switch independent of y and y is one-switch independent of x.

Theorem: x and y are mutually one-switch independent if and only if the following are all true:

1. The following equation holds:

$$
\begin{aligned}
u(x,y) = {} & a_1 u\left(x \mid y^0\right) + a_2 u\left(y \mid x^0\right) - k u\left(x \mid y^0\right) u\left(y \mid x^0\right) \\
& + \left(k - a_1\right) u\left(x \mid y^0\right) u\left(y \mid x^*\right) \\
& + \left(k - a_2\right) u\left(y \mid x^0\right) u\left(x \mid y^*\right) \\
& + (1-k) u\left(y \mid x^*\right) u\left(x \mid y^*\right)
\end{aligned} \tag{1.12}
$$

2. The following two equations are monotonic:

$$
\phi_y(y) \triangleq \frac{\left(k-a_2\right) u\left(y \mid x^0\right) + (1-k) u\left(y \mid x^*\right)}{a_1 + \left(1-a_1\right) u\left(y \mid x^*\right) - a_2 u\left(y \mid x^0\right)} \tag{1.13}
$$

$$
\phi_x(x) \triangleq \frac{\left(k-a_1\right) u\left(x \mid y^0\right) + (1-k) u\left(x \mid y^*\right)}{a_2 + \left(1-a_2\right) u\left(x \mid y^*\right) - a_1 u\left(x \mid y^0\right)} \tag{1.14}
$$

As a note: verifying monotonicity of (1.13) and (1.14) is a simple matter of finding $\frac{d}{dy}\phi_y(y)$ and $\frac{d}{dy}\phi_y(y)$ in turn and verifying that each is strictly positive or strictly negative over the domain $(0,1)$ (Abbas and Bell 2011). Abbas and Bell provided two specific functional forms where x and y are mutually one-switch independent.

Theorem: x and y are mutually one-switch independent in both of the following utility functions:

(1.15) $$u(x,y) = f_1(x) + f_2(y) + \phi_1(x)\phi_2(y)$$

(1.16) $$u(x,y) = u_1(x)u_2(y)\left[1 + \phi_1(x)\phi_2(y)\right]$$

where $\phi_1(x)$ and $\phi_2(y)$ are strictly monotonic and $u_1(x)$ and $u_2(y)$ have constant sign (Abbas and Bell 2011).

Additional One-Switch Approximating Functions

Switching focus, consider now a set of functions that approximate one-switch independence. Unlike the previous portions of this section, the following information was taken from Tsetlin and Winkler (2012) and, as previously indicated, utilizes a slightly different syntax. When expanding this to n attributes, as has been done for the remainder of this section, it is more convenient to utilize $u(x_1,\ldots,x_n)$ to represent a utility function over n attributes. In fact, from this point forward, both syntax depending on the context. The hope is that this will not be overly jarring to the reader but will in fact help to amplify understanding.

This subsection will look at approximations for one-switch multiattribute utility functions using a sumex utility, which is the sum of two multivariate exponential utility functions. Using a sumex utility approximation allows for the assessment of only $2n+2$ parameters, where n is the number of attributes. This subsection differs from the previous sections in that it suggests a functional form for the one-switch utility function and eliminates the elicitation of marginal utility functions for individual attributes.

It is beneficial to start with an alternative definition for one-switch independence, placing the emphasis on the decision maker:

Definition: One-switch independence exists for n attributes if for any gambles $\tilde{x}_1$ and $\tilde{x}_2$ and for any vectors $\alpha_1 \le \alpha_2 \le \alpha_3$, it cannot follow that $\tilde{x}_1 + \alpha_1 \succ \tilde{x}_2 + \alpha_1$, $\tilde{x}_1 + \alpha_2 \prec \tilde{x}_2 + \alpha_2$, and $\tilde{x}_1 + \alpha_3 \succ \tilde{x}_2 + \alpha_3$ (Tsetlin and Winkler 2012).

It is clear from the above definition that one-switch independence cannot exist if preferences between gambles switch more than once as the attributes increase in value, making this an analogous definition the one discussed prior. The above allows for the statement of the following theorem:

Theorem: A multiattribute utility function is one-switch independent if and only if it can be grouped into one of the following families of functions:

(1.17)
$$u(x_1, \ldots, x_n) = c_1 e^{\sum_{i=1}^{n} \lambda_i x_i} + c_2 e^{\sum_{i=1}^{n} \gamma_i x_i}$$

where the c_i are constants and for $i = 1, \ldots, n$, $\gamma_i \ge \lambda_i$.

(1.18) $$u\left(x_1,\ldots,x_n\right)=\sum_{i=1}^{n}\lambda_i x_i + c_2 e^{\sum_{i=1}^{n}\gamma_i x_i}$$

where for $i=1,\ldots,n$, $\gamma_i \geq 0$ or $\gamma_i \leq 0$.

(1.19) $$u\left(x_1,\ldots,x_n\right)=\left[c_1+\sum_{i=1}^{n}\lambda_i x_i\right]c_2 e^{\sum_{i=1}^{n}\gamma_i x_i}$$

where for $i=1,\ldots,n$, $\lambda_i \geq 0$ or $\lambda_i \leq 0$.

(1.20) $$u\left(x_1,\ldots,x_n\right)=\pm\left[\sum_{i=1}^{n}\lambda_i x_i\right]^2+\sum_{i=1}^{n}\gamma_i x_i$$

where for $i=1,\ldots,n$, $\lambda_i \geq 0$ or $\lambda_i \leq 0$ (Tsetlin and Winkler 2012).

It is clear from the above theorem that a multiattribute utility function is one-switch independent if and only if it is:

- sumex (the sum of two exponentials);
- a linear plus an exponential;
- the product of a linear and an exponential; or

- a quadratic (Tsetlin and Winkler 2012).

But are all four possible utility function forms necessary? The follow theorem addresses this directly:

Theorem: If a multiattribute utility function $u(x_1,\dots,x_n)$ defined such that for $i=1,\dots,n$, $x_i \in \left[x_i^0, x_i^*\right]$ and each of the x_i^0 and x_i^* are finite, then there exists a family of multiattribute utility functions $u_j(x_1,\dots,x_n)$ such that for all $x_i \in \left[x_i^0, x_i^*\right]$ and given $u_j(x_1,\dots,x_n)$ is sumex, $u(x_1,\dots,x_n) = \lim_{j\to 0^+} u_j(x_1,\dots,x_n)$ (Tsetlin and Winkler 2012).

From the above theorem all one-switch independent multiattribute utility functions can be approximated using a sumex function, as defined in (1.17). This is an important finding since it provides an overall structure under which one-switch functions exist.

1.2: Attribute Dominance Utility

The contents of the following section are taken from or follow from Abbas and Howard (2005). For the purpose of this section let's assume that all prospects are ordered, and that mutual preferential independence holds. That is, for x', x'', y', y'' the following is true:

(1.21) $$x'' \geq x' \Rightarrow (x'', y) \succeq (x', y), \forall y \in \left[y^0, y^*\right]$$

(1.22) $$y'' \geq y' \Rightarrow (y'', x) \succeq (y', x), \forall x \in [x^0, x^*]$$

Abbas and Howard (2005) also define:

(1.23) $$0 \leq u(x, y) \leq 1, \forall x \in [x^0, x^*], y \in [y^0, y^*]$$

(1.24) $$u(x^0, y^0) = 0, \text{ and } u(x^*, y^*) = 1$$

(1.25) $$u(x^0, y) = u(x, y^0) = 0, \forall x \in [x^0, x^*], y \in [y^0, y^*]$$

The above establishes a set of bounds and enforces that the utility function is well behaved. The last above defines mutual dominances of both x and y. All of these assume more is better, that is that each of the attributes is increasing in utility: Per convention, this is done without loss of generality. This leads us to two important definitions:

Definition: A multi-attribute utility function is an <u>attribute dominance utility function</u> if it satisfies (1.21), (1.22), (1.23), (1.24), and (1.25) (Abbas and Howard 2005).

For the remainder of Chapter 1, attribute dominance utility function are denoted with a superscript d.

Definition: A single attribute that forces the entire attribute dominance utility function to zero when it is below a specified level is a utility-dominant attribute (Abbas and Howard 2005).

Attribute dominance utility functions are not required to be utility independent though it is clear that due to the only dependence existing when an attribute is at its minimum level, attribute dominance utility is a special case of one-switch independence. This is due in large part to the enforcement of mutual preferential independence (Abbas and Howard 2005).

Attribute dominance utility functions can be applied in a number of situations. A simple example relates to a peanut butter a jelly sandwich. Imagine you love a peanut butter and jelly sandwich and are attempting to determine the best possible ratio. Imagine also that you will not eat a solely peanut butter sandwich or a solely jelly sandwich. In that case, you would use an attribute dominance utility function, since if either ingredient is not present, the total utility of the sandwich reduces to zero (Abbas and Howard 2005).

Marginal and Conditional Attribute Dominance Utility Functions

The following section aims to introduce new definitions related to attribute dominance utility functions.

Definition: A marginal attribute dominance utility function for a single attribute will be the value of that attribute when all other attributes are at their maximum value. Note that the marginal attribute dominance utility function is an attribute dominance utility function over one attribute. The two-attribute case follows:

(1.26)
$$u^d(x) \triangleq u^d(x, y^*)$$
$$u^d(y) \triangleq u^d(x^*, y).$$

This is a deviation from convention, which has typically defined the marginal utility function for one variable to be when the second variable is set at its lowest value. This is clearly necessary in this case since a marginal defined in this way would be trivial for attribute dominance. What's more, the conditional utility function also changes slightly (Abbas and Howard 2005).

Definition: A <u>conditional attribute dominance utility function</u> is a utility function of a single attribute given a fixed level of another attribute. In the two-dimensional case, this can be thought of as a slice of the multiattribute utility function for a fixed value of an attribute. As before, the two-attribute case follows:

(1.27)
$$u^d(y \mid x) \triangleq \frac{u^d(x, y)}{u^d(x)}, x \neq x^0$$
$$u^d(x \mid y) \triangleq \frac{u^d(x, y)}{u^d(y)}, y \neq y^0$$

However, by applying that original definition of the conditional yields these as they are (Abbas and Howard 2005). The difference here is the addition of the restriction on value. It should be clear that a conditional attribute dominance utility function is a normalized attribute dominance utility function over one attribute given a fixed value of the other attribute. Rearranging (1.27) yields:

(1.28)
$$u^d(x, y) = u^d(x)u^d(y \mid x)$$

Rewriting both equations in (1.27), Abbas and Howard (2005) shows that:

$$u^d\left(x \mid y\right) = \frac{u^d\left(y \mid x\right) u^d\left(x\right)}{u^d\left(y\right)} \quad . \tag{1.29}$$

Equation (1.29) is important because it allows for the solving for posterior utility functions if the other is easier to elicit. To that end, equation (1.29) acts as a form of Bayes' rule for attribute dominance utilities (Abbas and Howard 2005).

Attribute Dominance Utility Properties

The first property to explore is utility independence. An attribute-dominant utility x is utility independent of another attribute-dominant utility y if:

$$u^d\left(x \mid y\right) \triangleq \frac{u^d\left(x, y\right)}{u^d\left(y\right)} = u^d\left(x\right) \tag{1.30}$$

(Abbas and Howard 2005). Said another way, an attribute-dominant utility x is utility independent of another attribute-dominant utility y if the conditional attribute dominance utility function is the same no matter the set value of y. Similarly the reverse of (1.30) is defined as follows: An attribute-dominant utility y is utility independent of another attribute-dominant utility x if:

(1.31) $$u^d(y \mid x) \triangleq \frac{u^d(x,y)}{u^d(x)} = u^d(y)$$

What is further, utility independence is symmetric for attribute-dominant utilities (Abbas and Howard 2005). As a result utility independence of x from y directly implies utility independence of y from x. Abbas and Howard (2005) show that the attribute dominance utility can be written as follows when utility independence holds:

(1.32) $$u^d(x,y) = u^d(x)u^d(y)$$

This symmetric relationship makes attribute-dominant utilities very easy to use: The elicitation needed to implement these models is minor.

Expanding the attribute dominance utility to three attributes yields:

(1.33) $$\begin{aligned} u^d(x,y,z) &= u^d(x,y^*,z^*)\frac{u^d(x,y,z^*)}{u^d(x,y^*,z^*)}\frac{u^d(x,y,z)}{u^d(x,y,z^*)} \\ &\triangleq u^d(x)\frac{u^d(x,y)}{u^d(x)}\frac{u^d(x,y,z)}{u^d(x,y)} \\ &\triangleq u^d(x)u^d(y \mid x)u^d(z \mid x,y), \text{ where} \end{aligned}$$

$$\begin{aligned} u^d(x,y) &= u^d(x,y,z^*), \\ u^d(y \mid x) &= \frac{u^d(x,y)}{u^d(x)}, \text{ and} \\ u^d(z \mid x,y) &= \frac{u^d(x,y,z)}{u^d(x,y)} \end{aligned}$$

This can be expanded for n attributes using the same method (Abbas and Howard 2005).

I will now examine conditional utility independence for the attribute dominance utility. Abbas and Howard (2005) give us that:

Definition: Attributes y and z have <u>conditional utility independence</u> given x if

(1.34) $$u^d\left(y \mid x, z\right) = u^d\left(y \mid x\right) \text{ and } u^d\left(z \mid x, y\right) = u^d\left(z \mid x\right)$$

Given conditional utility independence (1.34) can be used to rewrite (1.33) as:

(1.35) $$u^d\left(x, y, z\right) = u^d\left(x\right) u^d\left(y \mid x\right) u^d\left(z \mid x\right)$$

When conditional utility independence holds, it helps to simplify the assessment of attribute dominance utility functions.

Finally, in this subsection, I will look at how to make tradeoffs with utility-dominant attributes. Tradeoffs can be assessed along a constant isopreference contour, such that any value along that curve has the same intrinsic value (Abbas and Howard 2005). Therefore, the change in value along this contour is zero. Hence, evaluating $u^d\left(x, y\right)$ along that contour, yields:

(1.36) $$du^d(x,y) = \frac{\partial u^d(x,y)}{\partial x}dx + \frac{\partial u^d(x,y)}{\partial y}dy = 0$$

This can then be restructured as:

(1.37) $$\frac{dy}{dx} = -\frac{\dfrac{\partial u^d(x,y)}{\partial x}}{\dfrac{\partial u^d(x,y)}{\partial y}} = -\frac{u^d(y)\dfrac{\partial u^d(x|y)}{\partial x}}{u^d(x)\dfrac{\partial u^d(x|y)}{\partial y}}$$

It is of note that if utility independence holds for the attributes, (1.37) can be reduced to:

(1.38) $$\frac{dy}{dx} = -\frac{u^d(y)\dfrac{d}{dx}u^d(x)}{u^d(x)\dfrac{d}{dx}u^d(y)} = -\frac{\dfrac{d}{dx}\ln\left(u^d(x)\right)}{\dfrac{d}{dx}\ln\left(u^d(y)\right)}$$

If desired, the tradeoffs between attributes can be found separately and then (1.37) or (1.38) (if applicable) can be used as a consistency check (Abbas and Howard 2005).

Decomposition of Two-Attribute Utility Functions

For this section assume that (1.21), (1.22), (1.23), and (1.24) hold and that (1.25) does not. With this in mind consider the following (notice that this is not specifically an attribute dominance utility):

Definition: A joint utility density function is second-order and (if it exists) is the mixed partial derivative as follows:

$$u'(x,y) \triangleq \frac{\partial^2}{\partial x \partial y} u(x,y) \tag{1.39}$$

(Abbas and Howard 2005). The following decomposition holds if (1.39) exists.

