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Some Intuition behind Large Cardinal Axioms, Their Characterization, and Related Results

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by

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Abstract

We aim to explain the intuition behind several large cardinal axioms, give characterization theorems for these axioms, and then discuss a few of their properties. As a capstone, we hope to introduce a new large cardinal notion and give a similar characterization theorem of this new notion. Our new notion of near strong compactness was inspired by the similar notion of near supercompactness, due to Jason Schanker.

Chapter 1

Introduction

1.1 Introduction

There are certain properties which characterize the largeness of ω which are provable in traditional set theory. Whether or not these same features of largeness can be applied to an uncountable cardinal κ is independent of the traditional axioms of set theory. Once we assume these larger infinities have such nice properties, the new set theory with such assumptions surprises us with its astonishing uniformity. Thus, large cardinal notions are not trivial and they play a crucial role in our goal of understanding the deeper aspects of mathematics.

We first give a high level sketch of the necessary set theoretic background knowledge to digest our main theorem while carefully explaining the intuition behind each step. Our result was inspired by a similar result due to Jason Schanker, which can be found in his paper [2]. Although it doesn't warrant a place in the bibliography, it should be mentioned that much of the discussion in Chapter 1 closely follows Spencer Unger's *2014 Forcing Summer School Lecture Notes*.

1.2 Set Theory Overview

Definition 1 *The* <u>*Zermelo-Fraenkel axioms*</u> *are the following statements:*

(i) Extensionality: $\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \to x = y]$

- *If two sets* x, y *have the same elements then they are equal.*

(ii) Pairing: $\forall x \forall y \exists z \forall w (w \in z \leftrightarrow w = x \vee w = y)$

- *For all sets* x, y *there is a set containing exactly* x, y *as elements, denoted* $\{x, y\}$.

(iii) *Replacement (for any functional predicate $\varphi(a)$):* $\forall x \exists y \forall z(z \in y \leftrightarrow \exists w[w \in x \wedge z = \varphi(w)])$

- *For any functional predicate $\varphi(a)$ and any set x there is a set y such that $z \in y$ if and only if there is an element $w \in x$ and z is the image of w under φ.*

(iv) *Union:* $\forall x \exists y \forall z[z \in y \leftrightarrow \exists w(w \in x \wedge z \in w)]$

- *For any set x there is a set y which contains all elements of elements of x, denoted $\bigcup x$.*

(v) *Powerset:* $\forall x \exists y \forall z(z \in y \leftrightarrow z \subseteq x)$

- *For all sets x there is a set y containing exactly all subsets of x, denoted $\mathcal{P}(x)$.*

(vi) *Empty set:* $\exists x \neg \exists y(y \in x)$

- *There is a set containing no elements, denoted $\emptyset$.*

(vii) *Infinity:* $\exists x[\emptyset \in x \wedge \forall y(y \in x \rightarrow \bigcup\{y, \{y\}\} \in x)]$

- *There exists an infinite set, namely the following set, denoted ω (sometimes called $\mathbb{N}$):*

$$\omega = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}...\}$$

(viii) *Separation (for any relation $R(a)$):* $\forall x \exists y \forall z(z \in y \leftrightarrow [z \in x \wedge R(z)])$

- *For any relation $R(a)$ and any set x there is a set y which is the restriction of x under R.*

(ix) *Foundation:* $\forall x[x \neq \emptyset \rightarrow \exists y(y \in x \wedge \forall z(z \in x \rightarrow \neg(z \in y)))]$

- *Every set contains a $\in$-minimal element, i.e. there are no infinite descending $\in$-chains.*

Definition 2 *The axiom of choice is the following claim: For all sets x there is a function f from $\mathcal{P}(x)$ to x such that for all $w \in \mathcal{P}(x)$ we have $f(w) \in w$.*

ZF refers to the standard Zermelo-Fraenkel axioms, while ZFC refers to the Zermelo-Fraenkel axioms with the addition of the axiom of choice. We can think about numbers in two different ways: *Ordinal numbers* are for when ordering matters (such as first, second, third, ...), while *cardinal numbers* refer to size (one, two, three, ...) or when ordering doesn't matter. Starting with the ordinals, we now define these two intuitive notions more rigorously.

Definition 3 *A binary relation R on a set S is well-founded if for every subset $A \subseteq S$ there is an element $c \in A$ such that for all $a \in A$ it is not the case that aRc, that is A has an R-minimal element.*

Definition 4 *A binary relation $\leqslant$ on a set S is a linear order if the following holds:*

(i) Antisymmetry: $\forall x, y \in S(x \leqslant y \wedge y \leqslant x \rightarrow x = y)$.

(ii) Transitivity: $\forall x, y, z \in S(x \leqslant y \wedge y \leqslant z \rightarrow x \leqslant z)$.

(iii) Comparability: $\forall x, y \in S(x \leqslant y \vee y \leqslant x)$.

Definition 5 *A set S is transitive if for for all $x \in S$ we also have $x \subseteq S$.*

Notice that the axiom of foundation from Definition 1 is simply the claim that $\in$ is a well-founded relation. Now we are ready to define ordinal numbers.

Definition 6 *A set α is an ordinal if $\in$ forms a well-founded linear order on α and α is transitive.*

We denote the class of all ordinals as ORD and define the following relation $<$ on all $\alpha, \beta \in \mathrm{ORD}$ as $\alpha < \beta$ if and only if $\alpha \in \beta$. Now we state a number of properties of ordinals as a proposition (the proofs can be found on pages 17-27 of Jech [1]).

Proposition 1

(i) $\emptyset \in \mathrm{ORD}$.

(ii) If $\alpha \in \mathrm{ORD}$ *and* $\beta \in \alpha$ *then* $\beta \in \mathrm{ORD}$.

(iii) If $\alpha, \beta \in \mathrm{ORD}$, $\alpha \neq \beta$, *and* $\alpha \subset \beta$ *then* $\alpha \in \beta$.

(iv) If $\alpha, \beta \in \mathrm{ORD}$ *then either* $\alpha \subset \beta$ *or* $\beta \subset \alpha$.

(v) $\in$ *linearly orders the class* ORD.

(vi) If $\alpha \in \mathrm{ORD}$ *then* $\alpha = \{\beta \mid \beta < \alpha\}$.

(vii) If $C \neq \emptyset$ *is a class of ordinals then* $\bigcap C \in \mathrm{ORD}$, $\bigcap C \in C$, *and* $\bigcap C = \inf C$.

(viii) If $X \neq \emptyset$ *is a set of ordinals, then* $\bigcup X \in \mathrm{ORD}$ *and* $\bigcup X = \sup X$.

(ix) If $\alpha \in \mathrm{ORD}$ *then* $\alpha \cup \{\alpha\} = \inf\{\beta \mid \alpha < \beta\}$.

If α is an ordinal, then we denote $\alpha+1 = \alpha \cup \{\alpha\}$ and call $\beta = \alpha+1$ the *successor* of α. If α is not a successor ordinal then we call α a *limit* ordinal and $\alpha = \sup\{\beta \mid \beta < \alpha\} = \bigcup \alpha$. Just to get the feel for things, we list some ordinals below along with their canonical representations.

$$0 = \emptyset \qquad 1 = \{\emptyset\} \quad 2 = \{\emptyset, \{\emptyset\}\} \qquad ... \qquad \omega = \{0, 1, 2, ...\} \qquad \omega + 1 = \{0, 1, 2, ..., \omega\} \quad ...$$

To finish up our explanation of ordinals we state and prove the following extremely useful theorem.

Theorem 1 *(Transfinite Induction) Let* $R(x)$ *be a relation, let* α *be any ordinal and suppose*

(i) $R(\emptyset)$

(ii) $\forall \beta \in ORD(\beta < \alpha \to R(\beta)) \to R(\alpha)$

then R *holds for the class of all ordinals.*

PROOF. Suppose γ is the first ordinal such that $\neg R(\gamma)$, then since $R(\beta)$ holds for all $\beta < \gamma$, by (ii) this implies $R(\gamma)$ holds, which contradicts our assumption. $\square$

Now we move our discussion towards cardinal numbers, which we introduce with the following definition.

Definition 7 *An ordinal* α *is a* <u>*cardinal*</u> *if for all* $\beta < \alpha$ *there is no surjection from* β *onto* α.

For any set A, we let $|A|$ denote the least ordinal κ such that there is a bijection from κ onto A, hence $|A|$ is also cardinal. If two sets A, B have the same cardinality, we denote this $|A| = |B|$, which means there is a bijection from A onto B. The following terminology will also prove extremely useful.

Definition 8 *(Cardinal Arithmetic) Let* κ, λ *be cardinals then we have the following:*

(i) $\kappa + \lambda = |A \cup B|$ *where* A, B *are disjoint and* $|A| = \kappa$ *and* $|B| = \lambda$.

(ii) $\kappa \cdot \lambda = |\kappa \times \lambda|$ *where* $\kappa \times \lambda = \{(x, y) \mid x \in \kappa, y \in \lambda\}$.

(iii) $\kappa^\lambda = |^\lambda \kappa|$ *where* $^\lambda \kappa = \{f \mid f : \kappa \to \lambda\}$.

We state the following lemma without proof.

Lemma 1 *(Cantor-Shröder-Bernstein Lemma) If* $|A| \leqslant |B|$ *and* $|B| \leqslant |A|$ *then* $|A| = |B|$.

For any infinite sets A, B we have $|A + B| = |A \cdot B| = |max\{A, B\}|$, however cardinal exponentiation is where things get interesting. Notice that for any set S we have $|\mathcal{P}(S)| = 2^{|S|}$; simply define a function f which maps subsets to their characteristic function.

Given the context of this paper, the following theorem of Cantor is especially significant. The axiom of infinity, thought of as the 'first' large cardinal axiom, when considered alongside the powerset axiom gives rise to the distinct infinite quantities set theory is famous for. The following theorem is a proof of this fact.

Theorem 2 *(Cantor's Theorem) Given a set* S*, there is no bijection between* S *and* $\mathcal{P}(S)$*, or in other words* $|S| < |\mathcal{P}(S)|$.

PROOF. Suppose $f : S \to \mathcal{P}(S)$ is a bijection. Then for every $s \in S$, we have $f(s) \subseteq S$. Define the set $D = \{s \in S \mid s \notin f(s)\}$. Clearly D is a subset of S, and because f is surjective there is an s_0 such that $f(s_0) = D$. Now ask, is s_0 in D? If $s_0 \in D$, then by definition $s_0 \notin f(s_0)$. However $f(s_0) = D$ and so $s_0 \notin D$ which is a contradiction. If $s_0 \notin D$ then $s_0 \notin f(s_0)$, but this is precisely what it means for $s_0 \in D$, which is also a contradiction. Thus no such bijection f can exist, and $|S| < |\mathcal{P}(S)|$. □

The above gives rise to the following corollary.

Corollary 1 *For any cardinal* κ, $2^\kappa > \kappa$.

Given a cardinal κ, we let κ^+ denote the smallest cardinal such that $\kappa < \kappa^+$. The infinite cardinals can be written as below.

1. $\aleph_0 = \omega$
2. $\aleph_{\alpha+1} = \aleph_\alpha^+$
3. $\aleph_\gamma = \cup_{\alpha<\gamma}\aleph_\alpha$ for γ a limit ordinal.

More often than not we will write ω_α in place of $\aleph_\alpha$, these are really the same object only the former is in the context of ordinals while the ladder is in the context of cardinals.

We also might want to take a moment and mention the *beth numbers*, defined as $\beth_0 = \omega$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$ for successor ordinals, and $\beth_\gamma = \sup\{\beth_\alpha \mid \alpha < \gamma\}$ for limit ordinals.

The two major definitions of this section are below.

Definition 9 *The Continuum Hypothesis* (CH) *is the the claim that given any set* S, *such that* $\omega \subseteq S \subseteq \mathcal{P}(\omega)$, *either* $|S| = \omega$ *or* $|S| = |\mathcal{P}(\omega)|$. *In other words* $\omega_1 = 2^\omega$.

Definition 10 *The Generalized Continuum Hypothesis* (GCH) *is the claim that given any ordinal* α, $\omega_{\alpha+1} = 2^{\omega_\alpha}$.

Notice that CH is equivalent to saying $\omega_1 = \beth_1$ while the GCH is equivalent to the $\omega_\alpha = \beth_\alpha$ for all $\alpha \in \mathrm{ORD}$. The investigation of the above two claims has historically pushed the best mathematicians to their limits, and naturally brings with it quite a bit of historical interest. The story behind them involves characters like Hilbert, Cantor, and Gödel, however, a full discussion of this topic will not be given here.

1.3 Models of Set Theory and Forcing Basics

When we use the *satisfies* (symbolically $\models$) relation there are two components. The left side is a model, usually some algebraic structure where the axioms of ZFC can be satisfied. The right side might contain formulas which can be satisfied by the model. Given a model of ZFC, M, suppose φ is an axiom of ZFC, then we'd have $M \models \varphi$. Now we ask, if $V \models$ ZFC what does V look like? Definition 11 and Theorem 3 follow page 64 of [1] closely.

Definition 11 *The levels of the* <u>*cumulative hierarchy*</u> *are defined by recursion:*

$$\begin{aligned} V_0 &= \emptyset \\ V_{\alpha+1} &= \mathcal{P}(V_\alpha) \\ V_\gamma &= \bigcup_{\alpha<\gamma} V_\alpha \qquad (\gamma \text{ a limit ordinal}) \end{aligned}$$

Theorem 3 *Given* $x \in V$ *there is an* $\alpha \in \mathrm{ORD}$ *such that* $x \in V_\alpha$.

