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Palindromic Products

Jamie Shive
Virginia Commonwealth University

DISSERTATION

PALINDROMIC PRODUCTS

A Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Virginia Commonwealth University.

by

JAMIE SHIVE

Bachelor of Science in Applied Mathematics at Virginia Commonwealth University

Director: Richard Hammack,

Professor, Department of Mathematics and Applied Mathematics

Virginia Commonwewalth University

Richmond, Virginia

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In this dissertation, we investigate a number of open problems related to products of palindromic graphs. The notion of a *palindromic graph* was first defined by Robert Beeler [1]. A graph G on n vertices is *palindromic* if there is a vertex-labeling bijection $f : V(G) \to \{1, 2, \dots, n\}$ with the property that for any edge $vw \in E(G)$ there is an edge $xy \in E(G)$ for which $f(x) = n - f(v) + 1$ and $f(y) = n - f(w) + 1$.

Beeler gives sufficient conditions on the factors of a Cartesian product of graphs that ensure the product is palindromic. He states that it is unknown whether the conditions are necessary. We prove that the conditions are indeed necessary. Further, we prove parallel results for the strong product and the lexicographic product.

Parallel results for the direct product hold when both factors are non-bipartite, however, they do not hold when at least one factor is bipartite. Instead, we use the notion of 2-fold involutions to help us address the case where one factor is bipartite and the other is non-bipartite.

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

In this dissertation, we investigate several problems related to products of special classes of graphs. Palindromic graphs are a class of graphs introduced by Robert Beeler [1]. The underlying inspiration for these graphs comes from the notion of palindromes in words and sequences. Palindromic graphs have an appealing, mirror-like symmetry. Beeler's paper introduced some basic conditions on palindromic graphs and explored some basic operations. This new class of graphs presents us with plenty of interesting open questions. Beeler covers sufficient conditions on the factors of a Cartesian product that ensure the product will be palindromic, namely, if G and H are palindromic graphs, then their product is palindromic. He states that the necessary conditions are unknown.

We are motivated to explore the necessary conditions on the Cartesian product, namely, if the Cartesian product of graphs is palindromic, then its factors will be palindromic, as well as investigate parallel palindromic conditions in the other three standard associative graph products - the strong product, the direct product, and the lexicographic product. At this time, we are only considering cases where the product is connected, and we leave any cases where the product is disconnected to conjecture.

The only published paper on palindromic graphs (prior to the projects in this dissertation) is by Robert Beeler [1]. In Beeler's paper, we are presented with the first major theorem that characterizes sufficient and necessary conditions on palindromic graphs. This theorem can be found in Section 1.3.

As Beeler notes in his paper, the necessary conditions on the factors of a palindromic Cartesian product are unknown. This is a clear gap in the literature, and the proof of which is nontrivial. We settle this question in Theorem 2.3 and also prove analagous results for the other standard graph products. As we expand the study of palindromic products to the remaining three standard graph products, the proof techniques become even more involved, and each requires a different approach.

Hammack [6] notes in his paper "On uniqueness of prime bipartite factors of graphs" that, given an involution in the direct product of two graphs where one of the factors is bipartite, it is surprisingly nontrivial to find an involution in the bipartite factor. The techniques and results found in Hammack's paper (which can be found in Section 1.5.3) are helpful in proving some of the main results in this dissertation.

1.2 Basic Definitions

Before introducing the projects, it will be helpful to introduce some basic definitions to establish notation. The standard definitions and other related terms can be found in [12], [3]. Unless stated, all graphs considered here are finite and undirected. The set of simple graphs will be denoted Γ, and the set of graphs with loops will be denoted Γ_0, where $\Gamma \subset \Gamma_0$.

A **subgraph** H of a graph G is a graph with vertex set $V(H) \subseteq V(G)$ and edge set $E(H) \subseteq E(G)$. Further, H is an **induced subgraph** of G if all of the edges between vertices of $V(H)$ from $E(G)$ are in $E(H)$. We will use the notation $\langle G \rangle$ to denote an induced subgraph of G. The **complement** of a graph G is denoted by $\overline{G}$, has the same vertex set as G, and edges are formed such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

The **open neighborhood** of a vertex $v \in V(G)$ is the set $N(v) = \{u \in V(G) \mid uv \in E(G)\}$, and the **closed neighborhood** of a vertex $v \in V(G)$ is the set $N[v] =$

$\{v\} \cup N(v)$. A graph is called **R-thin** if no two distinct vertices have the same open neighborhood, and a graph is called **S-thin** if no two distinct vertices have the same closed neighborhood.

The following two definitions and accompanying information come from the Handbook of Product Graphs [3]. We say that vertices x and y of a graph are in the **relation** S, written xSy, provided that each has the same closed neighborhood, that is, $N[x] = N[y]$. We define the quotient G/S to be the graph whose vertices are the S-classes of G, and for which $XY \in E(G/S)$ provided that $X \neq Y$ and G has an edge joining X to Y. Likewise, we say that vertices x and y of a graph are in the **relation** R, written xRy, provided that each has the same open neighborhood, that is, $N(x) = N(y)$. We define the quotient G/R to be the graph whose vertices are the R-classes of G, and for which $XY \in E(G/R)$ provided that G has an edge joining X to Y.

A graph G is called **bipartite** if its vertex set can be divided into two distinct sets G_1 and G_2, called partite sets, such that every edge of G connects a vertex of G_1 to a vertex of G_2.

We say a subset A of $V(G)$ is **externally related** if every vertex $x \in V(G)\backslash A$ that is adjacent to at least one vertex in A is adjacent to all vertices of A. In such a case, we will also say that the subgraph $\langle A \rangle$ is externally related.

We will denote by K_n and D_n the complete graph on n vertices and the edgeless graph on n vertices, respectively.

1.3 Palindromic Graphs

Before defining a palindromic graph, we will quickly review the basics of graph automorphisms. A **graph automorphism** is an isomorphism of a graph to itself, that is, it is a permutation φ of $V(G)$ such that uv is an edge if and only if $\varphi(u)\varphi(v)$

is an edge. The set of all automorphisms on a graph G, denoted $Aut(G)$, is called the **automorphism group** of G.

We say that a graph G on n vertices is a **palindromic graph** [1] if it has a vertex-labeling bijection $f : V(G) \to \{1, 2, ..., n\}$ with the property that $uv \in E(G)$ if and only if there is an edge $xy \in E(G)$ such that $f(x) = n - f(u) + 1$ and $f(y) = n - f(v) + 1$. Beeler [1] also notes that because a graph G and its complement $\overline{G}$ have the same automorphism groups, a graph G is palindromic if and only if its complement $\overline{G}$ is palindromic.

Before introducing Beeler's first major theorem, some definitions are required. A palindromic graph admits an **involution**, that is, an automorphism of order 2, so $\varphi^2(u) = u$ for all $u \in G$. If $\varphi(u) = u$ for some $u \in G$, then u is a **fixed point** of φ. From now forward, an involution that fixes at most one vertex is called a **palindromic involution**. Using the definition of a fixed point, we are able to classify palindromic involutions based on the parity of the vertex set, which impacts whether they have a fixed point or not. We say that a graph G is **even palindromic** if it is palindromic and has an even number of vertices, and a graph G is **odd palindromic** if it is palindromic and has an odd number of vertices. A palindromic involution that fixes no vertex is called an **even palindromic involution**, and one that fixes exactly one vertex is called an **odd palindromic involution**.