Theorem: Assuming x and y are mutually preferentially independent, their utility can be written as:

$$\begin{aligned} &u(x,y) = k_x u^d(x)|_{y^0} + k_y u^d(y)|_{x^0} + k_{xy} u^{gd}(x,y) \\ &\text{where} \\ &k_x = u(x^*, y^0),\ k_y = u(x^0, y^*),\ k_{xy} = 1 - k_y - k_x \\ &\text{and} \\ &u^{gd}(x,y) = \left(\frac{1}{k_{xy}}\right) \int_{y^0}^{y} \int_{x^0}^{x} u'(x,y)\,dx\,dy \end{aligned} \tag{1.40}$$ [3]

[3] $u(x)|_{y^0}$ and $u(y)|_{x^0}$ indicate that the utility functions are assessed at the minimum value of the indicated attribute.

(Abbas and Howard 2005). Abbas and Howard (2005) offer the above theorem without formal proof, but indicate that allowing $x = x^*$ and $y = y^*$ generates the following:

(1.41) $$1 = k_x + k_y + k_{xy}$$

which is known to be true.

Theorem: Assuming x and y are mutually preferentially independent, their utility can be written as:

(1.42) $$u(x,y) = k_y u^d(y)|_{x^0} + \left[(1-k_x)u^d(y)|_{x^*} + k_x - k_y u^d(y)|_{x^0}\right]u^d(x \mid y)$$

Abbas and Howard (2005) also offered the above without formal proof but noted that combining (1.40) and (1.42) yields:

(1.43) $$u^{gd}(x,y) = \frac{1}{k_{xy}}\left\{\left[(1-k_x)u^d(y)|_{x^*} + k_x - k_y u^d(y)|_{x^0}\right]u^d(x \mid y) - k_x u^d(x)|_{y^0}\right\}$$

To use (1.42), assessment of a condition utility and two marginal utilities is needed. It is of note that if $u^d(y)|_{x^0}\, u^d(x \mid y)$ and $u^d(y)|_{x^*}\, u^d(x \mid y)$ are non-decreasing in y then $u^d(y)|_{x^0}$, $u^d(y)|_{x^*}$, and $u^d(x \mid y)$ have the same properties as attribute dominance utility functions (Abbas and Howard 2005). It is possible to rewrite (1.42) as follows:

$$(1.44) \qquad u(x,y) = k_y u^d(y)|_{x^0} + \left[(1-k_x)u^d(y)|_{x^*} + k_x - k_y u^d(y)|_{x^0}\right]\frac{u^d(x,y)}{u^d(y)|_{x^*}},\ y \neq y^0$$

where $u^d(x,y)$ is the attribute dominance utility function (Abbas and Howard 2005). To see this substitute (1.27) into (1.42). Additionally, if $\left(\partial^2/\partial x \partial y\right)\left(u^d(y)|_{x^*}\, u^d(x \mid y)\right) \geq 0$ then the following is true:

$$(1.45)\ u(x,y) = k_y u^d(y)|_{x^0} + \left[(1-k_x)u^d(y)|_{x^*} + k_x - k_y u^d(y)|_{x^0}\right]\frac{\mathrm{C}\left(u^d(x)|_{y^*}, u^d(y)|_{x^*}\right)}{u^d(y)|_{x^*}},\ y \neq y^0$$

which has the same properties as a copula structure (Abbas and Howard 2005).

1.3: Multi-attribute Utility Trees

The following section is taken from Abbas (2011). This section introduces the concept of multi-attribute utility trees, a construct that is similar to the decision tree. The multi-attribute utility tree helps visually lay out which gambles will be needed to obtain the needed utility values. They also help ensure that any normalizing constraints are properly applied. Possibly the most important impact of the trees is that they may make the decision maker more comfortable by reversing the order of the utility assessments.

For the purposes of this section,

$$u(x_1, y_1) = p_{x_1,y_1} u(x^*, y^*) + (1 - p_{x_1,y_1})(x^0, y^0) = p_{x_1,y_1}, \tag{1.46}$$

where p_{x_1,y_1} is the probability of indifference for the gamble. Abbas 2011 shows that the utility value for given levels of x_1 and y_1 is equal to the probability of choosing the above gamble.

Utility Tree for a Single Two-Attribute Binary Consequence

Figure 1 shows a tree decomposition for a decision with two attributes.

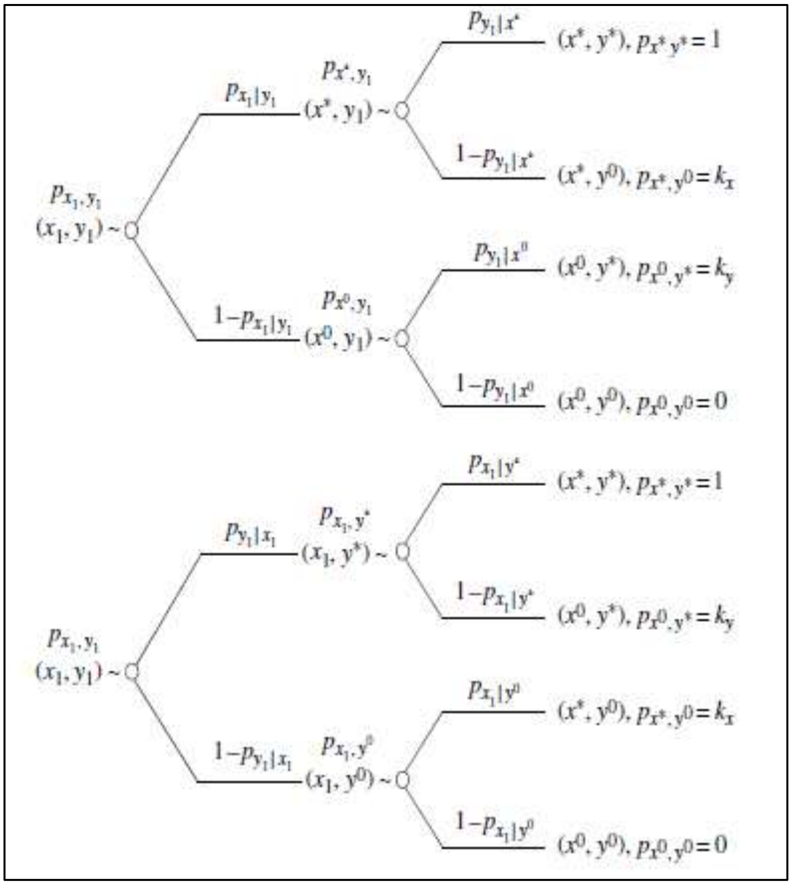

Figure 1: Utility Tree for a Decision with Two Attributes (Abbas 2011)

From this tree the following is true:

(1.47)

$$\begin{aligned} p_{x_1,y_1} = {} & p_{x_1|y_1} \times p_{y_1|x^*} \times p_{x^*,y^*} + p_{x_1|y_1} \times \left(1 - p_{y_1|x^*}\right) \times p_{x^*,y^0} \\ & + \left(1 - p_{x_1|y_1}\right) \times p_{y_1|x^0} \times p_{x^0,y^*} \\ & + \left(1 - p_{x_1|y_1}\right) \times \left(1 - p_{y_1|x^0}\right) \times p_{x^0,y^0}. \end{aligned}$$

By applying assumptions defined prior and by applying k_x and k_y as defined in Figure 1, Abbas (2011) derives the following two equations:

(1.48)
$$p_{x_1,y_1} = p_{x_1|y_1} \times p_{y_1|x^*} + p_{x_1|y_1} \times \left(1 - p_{y_1|x^*}\right) \times k_x + \left(1 - p_{x_1|y_1}\right) \times p_{y_1|x^0} \times k_y.$$

(1.49)
$$p_{x_1,y_1} = p_{y_1|x_1} \times p_{x_1|y^*} + p_{y_1|x_1} \times \left(1 - p_{x_1|y^*}\right) \times k_y + \left(1 - p_{y_1|x_1}\right) \times p_{x_1|y^0} \times k_x.$$

These follow directly from Figure 1 by replaying a single attribute at each step with a gamble whose components are the maximum and minimum values of the utility. It is important also to note that k_x and k_y are called corner assessments.

Theorem: The number of assessments required to solve a tree for n attributes is equal to the number of conditional assessments plus the number of corner assessments. This total is $\left(2^n - 2\right) + \left(2^n - 1\right) = 2^{n+1} - 3$ (Abbas 2011).

Utility Tree for a Single Two-Attribute Utility

The following shows an expansion of a utility function around a single attribute. To that end, this can be thought of as an expansion of the utility around x:

(1.50) $$u(x,y)=u(x^{*},y)u(x\mid y)+u(x^{0},y)\bar{u}(x\mid y)$$

where $\bar{u}(x\mid y)=1-u(x\mid y)$ and is called the disutility function (Abbas 2011).

In essence, treating each of the conditional utilities, which are restricted to be between zero and one and which obviously sum to one, as probabilities turns (1.50) into a structured gamble offering x at its maximum with some probability and offering x at its minimum with some complement probability. Notice the expansion is also around y:

(1.51) $$u(x,y)=u(x,y^{*})u(y\mid x)+u(x,y^{0})\bar{u}(y\mid x)$$

Figure 2 offers a tree representation of (1.50) (Abbas 2011).

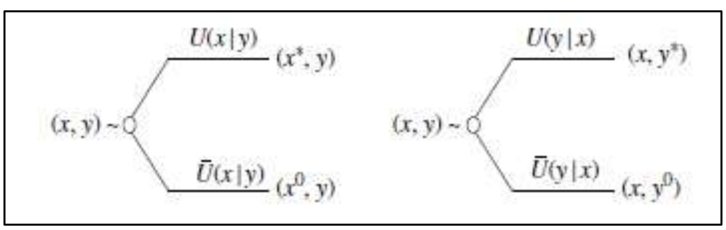

Figure 2: Utility Expansion around x (Abbas 2011)

It is a relatively simple step to now take the expansion around one attribute and extend it to an expansion around both attributes. This is accomplished with the following:

(1.52) $$u(x^{0},y)=u(x^{0},y^{*})u(y\mid x^{0})+u(x^{0},y^{0})\bar{u}(y\mid x^{0})$$

and

(1.53) $$u\left(x^*,y\right)=u\left(x^*,y^*\right)u\left(y\mid x^*\right)+u\left(x^*,y^0\right)\bar{u}\left(y\mid x^*\right)$$

(Abbas 2011). Combining (1.52) and (1.53) yields:

(1.54)

$$u\left(x,y\right)=u\left(x^*,y^*\right)u\left(x\mid y\right)u\left(y\mid x^*\right)+u\left(x^*,y^0\right)u\left(x\mid y\right)\bar{u}\left(y\mid x^*\right)+u\left(x^0,y^*\right)\bar{u}\left(x\mid y\right)u\left(y\mid x^0\right)$$
$$+u\left(x^0,y^0\right)\bar{u}\left(x\mid y\right)\bar{u}\left(y\mid x^0\right)$$

The assessment required is for three expressed conditional utilities and for the value of the four corner points: $u\left(x^*,y^*\right)=1$, $u\left(x^0,y^0\right)=0$, $u\left(x^0,y^*\right)\triangleq k_y$, and $u\left(x^*,y^0\right)\triangleq k_x$. Figure 3 shows a tree representation of (1.54).

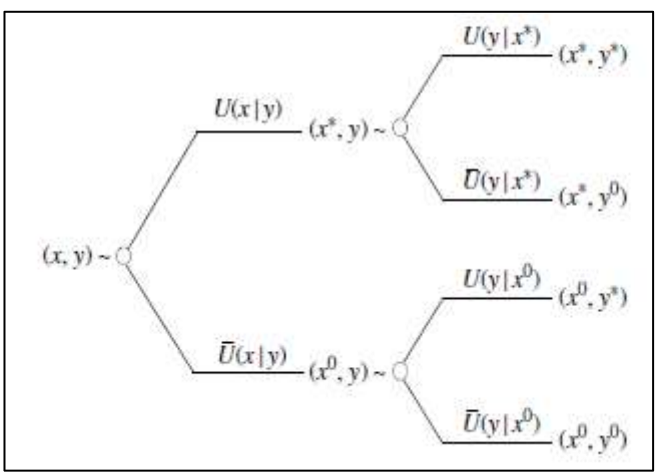

Figure 3: Utility Expansion around x and y (Abbas 2011)

Relation of Utility Trees to Types of Independence/Dependence

Note that equation (1.54) does not assume utility independence. In fact, different dependence structures and different types of independence make elicitation easier, not harder. If x is a utility-dominant attribute then only a corner value $u\left(x^*, y^0\right)$ and two conditional utilities $u(x \mid y), u\left(y \mid x^*\right)$ need to be assessed. The resulting equation for the utility is:

(1.55)
$$
\begin{aligned}
u(x, y) &= u\left(x^*, y^*\right) u(x \mid y) u\left(y \mid x^*\right) + u\left(x^1, y^0\right) u(x \mid y) \bar{u}\left(y \mid x^*\right) \\
&= u(x \mid y) u\left(y \mid x^*\right) + k_x u(x \mid y) \bar{u}\left(y \mid x^*\right)
\end{aligned}
$$

(Abbas 2011). If in fact both attributes are utility-dominant attributes, then none of the corner values need to be assessed, but both $u(x \mid y), u\left(y \mid x^*\right)$ still need to be (Abbas 2011). The resulting equation is:

(1.56)
$$
\begin{aligned}
u(x, y) &= u\left(x^*, y^*\right) u(x \mid y) u\left(y \mid x^*\right) \\
&= u(x \mid y) u\left(y \mid x^*\right)
\end{aligned}
$$

In instances where x is utility independent of y, then the corner assessments in conjunction with $u\left(x \mid y^0\right), u\left(y \mid x^0\right)$, and $u\left(y \mid x^*\right)$ are needed, with the following resulting equation:

(1.57)
$$\begin{aligned}u(x,y)=&u(x\mid y^0)u(y\mid x^*)u(x^*,y^*)+(x\mid y^0)\bar{u}(y\mid x^*)u(x^*,y^0)\\&+\bar{u}(x\mid y^0)u(y\mid x^0)u(x^0,y^*)\end{aligned}$$

(Abbas 2011). A new definition is now needed:

Definition: An attribute y is <u>boundary independent</u> of x if $u(y\mid x^0)=u(y\mid x^*)$

From the above definition it should be clear that if y is boundary independent of x then the following form is implied due to the need to only find the values for the corner assessments, $u(x\mid y)$, and $u(y\mid x^0)$:

(1.58)
$$u(x,y)=u(x\mid y)\left[u(y\mid x^0)(1-k_x-k_y)+k_x\right]+k_y u(y\mid x^0)$$

(Abbas 2011). Finally given that x is interpolation independent of y, the conditional utility $u(x\mid y)$ is simplified to be a function of its boundary values (Abbas 2011).

Reversing the Utility Tree for a Single Two-Attribute Utility

Using Figure 3, one can easily detect if x is utility independent of y and if y is boundary independent of x, but from Figure 3, can it be determined if y is utility independent of x? What is further, can $u(y|x)$ be found or does it need to be assessed separately? Notice that by rewriting (1.54) as follows yields a Bayes' rule style tool (Abbas 2011).