PROOF. Let C be the class of all x not in any V_α. If C is nonempty, then by foundation C has a $\in$-minimal element x_0. Because x_0 is $\in$-minimal in C, every $x \in x_0$ is not in C and thus for every $x \in x_0$ we have $x \in \bigcup_{\alpha \in \mathrm{ORD}} V_\alpha$, hence $x_0 \subseteq \bigcup_{\alpha \in \mathrm{ORD}} V_\alpha$. Replacement gives us a $\gamma \in \mathrm{ORD}$ such that $x_0 \subseteq \bigcup_{\alpha<\gamma} V_\alpha$. Hence $x_0 \in V_{\gamma+1}$, which contradicts our initial assumption.$\square$

The above theorem clearly implies $V = \bigcup_{\alpha \in \mathrm{ORD}} V_\alpha$. The model V is often called the *Von-Neumann universe* or simply *universe* for short. For any set S we can consider its Von-Neumann rank.

Definition 12 *Given a set* S*, the* <u>*Von-Neumann rank*</u> *of* S*, or* <u>*rank*</u> *of* S *when context is clear, is defined as* $\mathrm{rk}(S) = \alpha$ *where* α *is the least ordinal such that* $S \in V_{\alpha+1}$.

We will also need the following two definitions.

Definition 13 *Given a set* S*, the* <u>*transitive closure*</u> *of* S*, denoted* $\mathrm{TC}(S)$*, is defined recursively as follows:*

$$\begin{aligned} \mathrm{TC}_0(S) &= S \\ \mathrm{TC}_n(S) &= \bigcup \mathrm{TC}_{n-1}(S) \\ \mathrm{TC}(S) &= \bigcup_{n<\omega} \mathrm{TC}_n(S) \end{aligned}$$

Definition 14 *For an infinite cardinal* κ*, the* <u>*hereditarily of size* $< \kappa$*-sets*</u>*, denoted* H_κ*, is the collection of all sets* S *such that* $|\mathrm{TC}(S)| < \kappa$.

Recall how we mentioned that $V \models \mathrm{ZFC}$. If κ is a regular uncountable cardinal, then similarly we have $H_\kappa \models \mathrm{ZFC}^-$ where ZFC^- consists of the axioms of ZFC minus the powerset axiom.

The following theorem, stated without proof, is important regarding the theoretic basis for forcing arguments.

Theorem 4 *(Mostowski Collapse Theorem) Any well founded and extensional model* (P, E) *is isomorphic to transitive model* $(M, \in)$*. Also, such an isomorphism is unique.*

In 1939 Gödel published his first of two papers (the other in 1940) on the relative consistency of ZFC + CH with respect to ZF. In 1963 Cohen proved the relative consistency of ZFC + ¬CH with respect to ZFC. Here we shall hone in on Cohen's forcing technique. In fact, assuming that ZFC is consistent, Cohen's forcing technique can force models of both ZFC + CH and ZFC + ¬CH. We will need some definitions.

Definition 15 *A binary relation* $\leqslant$ *on a set* S *is a partial order if the following holds:*

(i) Antisymmetry: $\forall x, y \in S(x \leqslant y \wedge y \leqslant x \to x = y)$.

(ii) Transitivity: $\forall x, y, z \in S(x \leqslant y \wedge y \leqslant z \to x \leqslant z)$.

(iii) Reflexivity: $\forall x \in S(x \leqslant x)$.

Definition 16 *A partially ordered set, denoted* $(\mathbb{P}, \leqslant)$ *is a set* $\mathbb{P}$ *equipped with a partial order* $\leqslant$.

Definition 17 *A filter on a partially ordered set* $(\mathbb{P}, \leqslant)$ *is a subset* $S \subseteq \mathbb{P}$ *with the following properties:*

(i) Nonempty: $S \neq \emptyset$.

(ii) Pairwise extendable: $\forall x, y \in S\ \exists z \in S\ (z \leqslant x \wedge z \leqslant y)$.

(iii) Upward closure: $\forall x \in S\ p \in \mathbb{P}(x \leqslant p \Rightarrow p \in S)$.

(iv) Properness: $S \neq \mathbb{P}$.

Definition 18 *An atom in a partially ordered set* $(\mathbb{P}, \leqslant)$ *is a non empty element* $a \in \mathbb{P}$ *such that for all other non empty elements* $b \in \mathbb{P}$ *we have* $a \leqslant b$.

If a partial order contains no atoms then such a partial order is called *non-atomic*. Partial orders with atoms are not very interesting to force over, so we shall restrict our attention to the non-atomic ones.

To utilize the technique of forcing we do not need to fully master the intricate clockwork of Cohen's proofs, however the importance of filters on partially ordered sets cannot be overstated. Every forcing argument begins with a model of set theory, a partially ordered set in that model, and a special filter over this partially ordered set. This special type of filter is defined below.

Definition 19 *A subset* $D \subseteq \mathbb{P}$ *of partial order* $(\mathbb{P}, \leqslant)$ *is called dense if and only if* $\forall p \in \mathbb{P}\ \exists q \in D(q \leqslant p)$.

Definition 20 *Given* M*, a model of* ZFC*,* $\mathbb{P} \in M$ *where* $\mathbb{P}$ *is partial order, we say a filter* $G \subseteq \mathbb{P}$ *is* $\mathbb{P}$*-generic for* M *(or simply* $\mathbb{P}$*-generic when* M *is implied) if and only if for every subset* $D \subseteq \mathbb{P}$ *which* M *thinks is dense,* $D \cap G \neq \emptyset$*.*

Notice that these filters union up to be functions, i.e. $\bigcup G = f$. The partial orders $\mathbb{P}$ can be thought of as containing pieces of a total function from domain X onto Y, with X, Y defined very precisely. Thus a generic filter G through $\mathbb{P}$ defines a total function $f : X \to Y$ (which is how we will force both CH and $\neg$CH later on).

Starting with a model M, a partial order $\mathbb{P} \in M$, and given a generic filter G, we can then construct a new model $M[G]$ which agrees with a desired property. Oddly enough, this function G couldn't have been recognized by M as being a subset of $\mathbb{P}$. The significance of the generic filter is witnessed by the theorem below.

Theorem 5 *Given* M *a model of* ZFC*, suppose a partial order* $\mathbb{P}$ *is non-atomic and* $\mathbb{P} \in M$*. Let* G *be* $\mathbb{P}$*-generic for* M*, then* $G \notin M$*.*

PROOF. If $G \in M$ then $D = \mathbb{P} \backslash G \in M$. If we show that D is dense, then because G is $\mathbb{P}$-generic it should have a non-empty intersection with D, which would be a contradiction. For arbitrary $p \in \mathbb{P}$ because $\mathbb{P}$ is non-atomic there are two $q, r \leqslant p$ with neither $q \leqslant p$ nor $r \leqslant p$. G is a filter and thus only one of q, r can belong to G. Supposing $q \in G$ entails $r \in D$, which implies D is dense. By genericity we have $D \cap G \neq \emptyset$ but this is contradiction. □

The next question is perhaps "Do these generic filters actually exist?" and under certain reasonable assumptions the answer to this question is *yes* (this is where the Mostowski Collapse Theorem is important). By restricting our focus to countable transitive models, we can enumerate the dense sets $D_0, D_1, D_2, ...$ and apply the following famous lemma, arriving at the desired corollary.

Lemma 2 *(Rasiowa-Sikorski Lemma) Let* $\mathbb{P}$ *be a partially ordered set,* $\mathscr{D} \subset \mathcal{P}(\mathbb{P})$ *a countable collection of dense subsets of* $\mathbb{P}$*, then there exists a* $\mathscr{D}$*-generic filter* G *which intersects every element of* $\mathscr{D}$*.*

PROOF. We can enumerate the dense sets $D_0, D_1, ... \in \mathscr{D}$, so given any element p of $\mathbb{P}$ by density find a $p_0 \leqslant p$ with $p_0 \in D_0$. We can then create a $\leqslant$-chain

$$...p_2 \leqslant p_1 \leqslant p_0$$

with $p_i \in D_i$ for $i \in \mathbb{N}$. A $\mathscr{D}$-generic filter can then be constructed as $G = \{p \in \mathbb{P} : \exists p_i \text{ and } p \geqslant p_i\}$. □

Corollary 2 *(Cohen) Given M, a countable transitive model of ZFC,* $\mathbb{P} \in M$*, there does exists a* $\mathbb{P}$*-generic filter* G *for* M*.*

So we've been talking about this model $M[G]$ but we haven't really discussed what it is. In the rest of this section we will quickly describe $M[G]$ which should allow us to state the forcing theorems. Paralleling the construction of V we shall define the class of $\mathbb{P}$-names.

Definition 21 *The class of $\mathbb{P}$-names, denoted $V^{\mathbb{P}}$, is defined recursively as follows:*

$$\begin{aligned} V_0^{\mathbb{P}} &= \emptyset \\ V_{\alpha+1}^{\mathbb{P}} &= \mathcal{P}(V_\alpha^{\mathbb{P}} \times \mathbb{P}) \\ V_\gamma^{\mathbb{P}} &= \bigcup_{\alpha<\gamma} V_\alpha^{\mathbb{P}} \qquad (\gamma \textit{ a limit ordinal}) \end{aligned}$$

then we let $V^{\mathbb{P}} = \bigcup_{\alpha \in ORD} V_\alpha^{\mathbb{P}}$.

Definition 22 *Given $\tau \in V^{\mathbb{P}}$ we define the $\mathbb{P}$-name rank of τ as $\rho(\tau) = \alpha$ where α is the least ordinal such that $\tau \in V_{\alpha+1}^{\mathbb{P}}$.*

Note that if M is a countable transitive model of ZFC, then $M^{\mathbb{P}}$ denotes the collection of all $\mathbb{P}$-names τ such that $\tau \in M$, i.e. $M^{\mathbb{P}} = V^{\mathbb{P}} \cap M = (V^{\mathbb{P}})^M$.

Definition 23 *The interpretation under G is defined inductively. For every $x \in M^{\mathbb{P}}$ we define its interpretation $x[G]$ as:*

(i) $\emptyset[G] = \emptyset$

(ii) $x[G] = \{y[G] \mid x(y) \in G\}$

Then finally we can define the generic extension $M[G]$ as follows:

$$M[G] = \{x[G] \mid x \in M^{\mathbb{P}}\}$$

The following theorem tells us more about the structure $M[G]$ (Theorem 14.5 from [1]).

Theorem 6 *(Generic Model Theorem) Let M be a transitive model of ZFC and let $(\mathbb{P}, \leqslant)$ be notion of forcing in M. If $G \subset \mathbb{P}$ is $\mathbb{P}$-generic, then there exists a transitive model $M[G]$ with the following properties:*

(i) $M[G]$ *is a model of* ZFC.

(ii) $M \subset M[G]$ *and* $G \in M[G]$.

(iii) $ORD^{M[G]} = ORD^M$.

(iv) If N is a transitive model of ZF such that $M \subset N$ and $G \in N$, then $M[G] \subset N$.

We've been talking about forcing, but what does it mean? The following defines this terminology precisely.

Definition 24 *The forcing language consists of the symbols of first order logic, the symbol $\in$ as a binary relation, and constant symbols τ for each $\tau \in M^{\mathbb{P}}$. Let $\varphi(\tau_1, ..., \tau_n)$ be formula of the forcing language, with $\tau_1, ..., \tau_n$ all in $M^{\mathbb{P}}$. If $p \in \mathbb{P}$ we say p forces φ, denoted $p \Vdash \varphi(\tau_1, ..., \tau_n)$, if for every $\mathbb{P}$-generic filter G with $p \in G$ we have $M[G] \models \varphi(\tau_1[G], ..., \tau_n[G])$.*

The following proposition describes the properties of the forcing relation.

Proposition 2 *(Properties of Forcing). Let $(\mathbb{P}, <)$ be a notion of forcing in the ground model M, and let $M^{\mathbb{P}}$ be the class in M of all names.*

(i) (a) If $p \Vdash \varphi$ and $q \leqslant p$, then $q \Vdash \varphi$
(b) It is not the case that both $p \Vdash \varphi$ and $p \Vdash \neg\varphi$
(c) For every p there is a $q \leqslant p$ such that q decides φ, i.e. either $q \Vdash \varphi$ or $q \Vdash \neg\varphi$.

(ii) (a) $p \Vdash \neg\varphi$ if and only if no $q \leqslant p$ forces φ
(b) $p \Vdash \varphi \wedge \psi$ if and only if $p \Vdash \varphi$ and $p \Vdash \psi$
(c) $p \Vdash \forall x \varphi$ if and only if $p \Vdash \varphi(\dot{a})$ for every $\dot{a} \in M^{\mathbb{P}}$

(iii) If $p \Vdash \exists x \varphi$ then for some $\dot{a} \in M^{\mathbb{P}}, p \Vdash \varphi(\dot{a})$

1.4 Elementary Applications of Forcing

Now that we have given a sketch of the notion of forcing, we are ready to look at a few examples. We mentioned earlier that forcing arguments always involve a model of set theory M along with a partial order $\mathbb{P}$ in that model. A $\mathbb{P}$-generic filter G will give us a new model M[G] with a desired property, witnessed by $\bigcup G = f$. The following example is given as a warm up (Example 14.2 from [1]).

Example **1** *Here we construct a new real number not found in the ground model. Let M be a countable transitive model of ZFC, fix $\mathbb{P} \in M$ where $\mathbb{P} = \{p : p \in 2^{<\omega}\}$, ordered by inclusion (i.e. for $p, q \in \mathbb{P}$ we have $p \leqslant q$ if and only if $q \subseteq p$). Now we construct a new function $f : \omega \to 2$ that wasn't in the ground model M. Let $G \subset \mathbb{P}$ be $\mathbb{P}$-generic for M. Let $g = \bigcup G$.*

Notice that for any $n \in \omega$ we have $D_n = \{p \in \mathbb{P} : n \in \mathrm{dom}(p)\}$ is dense in $\mathbb{P}$, hence they each have non trivial intersection with G, so $\mathrm{dom}(g) = \omega$. Also notice that for any $f \in M \cap {}^{\omega}2$ we have the set $D_f = \{p \in P : f \restriction_{\mathrm{dom}(p)} \neq p\}$ is dense, therefore $g \neq f$ for any $f \in M \cap {}^{\omega}2$. So $M[G]$ has a real $g : \omega \to 2$ not in M.