In the following theorem, Beeler characterizes the circumstances under which a graph is palindromic. Figure 1 shows a graph that admits an even palindromic involution.

Theorem 1.3.1. *([1], Beeler) A graph of even order is palindromic if and only if it admits an involution with no fixed vertices. A graph of odd order is palindromic if and only if it admits an involution with exactly one fixed vertex.*

In other words, a graph is even palindromic if and only if admits an even palindromic involution, and a graph is odd palindromic if and only if it admits an odd palindromic involution. This definition grants us a certain mirror symmetry, demonstrated in Figure 2.

1.4 2-Fold Automorphisms

The following definition from [7] will be used regularly in our proof for the direct product.

Definition 1.4.1. *Given a graph G, a pair (α, β) of permutations of $V(G)$ that has the property $xy \in E(G) \iff \alpha(x)\beta(y) \in E(G)$ is called a* ***2-fold automorphism****.*

Because we are concerned with palindromic involutions, it is natural to extend the above definition of a 2-fold automorphism to that of a 2-fold involution whose definition can be found below.

Definition 1.4.2. *A* ***2-fold involution*** *is a 2-fold automorphism with permutations (α, β) such that $\alpha^2 = id$ and $\beta^2 = id$.*

Because we have classified palindromic involutions as either even or odd, meaning that either the involution has no fixed points or has one fixed point, we can extend the notion of a 2-fold involution to either have no or one fixed point, and we can classify them as even or odd, respectively. The following two definitions do just that.

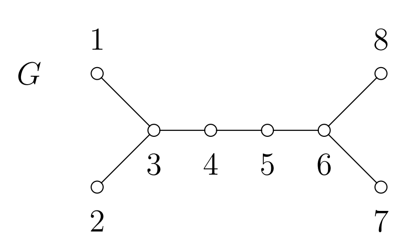


Fig. 1. A graph G that admits an even palindromic involution.

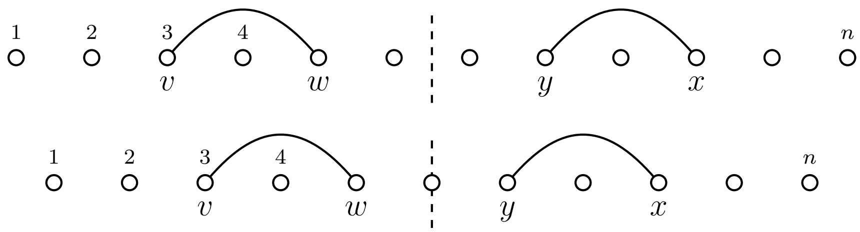


Fig. 2. Palindromic graphs of even order admit an involution with no fixed points, while palindromic graphs of odd order admit an involution with exactly one fixed point.

Definition 1.4.3. *An* ***even 2-fold involution*** *is a 2-fold involution with permutations (α, β) such that α and β have no fixed points.*

Definition 1.4.4. *An* ***odd 2-fold involution*** *is a 2-fold involution with permutations (α, β) such that at least one of α or β has exactly one fixed point.*

1.5 The Four Products

There are four standard associative graph products. What follows is a description of how to form each product, and examples of each product can be found in Figure 3. The subsections of this section provide more information about each product as well as relevant results. All of the following information and more can be found in the Handbook of Product Graphs [3].

The **Cartesian product** $G \square H$ of graphs G and H is constructed as follows:

$$V(G \square H) = V(G) \times V(H)$$

$$E(G \square H) = \{(x,y)(x',y') \mid (xx' \in E(G) \text{ and } y = y') \text{ or } (yy' \in E(H) \text{ and } x = x')\}.$$

The **strong product** $G \boxtimes H$ of graphs G and H is constructed as follows:

$$V(G \boxtimes H) = V(G) \times V(H)$$

$$E(G \boxtimes H) = \{(x,y)(x',y') \mid \big(xx' \in E(G) \text{ or } y = y'\big) \text{ and } \big(x = x' \text{ or } yy' \in E(H)\big)\}.$$

The **direct product** $G \times H$ of graphs G and H is constructed as follows:

$$V(G \times H) = V(G) \times V(H)$$

$$E(G \times H) = \{(x,y)(x',y') \mid xx' \in E(G) \text{ and } yy' \in E(H)\}.$$

The **lexicographic product** $G \circ H$ of graphs G and H is constructed as follows:

$$V(G \circ H) = V(G) \times V(H)$$

$$E(G \circ H) = \{(x,y)(x',y') \mid xx' \in E(G) \text{ or } (x' = x \text{ and } yy' \in E(H))\}.$$

1.5.1 The Cartesian Product

The Cartesian product is commutative and associative in the sense that the maps $(x,y) \mapsto (y,x)$ and $((x,y),z) \mapsto (x,(y,z))$ are isomorphisms from $G \,\square\, H \to H \,\square\, G$ and $(G \,\square\, H) \,\square\, K \to G \,\square\, (H \,\square\, K)$, respectively. Letting $H + K$ denote the disjoint union of graphs H and K, we also get the distributive law $G \,\square\, (H + K) = G \,\square\, H + G \,\square\, K$.

A nontrivial graph G is **prime** over the Cartesian product $\square$ if for any factoring $G \cong A \,\square\, B$, one of A or B is K_1 and the other is G. Certainly every graph can be factored into prime factors. Sabidussi and Vizing [8, 10] proved that each connected graph has a unique prime factoring up to order and isomorphism of the factors. More precisely, we have the following.

Theorem 1.5.1 ([3], Theorem 6.8)**.** *Let G and H be isomorphic connected graphs $G = G_1 \square \cdots \square G_k$ and $H = H_1 \square \cdots \square H_\ell$, where each factor G_i and H_i is prime. Then $k = \ell$, and*

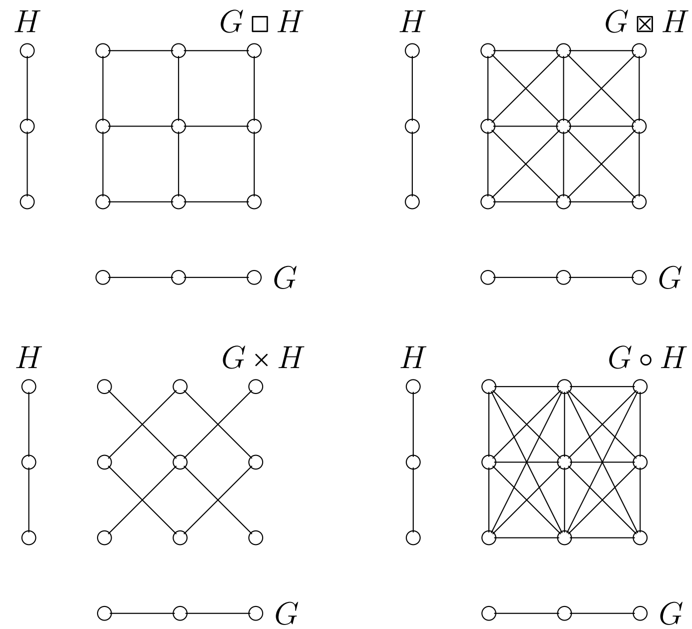


Fig. 3. The four standard associative graph products

for any isomorphism $\varphi : G \to H$*, there is a permutation* π *of* $\{1, 2, \ldots, k\}$ *and isomorphisms* $\varphi_i : G_{\pi(i)} \to H_i$ *for which* $\varphi(x_1, x_2, \ldots, x_k) = \big(\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), \ldots, \varphi_k(x_{\pi(k)})\big)$.