(1.59)
$$u(x,y)=\sum_{\forall x_i}\sum_{\forall y_i}u(x_i,y_i)\,g_{x_i}(x|y)\,g_{y_i}(y|x_i)$$

with

(1.60)
$$g_{x_i}(x|y)=\begin{cases}u(x|y), & x_i=x^*\\ \bar{u}(x|y), & x_i=x^0\end{cases}$$
$$g_{y_i}(y|x_i)=\begin{cases}u(y|x_i), & \mathrm{y}_i=y^*\\ \bar{u}(y|x_i), & \mathrm{y}_i=y^0\end{cases}$$

Theorem: Abbas (2011) provides that to reverse a two-attribute utility tree, use the following:

(1.61)
$$u(y|x)=\frac{\sum_{\forall x_i}\left[d_{x_i}(x,y)u(y|x_i)+m_{x_i}(x,y)\right]}{\sum_{\forall x_i}\left[d_{x_i}(x,y^*)+m_{x_i}(x,y^*)\right]}$$

with

(1.62)
$$d_{x_i}(x,y)=g_{x_i}(x|y)\left[u(x_i,y^*)-u(x_i,y^0)\right]$$
$$m_{x_i}(x,y)=u(x_i,y^0)\left[g_{x_i}(x|y)-g_{x_i}(x|y^0)\right]$$

Additional Independence/Dependence Relationships for Two-Attribute

Theorem: x is utility independent of y implies both of the following:

1. $m_{x^0}(x,y) = m_{x^*}(x,y) = 0$
2. $d_{x^0}(x,y)$ and $d_{x^*}(x,y)$ are independent of y (Abbas 2011).

Theorem: x is utility independent of y implies y is interpolation independent of x (Abbas 2011).

Theorem: x is utility independent of y and y is boundary independent of x implies y is utility independent of x (Abbas 2011).

This is important because it allows for the assessment of mutual utility independence of the two attributes directly from the tree without reversing the tree.

Theorem: y is a utility-dominant attribute implies $m_{x_i}(x,y) = 0, \forall i$ (Abbas 2011).

This significantly reduces the computational effort to reverse the tree for a utility-dominant attribute.

Utility Tree for Three-Attribute Utilities

Figure 4 shows a three-attribute tree.

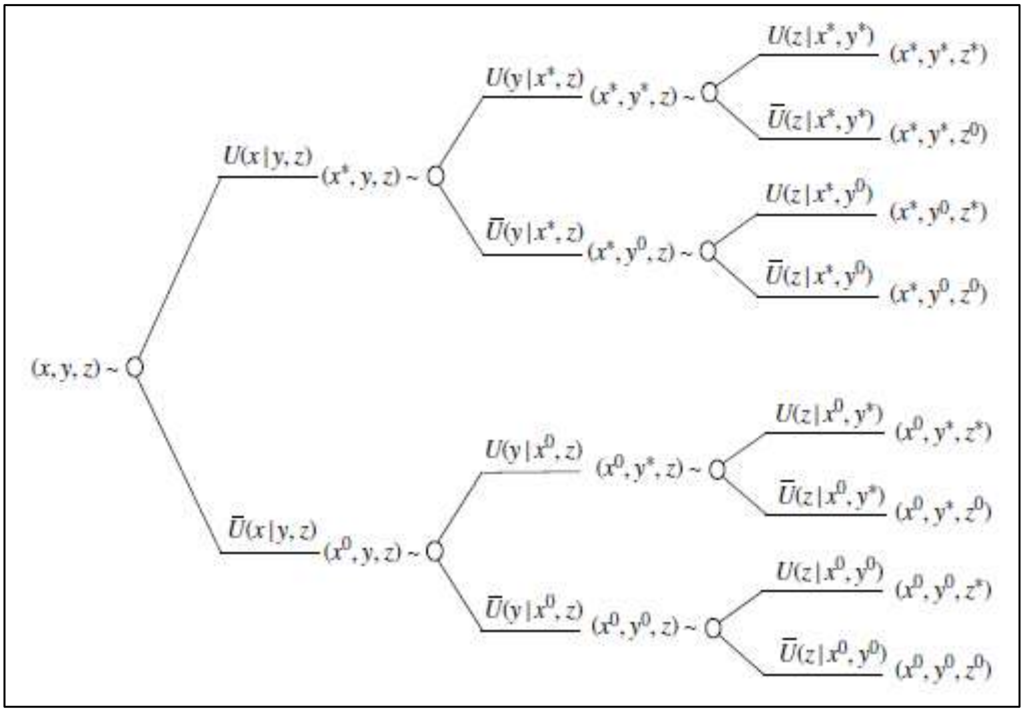

Figure 4: Utility Expansion around x, y, and z (Abbas 2011)

To start, assessing the conditional utility function of the first attribute is very straight forward because it can be read directly from the tree. To that end, I will cover how to get the conditional utility function of both the second and third attribute and omit anything further to do with assessing the first.

Theorem: The following equations allows for the direct calculation of the conditional utility function for the second attribute in a three-attribute tree:

(1.63) $$u\left(y \mid x,z\right)=\frac{\sum_{\forall x_i,z_k} d_{x_i z_k}\left(x,y,z\right)u\left(y \mid x_i,z\right)+\sum_{\forall x_i,z_k} m_{x_i z_k}\left(x,y,z\right)}{\sum_{\forall x_i,z_k}\left[d_{x_i z_k}\left(x,y^*,z\right)+m_{x_i z_k}\left(x,y^*,z\right)\right]}$$

where

(1.64) $$\begin{aligned} d_{x_i z_k}\left(x,y,z\right)&=g_{x_i}\left(x \mid y,z\right)\left[u\left(x_i,y^*,z_k\right)g_{z_k}\left(z \mid x_i,y^*\right)-u\left(x_i,y^0,z_k\right)g_{z_k}\left(z \mid x_i,y^0\right)\right] \\ m_{x_i z_k}\left(x,y,z\right)&=u\left(x_i,y^0,z_k\right)g_{z_k}\left(z \mid x_i,y^0\right)\left[g_{x_i}\left(x \mid y,z\right)-g_{x_i}\left(x \mid y^0,z\right)\right] \end{aligned}$$

(Abbas 2011). From these come three very powerful propositions that will help simplify the elicitation:

Theorem: x is conditionally utility independent of y given z implies:

1. $d_{x_i z_k}\left(x,y,z\right)$ does not depend on y.
2. $m_{x_i z_k}\left(x,y,z\right)=0,\forall x,z$
3. y is conditionally interpolation independent of x given z, which implies:

   $u\left(y \mid x,z\right)=w_{x^*}\left(x,z\right)u\left(y \mid x^*,z\right)+\left[1-w_{x^*}\left(x,z\right)\right]u\left(y \mid x^0,z\right)$, where

   $w_{x^*}\left(x,y\right)=d_{x^*}\left(x,y\right)/\left[d_{x^*}\left(x,y\right)+d_{x^0}\left(x,y\right)\right]$ (Abbas 2011).

Theorem: y is conditionally boundary independent of x given z, $u\left(y \mid x^*,z\right)=u\left(y \mid x^0,z\right)$,

and if x is conditionally utility independent of y given

z , $u(x \mid y, z) = u(x \mid y^0, z)$, then y is conditionally utility independent of x given z , $u(y \mid x, z) = u(y \mid x^0, z)$ (Abbas 2011).

Theorem: y is conditionally boundary independent of x given z , and if x is conditionally utility independent of y given z , and y is utility independent of z , at any boundary of x , $u(y \mid x_i, z) = u(y \mid x_i, z^0)$, for $x_i = x^*$ or $x_i = x^0$, then y is utility independent of both x and z (Abbas 2011).

Building off of (1.63) and (1.64), Abbas 2011 presented the following theorem:

Theorem: The following equations allows for the direct calculation of the conditional utility function for the third attribute in a three-attribute tree:

$$u(z \mid x, y) = \frac{\sum_{\forall x_i, y_j} d_{x_i y_j}(x, y, z) u(z \mid x_i, y_j) + \sum_{\forall x_i, y_j} m_{x_i y_j}(x, y, z)}{\sum_{\forall x_i, y_j} \left[d_{x_i y_j}(x, y, z^*) + m_{x_i y_j}(x, y, z^*) \right]} \tag{1.65}$$

where

$$\begin{aligned} d_{x_i y_j}(x, y, z) &= g_{x_i}(x \mid y, z) g_{y_j}(y \mid x_i, z) \left[u(x_i, y_j, z^*) - u(x_i, y_j, z^0) \right] \\ m_{x_i y_j}(x, y, z) &= u(x_i, y_j, z^0) \left[g_{x_i}(x \mid y, z) g_{y_j}(y \mid x_i, z) - g_{x_i}(x \mid y, z^0) g_{y_j}(y \mid x_i, z^0) \right] \end{aligned} \tag{1.66}$$

(Abbas 2011). Like before, there are a couple of very powerful propositions that will help simplify the elicitation:

Theorem: x is conditionally utility independent of z given y and y is conditionally utility independent of z at any boundary of x, $u\left(y \mid x_i, z\right) = u\left(y \mid x_i, z^0\right), \forall x$, then all of the following are true:

1. $d_{x_i y_j}\left(x, y, z\right)$ does not depend on z.
2. $m_{x_i y_j}\left(x, y, z\right) = 0, \forall x, y$
3. z is interpolation independent of x and y (Abbas 2011).

Theorem: x is conditionally utility independent of z given y, and z is boundary independent of y and x, and y is utility independent of z, at any boundary of x, then z is utility independent of both x and y (Abbas 2011).

With these theorems and propositions it is possible to assess all three of the conditional utility functions needed.

There is one additional definition of note because where there are three attributes, corner assessments become a function of that third attribute. Corner assessments are written as $k_x\left(z\right)$ and $k_y\left(z\right)$, With this information it is possible to define corner independence:

Definition: z is <u>corner independent</u> of both x and y if the values of $k_x(z)$ and $k_y(z)$ do not depend on z. Thus, the following two equations can be derived:

(1.67)
$$
\begin{aligned}
u(x^1, y^0, z) &= k_x(z)u(x^1, y^1, z) + (1 - k_x(z))u(x^0, y^0, z) \\
u(x^0, y^1, z) &= k_y(z)u(x^1, y^1, z) + (1 - k_y(z))u(x^0, y^0, z)
\end{aligned}
$$

where $k_x(z)$ and $k_y(z)$ are constants (Abbas 2011).

Chapter 2: Maximum Dominance

Chapters 2 and 3 consider a decision whose outcome has two attributes x and y, where $x \in \left[x^0, x^*\right]$ and $y \in \left[y^0, y^*\right]$. Preferences over the attribute x are assumed to be such that $x^0 < \cdots < x^* \rightarrow x^0 \prec \cdots \prec x^*$; however, this implication will not always hold for the attribute y. It will be necessary for y to fall into one of three cases:

Case 1: $y^0 \leq \cdots \leq y^* \rightarrow y^0 \preceq \cdots \preceq y^*$

Case 2: $y^0 \leq \cdots \leq y^* \rightarrow y^0 \succeq \cdots \succeq y^*$

Case 3: $y^0 \leq \cdots \leq y^* \rightarrow y^0 \preceq \cdots \preceq y^*, \forall x > x'$ and $y^0 \leq \cdots \leq y^* \rightarrow y^0 \succeq \cdots \succeq y^*, \forall x \leq x'$

Therefore, traditional assumptions, such as $\left(x^*, y^*\right)$ is the most preferred outcome and $\left(x^0, y^0\right)$ is the least preferred outcome, will not always apply here; however, at least one corner point will always be most and least preferred. I define a utility function over the two attributes $U\left(x, y\right) \in \left[0,1\right]$. To maintain maximum flexibility, assumptions about the use case of attribute y will be stated for each section; however, in many cases, the specific case will not influence the outcome. Additionally, certain well-established definitions will be modified to accommodate this added flexibility. In alignment with past work in this field I also define a conditional utility function:

$$U\left(x \mid y\right) = \frac{U\left(x, y\right) - U\left(x^0, y\right)}{U\left(x^*, y\right) - U\left(x^0, y\right)}.$$

Note the conditional utility function applies for all monotonically increasing and decreasing functions. I also use traditional notation for the corner points $U\left(x^*, y^0\right) = k_x$ and $U\left(x^0, y^*\right) = k_y$. Additionally, I apply similar notation for $U\left(x^0, y^0\right) = k_0$ and $U\left(x^*, y^*\right) = k_*$ since they may not have assumed values.

2.1: Dominance as a Concept

Applying Case 1, Abbas and Howard (2005) defines x as attribute dominant of y if $\left(x^0, y\right) \sim \left(x^0, y^0\right), \forall y$, which implies that $U\left(x^0, y\right) = 0, \forall y$. Thus if x is at its minimum level then the consequence (x, y) has minimum utility irrespective of the level of y, or x dominates y at its minimum level. I rename this concept <u>min-dominance</u> because the dominating attribute must be at its minimum level. The reader will recall in Chapter 1 that Abbas and Howard focused their work on situations in which both attributes dominate each other and defined an attribute dominance utility function. This work does not assume that both attributes dominate each other, instead recognizing that many situations exist where only one attribute may have some form of dominance.

Abbas and Howard offer several examples of attribute dominance at an attribute's minimum. In decisions about health and consumption, if the worst health state considered is death then at this level consumption gives no additional utility. However, one could also consider a pure gains framework with one attribute representing the amount of gain and another amplifying or existing only if that gain is greater than zero, e.g. press, notoriety. Abbas and Howard also consider the amount of peanut butter and jelly in a sandwich. If someone likes peanut butter alone and likes peanut butter and jelly, but does not like jelly alone, then the amount of peanut butter dominates the amount of jelly at its minimum level of zero.

One advantage of min-dominance is the simplification of the elicitation required for the utility function $U(x,y)$. To see this, first examine Abbas (2011)'s form for $U(x,y)$ with no assumption about the relationship between x and y, which from here forward will be called the standard form.

(2.1)
$$\begin{aligned}U(x,y) &= k_*U(x|y)U(y|x^*) + k_xU(x|y)\bar{U}(y|x^*)\\ &+ k_y\bar{U}(x|y)U(y|x^0) + k_0\bar{U}(x|y)\bar{U}(y|x^0)\end{aligned}$$

Where $\bar{U} = 1 - U$ is called a disutility. However, assuming x is min-dominant, the joint utility of x and y reduces to:

(2.2)
$$U(x,y) = k_*U(x|y)U(y|x^*) + k_xU(x|y)\bar{U}(y|x^*)$$

This saves the elicitation of two corner points and a conditional utility function. Further adding that y is min-dominant, then the joint utility reduces to:

(2.3)
$$U(x,y) = \begin{cases} k_*U(x|y)U(y|x^*), y > y^0 \\ 0, y = y^0 \end{cases},$$

which saves the elicitation of one additional corner point. The reader should notice a slight change in syntax here when compared to Abbas and Howard's form in Chapter 1. This is due to applying the definition of min-dominance to a standard utility function and not using the attribute dominance utility they defined.

New Definitions

Consider the symmetric condition to min-dominance: if x is max-dominant of y then $\left(x^{*}, y\right) \sim \left(x^{*}, y^{*}\right)$, ∀y, which implies that $U\left(x^{*}, y\right) = 1$ ∀y. Thus, if x is at its maximum level x^{*} then the consequence $\left(x, y\right)$ has maximum utility irrespective of the level of y, or x dominates y at its maximum level. For example: a car enthusiast might prefer the fastest car irrespective of fuel efficiency, but fuel efficiency matters for cars that are for commuting. Additionally, consider a pure loss framework, where one attribute represents the amount of loss and the other either amplifies or exists only if the loss is greater than zero. Note that in this example, the amount of attention a loss receives amplifies the loss, which decreases the amount of utility. In this case, more attention is not desirable and would be monotonically decreasing, as in Case 2.

I also define attributes x and y as mutually max-dominant attributes if $U\left(x^{*}, y\right) = U\left(x, y^{*}\right) = 1, \forall x, y$ and say if an attribute x both max-dominates and min-dominates the attribute y, then x full-dominates y. As a concept, full dominance can be applied to a number of decisions. Consider a decision about medical care. If one attribute is the chance of survival provided from a particular treatment and another attribute is the cost of the treatment, then for some people cost will only matter for intermediate chance values. Said another way, if death is near certain, cost would be of no consequence. Indeed, if survival is near certain, some people will pay any amount to ensure it.