In the above example, the partial order consisted of pieces of a function from ω onto 2, and thus it makes sense that the generic filter defines a function $f : \omega \to 2$ not in M. Familiarizing ourselves with certain partial orders will make our work later much easier.

Definition 25 *Let κ, λ be cardinals. Allow (i)-(iii) to be ordered by inclusion. The following partial orders are significant:*

(i) $\mathrm{Add}(\kappa, \lambda) = \{p \mid (p \textit{ is a function} \wedge \mathrm{dom}(p) \subseteq \lambda \times \kappa \wedge |p| < \kappa \wedge \mathrm{rang}(p) \subseteq \{0, 1\}\}$

(ii) $\mathrm{Add}(\kappa, 1) = \{p \mid \exists \alpha < \kappa \, p : \alpha \to 2\}$

(iii) $\mathrm{Col}(\kappa, \lambda) = \{p \mid \exists \alpha < \kappa \, p : \alpha \to \lambda\}$.

While the above partial orders might appear unmotivated, they are quite important. For example, starting with a model $M \models \mathsf{ZFC}$ we can force with $\mathrm{Add}(\omega_1, 1)$ to arrive at a model $M[G] \models \mathsf{ZFC} + \mathsf{CH}$, or we can force with $\mathrm{Add}(\omega, \omega_2)$ to arrive at a model $M[G] \models \mathsf{ZFC} + \neg\mathsf{CH}$. Before getting too ahead of ourselves, consider the following definition and proposition.

Definition 26 *A partial order $\mathbb{P}$ is <u>countably closed</u> if for every ω sequence $\langle p_n \mid n < \omega \rangle$ such that $p_{n+1} \leqslant p_n$ for all $n < \omega$ there is a $p \in \mathbb{P}$ such that $p \leqslant p_n$ for all $n < \omega$.*

Proposition 3 *If $\mathbb{P}$ is countably closed and G is $\mathbb{P}$-generic for M, then $\omega_1^M = \omega_1^{M[G]}$ and $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)^{M[G]}$.*

Which leads naturally to our next example.

Example **2** *(The Consistency of CH) Let $\mathbb{P} = \mathrm{Add}(\omega_1, 1)$, let G be $\mathbb{P}$-generic for M. Working in $M[G]$ let $g = \bigcup G$. Also for each $\alpha < \omega_1$ we define a subset of ω as $x_\alpha = \{n \mid g(\omega \cdot \alpha + n) = 1\}$. Now, if we show that every $x \in \mathcal{P}(\omega)^M = \mathcal{P}(\omega)^{M[G]}$ can be identified $x = x_\alpha$ for some $\alpha < \omega_1^{M[G]} = \omega_1^M$ then we would be finished.*

Working in M, let $x \subseteq \omega$ and define $D_x = \{p \in \mathbb{P} \mid \exists \alpha < \omega_1 \forall n < \omega [\chi_x(n) = p(\omega \cdot \alpha + n]\}$. To see that each D_x is dense, let $p \in \mathbb{P}$ with $\mathrm{dom}(p) = \beta$. If $\alpha > \beta$, then for all $n < \omega$ we can extend p to be defined on $\omega \cdot \alpha + n$. Thus each D_x is dense, and because G intersects every dense set, the mapping $\alpha \mapsto x_\alpha$ is a surjection from ω_1 onto $\mathcal{P}(\omega)$, i.e. $M[G] \models \mathsf{ZFC} + \mathsf{CH}$. Recall that Proposition 3 guarantees that we are talking about the same ω_1 and $\mathcal{P}(\omega)$ as in the ground model.

To complement the above example we will now work towards a similar example where we show the consitency of $\mathsf{ZFC} + \neg\mathsf{CH}$. Please allow the following definitions and propositions.

Definition 27 *Given $\mathbb{P}$ a partial order, $A \subseteq \mathbb{P}$ is an <u>antichain</u> if for every $p, q \in A$ neither $p \leqslant q$ nor $q \leqslant p$.*

Definition 28 *A partial order $\mathbb{P}$ satisfies the <u>countable chain condition</u> (ccc) if every antichain of $\mathbb{P}$ is countable. More generally $\mathbb{P}$ has the <u>κ-chain condition</u> (κ-cc) if every antichain of $\mathbb{P}$ has size size less than κ.*

Proposition 4 *Suppose $\mathbb{P}$ is a ccc partial order. Given a $\mathbb{P}$-generic filter G over M, and κ an ordinal, then*

$$M \models \kappa \textit{ is a cardinal} \Leftrightarrow M[G] \models \kappa \textit{ is a cardinal}$$

Proposition 5 *The partial order $\mathrm{Add}(\omega, \omega_2)$ has the ccc.*

Example **3** *(The Consistency of $\neg$CH) Let $\mathbb{P} = \mathrm{Add}(\omega, \omega_2)$. If G is $\mathbb{P}$-generic over M, then we will show $M[G] \models 2^\omega \geqslant \omega_2$. Notice that Proposition 4 implies $\omega_2^M = \omega_2^{M[G]}$ and $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)^{M[G]}$. From this follows the consistency of $\neg\mathrm{CH}$. It is possible to prove $M[G] \models 2^\omega = \omega_2$, but the full argument will not be given here.*

Working in $M[G]$, let $g = \bigcup G$. For each pair $n < \omega, \alpha < \omega_2$, we define $D_{n,\alpha} = \{p \in \mathbb{P} \mid \exists(n, \alpha) \in \mathrm{dom}(p)\}$. To see that each $D_{n,\alpha}$ is dense, given any $p \in \mathbb{P}$ either $(n, \alpha) \in \mathrm{dom}(p)$ already or we can extend p to a condition $p' \leqslant p$ where $(n, \alpha) \in \mathrm{dom}(p')$. Thus $\mathrm{dom}(g) = \omega \times \omega_2$. It might be useful to think about g as being a rectangle containing ω_2 many 0-1 sequences, each of length ω.

We need to show that each of these ω_2 many sequences are distinct. For $\alpha < \omega_2$, define $f_\alpha : \omega \to \{0, 1\}$ as $f_\alpha(n) = g(n, \alpha)$. Noting that $D = \{p \in \mathbb{P} : \exists n < \omega, \exists \alpha, \beta < \omega_2[(\alpha \neq \beta) \wedge (p(n, \alpha) \neq p(n, \beta))]\}$ is dense in $\mathbb{P}$ tells us each f_α is distinct. Thus in $M[G]$ it holds that g identifies ω_2 many distinct 0-1, each of legngth ω. Propositions 4 and 5 ensure us that cardinals are preserved, and hence $M[G] \models 2^\omega \geqslant \omega_2$. In other words $M[G] \models \mathrm{ZFC} + \neg\mathrm{CH}$.

Chapter 2

Preliminaries

2.1 Introduction to Large Cardinal Axioms

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Where do large cardinal assumptions come from? The intuition often comes from properties of ω provable in ZFC, which characterize the largeness of ω relative to all finite $n < \omega$. Whether or not an uncountable cardinal κ has same properties is often independent of the axioms of ZFC. Consider the following properties of ω provable in ZFC: (1) Given $n < \omega$, then because n is finite, $\mathcal{P}(n)$ is also finite, and hence $|\mathcal{P}(n)| < \omega$, and (2) for any $S \subset \omega$ with $|S| < \omega$ there is an $n < \omega$ with $s \subseteq n$. Asking whether or not these properties hold for uncountable κ leads to the following definitions.

Definition 29 *A cardinal κ is a <u>strong limit</u> if for every $\alpha < \kappa$ we have $|\mathcal{P}(\alpha)| < \kappa$.*

Definition 30 *A cardinal κ is a <u>regular</u> if for every subset $X \subset \kappa$ with $|X| < \kappa$ there is some $\alpha < \kappa$ such that $X \subseteq \alpha$.*

Definition 31 *An uncountable cardinal κ is <u>inaccessible</u> if and only if it is a regular strong limit cardinal.*

We now prove that $(\mathsf{ZFC} + \exists\kappa \text{ inaccessible})$ is strictly stronger in terms of consistency strength than ZFC alone. We first state without clarification Gödel's Second Incompleteness Theorem, its intuitive feel should suffice.

Theorem 7 *(Gödel's Second Incompleteness Theorem) If T is a consistent (first-order) theory which is recursively axiomatizable then T cannot prove its own consistency.*

$$T \nvdash \mathrm{Con}(T)$$

We follow Section 0.1 from [3] in proving the following interesting lemma and corollary.

Lemma 3 *If κ is inaccessible, then $V_\kappa \models \mathsf{ZFC}$.*

PROOF. Let κ be an inaccessible cardinal. Clearly the only case in which V_κ would fail to model ZFC would be in the case of the powerset axiom. However, inaccessible cardinals are defined to be strong limits and hence $V_\kappa \models \mathsf{ZFC}$. $\square$

The consistency strength result is now a corollary.

Corollary 3 *Given $\mathsf{ZFC} + \mathsf{Con}(\mathsf{ZFC})$ is consistent, then*

$$\mathsf{ZFC} \nvdash (\mathsf{Con}(\mathsf{ZFC}) \Rightarrow \mathsf{Con}(\mathsf{ZFC} + \exists\kappa \text{ inaccessible}))$$

PROOF. By Lemma 3, $\mathsf{Con}(\mathsf{ZFC} + \exists\kappa \text{ inaccessible})$ implies $\mathsf{Con}(\mathsf{ZFC} + \mathsf{Con}(\mathsf{ZFC}))$, which would entail that $\mathsf{ZFC} + \mathsf{Con}(\mathsf{ZFC})$ proves its own consistency. Clearly this contradicts Gödel's Second Incompleteness Theorem. $\square$

Notice that while the properties of being a strong limit and being regular are provable properties about ω in ZFC, for κ uncountable they are not.

Much of the intuition behind large cardinal axioms can be given in terms of filters. Many large cardinal axioms have additional formulations in terms of elementary embeddings, more will be said on this later.

Definition 32 *A filter on a set S is a subset $F \subseteq \mathcal{P}(S)$ such that*

(i) $A, B \in F \Rightarrow A \cap B \in F$

(ii) $A \in F \wedge A \subseteq B \Rightarrow B \in F$

Definition 33 *A filter F is principal if there is a nonempty set S such that for every $X \in F$, $S \subseteq X$. If F is not principal we say it is non-principal.*

Definition 34 *A filter F on a set S is κ-complete if for any collection $\mathscr{A} \subset F$ with $|\mathscr{A}| < \kappa$ we have $\bigcap \mathscr{A} \in F$.*

Definition 35 *A filter F on a set S is an ultrafilter if for any other filter $F' \subseteq \mathcal{P}(S)$ such that $F \subseteq F'$ implies $F = F'$.*

Similar to inacessibility, κ-completeness and being an ultrafilter are generalizations of properties provable in ZFC about ω. If we define a filter F on ω as $X \in F$ if and only if $|\omega \backslash X| < \omega$ then by applying Zorn's Lemma we arrive at a ω-complete non-principal ultrafilter on ω. The question is then whether or not there exists an uncountable cardinal κ with a κ-complete non-principal ultrafilter? As might be expected, such a result is not provable from ZFC alone.

However, it turns out that this notion of a κ-complete non-principal ultrafilter is not trivial, and has several equivalent characterizations. The standard definition is given below. Our development of measurable cardinals until Theorem 9 closely follows Section 0.2 of [3].

Definition 36 *An uncountable cardinal κ is called* <u>*measurable*</u> *if there is a κ-complete, non-principal ultrafilter on κ. A* <u>*measure*</u> *$U \subseteq \mathcal{P}(S)$ on a set S is a non-pincipal κ-complete ultrafilter on S.*

Later on, once enough machinery has been built up, we will prove the following lemma.

Lemma 4 *The existence of a measurable cardinal is strictly stronger in consistency strength than the existence of an inaccessible cardinal.*

Starting with V, suppose $I \in V$ is some infinite set and U an ultrafilter over I. By V^I we denote the class of functions $f : I \to V$. We modulo V^I by the ultrafilter to arrive at the structure $V^I/U = \{[f]_U \mid f \in V^I\}$ where $[f]_U \in [g]_U$ if and only if $\{x \mid f(x) \in g(x)\} \in U$ (and of course $[f]_U = [g]_U$ if and only if $\{x \mid f(x) = g(x)\} \in U$). However, the equivalence classes of V^I/U form proper classes, so we perform Scott's trick to arrive at the restricted equivalence classes of the form $(f)_U = \{g \in V_\alpha \mid f =_U g\}$, where α is of minimum Lévy rank such that the set is non-empty. It is clear that $(f)_U \in (g)_U$ if and only if $[f]_U \in [g]_U$. Thus from here on when we denote the *ultrapower* V^I/U we do not refer to the collection of proper classes, but instead we refer to the structure $V^{I*}/U = \{(f)_U \mid f \in V^I\}$. The following theorem assures us that the ultrapower works nicely.

Theorem 8 *(Łos) If U is an ultrafilter on a set I, then*

$$V^I/U \models \varphi[(f_1)_U, ..., (f_n)_U] \Leftrightarrow \{x \in I | \varphi(f_1(x), ..., f_n(x))\} \in U$$

The following definition will aid in our dialogue.