Cancellation is a consequence of unique prime factorization. For connected graphs A, B, and C, $A \,\square\, B \cong A \,\square\, C$ implies that $B \cong C$ [3].

1.5.2 The Strong Product

The strong product is commutative and associative. If $N_G[x]$ is the closed neighborhood of a vertex $x \in V(G)$, then

$$N_{G \boxtimes H}[(x, y)] = N_G[x] \times N_H[y]. \tag{1.1}$$

Also $K_1 \boxtimes G \cong G$ for all graphs G. A graph G is **prime** over the strong product $\boxtimes$ if for any factoring $G = A \boxtimes B$, one of A or B is K_1 and the other is isomorphic to G.

Recall from Section 1.2 that we say that vertices x and y of a graph are in relation S, written xSy, provided that each has the same closed neighborhood, that is, $N[x] = N[y]$. It is easy to check that S is an equivalence relation the graph's vertex set. Given a vertex x of G, we denote the S-equivalence class that contains x as $[x] = \{x' \in V(G) \mid N_G[x'] = N_G[x]\}$. We call an S-equivalence class of $V(G)$ an **S-class** of G. (Note that a graph is S-thin if and only if each S-class consists of a single vertex.) In general, if X is an S-class of graph G, then the subgraph of G induced on X is the complete graph $K_{|X|}$. Also, for any distinct S-classes X and Y, X and Y are externally related.

Regarding factoring over the strong product, we have the following result that is analogous to 1.5.1 with the minor caveat that it only applies to S-thin graphs.

Theorem 1.5.2 ([3], Theorem 7.16)**.** *Let φ be an automorphism of an S-thin connected graph G with prime factorization $G = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_k$. Then there is a permutation π of $\{1, 2, \ldots, k\}$ and isomorphisms $\varphi_i : G_{\pi(i)} \to G_i$ for which $\varphi(x_1, x_2, \ldots, x_k) = \big(\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), \ldots, \varphi_k(x_{\pi(k)})\big)$.*

Because S is defined in terms of the adjacency structure of a graph, any isomorphism $\varphi : G \to H$ sends S-classes of G bijectively onto S-classes of H. From the discussion above it should be clear that any isomorphism $\varphi : G \to H$ induces an isomorphism $\widetilde{\varphi} : G/S \to H/S$ where $\widetilde{\varphi}(X) = \varphi(X)$, that is, $\widetilde{\varphi}(X)$ is the image of the S-class X under φ.

But the existence of an isomorphism $\widetilde{\varphi} : G/S \to H/S$ does not necessarily mean that there is an isomorphism $\varphi : G \to H$. However, if $|X| = |\widetilde{\varphi}(X)|$ for each $X \in V(G/S)$, then we can lift $\widetilde{\varphi}$ to an isomorphism $\varphi : G \to H$ simply by declaring φ to restrict to a bijection $X \to \widetilde{\varphi}(X)$ for each X.

Using Equation (1.1), one can show that the S-classes of $G \boxtimes H$ are precisely the Cartesian products $X \times Y$ (of sets), where X is an S-class of G and Y is an S-class of H. In other words, the vertices of $(G \times H)/S$ are $X \times Y$, where $X \in V(G/S)$ and $Y \in V(H/S)$. Further, there is a natural isomorphism

$$\begin{array}{rcl} (G \boxtimes H)/S & \longrightarrow & G/S \boxtimes H/S \\ X \times Y & \longmapsto & (X, Y). \end{array} \tag{1.2}$$

1.5.3 The Direct Product

The following theorem is well-known and establishes the structure of the direct product given bipartite or non-bipartite factors.

Theorem 1.5.3 ([11], Weichsel)**.** *A direct product $A \times B$ of connected graphs is connected if and only if at least one factor has an odd cycle; if both factors are bipartite, then the product has exactly two components. If general, if both A and B have odd cycles, then so does $A \times B$. Moreover, if B is bipartite, with bipartition $X \cup Y$, then $A \times B$ is bipartite, with bipartition $(V(A) \times X) \cup (V(A) \times Y)$.*

Recall from Section 1.2 that we say two vertices x and y of a graph G are in the **relation** R, written xRy, provided that $N_G(x) = N_G(y)$ (where $N_G(x)$ is the open neighborhood of $x \in V(G)$). Given a vertex x of G, we denote the R-equivalence class that contains x as $[x] = \{x' \in V(G) \mid N_G(x') = N_G(x)\}$. We refer to an R-equivalence class as an ***R*-class**. We define the quotient G/R of G to be the graph whose vertices are the R-classes of G, and for which $XY \in E(G/R)$ provided G has an edge joining X to Y. Note that G is bipartite if and only if G/R is bipartite. As the relation R is defined entirely in terms of adjacencies, given an isomorphism $\varphi : G \to H$, we have $xRy \in G$ if and only if $\varphi(x)\varphi(y) \in H$. Thus φ maps R-classes of G bijectively to R-classes of H. In particular, $\varphi([x]) = [\varphi(x)]$. Moreover, any isomorphism $\varphi : G \to H$ induces an isomorphism $\widetilde{\varphi} : G/R \to H/R$ defined as $\widetilde{\varphi}([x]) = [\varphi(x)]$. Conversely, if $|X| = |\widetilde{\varphi(X)}|$ for all $X \in G/R$, then we can lift any automorphism $\widetilde{\varphi} : G/R \to G/R$ to an automorphism φ of G by simply declaring that each R-class X maps bijectively to some $\varphi(X)$.

Proposition 1.5.1 ([3], Proposition 8.4)**.** *Suppose $G, H \in \Gamma_0$. Then $G \cong H$ if and only if $G/R \cong H/R$ and there is an isomorphism $\widetilde{\varphi} : G/R \to H/R$ with $|X| = |\widetilde{\varphi}(X)|$ for each $X \in V(G/R)$. In fact, given an isomorphism $\widetilde{\varphi} : G/R \to H/R$, any map $\varphi : V(G) \to V(H)$ that restricts to a bijection $\varphi : X \to \widetilde{\varphi}(X)$ for every $X \in G/R$ is an isomorphism from G to H.*

Also proven in Section 8.2 of [3], we have that $\varphi : (G \times H)/R \to G/R \times H/R$ such that $[(v, w)] \mapsto ([v], [w])$. In general, $[(v, w)] = [v] \times [w]$ (as sets). Consequently, $|[(v, w)]| = |[v]| \cdot |[w]|$.