While it is possible to have mutual min-dominance and mutual max-dominance, the two conditions cannot co-exist in the same way.

Proposition 1: If an attribute x max-dominates another attribute y, y cannot be min-dominant of x.

Proof: To see that Proposition 1 and its mirrored condition are true, assume that $U\left(x, y^0\right)=0$ for all x and $U\left(x^*, y\right)=1$ for all y. If these are both true, then $U\left(x^*, y^0\right)$ has no definitive value and as such, the two conditions are not compatible. You can see the same incompatibility exists by setting $U\left(x, y^*\right)=1$ for all x and $U\left(x^0, y\right)=0$ for all y. ■

2.2: Min-Dominance, Max-Dominance, and Utility Independence

Of particular interest is the interaction of different dominance conditions and independence assumptions, as the more that is known or that can realistically be assumed about the relationship between two attributes, the simpler the elicitation. Abbas and Howard's work on attribute dominance focused on instances where both attributes were utility dominant. Recognizing that even min-dominance can exist for only one attribute and that multiple types of dominance can exits, the following starts to define some of the interactions between independence and dominance:

Proposition 2: Given an attribute x is utility independent of an attribute y and x is min-dominant of y, then the utility function takes the following form:

(2.4) $$U(x,y) = k_* U(x \mid y^*) U(y \mid x^*) + k_x U(x \mid y^*) \bar{U}(y \mid x^*)$$

Proposition 2 moved the form for min-dominance discussed in the opening of this article from requiring the elicitation of a conditional utility function, a marginal utility function, and a corner point to two marginal utility functions and a corner point. This is a substantial reduction.

Proof: Utility Independence give us $U(x,y) = c_1(y) + c_2(y) U(x,y^*)$. Notice that if x is min-dominant of y then $U(x^0,y) = 0$ for all y, so $c_1(y) = 0$. This simplifies the general form to $U(x,y) = c_2(y) U(x,y^*)$. But now, $U(x^*,y) = c_2(y) U(x^*,y^*)$ so $c_2(y) = (k_*)^{-1} U(x^*,y)$. Thus, $U(x,y) = (k_*)^{-1} U(x^*,y) U(x,y^*)$. Further,

$$U(x \mid y^*) = \frac{u(x,y^*) - u(x^0,y^*)}{u(x^*,y^*) - u(x^0,y^*)} = (k_*)^{-1} u(x,y^*) \text{ and}$$

$$U(y \mid x^*) = \frac{u(x^*,y) - u(x^*,y^0)}{u(x^*,y^*) - u(x^*,y^0)} = \frac{u(x^*,y) - k_x}{k_* - k_x} \text{, or}$$

$U(x^*,y) = k_x + (k_* - k_x) U(y \mid x^*)$. Combining these and placing the equation in standard form yields, $U(x,y) = k_* U(x \mid y^*) U(y \mid x^*) + k_x U(x \mid y^*) \bar{U}(y \mid x^*)$. ■

Based on the results above, mirrored relations based both on max-dominance and full dominance follow:

Proposition 3: Given an attribute x is utility independent of an attribute y and x is max-dominate of y, then the utility function takes the following form:

(2.5) $$U(x,y)=U(x\mid y^0)+k_y\bar{U}(x\mid y^0)U(y\mid x^0)+k_0\bar{U}(x\mid y^0)\bar{U}(y\mid x^0)$$

As with the min-dominate form in Proposition 2, the elicitation of this joint utility requires only the elicitation of two marginal utility functions and a corner point.

Proof: Starting with the standard form:

$$\begin{aligned}U(x,y)=&k_*U(x\mid y)U(y\mid x^*)+k_xU(x\mid y)\bar{U}(y\mid x^*)+k_y\bar{U}(x\mid y)U(y\mid x^0)\\&+k_0\bar{U}(x\mid y)\bar{U}(y\mid x^0)\end{aligned}.$$

If x is utility independent of y then the following must be true:

$$\begin{aligned}U(x,y)=&k_*U(x\mid y^0)U(y\mid x^*)+k_xU(x\mid y^0)\bar{U}(y\mid x^*)+k_y\bar{U}(x\mid y^0)U(y\mid x^0)\\&+k_0\bar{U}(x\mid y^0)\bar{U}(y\mid x^0)\end{aligned}$$

If x is max-dominant of y then $U(x^*,y)=1$ for all y then:

$$U(x,y)=U(x\mid y^0)+k_y\bar{U}(x\mid y^0)U(y\mid x^0)+k_0\bar{U}(x\mid y^0)\bar{U}(y\mid x^0).\blacksquare$$

Proposition 4: If an attribute x is utility independent of an attribute y and x full dominates y the resulting form is:

(2.6) $$U(x,\mathrm{y})=U(x,\mathrm{y}^0)=U(x,\mathrm{y}^*)$$

Proof: If x is utility independent of y then $U(x,y)=c_1(y)+c_2(y)U(x,y^*)$. If x is min-dominant of y then $U(x^0,y)=0$ for all y, so $c_1(y)=0$. This simplifies the general form to $U(x,y)=c_2(y)U(x,y^*)$. Further, if x is min-dominant of y then $U(x^*,y)=1$ for all y, so $c_2(y)=1$. Hence: $U(x,\mathrm{y})=U(x,\mathrm{y}^*)$. A similar argument yields: $U(x,\mathrm{y})=U(x,\mathrm{y}^0)$. ■

Note that proposition 4 shows that the joint utility value is not dependent on the value of Y.

Lemma 4: If an attribute x is utility independent of an attribute y and x full dominates y, y has no impact on the value of the joint utility of value. As a result, if x is utility independent and full dominant of y, sufficient conditions exist to eliminate y.

As an example: imagine a loved one is receiving medical care for a serious illness and you want to assess which treatment to pursue. You decide that the two attributes you will assess are treatment cost and effective years added to life. You assess that if a treatment can add 20 years of life then you do not care about the cost and will give that a value of one. You also assess that if a treatment will add zero years of life then you will give that a utility value of zero, regardless of cost. You also assess that tradeoffs on years of life do not change for fixed values of cost. In this example, (2.6) holds and cost should be eliminated from consideration.

The interaction between dominance and utility independence becomes more complex when for example x is utility independent of y and y is min-dominant of x. To explore this relationship, recall that x utility independent of y implies the conditional utility function $U(x\,|\,y)$

is not dependent on the level of y; however, if y is min-dominant of x then $U\left(x \mid y^{0}\right)$ does not exist. In fact, the two conditions in this configuration are not compatible at y's lowest value, i.e. x fails to be utility independent of y when y is at its lowest value. An interpretation of this relationship is as follows: *How I feel about the level of x may not depend on the level of y unless y isn't there at all.* It is easy to see that if y is max-dominant of x instead, then $U\left(x \mid y^{*}\right)$ does not exist and the same incompatibility occurs but at y's largest value. With this in mind:

Proposition 5: Given an attribute x is utility independent of an attribute y, except at y^{0}, and x and y are mutually min-dominant, then the utility function takes the following form:

(2.7)
$$U(x, \mathrm{y})=k_{*} U\left(x \mid \mathrm{y}^{*}\right) U\left(y \mid x^{*}\right)$$

Proof: From Proposition 2: $U(x, y)=k_{*} U\left(x \mid y^{*}\right) U\left(y \mid x^{*}\right)+k_{x} U\left(x \mid y^{*}\right) \overline{U}\left(y \mid x^{*}\right)$.

Applying y is min dominant of x yields $U(x, y)=k_{*} U\left(x \mid y^{*}\right) U\left(y \mid x^{*}\right)$. ■

Note that with mutual min-dominance (as will also be the case with mutual max-dominance) traditional bounds must hold; however, I intentionally leave the form in its more general form.

Proposition 6: If both x and y are max-dominate attributes then the resulting form is:

(2.8) $$U(x,y) = U(x \mid y) + \bar{U}(x \mid y)U(y \mid x^0) + k_0\bar{U}(x \mid y)\bar{U}(y \mid x^0)$$

Proof: The standard form gives:

$$U(x,y) = k_* U(x \mid y)U(y \mid x^*) + k_x U(x \mid y)\bar{U}(y \mid x^*) + k_y \bar{U}(x \mid y)U(y \mid x^0)$$
$$+ k_0\bar{U}(x \mid y)\bar{U}(y \mid x^0)$$

Applying mutual max dominance yields:

$$U(x,y) = U(x \mid y) + \bar{U}(x \mid y)U(y \mid x^0) + k_0\bar{U}(x \mid y)\bar{U}(y \mid x^0). \blacksquare$$

As with mutual min-dominance, this form requires the elicitation of a conditional utility function, a marginal utility function, and a corner point. Applying utility dependence as is done below, reduces the conditional utility to a marginal utility function.

Proposition 7: Given an attribute x is utility independent of an attribute y, except at y^*, and x and y are mutually max-dominant, then the utility function takes the following form:

(2.9) $$U(x,y) = U(x \mid y^0) + \bar{U}(x \mid y^0)U(y \mid x^0) + k_0\bar{U}(x \mid y^0)\bar{U}(y \mid x^0)$$

Proof: From proposition 6: $U(x,y)=U(x|y)+\bar{U}(x|y)U(y|x^0)+k_0\bar{U}(x|y)\bar{U}(y|x^0)$.

Applying x is utility independent of y gives the following:

$U(x,y)=U(x|y^0)+\bar{U}(x|y^0)U(y|x^0)+k_0\bar{U}(x|y^0)\bar{U}(y|x^0)$. ■

To see an example of Proposition 7, imagine you work in business continuity and you are preparing a series of staffing options to respond to potential business disruptions due to hurricanes. You measure the effectiveness of the response based upon the availability of service at two reserve locations. If either location is 100% available, the response is a complete success; however, this is unlikely. Some combinations of availability of the two reserve locations are able to meet customer needs; however, the transfer of goods back and forth is treacherous and there is overall a negative public view on having to move customers from one location to another. It is pretty clear that preferences do not switch for fixed levels of either x or y. So in this instance, mutual utility independence holds.

Let attribute x measure the availability of reserve location one and attribute y measure the availability of reserve location two. For this example it is clear that $k_0=0$, hence our form from (2.9) is $U(x,y)=U(x|y^0)+\bar{U}(x|y^0)U(y|x^0)$. Assume that due to location two's strategic location, $U(x|y^0)=x^2$ and that $U(y|x^0)=y$. It this is the case, the final form becomes:

$$U(x,y)=x^2+y(1-x^2)$$

and is shown in Figure 5.

With these utility values in place, the options can be evaluated to assess the one with highest expected utility (see Figure 6). Assessing these options gives expected utilities of:

- Option 1: .742
- Option 2: .717
- Option 3: .730
- Option 4: .810

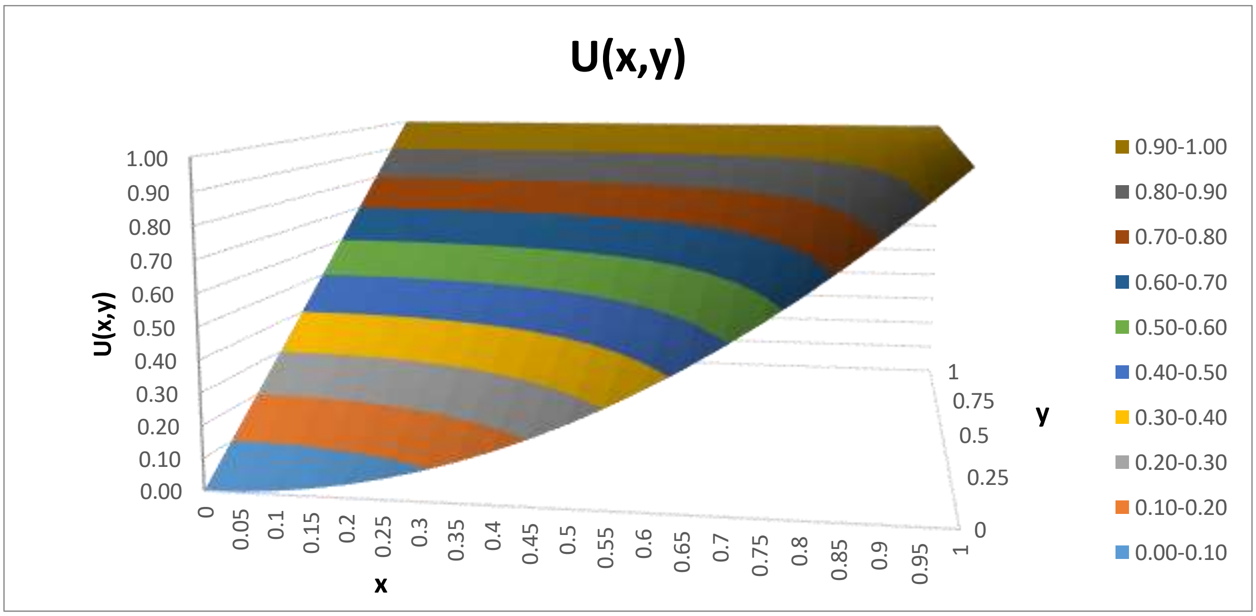

Figure 5: Graph of U(x,y) for reserve locations

Based upon this, Option 4 has a much higher expected utility and should be recommended. This example clearly give evidence that the functional form has a strong logical consistency. I feel this adds flexibility and ease of elicitation.

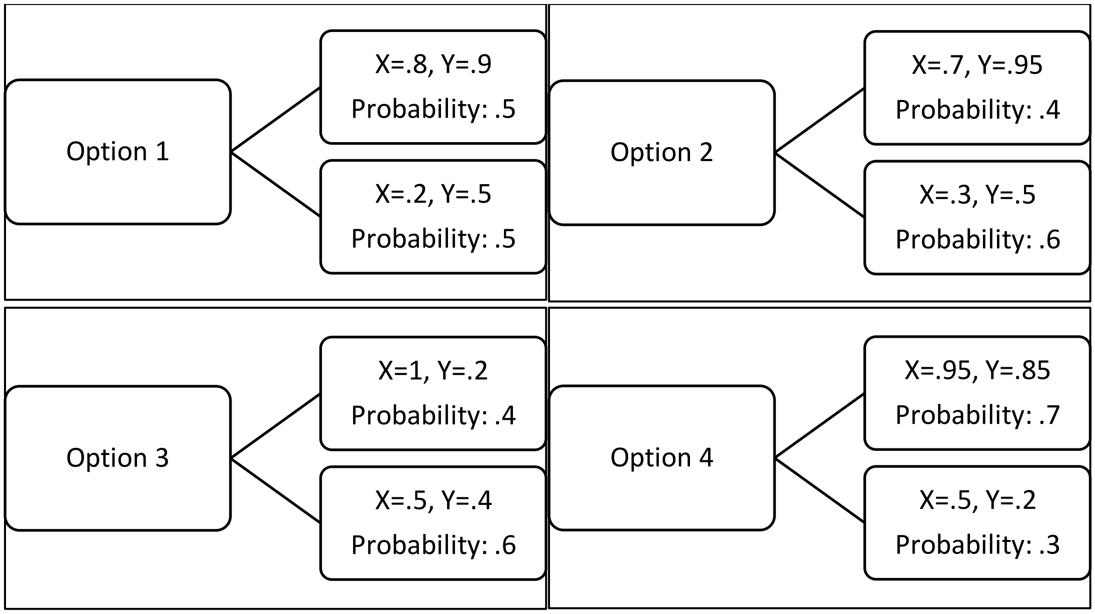

Figure 6: Staffing options for reserve locations

Recall in the previous example for full dominance and utility independence, I eliminated the cost attribute from the equation. If feelings about cost were to change slightly, it would not be eliminated. In Proposition 8, which differs slightly from Proposition 4 by changing which attribute is full-dominant., an attribute is not eliminated.