Definition 37 *Given two transitive models M, N of set theory, a function $j : M \to N$ is an* <u>*elementary embedding*</u> *if and only if for every first order φ, and all $x_1, ...x_n \in N$ we have*

$$M \models \varphi(x_1, ..., x_n) \Leftrightarrow N \models \varphi(j(x_1), ..., j(x_n))$$

From the above definition and theorem it can be shown that the canonical embedding $h : V \to V^I/U$ defined $x \mapsto (c_x)_U$ is elementary, where c_x is the constant function which maps elements of I to x.

We want to look at cases where U is a non-principal, κ-complete ultrafilter on I, i.e. where U witnesses the measurability of κ. If such is the case then $\in_U$ is well founded, and thus we can define the Mostowski collapse π on V^I/U defined $x \mapsto \{\pi(y) | y \in_U x\}$. Thus we have the class $M = \{\pi(f)_U | f : I \to V\}$ and the structure $(M, \in)$, sometimes denoted $\mathrm{Ult}(V, U)$. By Theorem 8, and the fact that the Mostowski collapse is an isomorphism, we know that if U witnesses the measurability of κ then $\varphi(\pi((f_1)_U), ..., \pi((f_n)_U))$ holds if and only if $\{x \in I | \varphi(f_1(x)), ..., f_n(x))\} \in U$. Because we will rarely talk about the entire class, from this point on rather than letting $[f]_U$ refer to the entire equivalence class modulo U, we let $[f]_U = \pi((f)_U))$. We also know $j : V \to M$ defined $j = \pi \circ h$ is elementary. Because of elementarity, we know that for every $\alpha \in \mathrm{ORD}$ we have $\alpha \leqslant j(\alpha)$. This naturally gives rise to the following definition and lemma.

Definition 38 *Given an elementary embedding $j : M \to N$ between two transitive class models of ZF, the critical point of the embedding is the least ordinal α such that $\alpha < j(\alpha)$, denoted $\mathrm{cp}(j)$. The emedding j is nontrivial if $j(x) \neq x$ for some x.*

Lemma 5 *Given $j : M \to N$ a nontrivial elementary embedding between transitive class models M and N of ZFC, then $\mathrm{cp}(j)$ exists.*

PROOF. Suppose that we have such an embedding but $\mathrm{cp}(j)$ doesn't exist, i.e. $j(\alpha) = \alpha$ for all $\alpha \in \mathrm{ORD}$. Let x be of minimal rank such that $j(x) \neq x$ occurs. Then enumerate the elements $x = \{x_\alpha | \alpha < |x|\}$. Because $j(\alpha) = \alpha$ for all $\alpha \in \mathrm{ORD}$ we have $j(x) = \{j(x_\alpha) | \alpha < |x|\}$. But because we assumed x was minimal, we know $j(x_\alpha) = x_\alpha$ and hence $j(x) = x$ for all x. Thus j was a trivial embedding, which contradicts our assumption. □

We are now able to give our first characterization theorem. The following shows that the property of measurability has equivalent characterizations in terms of particular filters and elementary embeddings.

Theorem 9 *A cardinal κ is measurable if and only if it is the critical point of a nontrivial elementary embedding $j : V \to M$ in V, where M is transitive.*

PROOF. For the forward direction, given a measurable cardinal κ and a measure U on κ, consider the corresponding ultrapower embedding $j : V \to M = \mathrm{Ult}(V, U)$. For any $\beta < \kappa$ note that $j(\beta)$ is the order type of elements $[f]_U \in j(\beta)$. By definition $j(\beta) = [c_\beta]_U$, and hence if $j \in_U c_\beta$, then $\{\alpha | f(\alpha) < \beta\} \in U$. But $\{\alpha \mid f(\alpha) < \beta\} = \bigcup_{\eta<\beta}\{\alpha \mid f(\alpha) = \eta\}$ and hence $[f]_U = [c_\eta]_U = j(\eta)$ for some $\eta < \beta$. Thus $j(\beta) \cong \beta$, and by definition of ordinal $j(\beta) = \beta$, which holds for all $\beta < \kappa$. To see $\kappa < j(\kappa)$, notice that for any $\beta < \kappa$, $[c_\beta]_U < [\mathrm{id}]_U$, because the identity function eventually exceeds β. So $\kappa \leqslant [\mathrm{id}]_U$. But $[\mathrm{id}]_U < [c_\kappa]_U = j(\kappa)$ because the $\mathrm{id} : \kappa \to \kappa$ is less than κ at every point. Thus $\kappa \leqslant [\mathrm{id}]_U < j(\kappa)$, hence $\kappa < j(\kappa)$.

For the other direction, assume κ is the critical point of a nontrivial elementary embedding. Clearly κ is uncountable because $j(\omega) = \omega$ by absoluteness. Let $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$. It is routine to show that this is a κ-complete non-principal ultrafilter on κ.

Hence κ is measurable. $\square$

We can also provide a proof of Lemma 4, i.e that the existence of a measurable cardinal is stronger in consistency strength than the existence of an inaccessible cardinal.

PROOF. (Lemma 4) Suppose κ is measurable. To show the forward direction we first show that κ is also inaccessible. By Theorem 9 assume κ is the critical point of an elementary embedding $j : V \to M$ between to models of set theory. To see that κ is regular we suppose that κ is not, and thus we can suppose there exists a set $A = \{\gamma_\alpha \mid \alpha < \beta\}$ with $A \subseteq \kappa$, $\beta < \kappa$ and $\sup A = \kappa$. Since $\beta < \kappa$ we know $j(A)$ has order type $j(\beta) = \beta$. Similarly, because $\alpha < \beta$ we have $j(\alpha) = \alpha$ and $j(\gamma_\alpha) = \gamma_\alpha$. However, this tells us $j(\kappa) = \sup j(A) = \sup A = \kappa$ which contradicts the fact that j witnessed the measurability of κ.

Now we show that κ is also a strong limit cardinal. To do this, first suppose not, then for some $\beta < \kappa$ we have $2^\beta \geqslant \kappa$. Let $\vec{a} = \langle a_\alpha \mid \alpha < \kappa \rangle$ be κ distinct subsets of β. Since $\beta < \kappa$, and $j(\beta) = \beta$ we have that $j(\vec{a})$ is a $j(\kappa)$-sequence of distinct subsets of β. Let $a = j(\vec{a})(\kappa)$ be the κ-th element of $j(\vec{a})$. Since $a \subseteq \beta < \kappa$ we have $j(a) = a$, and thus M thinks $j(a) = a$ appears on $j(\vec{a})$. Elementarity tells us a also appears on $\vec{a}$, say $a = a_\alpha$ for $\alpha < \kappa$. However this entials that a is also the $j(\alpha)$-th term of $j(\vec{a})$ which is a contradiction. Thus κ is a inaccessible.

To see the reverse implication doesn't hold, i.e. to see the consistency strenght is strictly stronger, we first show that every measurable cardinal has an inaccessible below it. If κ is measurable, witnessed by $j : V \to M$, then because $M^\kappa \subseteq M$ we have that κ is measurable in M as well. As shown above, every measurable cardinal is inaccessible. Since $\kappa < j(\kappa)$ by elementarity this tells us that in our original model there was an inaccessible below κ. In fact it can be shown that there are an unbounded number of inaccessibles below κ.

To actually see the reverse implication doesn't hold, suppose it did, then the existence of a model $(\mathsf{ZFC} + \exists\kappa \text{ inaccessible})$ would imply the existence of a model M of $(\mathsf{ZFC} + \exists\kappa \text{ measurable})$. Chop M off at the measurable cardinal κ and by the above M_κ has an inaccessible below it. We get $\mathsf{ZFC} + \exists\kappa \text{ inaccessible} \vdash \mathrm{Con}(\mathsf{ZFC} + \exists\kappa \text{ inaccessible})$ which contradicts Gödel's Second Incompleteness Theorem. $\square$

Generalizing the definition of measurable cardinals we arrive at supercompact cardinals. Technically, supercompactness arose out of generalizations of something called ω-compactness (see Section 3.1 from [3]). However, because of time, and because we are already within the context of filters, we will try to motivate supercompactness using filters alone.

Considering κ,λ a cardinal and ordinal respectively, whereas measurable cardinals involve ultrafilters on simply κ, supercompact cardinals involve ultrafilters on $\mathcal{P}_\kappa\lambda = \{X \subseteq \lambda \mid |X| < \kappa\}$. In set theory when we go up levels on the Von-Neumann universe, this is done through iterations of the powerset. Thus we could expect a κ-complete measure on $\mathcal{P}_\kappa\lambda$ to be of higher complexity than such a measure on κ alone. Our development towards a characterization theorem for supercompact cardinals was heavily inspired by Section 3.1 from [3]. The following series of statements will include some lemmas from seed theory which will be especially useful. The references for the statements from seed theory can be found in Section 0.3 from [3].

Definition 39 *Given a measure U on a set S, a function $f : S \to S$ is regressive on a set in U if* $\{\alpha \in S \mid f(\alpha) \in \alpha\} \in U$.

Definition 40 *A measure U on a set S is called normal if every function that is regressive on a set in U is constant on a set in U.*

Definition 41 *Given cardinal κ and ordinal λ, a measure on $\mathcal{P}_\kappa\lambda$ is fine if for every $\alpha < \lambda$ we have $\{\sigma \in \mathcal{P}_\kappa\lambda \mid \alpha \in \sigma\} \in U$.*

Definition 42 *Given a cardinal κ and an ordinal λ, we say κ is λ-supercompact if there is a normal fine measure on $\mathcal{P}_\kappa\lambda$. We say κ is supercompact if κ is λ-supercompact for all $\lambda \in \mathrm{ORD}$.*

We now mention the few definitions and results from seed theory.

Definition 43 *If $j : V \to M$ is an elementary embedding with $a \in j(I)$ for some $I \in V$, then a is a seed for the measure U on I defined by $X \in U$ if and only if $X \subseteq I \wedge a \in j(X)$. If $b = j(f)(a)$ for some $f \in V$, then we say a generates b by the embedding. If every element of M is generated by a, then we say a generates all of the embedding j.*

Lemma 6 *(Seed Lemma) An elementary embedding $j : V \to M$ is an ultrapower embedding if and only if there exists a seed a generating all of M. If such is the case then for all $[f]_U \in M$ we have $[f]_U = j(f)(a)$.*

Lemma 7 *(Unique Seed Lemma) If $j : V \to M$ is an ultrapower embedding by a measure U, then $[id]_U$ is the unique seed for U via j, where $[id]_U$ is the equivalence class of the identity function.*

Definition 44 *Given $j : V \to M$ an elementary embedding and $S \subseteq j(I)$ with $S \neq \emptyset$, then the seed hull of S via j in M is $X_S = \{j(f)(s) \mid f : I^{<\omega} \to V, f \in V, s \in S^{<\omega}\}$.*

Definition 45 *If M, N are models of set theory, and for all formulas $\varphi(x_1, ..., x_n)$ with free variables $x_1, ..., x_n$, then N is an elementary substucture of M, denoted $N \preceq M$, given the following holds for all $a_1, ..., a_n \in N$:*

$$N \models \varphi(a_1, ..., a_n) \Leftrightarrow M \models \varphi(a_1, ..., a_n)$$

.

Lemma 8 *(Seed Hull Lemma)* $X_S \preceq M$.

Note that for $X \in M$, $j"X$ refers to the pointwise image of X under j, symbolically $j"X = \{y \mid \exists x \in X(j(x) = y)\}$. We are now able to state and prove an elementary characterization theorem for supercompact cardinals.

Theorem 10 *Given cardinal κ and ordinal λ, then κ is λ-supercompact if and only if there is an elementary embedding $j : V \to M$ with $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, and $M^\lambda \subseteq M$.*

PROOF. To see the forward direction, let U be a normal fine measure on $\mathcal{P}_\kappa\lambda$. Let $M = \mathrm{Ult}(V, U)$ and $j : V \to M$ the canonical ultrapower embedding. That $\mathrm{cp}(j) = \kappa$ follows from κ-completeness of U. For each $x \in \mathcal{P}_\kappa\lambda$ let λ_x represent the order type of x. Since the order type of $j"\lambda$ is λ, λ is represented in M as the function $x \mapsto \lambda_x$. Since $\lambda_x < \kappa$ for all x, $j(\kappa) > \lambda$.

To see that $M^\lambda \subseteq M$, we need to show that whenever we are given $\langle a_\alpha \mid \alpha < \lambda\rangle$ with $a_\alpha \in M$ we have $\{a_\alpha \mid \alpha < \lambda\} \in M$. Let $\langle [f_\alpha] \mid \alpha < \lambda\rangle$ be such that $[f_\alpha] \in M$, with representatives $\langle f_\alpha \mid \alpha < \lambda\rangle$. Define f on $\mathcal{P}_\kappa\lambda$ as $f(x) = \{f_\alpha(x) \mid \alpha \in x\}$. To see that $[f] = \{a_\alpha | \alpha < \lambda\}$ notice that if $\alpha < \lambda$ then by fineness $\{x \mid \alpha \in x\} \in U$, and hence $[f_\alpha] \in [f]$ because $\{x \in \mathcal{P}_\kappa\lambda \mid f_\alpha(x) \in f(x)\} = \{x \in \mathcal{P}_\kappa\lambda \mid \alpha \in x\} \in U$. If $[g] \in [f]$ then $\{x \mid \exists\alpha \in x(g(x) = f_\alpha(x))\} \in U$. By normality, there is a single $\gamma < \lambda$ such that $\{x \mid g(x) = f_\gamma(x)\} \in U$, and hence $[g] = a_\gamma$.