1.5.4 The Lexicographic Product

The lexicographic product does not admit desirable factorization qualities like the other three products. The proof for this product will require a significantly different approach. The identity $\overline{G \circ H} = \overline{G} \circ \overline{H}$ [3, Section 10.2] will be needed in our proofs.

1.6 The Cartesian Skeleton

We will use the notion of the Cartesian Skeleton (see [2] for a thorough introduction and description) to link the direct product to the Cartesian product, allowing us to use some results on the Cartesian product to guide our proof. The Cartesian skeleton $S(G)$ is a graph on the vertex set of G that has the property $S(A \times B) = S(A) \,\Box\, S(B)$ in the class of R-thin graphs. What follows is a description of how to construct the Cartesian skeleton $S(G)$ of a graph $G \in \Gamma_0$ (where Γ_0 is the set of isomorphism classes of graphs that may have loops).

The Cartesian skeleton $S(G)$ is constructed as a subgraph of the Boolean square of G with specific properties. The **Boolean square** of G is the graph G^s with

$V(G^s) = V(G)$ and $E(G^s) = \{xy \mid N_G(x) \cap N_G(y) \neq \varnothing\}$. It follows that xy is an edge of G^s if and only if G has an $x-y$ walk of length two. Moreover, if G has an $x-y$ walk W of even length, then G^s has an $x-y$ walk of length $|W|/2$ on alternate vertices of W. Observe that G^s is connected if G is connected and has an odd cycle. If G is connected and bipartite, then G^s has exactly two components, and their respective vertex sets are the two partite sets of G.

As stated above, the Cartesian skeleton $S(G)$ is a certain spanning sugraph of the Boolean square G^s. The following is a description of how to construct the Cartesian skeleton $S(G)$ from G^s. An edge xy of G^s is **dispensable** if $x = y$ or there exists $z \in V(G)$ for which both of the following statements hold:

1. $N_G(x) \cap N_G(y) \subset N_G(x) \cap N_G(z)$ or $N_G(x) \subset N_G(z) \subset N_G(y)$,

2. $N_G(y) \cap N_G(x) \subset N_G(y) \cap N_G(z)$ or $N_G(y) \subset N_G(z) \subset N_G(x)$.

Now the **Cartesian skeleton** of a graph G is the spanning subgraph $S(G)$ of G^s obtained by removing all dispensable edges from G^s. The following proposition from [2] links the Cartesian and direct products together.

Proposition 1.6.1. *([2]) If G, H are R-thin graphs, then $S(G \times H) = S(G) \square S(H)$, provided that neither G nor H has any isolated vertices. This is equality, not mere isomorphism; the graphs $S(G \times H)$ and $S(G) \square S(H)$ have identical vertex and edge sets.*

As $S(G)$ is defined entirely in terms of the adjacency structure of G, the following result is an immediate consequence.

Proposition 1.6.2 (([2]), Hammack and Imrich)**.** *Any isomorphism $\varphi : G \to H$, as a map $V(G) \to V(H)$, is also an isomorphism $\varphi : S(G) \to S(H)$.*

We also have the following result from [2] which concerns the connectivity of Cartesian skeletons.

Proposition 1.6.3 (([2]), Hammack and Imrich)**.** *Suppose G is connected.*

1. *If G has an odd cycle, then $S(G)$ is connected.*
2. *If G is nontrivial bipartite, then $S(G)$ has two connected components. Their respective vertex sets are the two partite sets of G.*

CHAPTER 2

THE CARTESIAN PRODUCT OF PALINDROMIC GRAPHS

2.1 Summary of Project

We are motivated to prove that the sufficient conditions of the Cartesian product of palindromic graphs (proven by Beeler [1]) are necessary.

The following is the main theorem of this project.

Theorem 2.1.1. *Suppose G and H are connected:*

1. *$G \,\square\, H$ is even palindromic if and only if G or H is even palindromic*
2. *$G \,\square\, H$ is odd palindromic if and only if G and H are odd palindromic*

This project was included in a paper published with Richard Hammack in *The Art of Discrete and Applied Mathematics* [4].

2.2 Preliminary Information

In order to prove the necessary conditions, we can make use of the following theorem mentioned in 1.5.1.

Theorem 2.2.1 ([3], Theorem 6.8)**.** *Let G and H be isomorphic connected graphs $G = G_1 \square \cdots \square G_k$ and $H = H_1 \square \cdots \square H_\ell$, where each G_i and H_i is prime. Then $k = \ell$, and for any isomorphism $\varphi : G \to H$, there is a permutation π of $\{1, 2, \ldots, k\}$ and isomorphisms $\varphi_i : G_{\pi(i)} \to H_i$ for which $\varphi(x_1, x_2, \ldots, x_k) = \big(\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), \ldots, \varphi_k(x_{\pi(k)})\big)$.*

This theorem tells us that any automorphism φ permutes prime factors over the Cartesian product. We can prime factor the factors G and H of $G \,\square\, H$, and since any palindromic involution φ only permutes the prime factors of G and H, we can group prime factors together based on observations made about the way in which they must be permuted. Each group has an associated automorphism, so utilizing the behavior of the palindromic involution in the product, we can prove that the automorphism(s) in the factor(s) are palindromic involutions.

2.3 Major Theorems

We begin with a restatement of Beeler's theorem.

Theorem 2.3.1 ([1], Beeler)**.** *If G or H is even palindromic, then $G \square H$ is even palindromic. If G and H are odd palindromic, then $G \square H$ is odd palindromic.*

Proof. Let one of G or H (say G) be even palindromic. Theorem 1.3.1 yields an even palindromic involution $\alpha : G \to G$. Form the even palindromic involution $(x, y) \mapsto (\alpha(x), y)$ of $G \square H$. Thus the product is even palindromic because it has no fixed points. For the second statement, say both G and H are odd palindromic. By Theorem 1.3.1, G has an involution α with exactly one fixed point x_0. (That is, $\alpha(x_0) = x_0$). For the same reason, H has an involution β with exactly one fixed point y_0. Then $(x, y) \mapsto (\alpha(x), \beta(y))$ is an involution of $G \square H$ that has exactly one fixed point (x_0, y_0). Therefore $G \square H$ is odd palindromic. □

The following is the main theorem of this project.

Theorem 2.3.2. *Suppose G and H are connected:*

1. *$G \square H$ is even palindromic if and only if G or H is even palindromic.*
2. *$G \square H$ is odd palindromic if and only if G and H are odd palindromic.*

Proof. The forward direction is Theorem 2.3.1.

Conversely, suppose $G \square H$ is palindromic and let φ be a palindromic involution of it. Take prime factorings $G = G_1 \square \cdots \square G_j$ and $H = G_{j+1} \square \cdots \square G_k$, so we have an involution φ of $G \square H = (G_1 \square \cdots \square G_j) \square (G_{j+1} \square \cdots \square G_k)$.