Proposition 8: Given an attribute x is utility independent of an attribute y, except at y^* and y^0, and y is full-dominant, then the utility function takes the following form:

(2.10)
$$U(x,y) = U(x \mid y')U(y \mid x^*) + \bar{U}(x \mid y')U(y \mid x^0)$$

for some $y^0 < y' < y^*$.

Proof: If y is full-dominant of x, this implies: $U(y \mid x) = \dfrac{U(x,y) - U(x,y^0)}{U(x,y^*) - U(x,y^0)}$, which yields: $U(x,y) = U(y \mid x)$. This further implies: $U(x,y) = U(x \mid y)U(y \mid x^*) + \bar{U}(x \mid y)U(y \mid x^0)$. Applying that x is utility independent of Y, gives: $U(x,y) = U(x \mid y')U(y \mid x^*) + \bar{U}(x \mid y')U(y \mid x^0)$. Note that I choose a y' that is not at an end point because y full-dominates x. ■

If in the previous example attribute y is the years of added life with a maximum of 20 and a minimum of zero, consider now that attribute x is a measure of the quality of that life on a scale of zero to five. Assume that 20 years holds some significance and that if that is attainable, the quality of the life is irrelevant. Also assume that zero years of life is completely unacceptable. In this case, attribute y would full dominate attribute x. However, assume also that for fixed levels of life quality, tradeoffs on years of life are not constant. But that high life quality is always preferred regardless of fixed years of life. In this case, attribute x is utility independent of attribute y but not the other way around. To assess this, (2.10) from Proposition 8 holds.

Assume the following utilities were elicited:

$$U(y \mid x^*) = \frac{y^3}{8000}$$

$$U(y \mid x^0) = \frac{e^y}{e^{20}}$$

$$U(x \mid y') = \frac{x^2}{25}$$

From these the full functional form is:

$$U(x,y) = \frac{x^2 y^3}{200{,}000} + \left(1 - \frac{x^2}{25}\right)\frac{e^y}{e^{20}}$$

which is shown graphically in Figure 7. You find that two courses of treatment are offered as shown in Figure 8. In this case, Treatment 1 would be preferred with an expected utility of .590 to Treatment 2's .385. The key driver here is related to attribute x : quality of life.

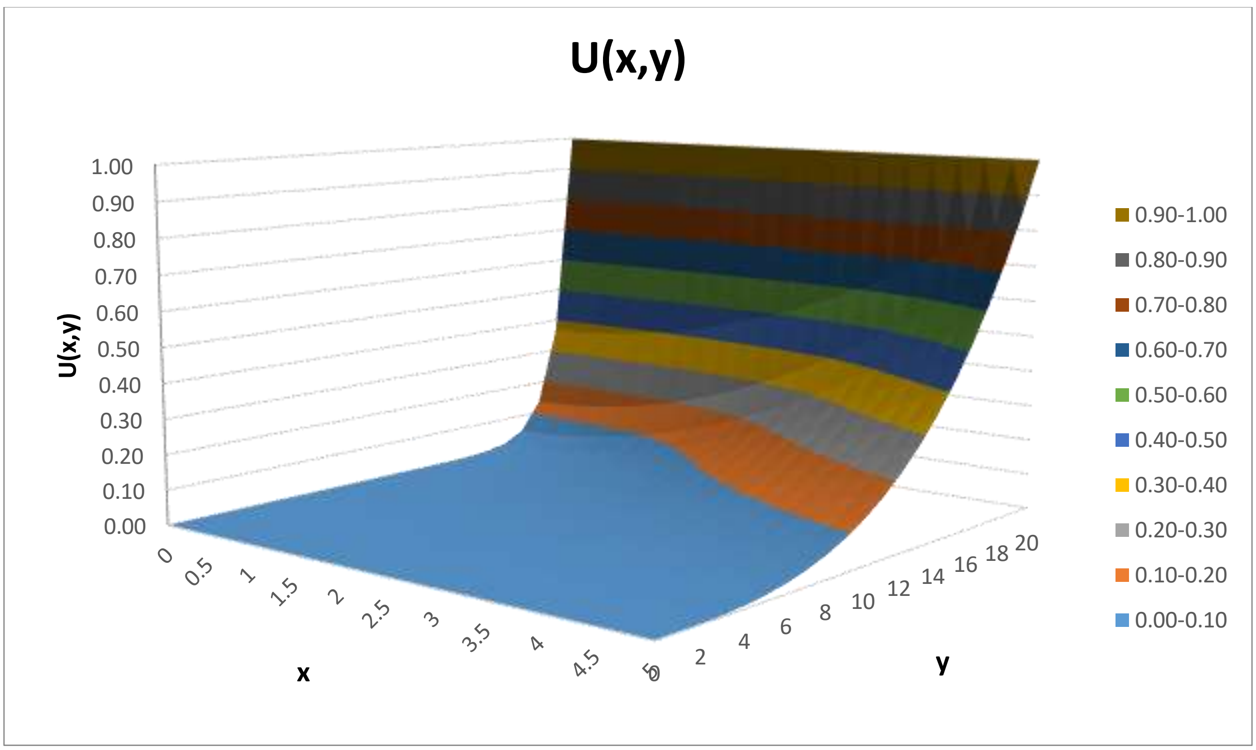

Figure 7: Graph of U(x,y)for severe illness

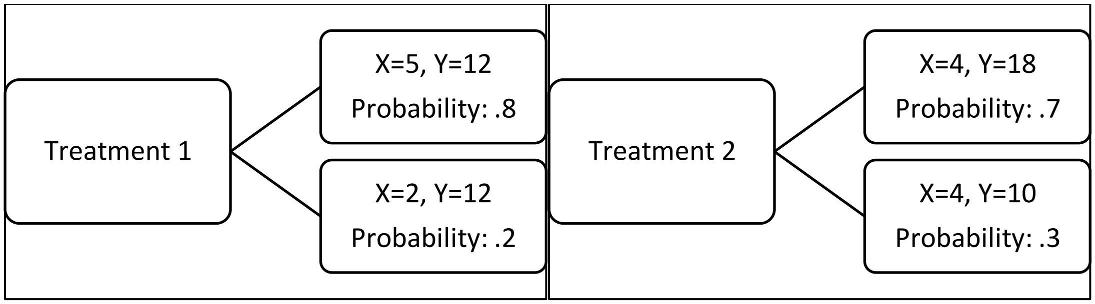

Figure 8: Treatment options for a severe illness

2.3: Min/Max-Dominance and One-Switch Independence

Abbas and Bell (2011) define one-switch independence for a multi-attribute utility function, the one-switch independence is defined as: x is one-switch independent of y if "preference between any pair of gambles on x can switch at most once as the level of y increase." From this I suggest the following:

(2.11) $$U(x \mid y) = U(x \mid y^*)\phi(\mathrm{y}) + U(x \mid y^0)\bar{\phi}(\mathrm{y}),$$

where

(2.12) $$\phi(y) \triangleq \frac{U(x \mid y) - U(x \mid y^0)}{U(x \mid y^*) - U(x \mid y^0)},$$

which is a utility function of y, monotonic, and elicitable for a single $x = x'$. Where one-switch independence gives a convenient way of evaluating $U(x \mid y)$, it does not reduce the standard form.

Proof: To see that (2.11) must hold, notice if x is one-switch independent of y, it implies:

$U(x, y) = g_0(y) + g_1(y)\left[f_1(x) + f_2(x)\phi(y)\right]$ with,

$$g_0(y) = U(x^0, y) = (k_y - k_0)U(y \mid x^0) + k_0$$

$$\begin{aligned} g_1(\mathrm{y}) &= \left[\mathrm{U}(\mathrm{x}^*,\mathrm{y}) - \mathrm{U}(\mathrm{x}^0,\mathrm{y})\right] \\ &= k_* U(y \mid x^*) + k_x \overline{U}(y \mid x^*) - k_y U(y \mid x^0) - k_0 \overline{U}(y \mid x^0) \end{aligned}$$

$f_1(x) = U(x \mid y^0)$, and

$f_2(\mathrm{x}) = U(x \mid y^*) - U(x \mid y^0)$.

Combining these yields:

$$U(x, y) = \begin{bmatrix} k_* U(y \mid x^*)\left[U(x \mid y^0)\overline{\phi}(\mathrm{y}) + U(x \mid y^*)\phi(\mathrm{y})\right] \\ + k_x \overline{U}(y \mid x^*)\left[U(x \mid y^0)\overline{\phi}(\mathrm{y}) + U(x \mid y^*)\phi(\mathrm{y})\right] \\ + k_y U(y \mid x^0)\left[1 - \left[U(x \mid y^0)\overline{\phi}(\mathrm{y}) + U(x \mid y^*)\phi(\mathrm{y})\right]\right] \\ + k_0 \overline{U}(y \mid x^0)\left[1 - \left[U(x \mid y^0)\overline{\phi}(\mathrm{y}) + U(x \mid y^*)\phi(\mathrm{y})\right]\right] \end{bmatrix}$$

Recalling the standard from, this directly implies:

$U(x \mid y) = U(x \mid y^0)\left(\overline{\phi}(\mathrm{y})\right) + U(x \mid y^*)\phi(\mathrm{y})$. ■

When discussing the interaction between utility dominance and one-switch independence, it is important to differentiate between one-switch independence as defined by Abbas and the one-switch independence that excludes the zero-switch case.

Definition: If x is one-switch independence of y but not zero-switch (i.e. utility independent), then x is <u>strong one-switch independent</u> of y.

This is an important distinction because fundamental incompatibilities exist between strong one-switch independence and dominance forms. Note: the form in (2.11) and (2.12) still applies to strong one-switch. I formalize the incompatibilities between strong one-switch and dominance as follows:

Proposition 9: If x is strong one-switch independent of y, then y can be neither min-dominant nor max-dominant.

Proof: From Abbas and Bell (2011) if x is one-switch independent of y, then the following form for the utility function is valid:

$U(x,y) = g_0(y) + g_1(y)\left[f_1(x) + f_2(x)\phi(y)\right]$ where:

$g_1(y) = \left[U(x^*,y) - U(x^0,y)\right] > 0$; however, if y is min-dominant, then:

$g_1(y^0) = \left[U(x^*,y^0) - U(x^0,y^0)\right] = 0 \not> 0$. Similarly, if y is max-dominant, then:

$g_1(y^*) = \left[U(x^*,y^*) - U(x^0,y^*)\right] = 0 \not> 0$. Also note that if y is min-dominant, then:

$f_1(x) = U(x \mid y^0) = \dfrac{U(x,y^0) - U(x^0,y^0)}{U(x^*,y^0) - U(x^0,y^0)} = \dfrac{0}{0}$, which is indeterminate. Similarly, if

y is max-dominant, then: $f_2(\mathrm{x}) = U(x \mid y^*) - U(x \mid y^0) = \frac{0}{0} - U(x, y^0)$, which is also indeterminate. ■

Note that if x is strong one-switch independent of y, x can still dominate y. Note also that it follows directly from Proposition 9 that if x and y are both strong one-switch independent neither x nor y can be either max or min dominant.

Beyond the proofs, it is not intuitive that Proposition 9 would be true; however, if you consider the simplified form in (2.11), the dynamic at play for one switch is one that assesses how attribute x behaves when attribute x is at its highest value and when attribute x is at its lowest value. Attribute y is then assessed within a ratio that measures its impact on the attribute x to find the inflection point. Dominance at the highest or lowest values for attribute x does not allow this assessment to occur. Whether this is inherent or a flaw in the one-switch form is not assessed here but is instead highlighted as potential future work.

Chapter 3: Mid-Dominance

Max- and min-dominance are not the only two cases, as one can also define x as mid-dominant of y at the value x' if $(x',y) \sim (x',y')$, ∀y, which implies that $U(x',y) = U(x',y') = k$ $\forall y$, for some value of $k \in [0,1]$. Thus, if x is at level x' then the consequence (x,y) has the same utility level k irrespective of the level of y, or x dominates y at an intermediate level. Mid-dominance occurs frequently when one attribute has both a chance of gain and loss and the other magnifies or is defined by that gain/loss. Consider a business owner who is trying to decide if she should invest in a new product. If the product is a success, the amount of press the product rollout receives would amplify the outcome of attribute x. A lot of press in conjunction with a product failure would make the failure worse; however, that same amount of press would make a product success better. By considering 'amount of press' as attribute y, and the "amount of gain/loss" to be attribute x, attribute y only matters (takes value) if attribute x moves from zero. In this case, x' represents the 'status quo' of attribute x. Note in this case, attribute y is increasing in utility when x represents a gain and decreasing in utility when x represents a loss and therefor falls into case 3.

As with other forms of dominance, at least one conditional utility function, $U(y \mid x')$, is indeterminate as the denominator is zero. This is because the joint utility function is constant along the line $x = x'$. $U(x' \mid y)$ does exist, however, as a function of y that effectively changes the reference points around k.

Two attributes x and y can only be mutually mid-dominant in an interesting special case. To see this, suppose $U(x',y) = k^1$ for all y and $U(x,y') = k^2$ for all x. Yet, $U(x',y')$ cannot equal both k^1 and k^2, unless $k^1 = k^2$. This implies that $U(x', y) = U(x, y') = k$ for all x and y, so the joint utility function is constant on a cross defined by the lines $x = x'$ and $y = y'$.

Proposition 10: If x is mid-dominant of y at x' and y is mid-dominant of x at y' then the joint utility function is constant on a cross defined by the lines $x = x'$ and $y = y'$.

Proof: To see that this is true, notice that a constant line is formed by the function $U(x', y) = k_1$. Additionally, notice that a constant line is formed by the function $U(x, y') = k_2$. Notice also that the intersection point $U(x', y') = k_1 = k_2$. ■

From a practical perspective, utility functions of this type are difficult to work with. In fact, I discuss in Chapter 4 a special case of mid-dominance that adheres to Proposition 10 and will argue that a class of problems have been unduly simplified as a result of this difficulty.

3.1: Mid-Dominance and Utility Independence

Moving the range of the joint utility such that $U(x, y) \in [-\lambda, 1]$ and fixing $k = 0$, one can see that utility independence and mid-dominance coexist nicely and generate a functional form that reduces to two marginal utility functions and the corner points. To see this, if x is utility independent of y, then $U(x, y) = c_1(y) + c_2(y)U(x, y^*)$. If x is also mid-dominant of y at x' then $U(x', y) = k = 0$ for all y, so $U(x', y) = c_1(y) + c_2(y)U(x', y^*)$ or $c_1(y) = 0$. Then solving $U(x^*, y) = c_2(y)k_*$, one can see that $c_2(y) = (k_*)^{-1}U(x^*, y)$. These together yield an identical form to utility independence and min-dominance, when the utility function range is set as aforementioned. I formalize this form as follows:

Proposition 11: For a utility function $U(x,y) \in [-\lambda, 1]$ and given an attribute x is utility independent of an attribute y and x is mid-dominant of y such that $U(x', y) = k = 0$, the utility function takes the following form:

(3.1) $$U(x,y) = k_* U(x \mid y^*) U(y \mid x^*) + k_x U(x \mid y^*) \bar{U}(y \mid x^*)$$

Note that $U(x \mid y^0) = U(x \mid y^*)$ due to utility independence. Notice that because of this, the following must be true:

(3.2) $$k_0 k_* = k_y k_x$$

This implies that $U(y \mid x^*) = U(y \mid x^0)$. This does not directly imply that y is utility independent of x; however, it does indicate that y is boundary independent of x.