For the reverse direction, let $j : V \to M$ be an embedding with $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, and $M^\lambda \subseteq M$. Because M is closed under λ-sequences we know $j"\lambda = \{j(\gamma) \mid \gamma < \lambda\} \in M$. Thus we may define an ultrafilter U on $\mathcal{P}_\kappa\lambda$ using $j"\lambda$ as a seed, i.e. $X \in U$ if and only if $j"\lambda \in j(X)$. To see that U is a measure is standard. Regarding fineness, we know $|j"\lambda| < j(\kappa)$ in M, so $j"\lambda \in (\mathcal{P}_{j(\kappa)}j(\lambda))^M$. This implies $\mathcal{P}_\kappa\lambda \in U$ and hence for any $\alpha < \lambda$ we have $\{x \mid \alpha \in x\} \in U$. Regarding U being normal, first notice that by $\in$-induction $[f] = j(f)(j"\lambda)$ for all $f \in V$. Furthermore, if we suppose $\{x \mid f(x) \in x\} \in U$ then $j(f)(j"\lambda) \in j"\lambda$. This implies that for some $\gamma < \lambda$ we have $j(f)(j"\lambda) = j(\gamma)$, or that $\{x \mid f(x) = \gamma\} \in U$. $\square$

There does exist an analog of Lemma 4, to see that the existence of a supercompact cardinal is strictly stronger in consistency strength than the existence of a measurable cardinal. To see that a supercompact cardinal is always measurable, set $\lambda = \kappa$ and apply Theorem 10. To see $\mathrm{Con}(\mathrm{ZFC} + \exists\kappa\ \mathrm{measurable}) \not\Rightarrow \mathrm{Con}(\mathrm{ZFC} + \exists\kappa\ \mathrm{supercompact})$, one can use a reflection argument similar to the proof of $\mathrm{Con}(\mathrm{ZFC} + \exists\kappa\ \mathrm{inaccessible}) \not\Rightarrow \mathrm{Con}(\mathrm{ZFC} + \exists\kappa\ \mathrm{measurable})$ in Lemma 4.

2.2 Easton Support Iterations

In Section 1.4 using forcing we showed the consistency of ZFC implied the consistency of $\mathrm{ZFC} + \mathrm{CH}$ and the consistency of $\mathrm{ZFC} + \neg\mathrm{CH}$, in otherwords we showed $\mathrm{Con}(\mathrm{ZFC}) \Rightarrow$

$\mathsf{Con}(\mathsf{ZFC}+\mathsf{CH})$ and $\mathsf{Con}(\mathsf{ZFC}) \Rightarrow \mathsf{Con}(\mathsf{ZFC}+\neg\mathsf{CH})$. Also, in Section 2.1 we gave a short characterization proof for measurable cardinals. One question might be as follows: Is there a model M of ZFC such that M thinks κ is measurable and $M \models 2^\kappa > \kappa^+$, in other words where GCH fails at κ? Before the advent of forcing, Dana Scott proved a very interesting result regarding measurable cardinals.

Theorem 11 *(Scott) Given κ measurable, if for every $\alpha < \kappa$ we have $2^\alpha = \alpha^+$, then $2^\kappa = \kappa^+$.*

The above theorem implies that if we would like GCH to fail at a measurable cardinal κ, then GCH must already fail below κ in several places (a measure one set). This implies that we would need more than a single forcing partial order for GCH to fail at a measurable cardinal. We first develop the idea of *product forcing* (Definitions 46-47 follow pages 229-30 of [1]).

Definition 46 *Given two forcing partial orders $\mathbb{P}, \mathbb{Q}$ the product $\mathbb{P}\times\mathbb{Q}$ is the partial order formed on the set product of $\mathbb{P}$ and $\mathbb{Q}$ ordered as follows (for $p_1, p_1 \in \mathbb{P}$ and $q_1, q_2 \in \mathbb{Q}$):*

$$(p_1, q_1) \leqslant (p_2, q_2) \textit{ if and only if } p_1 \leqslant p_2 \wedge q_1 \leqslant q_2$$

Note that given a $\mathbb{P}\times\mathbb{Q}$-generic filter G over some model M the sets $G_1 = \{p \in \mathbb{P} | (p, q) \in G \text{ for some } q \in \mathbb{Q}\}$ and $G_2 = \{q \in \mathbb{Q} | (p, q) \in G \text{ for some } p \in \mathbb{P}\}$ are $\mathbb{P}$-generic and $\mathbb{Q}$-generic (both over M) respectively. Similarly given G_1, G_2, $\mathbb{P}$-generic and $\mathbb{Q}$-generic (both over M) respectively, then $G_1 \times G_2$ is $\mathbb{P}\times\mathbb{Q}$-generic over M. In fact $M[G_1 \times G_2] = M[G_1][G_2] = M[G_2][G_1]$.

Definition 47 *Let $\{\mathbb{P}_i \mid i \in I\}$ be a collection of partially ordered sets, each with a common maximum element 1. The product $\mathbb{P} = \prod_{i\in I} \mathbb{P}_i$ consists of functions $p : I \to \bigcup_{i\in I} \mathbb{P}_i$ with $p(i) \in \mathbb{P}_i$, and $p(i) = 1$ for all bit finitely many $i \in I$. The ordering is given as follows for $p, q \in P$:*

$$p \leqslant q \quad \textit{if and only if} \quad \forall i \in I(p(i) \leqslant q(i))$$

(See Theorem 15.18 of [1] for a source on Definitions 48-49).

Definition 48 *Given $\mathbb{P} = \prod_{i\in I} \mathbb{P}_i$ as in the above the conditions p have the form $p = \langle p_i \mid i \in I\rangle \in \prod_{i\in I} \mathbb{P}$. The support of p is the set $s(p) = \{i \in I \mid p_i \neq \emptyset\}$.*

Definition 49 *Let F be a function defined on a set A of regular cardinals satisfying ($\mathrm{cf}(F(\kappa) > \kappa$) and monotonicity ($k < k' \to F(\kappa) \leqslant F(\kappa')$). For each $\kappa \in \mathrm{dom}(F)$, let $(\mathbb{P}_\kappa, \supset)$ be $\mathrm{Add}(\kappa, F(\kappa))$. The conditions p of the Easton product $\mathbb{P} = \prod_{k\in A} \mathbb{P}_\kappa$ have the following*

$$\textit{for every regular cardinal } \kappa,\ |s(p) \cap \kappa| < \kappa.$$

Similarly we can develop *Easton support iterations* below (see Definitions 16.29 and 21.5 from [1]).

Definition 50 *Let $\alpha \geqslant 1$. A forcing notion $\mathbb{P}_\alpha$ is an* <u>*iteration*</u> *(of length α) if it is a set of α-sequences with the following properties:*

1. *If $\alpha = 1$ then for some forcing notion $\mathbb{Q}_0$*
   - *(a) $\mathbb{P}_1$ is the set of all 1-sequences $\langle p(0)\rangle$ where $p(0) \in \mathbb{Q}_0$;*
   - *(b) $\langle p(0)\rangle \leqslant_1 \langle q(0)\rangle$ if and only if $p(0) \leqslant q(0)$.*
2. *If $\alpha = \beta + 1$ then $\mathbb{P}_\beta = \mathbb{P}_\alpha \restriction_\beta = \{p \restriction_\beta \mid p \in \mathbb{P}_\alpha\}$ is an iteration of length β, and there is some forcing notion $\dot{\mathbb{Q}}_\beta \in V^{\mathbb{P}_\beta}$ such that*
   - *(a) $p \in \mathbb{P}_\alpha$ if and only if $p \restriction_\beta \in \mathbb{P}_\beta$ and $\Vdash_\beta p(\beta) \in \dot{\mathbb{Q}}_\beta$;*
   - *(b) $p \leqslant_\alpha q$ if and only if $p \restriction_\beta \leqslant_\beta q \restriction_\beta$ and $p \restriction_\beta \Vdash_\beta p(\beta) \leqslant q(\beta)$.*
3. *If α is a limit ordinal, then for every $\beta < \alpha$, $\mathbb{P}_\beta = \mathbb{P}_\alpha \restriction_\beta = \{p \restriction_\beta \mid p \in \mathbb{P}_\alpha\}$ is an iteration of legnth β and*
   - *(a) The α-sequence $\langle 1, 1, ..., 1, ...\rangle$ is in $\mathbb{P}_\alpha$;*
   - *(b) if $p \in \mathbb{P}_\alpha, \beta < \alpha$ and if $q \in \mathbb{P}_\beta$ is such that $q \leqslant_\beta p \restriction_\beta$, then $r \in \mathbb{P}_\alpha$ where for all $\delta < \alpha, r(\delta) = q(\delta)$ if $\delta < \beta$ and $r(\delta) = p(\delta)$ if $\beta \leqslant \delta < \alpha$.*
   - *(c) $p \leqslant_\alpha q$ if and only if $\forall \beta < \alpha (p \restriction_\beta \leqslant_\beta q \restriction_\beta)$*

Definition 51 *Let $\alpha \geqslant 1$, and let $\mathbb{P}_\alpha$ be an iterated forcing of legnth α. Then $\mathbb{P}_\alpha$ is an* <u>*Easton support iteration*</u> *if for every $p \in \mathbb{P}_\alpha$ and every regular cardinal $\gamma \leqslant \alpha, |s(p) \cap \gamma| < \gamma$.*

(See Lemma 21.8 from [1] for a source on the following theorem).

Theorem 12 *Let $\mathbb{P}_{\alpha+\beta}$ be a forcing iteration of $\langle \dot{\mathbb{Q}}_\delta \mid \delta < \alpha + \beta\rangle$, where each $\mathbb{P}_\delta$, $\delta \leqslant \alpha + \beta$ is an Easton support iteration. In $V^{\mathbb{P}_\alpha}$, let $\dot{\mathbb{P}}^{(\alpha)}_\beta$ be the forcing iteration of $\langle \dot{\mathbb{Q}}_{\alpha+\delta} \mid \delta < \beta\rangle$ such that for every limit ordinal $\delta < \beta, \dot{\mathbb{P}}^{(\alpha)}_\delta$ uses bounded or full-support, according to whether $\mathbb{P}_{\alpha+\delta}$ used bounded of full-support. If $\mathbb{P}_{\alpha+\delta}$ is defined using full support for every limit ordinal $\delta \leqslant \beta$ such that $\mathrm{cf}\delta \leqslant |\mathbb{P}_\alpha|$, then $\mathbb{P}_{\alpha+\beta}$ is isomorphic to $\mathbb{P}_\alpha * \dot{\mathbb{P}}_\beta$.*

2.3 Failure of GCH at Measurable Cardinals

In Section 1.4 we proved $\mathrm{Con(ZFC)} \Rightarrow \mathrm{Con(ZFC{+}CH)}$ as well as $\mathrm{Con(ZFC)} \Rightarrow \mathrm{Con(ZFC{+}\neg CH)}$. So clearly we can break the GCH at ω. Now that we have introduced measurable cardinals, a natural question might be whether or not we can break GCH at κ where κ is measurable. It turns out, that given κ a κ^{++}-supercompact cardinal we can then find a model where GCH fails at a measurable cardinal, a proof which is developed below (for references on lemmas 9-11 see sections 1.3, 1.4, and Theorem 53 from [3]).

Lemma 9 *(Lifting Criterion) Suppose* $j : M \to N$ *is an elementary embedding of two models of* ZF *with forcing extensions* $M \subseteq M[G]$ *and* $N \subseteq N[H]$ *by* $\mathbb{P}$ *and* $j(\mathbb{P})$ *respectively. Then* j *lifts to an embedding* $j^* : M[G] \to N[H]$ *with* $j^*(G) = H$*, if and only if* $j"G \subseteq H$*. Additionally, such a* j^* *is unique.*

Lemma 10 *(Diagonalization Criterion) Given* $\mathbb{P}$ *a partial order,* M *a model of set theory with* $\mathbb{P} \in M$*, if for some cardinal* λ *the following are satisfied*

1. $M^\lambda \subseteq M$
2. $\mathbb{P}$ *is* $\leqslant \lambda$*-closed in* M
3. M *has at most* λ^+ *many maximal antichains for* $\mathbb{P}$*,*

then for any $p \in \mathbb{P}$ *there is an* M*-generic filter* $H \subseteq \mathbb{P}$ *with* $p \in H$*.*

Lemma 11 *Suppose that* $M^\lambda \subseteq M$ *in* V *and there is in* V *an* M*-generic filter* $H \subseteq \mathbb{Q}$ *for some forcing* $\mathbb{Q} \in M$*. Then* $M[H]^\lambda \subseteq M[H]$ *in* V*.*

(See Section 3.3 of [3] as a source for Theorem 13).

Theorem 13 *Let GCH hold and* κ *be* κ^{++}*-supercompact, then there is a cardinal-preserving forcing extension in which* κ *is measurable and* $2^\kappa = \kappa^{++}$*.*

PROOF. Let $j : V \to M$ be a witness to the κ^{++}-supercompactness of κ. Let $\mathbb{P}_{\kappa+1}$ be an Easton support iteration length $\kappa + 1$ defined as follows:

1) if $\gamma \leqslant \kappa$ is inaccessible then $\dot{\mathbb{Q}}_\gamma$ is a $\mathbb{P}_\gamma$-name for $\mathrm{Add}(\gamma, \gamma^{++})$.
2) if $\gamma < \kappa$ is not inaccessible then let $\dot{\mathbb{Q}}_\gamma$ be trivial.

Note that $|\mathbb{P}_\kappa| \leqslant \kappa$, and therefore $\mathbb{P}_\kappa$ satisfies the κ-cc, while $|\mathrm{Add}(\kappa, \kappa^{++})| \leqslant \kappa^{++<\kappa} = \kappa^{++}$ which implies $\mathrm{Add}(\kappa, \kappa^{++})$ is $< \kappa$-closed and has the κ^+-cc By the Diagonalization Criterion we may form a V generic $G_\kappa \cong G * H \subseteq \mathbb{P}_\kappa * \dot{\mathbb{Q}} = \mathbb{P}_\kappa * \mathrm{Add}(\kappa, \kappa^{++})$. By previous results $V[G * H] \models 2^\kappa = \kappa^{++}$.