The involution φ permutes the prime factors of this product in the sense of Theorem 1.5.1, where the permutation π satisfies $\pi^2 = \mathrm{id}$. Using commutativity of $\square$, group together the prime factors G_i of G for which $1 < \pi(i) \le j$, and call their product A. (By convention, $A = K_1$ if there are no such factors G_i. The same applies for the graphs B and D defined below.) Let B be the product of the remaining factors G_i of G. Also group together the prime factors G_i of H for which $j+1 < \pi(i) \le k$, and call their product D. The Cartesian product of the remaining factors of H is then a graph isomorphic to B. The structure of φ under this scheme is as indicated below, where the arrows represent isomorphisms $\varphi_i : G_{\pi(i)} \to G_i$ between factors.

$$\begin{tikzcd}[column sep=tiny]
 & & & & G & & & & & & H & & \\
 & & & A & & B & & B & & & D & & \\
G \square H \arrow[dd, "\varphi"'] & = & (G_1 \square \arrow[ddr] & G_2 \square \arrow[ddl] & G_3 \square \arrow[dd] & G_4 \square \arrow[ddrr] & G_5) \square \arrow[ddrr] & (G_6 \square \arrow[ddll] & G_7 \square \arrow[ddll] & G_8 \square \arrow[dd] & G_9 \square \arrow[ddr] & G_{10} \square \arrow[ddl] & G_{11}) \arrow[dd] \\
 & & & & & & & & & & & & \\
G \square H & = & (G_1 \square & G_2 \square & G_3 \square & G_4 \square & G_5) \square & (G_6 \square & G_7 \square & G_8 \square & G_9 \square & G_{10} \square & G_{11}) \\
 & & & A & & B & & B & & & D & &
\end{tikzcd}$$

We have now coordinatized G and H as $G = A \square B$ and $H = B \square D$, and φ is an involution of $G \square H = (A \square B) \square (B \square D)$ for which $\varphi\big((a,b),(b',d)\big) = \big((\alpha(a),\beta(b')),(\gamma(b),\delta(d))\big)$, for automorphisms $\alpha : A \to A$, $\beta, \gamma : B \to B$ and $\delta : D \to D$. But because φ^2 is the identity, it must be that $\alpha^2 = \mathrm{id}$, $\gamma = \beta^{-1}$ and $\delta^2 = \mathrm{id}$. Thus we have involutions α and δ of A and D, respectively, and

$$\varphi\big((a,b),(b',d)\big) = \big((\alpha(a),\beta(b')),(\beta^{-1}(b),\delta(d))\big), \tag{2.1}$$

From (2.1) it is evident that the fixed points of φ (if any) are precisely

$$\big((a_0, \beta(b)), (b, d_0)\big) \quad \text{with } \alpha(a_0) = a_0,\ \delta(d_0) = d_0, \text{ and } b \in V(B). \tag{2.2}$$

Thus φ has a fixed point if and only if both α and δ have fixed points. Further, if φ has a fixed point, then it has exactly $|V(B)|$ of them.

Now suppose $G \square H$ is even palindromic. Let φ be an even palindromic involution of $G \square H$ (having no fixed point). From (2.2), at least one of α or δ has no fixed point; say it is α. Then α is an even palindromic involution of A, so A is even palindromic. By the first part of the theorem, $G = A \square B$ is even palindromic. Similarly H is even palindromic if δ has no fixed points.

Suppose $G \square H$ is odd palindromic. Let φ be an odd palindromic involution whose sole fixed point is $\big((a_0, \beta(b_0)), (b_0, d_0)\big)$. The remark following (2.2) implies φ has at least $|V(B)|$ fixed points, so $B = K_1$. Thus we can drop B from our discussion, so $G = A$, $H = D$ and $\varphi(a, d) = \big(\alpha(a), \delta(d)\big)$. We now have involutions $\alpha : G \to G$ and $\delta : H \to H$ with fixed points a_0 and d_0, respectively. Also (a_0, d_0) is a fixed point of φ. If the involution α of G had a second fixed point a_1, then (a_0, d_0) and (a_1, d_0) would be two distinct fixed points of φ. Thus a_0 is the only fixed point of α, so α (hence also G) is odd palindromic. By the same reasoning H is odd palindromic. $\square$

CHAPTER 3

THE STRONG PRODUCT OF PALINDROMIC GRAPHS

3.1 Summary of Project

The structure of strong product is similar to the Cartesian product with respect to their edge sets, so after proving the main theorem on the Cartesian product, we are inspired to prove analogous results for the strong product.

The following is the main theorem of this project.

Theorem 3.1.1. *Suppose G and H are connected graphs. Then:*

(1) *G or H is even palindromic if and only if $G \boxtimes H$ is even palindromic.*

(2) *G and H are odd palindromic if and only if $G \boxtimes H$ is odd palindromic.*

This project was included in a paper published with Richard Hammack in *The Art of Discrete and Applied Mathematics* [4].

3.2 Preliminary Information

Recall from the definitions of the Cartesian and strong product that the edge set of the Cartesian product is contained in the edge set of the strong product. This suggests that proving analogous results for the strong product will be similar to the proof for the Cartesian product. However, the extra edges in the strong product provides concerns around the neighborhoods in the graph. When two vertices have the same open or closed neighborhood, an automorphism may swap them. This introduces an extra restriction in Theorem 3.2.1 (analogous to Theorem 2.2.1) shown below.

Theorem 3.2.1 ([3], Theorem 7.16)**.** *Let φ be an automorphism of an S-thin connected graph G with prime factorization $G = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_k$. Then there is a permutation π of $\{1, 2, \ldots, k\}$ and isomorphisms $\varphi_i : G_{\pi(i)} \to G_i$ for which $\varphi(x_1, x_2, \ldots, x_k) = \big(\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), \ldots, \varphi_k(x_{\pi(k)})\big)$.*

Utilizing this theorem will help us get a handle on how the proof works for an S-thin strong product and S-thin factors, however, we would like it to work for all general strong products and factors, and the existence of an isomorphism $\widetilde{\varphi} : G/S \to H/S$ does not necessarily mean that there is an isomorphism $\varphi : G \to H$. However, as we noted in 1.5.2, if $|X| = |\widetilde{\varphi}(X)|$ for each $X \in V(G/S)$, then we can lift $\widetilde{\varphi}$ to an isomorphism $\varphi : G \to H$ simply by declaring φ to restrict to a bijection $X \to \widetilde{\varphi}(X)$ for each X. Thus to complete the proof, we will need to show that $|X| = |\widetilde{\varphi}(X)|$ for each $X \in V(G/S)$, and then we can lift a palindromic involution $\widetilde{\varphi}$ to a palindromic involution φ using the method described above.

3.3 Major Theorems

What follows is the main theorem of this project.

Theorem 3.3.1. *Suppose G and H are connected graphs. Then:*

(1) *G or H is even palindromic if and only if $G \boxtimes H$ is even palindromic.*

(2) *G and H are odd palindromic if and only if $G \boxtimes H$ is odd palindromic.*

Proof. If G or H (say G) is even palindromic, then there exists an even palindromic involution α of G, so $(x, y) \mapsto (\alpha(x), y)$ is an even palindromic involution of $G \boxtimes H$. Next suppose G and H are odd palindromic. Then G has an odd palindromic involution α with fixed point x_0, and H has an odd palindromic involution β with fixed point y_0. Then $(x, y) \mapsto (\alpha(x), \beta(y))$ is an odd palindromic involution of $G \boxtimes H$ whose sole fixed point is (x_0, y_0).