Proof: Starting with the standard form:

$$U(x,y) = k_* U(x \mid y) U(y \mid x^*) + k_x U(x \mid y) \bar{U}(y \mid x^*) + k_y \bar{U}(x \mid y) U(y \mid x^0) + k_0 \bar{U}(x \mid y) \bar{U}(y \mid x^0)$$

Given that attribute x is utility independent of an attribute y, the standard form simplifies to:

$$U(x,y)=k_*U(x\mid y^0)U(y\mid x^*)+k_xU(x\mid y^0)\bar{U}(y\mid x^*)+k_y\bar{U}(x\mid y^0)U(y\mid x^0)$$
$$+k_0\bar{U}(x\mid y^0)\bar{U}(y\mid x^0)$$

Mid-point dominance gives: $U(x',y)=0$, which implies:

$U(x'\mid y)=\dfrac{-U(x^0,y)}{U(x^*,y)-U(x^0,y)}$. The value for each of these is known and therefore

$U(x'\mid y)=\dfrac{-k_0}{k_x-k_0}=\dfrac{-k_y}{k_*-k_y}$ is a constant function. The primary impact of joining together midpoint dominance in this way and utility independence is a relationship among the corner points: $k_0k_*=k_yk_x$. A side effect of this relationship is:

$U(x^0,y)=\dfrac{-U(x^*,y)U(x'\mid y)}{1-U(x'\mid y)}$, which, applying utility independence yields

$U(x^0,y)=\dfrac{-U(x^*,y)U(x'\mid y^0)}{1-U(x'\mid y^0)}$. From this one can see:

$$U(y\mid x^0)=\frac{\frac{k_0}{k_x}\left[(k_*-k_x)U(y\mid x^*)+k_x\right]-k_0}{k_y-k_0}$$
$$\frac{\left[(k_*k_0-k_xk_0)U(y\mid x^*)\right]}{k_xk_y-k_xk_0}$$
$$=\frac{(k_0k_*-k_0k_x)U(y\mid x^*)}{(k_xk_y-k_xk_0)}.$$

With $k_0k_*=k_yk_x$, this shows that $U(y\mid x^0)=U(y\mid x^*)$, which is boundary independence. Applying this, the standard form further reduces to:

$U(x,y)=k_*U(x\mid y^0)U(y\mid x^0)+k_xU(x\mid y^0)\bar{U}(y\mid x^0)$. ■

Where y boundary independent of x does hold, $U\left(y \mid x^0\right)$ is not interpretable because it is forced to be an increasing function, where $U\left(x^0, y\right)$ is decreasing. As found in Chapter 1's utility trees section, if x is utility independent of y and y is boundary independent of x, then y is utility independent of x. In this case, that finding is not accurate. This switch to a decreasing $U\left(x^0, y\right)$ actually enforces strong one-switch independence. This is a counter example to Abbas's theorem in his 2011 utility trees paper. I formalize this as follows:

Proposition 12: For a utility function $U\left(x, y\right) \in \left[-\lambda, 1\right]$ and given an attribute x is utility independent of an attribute y and x is mid-dominant of y such that $U\left(x', y\right) = k = 0$, this forces y to be boundary independent of x but not utility independent of x. This in fact forces y to be strong one-switch independent of x.

To see that Proposition 12 is true, see the proof for Proposition 11. What is further, the implication of (3.2) is that since two corner points must be known, the elicitation of only one additional point is needed. This at least cuts the elicitation by one additional point but also shows there is a way of checking the underlying assumptions using this equality.

Consider the prior example where one attribute has both a chance of gain and loss and the other magnifies or is defined by that gain/loss. Let x be the amount of revenue a new product produces (which can be negative) and let y be the amount of press coverage that exists around the product, which will amplify the gains/losses of the product. For this example, notice that tradeoffs on revenue are not dependent on the level of notoriety: that is, more money is always better. This implies that x is utility independent of y. Clearly if the product is not marketed at all, the amount of press is irrelevant to the problem and utility is fixed at zero. This fits exactly proposition 11.

For this example, assume that the company making the decision can make up to \$5 million in the first year from product sales but could lose up to \$8 million over the same period. Also assume that the distribution of potential outcomes is well defined: this decision is being made under risk. Press will be measured by the number of mentions in print, radio, and television spots, up to 1000 possible. To apply the form in proposition 11, first assess where in the range the midpoint dominance will be. Said another way, start by setting the value of $-\lambda$. This value can be set by assessing by magnitude how much worse the worst possible scenario is than the best possible scenario. To do this, I recommend using tried and true techniques, e.g. swing weighting, to assess where in a 0, … ,1 range the midpoint dominant value, k, resides and then scale as follows: $k \to 0$ and $0 \to -\lambda = \frac{k-1}{k}$. For the purposes of this example, assume k=.333, which means $-\lambda = -2$ when rescaled.

Applying the scaling above, the value of $k_y = U\left(x^0, y^*\right) = -2$ and $k_* = U\left(x^*, y^*\right) = 1$. The remaining corner point, k_x, and the marginal utilities, $U\left(x \mid y^*\right)$ and $U\left(y \mid x^*\right)$, must be assessed. For this example, assume a linear assignment of utility to gains and losses. Notice that the piecewise function is continuous but gains and losses are allowed to have different slopes, representing different attitudes toward gains and losses:

$$U\left(x \mid y^*\right) = \begin{cases} \dfrac{x}{5,000,000}, x \geq 0 \\ \dfrac{x}{4,000,000}, x < 0 \end{cases}$$

Conversely, assume that the function on press is polynomial:

$$U\left(y \mid x^{*}\right)=\frac{y^{2}}{1,000,000}$$

If the value of $k_x = .7$ is elicited, this yields $k_0 = -2 \times .7 = -1.4$, if it is needed. Note that this is logically consistent. These yield a final form as follows:

$$U\left(x, y\right)=\begin{cases}\frac{x}{5,000,000}\frac{y^{2}}{1,000,000}+.7\frac{x}{5,000,000}\left[1-\frac{y^{2}}{1,000,000}\right], x \geq 0 \\ \frac{x}{4,000,000}\frac{y^{2}}{1,000,000}+.7\frac{x}{4,000,000}\left[1-\frac{y^{2}}{1,000,000}\right], x < 0\end{cases}$$

which graphically is represented in Figure 9. Lastly, imagine their market research and accounting functions generated three different potential strategies and identified what they believe to be the potential outcomes for each of the strategies. These are presented in Figure 10.

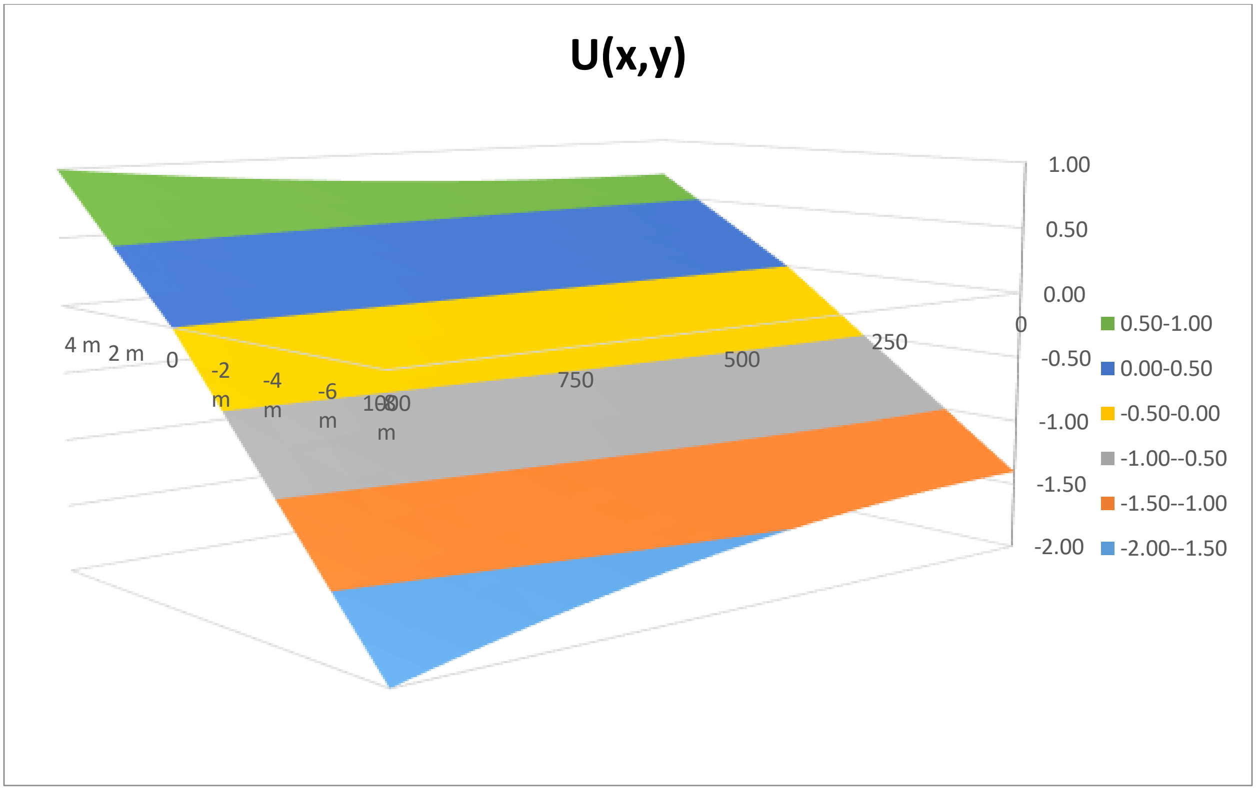

Figure 9: Graph of U(x,y) for use of press in marketing

Ignoring the inherent good/ill of press on the broader company and focusing only on expected value, Strategy 3 edges out Strategy 1 as the best option, with a $750,000 expected gain as compared to $700,000 for Strategy 1 and $325,000 for Strategy 2. However, using the utility values, which include both the effect of press on the broader company and built-in loss aversion, Strategy 1 has the largest expected utility of .07, edging out Strategy 3 at .06 and Strategy 2 at .02. This change in preference makes sense, as the potential loss for Strategy 3 is higher and the impact of press on the broader business if that loss were to occur is also higher. A data table is included in the appendix.

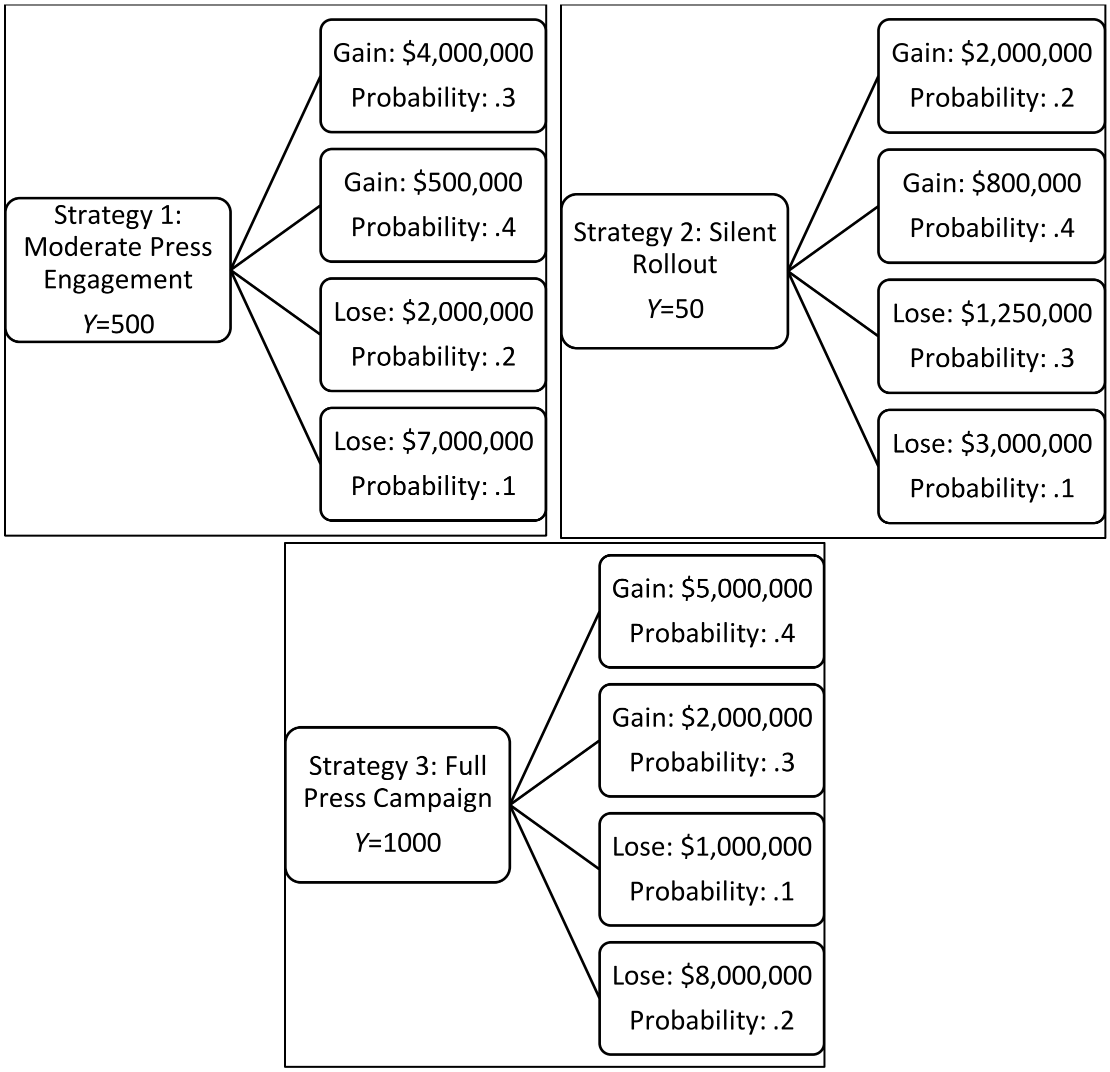

Figure 10: Strategy options for use of press in marketing

3.2: Mid-Dominance and One-Switch Independence

Maintaining the revised range for the joint utility, suppose that y is mid-dominant over x at y' and x is one-switch independent of y. The mid-dominance condition implies that $(x, y') = k = 0 \quad \forall x$, for some value of $k \in [0,1]$. Further, as stated prior, the form for one-switch is the standard form where the following is also true:

$$U(x \mid y) = U(x \mid y^0)(\bar{\phi}(\mathrm{y})) + U(x \mid y^*)\phi(\mathrm{y})$$

where

$$\phi(y) = \frac{u(x \mid y) - u(x \mid y^0)}{u(x \mid y^*) - u(x \mid y^0)}.$$

Though $\phi(y')$ is technically indeterminate, Chapter 4 shows that this does not prove to be an issue in practice. This is also consistent with all literature on dominance to date. Thus, this combination of independence conditions is possible, but does not provide much specificity to the form, as unlike utility independence, the ability to switch preference for attribute x as the value of attribute y varies, requires more elicitation in practice.

It is noteworthy to indicate that for at least once class of problems x being mid-dominant of y means that gambles on y for fixed x, vary exactly once as the value of x is changed. An example of this is the example described above in figures 9 and 10. To see that this is true, consider the following preferences of (x, y):

$(\tilde{x}, \tilde{y}) \prec (\tilde{x}, y^*)$ for all $\tilde{y} < y^*$ and $\tilde{x} > 0$; however, $(\tilde{x}, \tilde{y}) \succ (\tilde{x}, y^*)$ for all $\tilde{y} < y^*$ and $\tilde{x} < 0$. Notice that indifference occurs at $\tilde{x} = 0$ since x dominates y at this point. This is formalized as follows:

Proposition 13: For the class of problems where attribute x represents the chance for both a gain and a loss and attribute y magnifies that gain/loss, attribute, x mid-dominant of attribute y at $x=0$ implies that attribute y is strong one-switch independent of attribute x with preferences on y switching only between $x>0$ and $x<0$.