Now, we need to show that $V[G * H] \models (\kappa \text{ is } \mathtt{measurable})$, i.e. we need to show that in $V[G * H]$ there is an elementary embedding with critical point κ. In $V[G][H]$ fix an arbitrary $j : V \to M$, which we will lift to the desired type of elementary embedding in two stages.

Firstly, we need to lift j to $j : V[G] \to M[\hat{G}]$. Without loss of generality, assume M is an ultrapower by a normal fine measure U on $\mathcal{P}_\kappa \kappa^{++}$. To apply the Lifting Criterion we need an M-generic subset $\hat{G} \subseteq j(\mathbb{P}_\kappa)$ such that $j"G \subseteq \hat{G}$. Let $\mathbb{P}_\kappa$ be the Easton support iteration which at stage $\gamma < \kappa$ forces with $\mathbb{Q}_\gamma = \mathrm{Add}(\gamma, 1)^{V^{\mathbb{P}_\gamma}}$, given that γ is regular in $V^{\mathbb{P}_\gamma}$, and forces with $\emptyset$ otherwise. We see it maps to $j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \mathrm{Add}(\kappa, \kappa^{++}) * \mathbb{P}_{\mathrm{tail}}$, which is an Easton support iteration of length $j(\kappa)$. Since M and V agree up to κ^{++}, it follows that for stages $\leqslant \kappa$ of the Easton-support iteration, M and V agree. Because $G * H$ is V-generic we know $G * H$ is M-generic, thus we may form $M[G][H]$.

To complete the first stage, we must build an $M[G][H]$-generic filter $G_{tail} \subseteq \mathbb{P}_{tail}$. The next stage in the forcing beyond κ in $j(\mathbb{P}_\kappa)$ is inaccessible, which tells us $\mathbb{P}_{tail}$ is $\leqslant \kappa^{++}$-closed in $M[G][H]$. Since $M^{\kappa^{++}} \subseteq M$ in V, and since clearly $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ is κ^+-cc, we know $M[G][H]^\lambda \subseteq M[G][H]$ in $V[G][H]$. Finally, the number of dense sets of $\mathbb{P}_{tail}$ can be calculated $|j(2^\kappa)| \leqslant (2^\kappa)^{\kappa^{++<\kappa}} = \kappa^{+++}$ in V. Applying the Diagonalization Criterion gives us the $M[G][H]$-generic filter we were looking for. So $\hat{G} \cong G * H * G_{tail}$ is M-generic for $j(\mathbb{P}_\kappa)$ and $j"G \subseteq \hat{G}$ trivially, so by the Lifting Criterion we can lift j to $j : V[G] \to M[\hat{G}]$ in $V[G][H]$. By Lemma 11 we know $M[\hat{G}]^{\kappa^{++}} \subseteq M[\hat{G}]$ in $V[G][H]$.

For the second stage, we need to lift j to $j : V[G * H] \to M[\hat{G} * \hat{H}]$, that is we need to find an $M[\hat{G}]$-generic filter $j(H) \subseteq j(\mathbb{Q}_\kappa)$ which satisfies $j"H \subseteq j(H)$. The key to this construction is a so called *master condition* which will be given as follows. First note $H \in M[\hat{G}]$ by definition, and also $j \restriction_{\mathbb{Q}_\kappa} \in M[\hat{G}]$ in virtue of $M[\hat{G}]$ being κ^{++}-closed. Since $H = \mathrm{Add}(\kappa, \kappa^{++})$ and thus $|H| = \kappa^{++}$ we have that $j"H$ is a κ^{++} sequence, and hence $j"H = j \restriction_{\mathbb{Q}_\kappa} "H \in M[\hat{G}]$. The master condition will be $m = \bigcup j"H \in M[\hat{G}]$ because any $M[\hat{G}]$-generic filter containing it will automatically contain $j"H$ as a subset. Since $H \subseteq \mathbb{Q}_\kappa$ by elementarity we have $j"H \subseteq j(\mathbb{Q}_\kappa) = j(\mathrm{Add}(\kappa, \kappa^{++}))$. Also since $j"H \subseteq j(\mathbb{Q}_\kappa)$ and $|j"H| < \kappa^{++}$ we have $m \in j(\mathbb{Q}_\kappa)$. To apply the Diagonalization Criterion, notice $j(\mathbb{Q}_\kappa)$ is $< j(\kappa)$-closed in $M[\hat{G}]$, and hence also κ^{++}-closed in $M[\hat{G}]$. Also, recall that $M[\hat{G}]^{\kappa^{++}} \subseteq M[\hat{G}]$. Lastly, we check the number of open dense sets in $j(\mathbb{Q}_\kappa)$ which is $|j(2^{\kappa^{++}})| \leqslant (2^{\kappa^{++}})^{\kappa^{++<\kappa}} = 2^{\kappa^{++}} = \kappa^{+++}$ which allows us to apply the Diagonalization Criterion to construct an $M[\hat{G}]$-generic filter $\hat{H} \subseteq j(\mathbb{Q}_\kappa)$ below m. Since $j"H \subseteq \hat{H}$ we can apply the Lifting Criterion to lift j to $j : V[G][H] \to M[\hat{G}][\hat{H}]$, which preserves supercompactnesss. □

2.4 An Introduction to Indestructibility

We have the following proposition rather trivially (see Section 3 from [4]).

Proposition 6 *(Silver) Given the existence of a measurable cardinal κ, then there is a forcing extension where κ remains measurable, and remains measurable under further* $\mathrm{Add}(\kappa, 1)$ *forcing.*

In the above proposition, we say that the measurability of κ has been made *indestructible* relative to $\mathrm{Add}(\kappa, 1)$ forcing. Naturally, we might ask if a similar indestructibility property can be derived for κ supercompact. It turns out that we can prove an analog of Proposition 6, but for κ supercompact, where the class of forcing partial orders for which κ is made indestructible is far larger than simply $\mathrm{Add}(\kappa, 1)$.

However there are deeper reasons for us to be interested in the indestructibility phenomenon. The details escape our immediate interest, so instead we give this paragraph long survey. First, the existence of a Laver function, as utilized in the indestructibility proof below, can be used to prove the consistency of the *proper forcing axiom*, a tool very important to contemporary set theory. Secondly, we can obtain the failure of GCH at a measurable cardinal as a corrollary to Laver's Indestructibility Theorem, then apply a special type of forcing called *Prikry forcing* to obtain a failure of the *Singular Cardinal Hypothesis*. The rest of this section closely follows sections 3.4-3.5 from [3].

Definition 52 *Given a model* M *of set theory, an* <u>*inner model*</u> *is a substructure* $M' \prec M$ *such that* M' *models set theory and contains all the ordinals of* M.

Lemma 12 *If* M *is a transitive inner model,* $M \subseteq V$, *and* $M^{\beth_\lambda} \prec M$, *then we have* $V_\lambda^M = V_\lambda$.

PROOF. Let $j : V \to M$ be a λ-supercompactness embedding. We need to show $V_\lambda^M = V_\lambda$, or that M agrees with V about V_λ. This will be done by induction.

To see $V_\lambda^M \subseteq V_\lambda$, the base case gives us $V_\emptyset^M \subseteq V_\emptyset$. For successor stages $\beta + 1 \leqslant \lambda$ suppose that for all $\alpha \leqslant \beta$ we have $V_\alpha^M \subseteq V_\alpha$. To see $V_{\beta+1}^M \subseteq V_{\beta+1}$, first note $V_{\beta+1}^M = \mathcal{P}(V_\beta)^M$, and because M is transitive this tells us that if $X \in \mathcal{P}(V_\beta)^M$ then $X \subseteq V_\beta^M$. By induction we assumed $V_\beta^M \subseteq V_\beta$, therefore $X \in \mathcal{P}(V_\beta)$, and hence $X \in V_{\beta+1}$. For limit stages $\beta \leqslant \lambda$ suppose for all $\alpha < \beta$ we have $V_\alpha^M \subseteq V_\alpha$. Since $V_\beta^M = \bigcup_{\alpha<\beta} V_\alpha^M$, if $X \in V_\beta^M$ then $X \in V_\alpha^M$ for some $\alpha < \beta$. By induction we have $X \in V_\alpha$, and because $V_\beta = \bigcup_{\alpha<\beta} V_\alpha$, this gives us $X \in V_\beta$ as desired.

To see $V_\lambda^M \supseteq V_\lambda$, the base stage is $V_\emptyset^M \supseteq V_\emptyset$. For successor stages $\beta + 1 \leqslant \lambda$ suppose that for all $\alpha \leqslant \beta$ we have $V_\alpha^M \supseteq V_\alpha$. Given an $X \in V_{\beta+1}$ then in V we know $X \subseteq V_\beta$, and by induction we have $X \subseteq V_\beta^M$. Because $|X| \leqslant \beth_\beta \leqslant \beth_\lambda$ we can view X as a $\beth_\lambda$-sequence. Since $M^{\beth_\lambda} \subseteq M$ this implies $X \in M$, which entails $X \in V_{\beta+1}^M$. For limit stages $\beta \leqslant \lambda$ suppose for all $\alpha < \beta$ we have $V_\alpha^M \supseteq V_\alpha$. Then if $X \in V_\beta$, since $V_\beta = \bigcup_{\alpha<\beta} V_\alpha$ we have $X \in V_\alpha$ for some $\alpha < \beta$. By induction we have $X \in V_\alpha^M$ and hence $X \in V_\beta^M$ as desired. $\square$

The next celebrated result proves the existence of a rather interesting function, which will be useful in our attempt to find a forcing extension where supercompactness has been made indestructible.

Theorem 14 *(Laver) If* κ *is supercompact then there is a function* $\ell : \kappa \to V_\kappa$ *such that for every* x *and every* λ *with* $x \in H_{\lambda^+}$ *there is a* λ*-supercompactness embedding* $j : V \to M$ *with* $j(\ell)(\kappa) = x$.

PROOF. Suppose that up to some $\gamma < \kappa$ the restriction $\ell \restriction_\gamma$ has been defined. We define $\ell(\gamma)$ as follows: If there is a least λ such that for some $x \in H_{\lambda^+}$ there is no λ-supercompactness embedding $j : V \to M$ with critical point γ such that $j(\ell \restriction_\gamma)(\gamma) = x$, then let $\ell(\gamma)$ be such an x. If no such λ exists, let $\ell(\gamma) = \emptyset$.

Now we need to check that ℓ defined above really is a function from κ to V_κ. Towards a contradiction suppose there is a γ such that $\ell(\gamma) = x$ and $x \notin V_\kappa$. Then we know $x \in V_\theta$ for some limit ordinal θ such that $\theta > \kappa$. Choose $\lambda = \beth_\theta$ and let $j : V \to M$ be a

λ-supercompactness embedding with critical point κ. By Lemma 12 we know $V_\theta^M = V_\theta$. Because M and V agree about V_θ, we have $M \models \ell(\gamma) = x$ or that M thinks x is not anticipated by $\ell \restriction_\gamma$ by any supercompactness embedding for γ. Because $j(\ell \restriction_\gamma) = \ell \restriction_\gamma$ and $V_\theta \subseteq j(V_\kappa)$ we know M thinks there exists a y in $j(V_\kappa)$ not anticipated by $\ell \restriction_\gamma$. By elementarity, V thinks there is a y in V_κ not anticipated by $\ell \restriction_\gamma$ which is a contradiction. Therefore $\ell(\gamma) \in V_\kappa$ as desired.

So ℓ has the correct domain and range. Now we need to make sure ℓ has the desired feature. If we assume ℓ doesn't have the desired feature, then there is a minimal λ such that there is an $x \in H_{\lambda^+}$ not anticipated by any λ-supercompactness embedding for κ. Let $j : V \to M$ be a $2^{\lambda^{<\kappa}}$-supercompactness embedding for κ. Since $M^{2^{\lambda^{<\kappa}}} \subseteq M$ we know M and V agree on supercompactness measures on $\mathcal{P}_\kappa\lambda$. Similarly, M and V agree about functions from $P_\kappa\lambda$ to V_κ. Thus M and V agree that x is not anticipated by ℓ for any λ-supercompactness measure for κ, and M and V agree that λ is the least cardinal with such a non-anticipated set. Since $\ell = j(\ell) \restriction_\kappa$, this implies $j(\ell)(\kappa) = y$ where $y \in H_{\lambda^+}$ is not anticipated by ℓ for any λ-supercompactness embedding. Let $X = \{j(f)(j"\lambda) : \mathrm{dom}(f) = \mathcal{P}_\kappa\lambda, f \in V\}$. Because $j"\lambda \subseteq \mathrm{ran}(j)$ we can apply the Seed Hull Lemma to show $X \prec M$. Let $\pi : X \cong M_0$ be the Mostowski collapse of X. Let $j_0 = \pi \circ j$. To see that j_0 is well defined we need to ensure $\mathrm{ran}(j) \subseteq X$, but given an $j(x) \in \mathrm{ran}(j)$ the fact that j_0 is well defined follows from:

$$j(x) = [c_x]_\mu = j(c_x)(j"2^{\lambda^{<\kappa}}) = j(c_x \restriction_{\mathcal{P}_\kappa\lambda})(j"\lambda) \in X$$

By defining $k = \pi^{-1}$ we obtain the commutative diagram below:

$$\begin{tikzcd} V \arrow[d, "j_0"] \arrow[dr, "j"] & \\ M_0 \arrow[r, "k"] & M \end{tikzcd}$$

We will now show that y is fixed by the collapse π, and hence $k(y) = y$. This will be an essential component in contradicting our assumption that y is not anticipated by any λ-supercompactness embedding. To show that y is fixed, we need to first assure ourselves that $y \in X$ and $\lambda \subseteq X$.