It remains to prove the converses of the two statements. We will do this in three parts. The first part codifies the structure of involutions of $G \boxtimes H$.

Part I (Involution structure) Let $\varphi : G \boxtimes H \to G \boxtimes H$ be an involution. By the remarks preceding this theorem, φ induces an automorphism $\widetilde{\varphi}$ of the S-thin graph $(G \boxtimes H)/S \cong G/S \boxtimes H/S$. Because φ is an involution, we have $\widetilde{\varphi}^2 = \mathrm{id}$. (Note that $\widetilde{\varphi}$ could be the identity even if φ is not. This is the case if φ fixes each S-class, i.e., it restricts to a permutation on each S-class.)

Take prime factorings $G/S = G_1 \boxtimes \cdots \boxtimes G_j$ and $H/S = G_{j+1} \boxtimes \cdots \boxtimes G_k$. Then $\widetilde{\varphi}$ is an automorphism (of order 1 or 2) of the graph

$$G/S \boxtimes H/S = (G_1 \boxtimes \cdots \boxtimes G_j) \boxtimes (G_{j+1} \boxtimes \cdots \boxtimes G_k).$$

Now, $\widetilde{\varphi}$ permutes the prime factors of this product in the sense of Theorem 1.5.2, where the permutation π satisfies $\pi^2 = \mathrm{id}$. As in the proof of Theorem 2.3, group together the prime factors G_i of G/S for which $1 < \pi(i) \leq j$, and call their product A. Let B be the product of the remaining factors of G/S. Also group together the prime factors G_i of H/S for which $j+1 < \pi(i) \leq k$, and call their product D. The product of the remaining factors of H/S is then a graph isomorphic to B. Now we have $G/S = A \boxtimes B$ and $H/S = B \boxtimes D$, and $\widetilde{\varphi}$ is an automorphism of

$$G/S \boxtimes H/S = (A \boxtimes B) \boxtimes (B \boxtimes D)$$

satisfying $\widetilde{\varphi}^2 = \mathrm{id}$, and for which (as in the proof of Theorem 2.3) we have

$$\widetilde{\varphi}\big((a,b),(b',d)\big) = \big((\alpha(a),\beta(b')),(\beta^{-1}(b),\delta(d))\big) \tag{3.1}$$

for automorphisms $\alpha : A \to A$, $\beta : B \to B$ and $\delta : D \to D$, for which $\alpha^2 = \mathrm{id}$ and $\delta^2 = \mathrm{id}$.

In Equation (3.1) the ordered pairs (a, b) and $(\alpha(a), \beta(b'))$ label vertices of G/S, which are S-classes of G, and hence they have cardinalities $|(a, b)|$ and $|(\alpha(a), \beta(b'))|$. Similarly, (b', d) and $(\beta^{-1}(b), \delta(d))$ are S-classes of H/S.

By the remarks preceding this theorem, the involution φ of $G \boxtimes H$ sends the S-class $(a, b) \times (b', d)$ bijectively to S-class $(\alpha(a), \beta(b')) \times (\beta^{-1}(b), \delta(d))$, so

$$|(a, b)| \cdot |(b', d)| = \left|\big(\alpha(a), \beta(b')\big)\right| \cdot \left|\big(\beta^{-1}(b), \delta(d)\big)\right| \tag{3.2}$$

for all $a \in V(A)$, $b, b' \in V(B)$ and $d \in V(D)$. Putting $b' = \beta^{-1}(b)$ yields

$$|(a, b)| \cdot |(\beta^{-1}(b), d)| = \left|\big(\alpha(a), b\big)\right| \cdot \left|\big(\beta^{-1}(b), \delta(d)\big)\right|. \tag{3.3}$$

In (3.3) replace d with $\delta(d)$ (and use $\delta^2 = \mathrm{id}$) to get

$$|(a, b)| \cdot |(\beta^{-1}(b), \delta(d))| = \left|\big(\alpha(a), b\big)\right| \cdot \left|\big(\beta^{-1}(b), d\big)\right|. \tag{3.4}$$

Equations (3.3) and (3.4) imply $|(a, b)| = |(\alpha(a), b)|$. Form an automorphism $\widetilde{\alpha} : A \boxtimes B \to A \boxtimes B$ as $\widetilde{\alpha}(a, b) = \big(\alpha(a), b\big)$. Then $\widetilde{\alpha}^2 = \mathrm{id}$, so we have an involution (if it is not the identity map) $\widetilde{\alpha} : G/S \to G/S$ that maps each vertex (S-class) (a, b) to the vertex (S-class) $(\alpha(a), b)$ of the same cardinality.

Also (3.3) and (3.4) yield $\left|\big(\beta^{-1}(b), \delta(d)\big)\right| = \left|\big(\beta^{-1}(b), d\big)\right|$, and this implies $|(b, \delta(d))| = |(b, d)|$ for all $b \in V(B)$ and $d \in V(D)$. Form the automorphism $\widetilde{\delta} : B \boxtimes D \to B \boxtimes D$ where $\widetilde{\delta}(b, d) = \big(b, \delta(d)\big)$. Then $\widetilde{\delta}^2 = \mathrm{id}$, so we have an involution (if not the identity map) $\widetilde{\delta} : H/S \to H/S$ mapping each S-class (b, d) to the S-class $(b, \delta(d))$ of the same cardinality.

In summary, given an involution φ of $G \boxtimes H$, we have constructed automorphisms $\widetilde{\alpha}$ and $\widetilde{\delta}$ of G/S and H/S, respectively, satisfying $\widetilde{\alpha}^2 = \mathrm{id}$ and $\widetilde{\delta}^2 = \mathrm{id}$. Further, $|\widetilde{\alpha}((a, b))| = |(a, b)|$ for any S-class (a, b) of G. Thus we can lift $\widetilde{\alpha}$ to an automorphism $\lambda : G \to G$ by declaring that λ restricts to a bijection $(a, b) \to \big(\alpha(a), b\big)$, for each

S-class (a, b) of G. Similarly, $|\widetilde{\delta}((b,d))| = |(b,d)|$ for any S-class (b, d) of H, so we can lift $\widetilde{\delta}$ to an automorphism $\mu : H \to H$. In parts II and III of the proof these lifts will be palindromic involutions.

To carry out this plan we will need to consider S-classes of $G \boxtimes H$ that are fixed by φ (i.e. the S-classes whose vertices are permuted by φ.) By Equation (3.1), the fixed points of $\widetilde{\varphi}$ (respectively, the fixed S-classes of φ) are

$$\big((a_0, \beta(b)), (b, d_0)\big) \qquad \text{where } \alpha(a_0) = a_0,\ \delta(d_0) = d_0 \text{ and } b \in V(B) \tag{3.5}$$

$$(a_0, \beta(b)) \times (b, d_0) \qquad \text{where } \alpha(a_0) = a_0,\ \delta(d_0) = d_0 \text{ and } b \in V(B). \tag{3.6}$$

We call an S-class **even** (**odd**) if it has even (odd) cardinality.