The class of problems described in Proposition 13 comprise a superset that include single attribute decision analysis with the probability is treated as an attribute. They also stand as a counter-example described in Proposition 12. The proof for Proposition 11 stands as sufficient evidence to conclude the findings in Proposition 13.

Chapter 4: Application of Mid-Dominance to the Standard Gamble

Sarin and Weber (1993) introduced a class of risk-value models where preferences between lotteries are based on two attributes, the value of the lottery and the risk of the lottery. Specifically, for the lottery X, the function $V(X)$ represents the value of the lottery and the function $R(X)$ represents the risk . When choosing between lotteries X and Y, preferences can be represented by a risk-value model if

(4.1) $$X \succ Y \Leftrightarrow f\left(V(X), R(X)\right) > f\left(V(Y), R(Y)\right),$$

where the function f reflects the trade-offs between value and risk. The value of the lottery V could be the expected value or some other measure of central tendency; however, several options have been proposed for R (see Sarin and Webber (1993), Dyer and Jia (1997), and Butler et al. (2005) for a range of options). Risk-value models have also been studied from a theoretical and axiomatic perspective by Schmidt (2003) and Mitchell and Gelles (2003).

I follow this approach of representing preferences over lotteries in terms of the lottery's attributes. Instead of looking at overall risk and overall value, I consider at a more foundational level the probabilities and outcomes as attributes of a standard gamble (see Figure 11). Expected utility (von Neumann and Morgenstern 1944), rank-dependent utility (Quiggin 1982, 1991), and prospect theory (Kahneman and Tversky 1979, Tversky and Kahneman 1992) follow this approach in spirit, but this explicit use of multi-attribute utility theory allows for application of specific results and independence conditions developed in the multi-attribute literature.

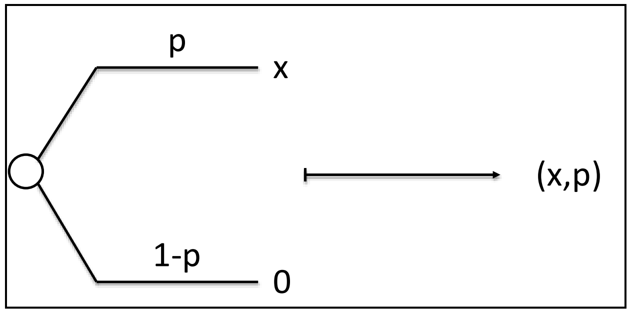

Figure 11: Representing the standard gamble as a two-attribute alternative

In this chapter, I consider the development of first a joint utility function as a multi-attribute problem, where the probability an event occurs is incorporated as attribute. I then speak to the necessary and sufficient condition to aggregate these join utilities using a representing function over the utilities. I consider the implications on the form of the representing function of assumptions of utility independence (Keeney and Raiffa 1976) and one-switch independence (Abbas and Bell 2011, Tsetlin and Winkler 2012) through the use mid-dominance, as established in Chapter 3.

4.1: Modeling the Standard Gamble using Multiple Attributes

Consider the standard gamble in Figure 11. There is a probability p of receiving the outcome x, otherwise the decision maker receives 0, the status quo. Clearly, non-zero reference points can be selected; however, I choose zero without loss of generality. Consider (x, p) as the notation for the standard gamble with bounds for the attributes $x \in \left[x^0, x^*\right]$ and $p \in \left[p^0, p^*\right]$, where $p^0 = 0$ and $p^* = 1$. The joint utility function is then

(4.2) $$u(x, p): \mathbb{R} \times [-\lambda, 1].$$

As previously stated, expected utility theory is the leading prescriptive theory for choice under risk. Descriptive generalizations of EU have been proposed including Rank Dependent Utility (RDU) (Quiggin 1982, 1991) and Prospect Theory (Kahneman and Tversky 1979, Tversky and Kahneman 1992). By assuming that x and p are mutually utility independent and that both attributes are min-dominant attributes, applying Abbas and Howard (2005) shows that $u(x, p) = u(x)u(p)$. If $u(p)$ is linear in p, the form reduces to $u(x, p) = u(x)p$, the expected utility of the standard gamble.

The assumption of mutual utility independence for x and p is questionable in a descriptive setting. When $p > p^0$ most decision makers will prefer $(x, p) \succ (x', p)$ for $x > x'$. Thus, preferences over x do not depend on p for $p > p^0$. However, when $p = p^0$ decision makers will be indifferent $(x, p) \sim (x', p)$. Thus, say x can be considered utility independent of p for $p > p^0$ (Keeney and Raiffa 1976). Also, p is utility independent of x when dealing with only gains (or symmetrically only losses), but not in the general case. Further, if $p > p'$ then most decision makers would prefer $(x, p) \succ (x, p')$ when x is a gain ($x > 0$), but $(x, p) \prec (x, p')$ when x is a loss ($x < 0$). In decision making under both risk and uncertainty this is known as reference dependence, but in a multi-attribute sense this implies that p is one-switch independent of x (Abbas and Bell 2011, Tsetlin and Winkler 2012) when dealing with both gains and losses. As these preferences hold whether x is certain or uncertain (i.e. a nested standard gamble), p is also contextual one-switch independent of x (Abbas and Bell 2012).

The assumption of min-dominance can also be questioned from a descriptive viewpoint. It is clear that $(x, p^0) \sim (x', p^0)$ for all x and x' as there is no chance of receiving the payoff. Further, $(0, p) \sim (0, p')$ for all p and p' as the payoff is always zero no matter what the probability may be. This same reasoning yields $(x, p^0) \sim (0, p)$ for all x and p. Thus, x mid-dominates p at $x = 0$.

Given this treatment of the standard gamble in a multi-attribute setting, the assumptions necessary to obtain a utility function with the form of expected utility are not descriptively appropriate. In the next section, I use multi-attribute concepts to develop a more appropriate representing function.

4.2: A One-Switch Representing Function for the Standard Gamble

For a standard gamble denoted (x, p), it has been shown that x is utility independent of p for all $p > p^0$, p is one-switch independent of x, p is min-dominant over x, and x mid-dominates p at $x = 0$. What form does this imply for $u(x, p)$? It is standard in the multi-attribute literature to scale the utility function so that $u(x^0, p^0) = 0$ and $u(x^*, p^*) = 1$. I follow this convention, but $(x^0, p^0) \succ (x^0, p)$ when $p > p^0$, so the utility function is bounded on $[-\lambda, 1]$, not $[0,1]$. As p is min-dominant over x, $u(x, p^0) = u(x^0, p^0) = 0$ for all x. Further, as x mid-dominates p at $x = 0$, $u(0, p) = u(0, p^0) = 0$.

Proposition 14: Given x utility independent of p for $p > p^0$, p is one-switch independent of x, p is min-dominant over x, and x mid-dominates p at $x = 0$, the joint utility function for the standard gamble is (note that : x utility independent of p for $p > p^0$ and x mid-dominates p at $x = 0$ gives use that p is also boundary independent of x):

(4.3) $$U(x, p) = (1+\lambda)U(x \mid p^*)U(p \mid x^*) - \lambda U(p \mid x^*)$$

From inspection it is easily seen that if the weighting on p is linear such that $U(p \mid x^*) = p$, the above reduces to a scaled form of EU with a weighting on λ to adjust the utility down by $-p\lambda$.

Proof: I start with the known corner points based on dominance:

$U(x^*, p^*) = 1, U(x^0, p^*) = -\lambda, U(x^*, p^0) = k_x = 0 = U(x^0, p^0)$. The following comes from Abbas and Bell (2011) and I solve for individual pieces.

$U(x, p) = g_0(x) + g_1(x)\left[f_1(p) + f_2(p)\phi(x)\right]$, with:

$$g_0(x) = U(x, p^0) = 0$$

$$g_1(x) = U(x, p^*) - U(x, p^0) = U(x, p^*)$$

$$f_1(p) = U(p \mid x^0) = \frac{U(x^0, p) - U(x^0, p^0)}{U(x^0, p^*) - U(x^0, p^0)} = (-1)\frac{U(x^0, p)}{\lambda}$$

$$U(p \mid x^*) = \frac{U(x^*, p) - U(x^*, p^0)}{U(x^*, p^*) - U(x^*, p^0)} = U(x^*, p)$$, and

$$f_2(p)=\frac{\lambda U\left(x^*,p\right)+U\left(x^0,p\right)}{\lambda}.$$

From Abbas and Bell (2011), I know that:

$$\begin{aligned}U(x,p')&=U\left(x,p^0\right)+\left[U\left(x,p^*\right)-U\left(x,p^0\right)\right]U(p'\mid x)\\&=U\left(x,p^*\right)U(p'\mid x)\\U(x,p')&=U\left(x^0,p'\right)+\left[U\left(x^*,p'\right)-U\left(x^0,p'\right)\right]U(x\mid p')\end{aligned}.$$

From this: $U\left(x,p^*\right)U(p'\mid x)=U\left(x^0,p'\right)+\left[U\left(x^*,p'\right)-U\left(x^0,p'\right)\right]U(x\mid p')$.

Hence: $U(p'\mid x)=\dfrac{U\left(x^0,p'\right)+\left[U\left(x^*,p'\right)-U\left(x^0,p'\right)\right]U(x\mid p')}{U\left(x,p^*\right)}$,

$U\left(p'\mid x^*\right)=U\left(x^0,p'\right)+\left[U\left(x^*,p'\right)-U\left(x^0,p'\right)\right]U\left(x^*\mid p'\right)$, and

$$U\left(p'\mid x^0\right)=(-1)\frac{U\left(x^0,p'\right)+\left[U\left(x^*,p'\right)-U\left(x^0,p'\right)\right]U\left(x^0\mid p'\right)}{\lambda}.$$

Applying that x is utility independent of p yields:

$$U(x\mid p')=U\left(x\mid p^*\right)=\frac{U\left(x,p^*\right)-U\left(x^0,p^*\right)}{U\left(x^*,p^*\right)-U\left(x^0,p^*\right)}=\frac{U\left(x,p^*\right)+\lambda}{1+\lambda},$$

$$U\left(x^*\mid p'\right)=U\left(x^*\mid p^*\right)=\frac{U\left(x^*,p^*\right)-U\left(x^0,p^*\right)}{U\left(x^*,p^*\right)-U\left(x^0,p^*\right)}=1, \text{ and}$$

$$U\left(x^0\mid p'\right)=U\left(x^0\mid p^*\right)=\frac{U\left(x^0,p^*\right)-U\left(x^0,p^*\right)}{U\left(x^*,p^*\right)-U\left(x^0,p^*\right)}=0.$$

Hence:

$$U\left(p' \mid x\right)=\frac{U\left(x^{0}, p'\right)+\left[U\left(x^{*}, p'\right)-U\left(x^{0}, p'\right)\right]\left[\frac{U\left(x, p^{*}\right)+\lambda}{1+\lambda}\right]}{U\left(x, p^{*}\right)}$$

$$=\frac{(1+\lambda) U\left(x^{0}, p'\right)+\left[U\left(x^{*}, p'\right)-U\left(x^{0}, p'\right)\right]\left[U\left(x, p^{*}\right)+\lambda\right]}{(1+\lambda) U\left(x, p^{*}\right)}$$

Since, $U\left(p' \mid x^{*}\right)=U\left(x^{*}, p'\right)$ and $U\left(p' \mid x^{0}\right)=(-1) \frac{U\left(x^{0}, p'\right)}{\lambda}$, these can be

substituted into: $\phi(x)$. Solving this yields:

$$\phi(x)=\frac{\frac{(1+\lambda) U\left(x^{0}, p'\right)+\left[U\left(x^{*}, p'\right)-U\left(x^{0}, p'\right)\right]\left[U\left(x, p^{*}\right)+\lambda\right]}{(1+\lambda) U\left(x, p^{*}\right)}+\frac{U\left(x^{0}, p'\right)}{\lambda}}{\frac{\lambda U\left(x^{*}, p'\right)+U\left(x^{0}, p'\right)}{\lambda}},$$

which simplifies to: $\phi(x)=\frac{\left[U\left(x, p^{*}\right)+\lambda\right]}{(1+\lambda) U\left(x, p^{*}\right)}$.

Knowing that $\phi(x)$ must be monotonic, to verify this:

$$\frac{d \phi(x)}{d x}=\frac{(1+\lambda) U\left(x, p^{*}\right) U'\left(x, p^{*}\right)-\left[U\left(x, p^{*}\right)+\lambda\right](1+\lambda) U'\left(x, p^{*}\right)}{(1+\lambda)^{2} U\left(x, p^{*}\right)^{2}}$$, which

simplifies to: $\frac{d \phi(x)}{d x}=\frac{-\lambda U'\left(x, p^{*}\right)}{(1+\lambda) U\left(x, p^{*}\right)^{2}}$.

This derivative is always negative, hence $\phi(x)$ is monotonic. To get the final form, I

substitute all components back into the original form:

$$U(x,p) = U(x,p^*)\left[(-1)\frac{U(x^0,p)}{\lambda} + \frac{\lambda U(x^*,p) + U(x^0,p)}{\lambda}\frac{\left[U(x,p^*)+\lambda\right]}{(1+\lambda)U(x,p^*)}\right]$$

$$= \frac{U(x^*,p)\left[U(x,p^*)+\lambda\right] + U(x^0,p)\left[1-U(x,p^*)\right]}{(1+\lambda)}$$

$$= U(p \mid x^*)U(x \mid p^*) - \lambda U(p \mid x^0)\bar{U}(x \mid p^*)$$

Finally, applying boundary independence, yields:

$$U(x,p) = (1+\lambda)U(x \mid p^*)U(p \mid x^*) - \lambda U(p \mid x^*). \blacksquare$$

4.3: Restrictions on the form of the representing function

As you can see, I question the assumption of mutual utility independence between x and p in a descriptive sense. When $p > p^0$ most decision makers will prefer $(x,p) \succ (x',p)$ for $x > x'$. Thus, preferences over x do not depend on p for $p > p^0$. However, when $p = p^0$ decision makers will be indifferent $(x,p) \sim (x',p)$. Thus, x can be considered utility independent of p for $p > p^0$, but not for all p, (Keeney and Raiffa 1976). Also, if $p > p'$ then most decision makers would prefer $(x,p) \succ (x,p')$ when $x > 0$, a gain, but $(x,p) \prec (x,p')$ when $x < 0$, a loss. In decision making under risk this is known as reference dependence, but in a multi-attribute sense p is one-switch independent of x (Abbas and Bell 2011) when dealing with both gains and losses. In fact, as I would like to allow uncertainty about x and p then I assume that p is strong one-switch independent of x (Abbas and Bell 2012), leading to the following theorem.

Theorem: For the attributes x and p, let p be strong one-switch independent of x, x be utility independent of p for $p > p^0$, and assume that x is mid-dominant at 0 and p is min-dominant at p^0. Further, $u(x, p)$ is scaled $u(x, p)$ so that $u(0, p) = u(x, p^0) = 0$ for all x and p and $u(x^*, p^*) = 1$. Any utility function satisfying these conditions must take the form $u(x, p) = w(p)u(x)$ where $w(p^0) = 0$, $w(p^*) = 1/u(x^*)$, and $u(x)$ takes one of the following forms

$$u(x) = \begin{cases} x \\ 1 - e^{cx} \\ xe^{cx} \\ (1 - e^{cx})e^{dx} \end{cases}.$$

Proof: If p is strong one-switch independent of x then Abbas and Bell (2012) show that $u(x, p)$ must take one of the following forms:

$$u(x, p) = \begin{cases} ax^2 + [f_1(p) + f_2(p)x] \\ ax + [f_1(p) + f_2(p)e^{cx}] \\ [f_1(p) + f_2(p)x]e^{cx} + m \\ [f_1(p) + f_2(p)e^{cx}]e^{dx} + h \end{cases}$$

Case 1: Assuming $u(x, p) = ax^2 + \left[f_1(p) + f_2(p)x \right]$ then requiring that

$u(0, p) = a0^2 + \left[f_1(p) + f_2(p)0 \right] = 0$ for all p implies that $f_1(p) = 0$, which

reduces the form to $u(x, p) = ax^2 + f_2(p)x$. Further requiring that

$u(x, p^0) = ax^2 + f_2(p^0)x = 0$ for all x implies that $a = 0$ and $f_2(p^0) = 0$,

reducing the form to $u(x, p) = f_2(p)x$ where $f_2(p^0) = 0$. Lastly, given that

$u(x^*, p^*) = 1$ then $f_2(p^*) = \frac{1}{x^*}$, leaving the final form as $u(x, p) = f_2(p)x$, where

$f_2(p^0) = 0$ and $f_2(p^*) = \frac{1}{x^*}$, a form that satisfies x utility independent of p for

$p > p^0$.