To see that $y \in X$, find an $f : P_\kappa\lambda \to V_\kappa$ such that if $\sigma \cap \kappa \in \kappa$ we have $f(\sigma) = \ell(\sigma \cap \kappa)$. Because $\lambda > \kappa$ we know that for all $x \in j"\lambda$ either $x \geqslant j(\kappa)$ or $x \leqslant \kappa$, and hence $j"\lambda \cap j(\kappa) = \kappa$. We can then show

$$j(f)(j"\lambda) = j(\ell)(j"\lambda \cap j(\kappa)) = j(\ell)(\kappa) = y$$

and because $j(f)(j"\lambda) \in X$ we have the desired claim.

To see that $\lambda \subseteq X$, first note that if $\alpha < \kappa$ and if we let f be the constant function

α defined on $\mathcal{P}_\kappa\lambda$, then $j(f)(j"\lambda) = \alpha$, and hence $\kappa \subseteq X$. Now, if we can show that X is closed under λ-sequences then $\kappa + 1 \subseteq X, \kappa + 2 \subseteq X, \ldots$ and so forth until $\lambda \subseteq X$, which would give us the desired claim. Suppose $\vec{F} = \langle j(f_\alpha)(j"\lambda) : \alpha < \lambda\rangle \in X^\lambda$ is a λ-sequence, our goal is to show that $\vec{F} \in X$. Consider $\vec{f} = \langle f_\alpha : \alpha < \lambda\rangle \in V$ and recall that $\mathrm{dom}(f_\alpha) = \mathcal{P}_\kappa\lambda$ for all $\alpha < \lambda$. By elementarity, $j(\vec{f}) = \langle \overline{f}_\alpha : \alpha < j(\lambda)\rangle$ is a $j(\lambda)$-sequence in M. En route to showing $\vec{F} \in X$ we will first show $j(\vec{f}) \in X$ by finding a function f such that $j(f)(j"\lambda) = j(\vec{f})$. Let f be a function with domain $\mathcal{P}_\kappa\lambda$ defined as follows:

$$f(\sigma) = \vec{f}\restriction_\sigma = \langle f_\alpha : \alpha < \sigma\rangle \qquad\qquad \forall\sigma \in \mathcal{P}_\kappa\lambda$$

By elementarity $j(f)(\sigma) = j(\vec{f})\restriction_\sigma$ and hence:

$$j(f)(j"\lambda) = j(\vec{f})\restriction_{j"\lambda} = \langle \overline{f}_\alpha : \alpha \in j"\lambda\rangle = \langle j(f_\alpha) : \alpha \in \lambda\rangle \in X$$

Finally, because $j(id)(j"\lambda) = j"\lambda$ tells us $j"\lambda \in X$, it follows that $\vec{F} \in X$

Now, we can show that y is fixed by the collapse. Clearly $\emptyset \in TC(y)$ and $\emptyset \in X$ as well. Suppose that $z \in TC(y)$ and for all $w \in TC(z)$ we have $w \in X$. Since $TC(z) \subseteq X$ and $|TC(z)| \leqslant \lambda$ we get $z \in X$. This entails $TC(y) \subseteq X$ and since $\pi : X \cong M_0$ we get $\pi(y) = y$.

Now we know that $y \in M_0$ and we are ready to derive the contradiction. Since $j(\ell)(\kappa) = y$ this implies:

$$\pi(j(\ell)(\kappa)) = \pi(y) = j_0(\ell)(\kappa) = y$$

or that y is indeed anticipated by a λ-supercompactness embedding, namely j_0, which contradicts our earlier assumption. □

We will now utilize the function ℓ introduced in the previous theorem to prove there is a far more powerful analog of Proposition 6, but for supercompactness.

Definition 53 *Given a partial order $\mathbb{P}$, a subset $D \subseteq \mathbb{P}$ is directed if given any two elements $r, q \in D$ there is a $p \in D$ such that $p \leqslant r$ and $p \leqslant q$.*

Definition 54 *Given a partial order $\mathbb{P}$, a subset $D \subseteq \mathbb{P}$ is bounded below if there is a $p \in \mathbb{P}$ such that for every $q \in D$ we have $p \leqslant q$.*

Definition 55 *A partial order $\mathbb{P}$ is $<\kappa$-directed closed if every directed subset $D \subseteq \mathbb{P}$ with $|D| < \kappa$ has a lower bound.*

Theorem 15 *(Laver's Indestructibility Theorem) If κ is supercompact, then there is a forcing extension where κ remains supercompact, and the supercompactness of κ is indestructible by any further $<\kappa$-directed closed forcing.*

PROOF. Suppose $\ell : \kappa \to V_\kappa$ is a Laver function for κ, where κ is supercompact. We will define $\mathbb{P}_\kappa$, a special Easton support iteration of length κ, often called a *Laver preparation*.

We define the stages of this forcing recursively. If $\mathbb{P}_\alpha$ has been defined for $\alpha < \kappa$, α-inaccessible, and $\ell(\alpha)$ is a $\mathbb{P}_\alpha$-name for a $<\alpha$-directed closed poset, let $\dot{\mathbb{Q}}_\alpha$ be this poset. Otherwise let $\dot{\mathbb{Q}}_\alpha$ be $\emptyset$.

Given a V-generic $G \subseteq \mathbb{P}_\kappa$, we need to show $V[G] \models \kappa$ is supercompact and indestructible by any further $<\kappa$-directed closed forcing. It suffices to show that if $\mathbb{Q} \in V[G]$ is $<\kappa$-directed closed and $H \subseteq \mathbb{Q}$ is $V[G]$-generic then κ is supercompact in $V[G][H]$ (since if $\mathbb{Q} = \emptyset$ then $V[G][H] \cong V[G]$). Therefore suppose $\mathbb{Q} \in V[G]$ is a $<\kappa$-directed closed poset, $H \subseteq \mathbb{Q}$ is $V[G]$-generic, then $\mathbb{Q} = \dot{\mathbb{Q}}_{\mathbb{G}}$ for some $\mathbb{P}_\kappa$-name $\dot{\mathbb{Q}}$. Fix $\lambda > \kappa$ big enough such that $\dot{\mathbb{Q}} \in H^V_{\lambda^+}$. Since κ is supercompact in V, let $j : V \to M$ be a $2^{\lambda^{<\kappa}}$-supercompactness embedding with $j(\ell)(\kappa) = \dot{\mathbb{Q}}$. By elementarity, and our definition of $\mathbb{P}_\kappa$, the κ stage forcing in $j(\mathbb{P}_\kappa)$ is $\dot{\mathbb{Q}}$. Thus we can factor the poset $j(\mathbb{P}_\kappa)$ in M as $j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{P}}_{\text{tail}}$.

First we must lift j to $j : V[G] \to j[\hat{G}]$ with $\hat{G} \subseteq j(\mathbb{P}_\kappa)$ generic over M and $j\text{''}G \subseteq G$. Since $\mathbb{P}_{\text{tail}}$ is a poset in $V[G][H]$ let $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ be $V[G][H]$-generic and $\hat{G} = G * H * G_{\text{tail}}$. Our forcing poset $j(\mathbb{P}_\kappa)$ has bounded support, so clearly $j\text{''}G \subseteq \hat{G}$ and hence we may lift to $j : V[G] \to M[\hat{G}]$.

Now we must lift $j : V[G] \to V[\hat{G}]$ to $j : V[G][H] \to M[\hat{G}][\hat{H}]$ where $\hat{H} \subseteq j(\mathbb{Q})$ is $M[\hat{G}]$-generic and $j\text{''}H \subseteq \hat{H}$. Because H is a filter $j\text{''}H \subseteq j(\mathbb{Q})$ is directed, by elementarity $j(\mathbb{Q})$ is $< j(\kappa)$-directed closed. By Lemma 11 we know $M[G]^{2^{\lambda^{<\kappa}}} \subseteq M[G]$, and since $H \in M[\hat{G}]$ and $j \restriction_{\mathbb{Q}} \in M[\hat{G}]$ we know $j\text{''}H \in M[\hat{G}]$. Since $j\text{''}H$ has size $< j(\kappa)$ there is a single $p^* \in j(\mathbb{Q})$ such that $p^* \leqslant j(p)$ for all $p \in H$. Let $\hat{H} \subseteq j(\mathbb{Q})$ be $V[G * H * G_{\text{tail}}]$-generic with $p^* \in \hat{H}$. Thus, because $p \in \hat{H}$ this ensures $j\text{''}H \subseteq \hat{H}$. Working in $V[G * H * G_{\text{tail}} * \hat{H}]$ we may lift $j : V[G] \to V[\hat{G}]$ to $j : V[G][H] \to M[\hat{G}][\hat{H}]$.

As our final observation, working in $V[G * H * G_{\text{tail}} * \hat{H}]$ define the measure

$$U = \{X \in \mathcal{P}(\mathcal{P}^\lambda_\kappa)^{V[G*H]} : j\text{''}\lambda \in j(X)\}$$

Since $\mathbb{P}_{\text{tail}} * j(\mathbb{Q})$ is $\leqslant 2^{\lambda^{<\kappa}}$-closed in $V[G*H]$ and $|U|^{V[G*H]} \leqslant 2^{\lambda^{<\kappa}}$ we know $U \in V[G*H]$ and hence the supercompactness of κ comes from $V[G * H]$ in our lift. □

We give a list of some common $<\kappa$-directed closed forcing partial orders.

Proposition 7 *Let κ, λ be cardinals. The following partial orders are $<\kappa$-directed closed.*

- $\mathrm{Add}(\kappa, 1)$ *to add one subset to* κ.
- $\mathrm{Add}(\kappa, \lambda)$ *to add any number of subsets to* κ.
- $\mathrm{Col}(\kappa, \lambda)$ *to collapse any cardinal to* κ.
- *Iterations and products of the above.*

Chapter 3

Characterization Theorems for Large Cardinals

3.1 Additional Large Cardinal Axioms

In the previous chapter we showed how natural filter definitions give rise to distinct large cardinals, κ-complete non-principal ultrafilters in the case of measurability and normal fine measures on $\mathcal{P}_\kappa\lambda$ in the case of λ-supercompactness. However we can loosen or tweak these filter definitions to arrive at other large cardinal definitions, distinct from either measurability or supercompactness (although not quite as intuitive).

Similar to how supercompactness was originally motivated by generalizations of ω-compactness, weak and strong compactness were also originally motivated by compactness results for certain infinitary logics, hence the name. However, there is not enough time to properly motivate these cardinals in terms of compactness results, so instead we continue to define our cardinals in terms of filters (see sections 4.1 and 6.1 from [3]).

Definition 56 *Given $\kappa^{<\kappa} = \kappa$, then κ is weakly compact if and only if for every set $\mathscr{A}$ containing at most κ-many subsets of κ, then there is a κ-complete non-principal filter F measuring every set in $\mathscr{A}$.*

Definition 57 *Given $\kappa \leqslant \lambda$ are both uncountable cardinals, then κ is λ-strongly compact if there is a κ-complete fine measure on $\mathcal{P}_\kappa\lambda$.*

In a paper titled *Partial near supercompactness* Jason Schanker introduces the following notion [2]. Notice that it is similar to the traditional definition of supercompactness, but rather than having an ultrafilter measuring all if $\mathcal{P}_\kappa\lambda$ we have a filter measuring all sets in a large collection $\mathscr{A}$.

Definition 58 *Given $\lambda^{<\kappa} = \lambda$, cardinal κ is nearly λ-supercompact if for every collection $\mathscr{A}$ of at most λ-many subsets of $\mathcal{P}_\kappa\lambda$ and collection $\mathscr{F}$ of at most λ many functions from $\mathcal{P}_\kappa\lambda$ into λ there exists a κ-complete fine filter F on $\mathcal{P}_\kappa\lambda$ measuring all sets in $\mathscr{A}$ and which is $\mathscr{F}$-normal (i.e. for every regressive $f \in F$, there is an $\alpha_f < \lambda$ such that $\{\sigma \in \mathcal{P}_\kappa\lambda \mid f(\sigma) = \alpha_f\} \in F$).*

With the help of Dr. Brent Cody of Virginia Commonwealth University, we were able to define a similar notion to Schanker's near λ-supercompactness. Notice how the following definition is similar to the definition of strong compactness, but rather than a filter measuring all of $\mathcal{P}_\kappa\lambda$ we have one measuring all sets in a large collection $\mathscr{A}$. Because our definition has a similar flavor to Schanker's near supercompactness, we have utilized the *nearly* terminology as well.

Definition 59 *Given $\lambda^{<\kappa} = \lambda$, then cardinal κ is nearly λ-strongly compact if for every collection $\mathscr{A}$ of λ-many subsets of $\mathcal{P}_\kappa\lambda$ there is a non-principal κ-complete fine filter U measuring all sets in $\mathscr{A}$.*

3.2 Characterization of Strongly Compact and Weakly Compact Cardinals

In this section we will prove two characterization results involving weak and strong compactness. It should be noted, that while we call these theorems, they are by no means complete characterizations theorems. As mentioned before, there are ways to express these cardinals in terms of branches through trees, colorings on graphs, compactness results, among other. Because we defined our large cardinals in terms of filters and the only alternate framework we have developed is elementary embeddings, our characterization results will be to show an equivalent elementary embedding characterization of each filter definition (see sections 4.1 and 6.1 from [3] for more complete characterization theorems).

Theorem 16 *Given $\kappa^{<\kappa} = \kappa$, κ is weakly compact if and only if for every $A \subseteq \kappa$ there is a transitive set M closed under $< \kappa$ sequences with $A, \kappa \in M$, $M \models \mathsf{ZFC}^-$ and an elementary embedding $j : M \to N$ with $\mathrm{cp}(j) = \kappa$.*

PROOF. To see the $\Leftarrow$ direction, we need to show that if for every $A \subseteq \kappa$ there is a transitive set M closed under $< \kappa$ sequences with $A, \kappa \in M$, $M \models \mathsf{ZFC}^-$ and an elementary embedding $j : M \to N$ with $\mathrm{cp}(j) = \kappa$, then for every set $\mathscr{A}$ containing at most κ-many subsets of κ there is a κ-complete non-principal filter F measuring every set in $\mathscr{A}$.