Part II (Converse of Statement (1)) Suppose $G \boxtimes H$ is even palindromic. Then there is an even palindromic involution φ of $G \boxtimes H$. We adopt the development and notation of Part I of the proof.

Our strategy is to show that one of $\widetilde{\alpha} : G/S \to G/S$ or $\widetilde{\delta} : H/S \to H/S$ has no odd fixed point (S-class). For if this is the case for (say) $\widetilde{\alpha}$, then $\widetilde{\alpha}$ can be lifted to an automorphism $\lambda : G \to G$ sending any S-class (a, b) bijectively to $(\alpha(a), b)$. Whenever $\widetilde{\alpha}$ fixes an S-class (a, b), we can arrange for λ to restrict to an order-2 fixedpoint-free permutation of the even set (a, b). Then λ will be an even palindromic involution of G, so G is even palindromic.

Suppose to the contrary that $\widetilde{\alpha}$ had an odd fixed point (a, b) and $\widetilde{\delta}$ had an odd fixed point (b', d). (So $\alpha(a) = a$ and $\delta(d) = d$.) By (3.2),

$$\underbrace{\big|(a,b)\big|}_{\text{odd}} \cdot \underbrace{\big|(b',d)\big|}_{\text{odd}} = \big|(a, \beta(b'))\big| \cdot \big|(\beta^{-1}(b), d)\big|.$$

Then $(a, \beta(b'))$ is odd, so $(a, \beta(b')) \times (b', d)$ is an odd S-class of $G \boxtimes H$. But the involution φ fixes this odd S-class, by (3.6). Thus φ fixes some point of this S-class, contradicting the fact that φ is even palindromic.

Part III (Converse of Statement (2)) Suppose $G \boxtimes H$ is odd palindromic. Then there is an odd palindromic involution φ of $G \boxtimes H$ with fixed point (x_0, y_0). Then φ fixes the S-class X that contains (x_0, y_0), which necessarily has form $X = (a_0, \beta(b_0)) \times (b_0, d_0)$, where $\alpha(a_0) = a_0$ and $\delta(d_0) = d_0$. (See (3.6) in Part I.) As the involution φ fixes exactly one vertex, which is in X, we know X has odd cardinality. Thus $(a_0, \beta(b_0))$ is an odd S-class of G/S, and (b_0, d_0) is an odd S-class of H/S. Note that $(a_0, \beta(b_0))$ is a fixed point of $\widetilde{\alpha}$ and (b_0, d_0) is a fixed point of $\widetilde{\delta}$. Suppose $\widetilde{\delta}$ had another odd fixed point (b_1, d_1). Then $\delta(d_1) = d_1$ and by Equation (3.2),

$$\underbrace{\big|(a_0, \beta(b_0))\big|}_{\text{odd}} \cdot \underbrace{\big|(b_1, d_1)\big|}_{\text{odd}} = \big|(a_0, \beta(b_1))\big| \cdot \big|(b_0, d_1)\big|.$$

Therefore $\big|(a_0, \beta(b_1))\big|$ and $|(b_0, d_1)|$ are odd. Then $(a_0, \beta(b_1)) \times (b_1, d_1)$ and $(a_0, \beta(b_0)) \times (b_0, d_1)$ are odd S-classes of $G \times H$ that are fixed by φ. But $X = (a_0, \beta(b_0)) \times (b_0, d_0)$ is the only such S-class, hence $\beta(b_1) = \beta(b_0)$ and $d_1 = d_0$. This means $(b_1, d_1) = (b_0, d_0)$. Conclusion: (b_0, d_0) is the only odd S-class of H/S that is fixed by $\widetilde{\delta}$. Therefore we can lift $\widetilde{\delta} : H/S \to H/S$ to an odd palindromic involution $\mu : H \to H$ sending each S-class (b, d) bijectively to $(b, \delta(d))$, having only one fixed vertex on the odd fixed class (b_0, d_0) and no fixed points on any other fixed (even) S-class. Thus H is odd palindromic.

By a symmetric argument, G is also odd palindromic. $\square$

CHAPTER 4

THE DIRECT PRODUCT OF PALINDROMIC GRAPHS

4.1 Summary of Project

After successfully proving results for the Cartesian and strong product, we would like to prove analogous results for the direct product. Unfortunately, the following conjecture has been disproven, but it holds true for the case where G and H are nonbipartite.

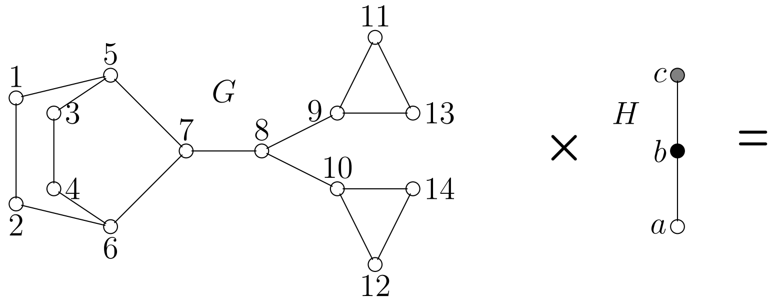
Conjecture 4.1.1. *Suppose G and H are connected. Then:*

1. *$G \times H$ is even palindromic if and only if G or H is even palindromic.*
2. *$G \times H$ is odd palindromic if and only if G and H are odd palindromic.*

The conjecture breaks down when the direct product has a bipartite factor. Figure 4 is an even palindromic direct product for which neither factor is even palindromic. In the figure, the graph G, a nonbipartite graph, is not even palindromic because any involution will fix both vertices 7 and 8, and the graph H is bipartite with an odd number of vertices, so neither factor is even palindromic. However, the product $G \times H$ is even palindromic, displaying a mirror symmetry about the dashed line, so this is a counterexample to the conjecture. However, G does admit the following even 2-fold involution (α, β):

$$\alpha : (1,13)(2,14)(3,11)(4,12)(5,10)(6,9)(7,8)$$

$$\beta : (1,12)(2,11)(3,14)(4,13)(5,9)(6,10)(7,8)$$



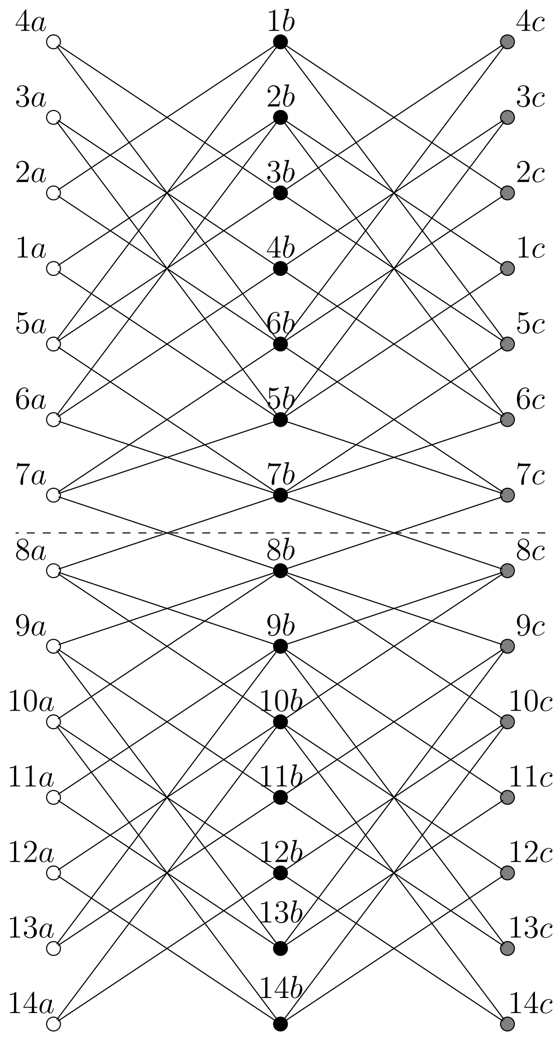


Fig. 4. Counterexample to Conjecture 4.1.1

We modify the conjecture in the case where one factor is nonbipartite and the other is bipartite, found below in the following proposition.