Case 2: Assuming $u(x, p) = ax + \left[f_1(p) + f_2(p)e^{cx} \right]$ then requiring that

$u(0, p) = a0 + \left[f_1(p) + f_2(p)e^{c0} \right] = 0$ for all p implies that $f_1(p) = 0$ and

$f_2(p) = 0$ for all p or $f_1(p) + f_2(p) = 0$. If $f_1(p) = 0$ and $f_2(p) = 0$ for all p

then it is not a proper representing function, so I discard this case, leaving

$u(x, p) = ax + f_1(p)(1 - e^{cx})$. Further requiring that

$u(x, p^0) = ax + f_1(p)(1 - e^{cx}) = 0$ for all x implies that $a = 0$ and $f_1(p^0) = 0$.

Lastly, given that $u(x^*, p^*) = 1$ then $f_1(p^*) = \frac{1}{x^*}$. So, $u(x, p) = f_1(p)(1 - e^{cx})$

where $f_1(p^0) = 0$ and $f_1(p^*) = \frac{1}{x^*}$.

Case 3: Assuming that $u(x,p)=\left[f_1(p)+f_2(p)x\right]e^{cx}+m$ then requiring that $u(0,p)=\left[f_1(p)+f_2(p)0\right]e^{c0}+m=0$ for all p implies that $m=0$ and $f_1(p)=0$ for all p, reducing the form to $u(x,p)=f_2(p)xe^{cx}$. Further requiring that $u(x,p^0)=\left[f_2(p^0)x\right]e^{cx}=0$ for all x implies that $f_2(p^0)=0$. Lastly, requiring that $u(x^*,p^*)=f_2(p^*)x^*e^{cx^*}=1$ implies that $f_2(p^*)=\frac{1}{x^*e^{cx^*}}$. So, $u(x,p)=f_2(p)xe^{cx}$, where $f_2(p^0)=0$ and $f_2(p^*)=\frac{1}{x^*e^{cx^*}}$, a form that satisfies x utility independent of p for $p>p^0$.

Case 4: Assuming that $u(x,p)=\left[f_1(p)+f_2(p)e^{cx}\right]e^{dx}+h$ then requiring that $u(0,p)=\left[f_1(p)+f_2(p)e^{c0}\right]e^{d0}+h=0$ for all p implies that $h=0$ and either $f_1(p)=0$ and $f_2(p)=0$ for all p or $f_1(p)+f_2(p)=0$ for all p. If $f_1(p)=0$ and $f_2(p)=0$ for all p then it is not a proper representing function, so I discard this case, leaving $u(x,p)=f_1(p)\left(1-e^{cx}\right)e^{dx}$. Further requiring that $u(x,p^0)=f_1(p^0)\left(1-e^{cx}\right)e^{dx}=0$ for all x implies that $f_1(p^0)=0$. Lastly, requiring that $u(x^*,p^*)=f_1(p^*)\left(1-e^{cx^*}\right)e^{dx^*}=1$ implies that $f_1(p^*)=\frac{1}{\left(1-e^{cx^*}\right)e^{dx^*}}$. So, $u(x,p)=f_1(p)\left(1-e^{cx}\right)e^{dx}$, where $f_1(p^0)=0$ and

$$f_1\left(p^*\right)=\frac{1}{\left(1-e^{cx^*}\right)e^{dx^*}}$$, a form that satisfies x utility independent of p for $p > p^0$.

■

The form in the above theorem is recognizable as the original form of prospect theory (Kahneman and Tversky 1979), but with restrictions on the form of the value (utility) function. In fact, it is also equivalent to CPT (Tversky and Kahneman 1992) as I am restricting attention to the standard gamble in this section. The possible forms for the value function are linear, exponential, linear-exponential, and double-exponential. The form for $w(p)$ is not restricted, however, beyond the scaling requirements $w\left(p^0\right)=0$, $w\left(p^*\right)=1/u\left(x^*\right)$. This connection clearly demonstrates both the potential use of mid-dominance as a concept and the link between mid-dominance, one-switch independence, and prospect theory.

4.4: Gambles with *n* Outcomes

As I mentioned earlier, cumulative prospect theory, as with prospect theory and rank dependent utility, assume that the x's are additive to apply a modified form of expectation. In essence, this is not an altogether strange thing to do. Consider, you have a series of potential outcomes $x_1,\dots,x_n$ and probabilities $p\left(x_1\right),\dots,p\left(x_n\right)$ that those outcomes occur, forming a probability distribution over the potential outcomes. Any weightings on the probabilities $w\left[p\left(x_1\right)\right]=q\left(x_1\right),\dots,w\left[p\left(x_n\right)\right]=q\left(x_n\right)$ still function as a distribution over the potential outcomes[4]. This is stated without loss of generality, even though cumulative prospect theory and rank dependent utility both utilize a ranking system. However, this system still proxies a weighting on the probability, generating a distribution over the outcomes. As such, each of these systems calculates a type of expectation.

This system integrates the probability into the utility as an attribute. As such, simply applying a distribution over the potential outcomes is not appropriate. In fact, the incorporation in this way changes the nature of the utility on two attributes such that calculating a traditional expectation doesn't work; however, I still seek a measure of center. As such, I define a value function over the joint utilities:

$$V\left[U\left(x_1, p_1\right), \ldots, U\left(x_n, p_n\right)\right] \tag{4.4}$$

The nature of this value function is knowable. Consider, all joint utilities $U\left(x_i, p_i\right)$ are trade off consistent, that is global trade off consistency exists. To see that this is true, recall the nature of a cardinal utility function, such as this joint utility. Cardinal utility functions preserve not only preference order but also the magnitude of those differences in preference. Since this form also includes the probability, even that is baselined in magnitude. This directly implies trade off consistency. Keeney and Raiffa (1976) observes that if global trade off consistency holds for a set of arguments, $y_1, \ldots, y_n$, then a value function defined over those arguments will be additive. This is significant, as it allows for the following:

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[4] Note that for Cumulative Prospect Theory the weighting function on the probabilities are not required to sum to one.

Proposition 15: Given a set of joint utility values $U(x_1, p_1), \ldots, U(x_n, p_n)$, defined over a set of prospects $(x_1, p_1), \ldots, (x_n, p_n)$, that fully define both the attribute x and the probability that x occurs, an additive value function allows for an expectation-like measure of center:

(4.5) $$V\left[U(x_1, p_1), \ldots, U(x_n, p_n)\right] = \sum_{i=1}^{n} U(x_i, p_i)$$

Proof: To see that this is true, assume $U(x_t, p_t) = \alpha_t$ and $U(x_w, p_w) = \alpha_w$ are two branches of a lottery l_1. Since these are cardinal utility functions, if I am able to change the values of x_t and/or p_t such that the joint utility value increases to $\alpha_t + \delta$ and I am able to change the values of x_w and/or p_w such that the joint utility value decreases to $\alpha_w - \delta$ the net impact is a lottery l_2 such that I am indifferent between the two lotteries. This is global tradeoff consistency. To see that this implies an additive value function over the utilities, note that for lottery l_1, $V\left[U(x_1, p_1), \ldots, U(x_n, p_n)\right] = \alpha_1 + \ldots + \alpha_t + \ldots + \alpha_w + \ldots + \alpha_n$ and that for lottery l_2,

$$\begin{aligned} V\left[U(x_1, p_1), \ldots, U(x_n, p_n)\right] &= \alpha_1 + \ldots + \alpha_t + \delta + \ldots + \alpha_w - \delta + \ldots + \alpha_n \\ &= \alpha_1 + \ldots + \alpha_t + \ldots + \alpha_w + \ldots + \alpha_n. \end{aligned}$$

This generates equal value function totals for each lottery. ■

As an example, returning to the standard gamble yields:

$$V\left[U\left(x,p\right),U\left(0,1-p\right)\right]=\left(1+\lambda\right)\left[U\left(x\mid p^{*}\right)U\left(p\mid x^{*}\right)+U\left(0\mid p^{*}\right)U\left(1-p\mid x^{*}\right)\right]$$
$$-\lambda\left[U\left(p\mid x^{*}\right)+U\left(1-p\mid x^{*}\right)\right]$$

The first piece of this is original prospect theory. The latter part simplifies to λ if I force the marginal utilities on p to sum to one. Applying this yields:

$$V\left[U\left(x,p\right),U\left(0,1-p\right)\right]=-\lambda+\left(1+\lambda\right)\left[U\left(x\mid p^{*}\right)U\left(p\mid x^{*}\right)+U\left(0\mid p^{*}\right)U\left(1-p\mid x^{*}\right)\right]$$

Which is a scaled and adjusted form of original prospect theory. As such, applying a linear weighting on p, such that $U\left(p_i\mid x^{*}\right)=p_i$, yields a scaled and adjusted expected utility. This strongly implies that the joint utility on x and p in a multiattribute sense accounts for both the utility on x and the modified distribution over x. This should be quite natural. Extending this to n potential outcomes, provides:

(4.6) $$V\left[U\left(x_1,p_1\right),\ldots,U\left(x_n,p_n\right)\right]=\left(1+\lambda\right)\sum_{i=1}^{n}U\left(x_i\mid p^{*}\right)U\left(p_i\mid x^{*}\right)-\lambda\sum_{i=1}^{n}U\left(p_i\mid x^{*}\right)$$

The first piece of (4.6) is clearly original prospect theory with a scaling to $\left[-\lambda,1\right]$. The latter piece moves the starting value down. This argues heavily for $\sum_{i=1}^{n} U\left(p_i \mid x^*\right)=1$, which is not fixed as part of original prospect theory. Setting this however is consistent with the form and allows for a fully established distribution over x. Adding this, yields:

(4.7) $$V\left[U\left(x_1, p_1\right),\ldots,U\left(x_n, p_n\right)\right]=-\lambda+\left(1+\lambda\right)\sum_{i=1}^{n} U\left(x_i \mid p^*\right)U\left(p_i \mid x^*\right)$$

Finally, I can allow the utility on x to be piecewise, which adjusts for differing attitudes on gains and losses:

(4.8) $$U\left(x_i \mid p^*\right)=\begin{cases} u^+\left(x_i \mid p^*\right), x_i \geq 0 \\ u^-\left(x_i \mid p^*\right), x_i < 0 \end{cases}$$

Applying this gives a final form that succeeds in separating gains from losses for utility on x ; however, the separation of weighting functions each for gains and losses is not supported by the theory. Recall that I previously established in Proposition 14 that p is boundary independent of x. This means that $U\left(p \mid x^*\right)=U\left(p \mid x^0\right)$. As such, there can be no difference between the weighting of the probabilities for gains and losses. Said another way, the utility function over the probability for the largest gain must be equal to the utility function over the probability for the largest loss. This gives theoretical support in favor of criticisms of Cumulative Prospect Theory that Wu and Markle 2008 presented. They acknowledge this is understudied in the literature up to 2008.

Bromiley 2010 underscores what Wu and Markle found and both argue that the theoretical underpinning of Prospect Theory is in gambles evaluated separately for gains and losses.

This theory is compatible with the type of cumulative probabilities included in RDU and CPT that seek to ensure the desired outcome of obeying stochastic dominance. As such, I present an alternative form to CPT, namely One-Switch Prospect Theory (OSPT) as such:

(4.9)
$$OSPT = -\lambda + (1+\lambda)\sum_{i=1}^{n} U\left(x_i \mid p^*\right) W\left(p_i\right)$$

where $W\left(p_i\right) = U\left(p_1 + \cdots + p_i \mid x^*\right) - U\left(p_1 + \cdots + p_{i-1} \mid x^*\right)$ and $U\left(x_i \mid p^*\right)$ may be piecewise as proposed in (4.8).

This form includes, reference dependence, loss aversion and cumulative weighting functions on the probabilities that speaks to interpretation of or reaction to chance while obeying stochastic dominance. It maintains gain/loss separability for the utility on x but applies a single weighting function to the probabilities. It is axiomatically linked back to MAUT in a way that cumulative prospect theory is not, while also restricting the representing function shape.

Chapter 5: Conclusion and Next Steps

I extended the concept of attribute dominance (Abbas and Howard 2005) to max-dominance, where an attribute at its maximum level makes the levels of the other attributes irrelevant, and mid-dominance, where an attribute at a mid-level makes the levels of the other attributes irrelevant. I studied the interactions of the three forms of dominance with utility independence and used them to develop a representing function for standard gambles along with one-switch independence (Abbas and Bell 2011, 2012). The representing function developed is the same as prospect theory (Kahneman and Tversky 1979, Tversky and Kahneman 1992), but the utility function must take either linear, exponential, linear-exponential, or double-exponential forms. Lastly, I was successful in creating a full alternative to Cumulative Prospect Theory, derived from within the Multiattribute literature.

There are numerous areas for further research. First, I would seek to expand the view of max and full dominance to more than two attributes. This would allow a fuller definition of the interaction with other forms of independence (e.g. corner independence). Second, a deeper look at the one-switch form proposed by Abbas and Bell is needed. Third, an interesting extension would be to consider further the implications of a single weighting function to CPT (Tversky and Kahneman 1992).

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Appendix: Data Tables:

The following table documents the values and calculations for the business continuity example:

x	y	U(x,y)	p	EU
0.800	0.900	0.964	0.500	0.742
0.200	0.500	0.520	0.500	
0.700	0.950	0.975	0.400	0.717
0.300	0.500	0.545	0.600	
1.000	0.200	1.000	0.400	0.730
0.500	0.400	0.550	0.600	
0.950	0.850	0.985	0.700	0.810
0.500	0.200	0.400	0.300	

The following table gives the values and calculations for the quality of life example:

x	y	U(x,y)	p	EU
5.000	18.000	0.729	0.800	0.590
2.000	12.000	0.035	0.200	
4.000	18.000	0.515	0.700	0.385
4.000	10.000	0.080	0.300	

This table lists values and calculations for the marketing example:

X	Y	U(X|Y)	U(Y|X)	U(X,Y)	P	EV	EU
$4,000,000	500	0.800	0.250	0.620	0.300	$700,000.00	0.07
$1,500,000	500	0.300	0.250	0.233	0.400		
-$2,000,000	500	-0.500	0.250	-0.388	0.200		
-$7,000,000	500	-1.750	0.250	-1.356	0.100		
$3,000,000	50	0.600	0.003	0.420	0.200	$325,000.00	0.02
$1,000,000	50	0.200	0.003	0.140	0.400		
-$1,250,000	50	-0.313	0.003	-0.219	0.300		
-$3,000,000	50	-0.750	0.003	-0.526	0.100		
$5,000,000	1000	1.000	1.000	1.000	0.400	$750,000.00	0.06
$1,500,000	1000	0.300	1.000	0.300	0.300		
-$1,000,000	1000	-0.250	1.000	-0.250	0.100		
-$8,000,000	1000	-2.000	1.000	-2.000	0.200		

Vita

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