Fix $\mathscr{A}$, and because $\mathscr{A}$ contains at most κ-many subsets of κ, we can code $\mathscr{A}$ into a single $A \subseteq \kappa$. By assumption we know there is a set M with $A, \kappa \in M$, $M \models \mathsf{ZFC}^-$, and an elementary embedding $j : M \to N$. Set $F = \{X \in \mathcal{P}(\kappa)^M \mid \kappa \in j(X)\}$. To see that F measures every set in $\mathscr{A}$, since $\mathscr{A} \subseteq \mathscr{P}(\kappa)^M$, we know that if $X \in \mathscr{A}$ then either $\kappa \in j(X)$ or $\kappa \in j(\kappa \setminus X)$, hence $X \in F$ or $\kappa \setminus X \in F$.

To see the $\Rightarrow$ direction, fix $A \subseteq \kappa$. Using repeated applications of the Löwenheim Skolem Theorem, we can create such a transitive set M closed under $< \kappa$ sequences, with $\kappa, A \in M$ and $M \models \mathsf{ZFC}^-$. Let $\mathscr{A} = \mathcal{P}(\kappa) \cap M = \mathscr{P}(\kappa)^M$. By assumption, we know in V there is a filter F measuring every set in $\mathcal{P}(\kappa)^M$. Look at the M-ultrapower defined $j : M \to N = \mathrm{Ult}(M, F) = (M^\kappa \cap M)/F$.

We now need to show that $\mathrm{cp}(j) = \kappa$. First we show that $j \upharpoonright_\kappa = \mathrm{id} \upharpoonright_\kappa$. For every $\alpha < \kappa$ by elementarity, $j(\alpha)$ is the order type of its elements. Thus if $[f]_F \in j(\alpha) = [c_\alpha]_F$, we know that $\{\beta < \kappa \mid f(\beta) \in \alpha\} \in F$. But clearly we have

$$\{\beta < \kappa \mid f(\beta) \in \alpha\} = \bigcup_{\eta < \alpha} \{\beta < \kappa \mid f(\beta) = \eta\}$$

By κ-completeness there is a single $\eta < \alpha$ such that $\{\beta < \kappa \mid f(\beta) = \eta\} \in F$. Hence $[f]_F = [c_\eta]_F = j(\eta)$. Thus for some arbitrary $[f]_F \in j(\alpha)$ we have $[f]_F = j(\eta)$ for some $\eta < \alpha$, and hence $j(\alpha) \cong \alpha$, and by definition of ordinal $j(\alpha) = \alpha$. Secondly, we will show that $\kappa \leqslant [\mathrm{id}]_F < j(\kappa)$. The first inequality comes from the fact that for all $\alpha < \kappa$ we have $j(\alpha) = [c_\alpha]_F < [\mathrm{id}]_F$. The second comes from the fact that $\{\alpha < \kappa \mid \mathrm{id}(\alpha) < c_\kappa(\alpha)\} = \kappa \in F$. Thus $\mathrm{cp}(j) = \kappa$ which completes the proof. $\square$

Theorem 17 *κ is λ-strongly compact if and only if there is an elementary embedding $j : V \to M$, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and there is an $s \in M$ with $j``\lambda \subseteq s$ and $|s|^M < j(\kappa)$.*

PROOF. To see the $\Leftarrow$ direction, we need to show that if there is an elementary embedding $j : V \to M$, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and there is an $s \in M$ with $j``\lambda \subseteq s$ and $|s|^M < j(\kappa)$, then there is a κ-complete fine measure on $\mathcal{P}_\kappa\lambda$. To do this, start by taking $j : V \to M$ as in the assumption, let $U = \{X \subseteq P_\kappa\lambda \mid s \in j(X)\}$. Without loss of generality $s \subseteq j(\lambda)$.

To see that U is κ-complete, suppose we had $\langle X_\alpha \mid \alpha < \gamma\rangle$ for $\gamma < \kappa$ with each $X_\alpha \in U$. We need to show $s \in j(X_\alpha)$ for all $\alpha < \gamma$. But since $\gamma < \mathrm{cp}(j)$ we know $s \in j(\cap_{\alpha<\gamma} X_\alpha) = \cap_{\alpha<\gamma} j(X_\alpha)$ which shows the κ-completeness of U. To see that U is fine we need that for every $\alpha < \lambda$ we have $\{x \in \mathcal{P}_\kappa\lambda \mid \alpha \in x\} \in U$. For this to hold, we need that for all $\alpha < \lambda$ we have $s \in j(\{x \in P_\kappa\lambda \mid \alpha \in x\}) = \{x \in j(\mathcal{P}_\kappa\lambda) \mid j(\alpha) \in x\}$. But $s \in j(\mathcal{P}_\kappa\lambda) = \mathcal{P}_{j(\kappa)} j(\lambda)^M$ and since $j``\lambda \subseteq s$ for all $\alpha < \lambda$ we have $j(\alpha) \in s$. Hence for every $\alpha < \lambda$ we have $\{x \in \mathcal{P}_\kappa\lambda \mid \alpha \in x\} \in U$, or that U is fine. To see that U is non-principal, if it was then for a single $a \in \mathcal{P}_\kappa\lambda$ we have $X \in U \Leftrightarrow a \in X$, but then U would be $(\lambda + 1)$-complete which contradicts fineness.

For the $\Rightarrow$ direction let U be a non-principal κ-complete fine filter on $\mathcal{P}_\kappa\lambda$. To see that $\mathrm{cp}(j) = \kappa$ we first show that $j \upharpoonright_\kappa = \mathrm{id} \upharpoonright_\kappa$. If $\alpha < \kappa$ then $j(\alpha)$ is the order tpe of its elements. Thus if $[f]_U \in j(\alpha) = [c_\alpha]_U$, then $\{x \in \mathcal{P}_\kappa\lambda \mid f(x) \in \alpha\} \in U$. However, since $\{x \in \mathcal{P}_\kappa\lambda \mid f(x) = \alpha\} = \bigcup_{\eta<\alpha}\{x \in \mathcal{P}_\kappa\lambda \mid f(x) = \eta\}$ and due to the κ-completeness of U there is a single $\eta < \alpha$ such that $[f]_U = [c_\eta]_U = j(\eta)$. Hence for any $[f]_U \in j(\alpha)$ we have $[f]_U = j(\eta)$ for some $\eta < \alpha$, or that $j(\alpha) = \alpha$.

To see that $\kappa < j(\kappa)$ notice that

$$\kappa \leqslant \lambda \leqslant |[\mathrm{id}]_U|^M < j(\kappa)$$

The middle inequality above follows from the fact that $j"\lambda \subseteq [id]_U$. To see this, consider $j(\alpha) \in j"\lambda$. Then since $\{x \in \mathcal{P}_\kappa\lambda \mid c_\alpha(x) \in id(x)\} = \{x \in \mathcal{P}_\kappa\lambda \mid \alpha \in x\}$ by fineness we know $j(\alpha) = [c_\alpha]_U \in [id]_U$. □

3.3 Characterization of Near Strong Compact Cardinals

In this section we hope to state and prove a preliminary characterization result involving near strong compactness. Before stating and proving the preliminary result, we will state a related result due to Jason Schanker [2].

Theorem 18 *(J. Schanker) Given* $\lambda^{<\kappa} = \lambda$, κ *is near* λ*-supercompact if and only if for every* $A \subseteq \lambda$ *there is a transitive* $M \models ZFC^-$ *closed under* $< \kappa$ *sequences,* $A, \lambda \in M$*, and a transitive* N *with an elementary embedding* $j : M \to N$, $cp(j) = \kappa$, $j(\kappa) > \lambda$ *and* $j"\lambda \in N$.

We can now state our preliminary result. We state it as a theorem, but surely there are more ways to characterize near strong compactness. Hopefully we will continue our research and find additional equivalent characterizations in order to add more substance to the following theorem.

Theorem 19 *(B. Cody, P. White) Given* $\lambda^{<\kappa} = \lambda$, κ *is nearly* λ*-strongly compact if and only if for every* $A \subseteq \lambda$ *there is a transitive* $M \models ZFC^-$*, with* $\lambda, A \in M$, $M^{<\kappa} \cap V = M$, $|M| = \lambda$*, and there is an elementary embedding* $j : M \to N$ *with* $cp(j) = \kappa$, $j(\kappa) > \lambda$ *and there is an* $s \in N$ *such that* $j"\lambda \subseteq s$ *and* $|s|^N < j(\kappa)$.

PROOF. To see the $\Rightarrow$ direction, fix $A \subseteq \lambda$, and build a transitive M closed under $< \kappa$-sequences with $A, \lambda \in M$. Let $\mathscr{A} = \mathcal{P}(\mathcal{P}_\kappa\lambda)^M$. By assumption, we know there is a non-principal κ-complete fine filter U on $\mathcal{P}_\kappa\lambda$ measuring all sets in $\mathscr{A}$. Let $j : M \to N$ where $N = (M^{\mathcal{P}_\kappa\lambda} \cap M)/U$. Proving $j\restriction_\kappa = id\restriction_\kappa$ follows the same argument as in Theorem 17 and Theorem 16. To see that $j(\kappa) > \kappa$ notice

$$\kappa < \lambda \leqslant |[id]_U|^N < j(\kappa)$$

As in Theorem 17, the middle inequality above follows from the fact that $j"\lambda \subseteq [id]_U$. To see this consider $j(\alpha) \in j"\lambda$. As in the case for strong compactness, since $\{x \in \mathcal{P}_\kappa\lambda \mid c_\alpha(x) \in id(x)\} = \{x \in \mathcal{P}_\kappa\lambda \mid \alpha \in x\}$ by fineness we know $j(\alpha) = [c_\alpha]_U \in [id]_U$.

To see the $\Leftarrow$ direction, fix $\mathscr{A}$. Because $\mathscr{A}$ contains at most λ many subsets of $\mathscr{P}_\kappa\lambda$ we can encode $\mathscr{A}$ into a single subset $A \subseteq \lambda$. By assumption, we know there is a transitive M closed under $< \kappa$-sequences with $M \models ZFC^-$, both $\lambda, A \in M$, and an embedding $j : M \to N$ with $cp(j) = \kappa, j(\kappa) > \lambda, j"\lambda \subseteq s, |s|^N < j(\kappa)$. Since $A \in M$ via the encoding we have $\mathscr{A} \in M$. Define $U = \{X \in \mathcal{P}(\mathcal{P}_\kappa\lambda)^M \mid s \in j(X)\}$. Since $\mathscr{A} \subseteq \mathcal{P}(\mathcal{P}_\kappa\lambda)^M$ we have that U measures all elements of $\mathscr{A}$.

To show U is fine we need for all $\alpha < \lambda$ that $s \in j(\{x \in \mathcal{P}_\kappa\lambda \mid \alpha \in x\}) = \{x \in j(\mathcal{P}_\kappa\lambda) \mid j(\alpha) \in x\}$. But $s \in j(\mathcal{P}_\kappa\lambda) = \mathcal{P}_{j(\kappa)}j(\lambda)^N$ and $j(\alpha) \in s$ because $j``\lambda \subseteq s$. We also can be assured that $\{x \in \mathscr{P}_\kappa\lambda \mid \alpha \in x\} \in M$ because $\alpha \in M$ and $\mathcal{P}_\kappa\lambda \in M$. To see that U is non-principal, for $\alpha < \lambda$ define $X_\alpha = \{x \in P_\kappa\lambda \mid \alpha \in x\}$ and notice $X_\alpha \in U$ and $\bigcap_{\alpha<\kappa} X_\alpha = \emptyset$. □

It is also interesting to note how similar Theorem 19 is to Theorem 17. Clearly we have that (1) if κ is λ-strongly compact then κ is nearly λ-strongly compact, (2) if κ is nearly λ-supercompact compact then κ is nearly λ-strongly compact, and (3) if κ is nearly λ-strongly compact then κ is weakly compact.

Lastly, of course there are several loose ends ripe for further research, which we will mention briefly. It would be interesting to continue our research and discover which characterizations of weak compactness and λ-strong compactness generalize to near λ-strong compactness. An alternate route of inquiry might involve the GCH; if κ is nearly λ-strongly compact is there a forcing extension in which κ remains nearly λ-strongly compact and $2^\kappa = \kappa^{++}$?

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Vita

Philip Alexander White was born in 1990, in Würzburg Bavaria, but has spent most of his life thus far in central Virginia. He is the son of Philip James White and Elsa Yaneth White, and is the brother of James Isaac White.

Since his youth, Mr. White has always been interested in philosophy, but was quite a late bloomer regarding mathematics. In fact, he made a C in his first logic course (Phil 222). After reading a book titled *What is Mathematical Logic* by John Newsome Crossley in the latter part of his undergraduate years, Mr. White became intensely interested in mathematical logic. He was eventually very active in the Virginia Commonwealth University (VCU) Philosophy and Logic Club (from years 2011-2013), and helped bring about the name change from simply the VCU Philosophy Club. He graduated in December of 2013 with a B.A. in Philosophy and a B.S. in Mathematics from VCU. After finishing his undergraduate education, Mr. White was a high school mathematics instructor (from 2014-2017). In the fall of 2017 Mr. White enrolled at VCU once again, but this time to study set theory under Dr. Brent Cody. Although Mr. White's thesis is on the topic of set theory, his true love is the philosophy of logic. His favorite philosophers are Plato and Leibniz. During his free time, which rarely happens, Mr. White enjoys playing music.

Mr. White plans on continuing his education at George Washington University towards a Ph.D. in mathematical logic.