Proposition 4.1.1. *Let G and H be connected where G is nonbipartite and H is bipartite. Then:*

1. *$G \times H$ is even palindromic if and only if:*
   - *G has an even 2-fold involution, or*
   - *H is even palindromic, or*
   - *H has an involution η that preserves partite sets where all fixed points of η are in the same partite set, and G has a 2-fold involution (α, β) where only one of α or β is fixed-point free.*
2. *$G \times H$ is odd palindromic if and only if G has an odd 2-fold involution and H is odd palindromic.*

4.2 Preliminary Information

Similar to the connection between the Cartesian and strong product, there is a connection between the direct and strong products. The edge set of the direct product is contained in the edge set of the strong product (in fact, the edge set of the strong product is the edge set of the Cartesian product unioned with the edge set of direct product). After successfully proving results on the strong product, one might conjecture that proving results on the direct product would be very similar to that of the Cartesian and strong products. It would be convenient to have a theorem that is analogous to 1.5.1 and 1.5.2, and indeed, we have the following:

Theorem 4.2.1 ([3], Theorem 8.18)**.** *Suppose φ is an automorphism of a connected nonbipartite R-thin graph G that has a prime factorization $G = G_1 \times G_2 \times \ldots \times G_k$.*

Then there exists a permutation of π *of* $\{1, 2, ..., k\}$*, together with isomorphisms* $\varphi_i : G_{\pi(i)} \to G_i$ *such that* $\varphi(x_1, x_2, ..., x_k) = (\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), ..., \varphi_k(x_{\pi(k)}))$*. Thus* $Aut(G)$ *is generated by the automorphisms of the prime factors and transpositions of isomorphic factors. Consequently,* $Aut(G)$ *is isomorphic to the automorphism group of the disjoint union of the prime factors of* G*.*

Unfortunately, this theorem only applies to connected, nonbipartite, R-thin graphs. However, if we work exclusively in the case of two nonbipartite factors (resulting in a nonbipartite product), then the proof of this case will be remarkably similar to the proof of the strong product.

While the above result will help us prove the case of a palindromic direct product resulting from two nonbipartite factors, it does not apply to the cases where one or both factors are bipartite. As a result, we will need to turn our attention to a different technique for proving such cases. Further, because the direct product of two bipartite graphs is disconnected, we will only provide results on the case where the direct product is formed by one bipartite factor and one nonbipartite factor. We will briefly discuss the case where both factors are bipartite, leaving much to conjecture.

Regarding the case where one factor is bipartite and the other is nonbipartite, observe that an automorphism of any connected bipartite graph will either preserve or reverse its partite sets, and an involution that reverses the partite sets of a bipartite graph is, by definition, an even palindromic involution. We can break up the aforementioned case even further and focus on the instance where a palindromic involution preserves the products partite sets and the instance where it reverses the partite sets. The following theorem by Hammack [5] will help us quickly address the case where a palindromic product, formed by one nonbipartite factor and one bipartite factor, admits a partition-reversing involution.

Theorem 4.2.2 ([5], Hammack)**.** *Suppose G and H are connected graphs, and H is bipartite but G is not. If $G \times H$ admits an involution that reverses its partite sets, then H also admits an involution that reverses its partite sets.*

Moreover, the technique used to prove Theorem 4.2.2 is the key to proving the necessary conditions presented in Section 4.4. Using the Cartesian skeleton to link the direct product to the Cartesian product, as discussed in Section 1.6, we can use Theorem 1.5.1 in addition to the techniques from strong product and the two nonbipartite factors case to help us prove the main results in that section.

4.3 Major Theorems for Two Nonbipartite Factors

The following proposition describes the sufficient conditions for a palindromic direct product.

Proposition 4.3.1. *If G or H is even palindromic, then $G \times H$ is even palindromic. If G and H are odd palindromic, then $G \times H$ is odd palindromic.*

Proof. Suppose, without loss of generality, that G is even palindromic. Let $\alpha : G \to G$ defined by $x \mapsto \alpha(x)$ be even palindromic. Then $\varphi : G \times H \to G \times H$ defined by $(x, y) \mapsto (\alpha(x), y)$ is an even palindromic involution. Since α has no fixed points, $G \times H$ is even palindromic.

Suppose G and H are odd palindromic. Then G has an involution α with exactly one fixed vertex x_0, so $\alpha(x_0) = x_0$. Likewise, H has an involution β with exactly one fixed vertex y_0, so $\beta(y_0) = y_0$. Then $\varphi : G \times H \to G \times H$ defined by $(x, y) \mapsto (\alpha(x), \beta(y))$ is an involution of $G \times H$ with exactly one fixed vertex (x_0, y_0), so $G \times H$ is odd palindromic. □

The following proposition will be used in proving the case of the main theorem where G and H are nonbipartite and $G \times H$ is palindromic.

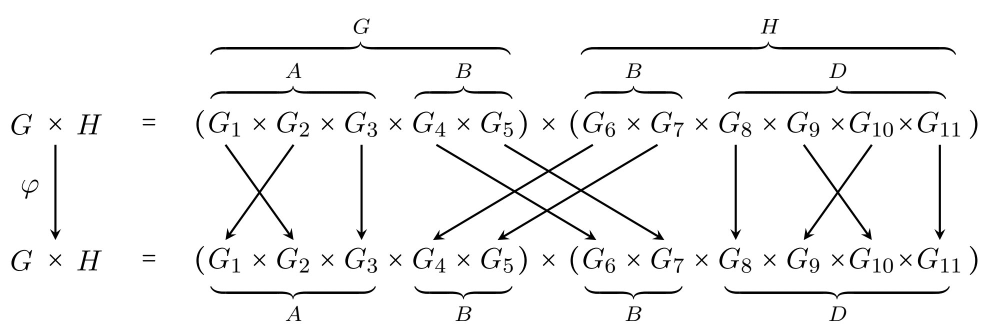
Proposition 4.3.2. *Suppose G and H are connected, R-thin, nonbipartite graphs. Then:*

(1) *G or H is even palindromic if and only if $G \times H$ is even palindromic.*

(2) *G and H are odd palindromic if and only if $G \times H$ is odd palindromic.*

Proof. Suppose $G \times H$ is palindromic with palindromic involution φ. Consider the connected, nonbipartite, R-thin, prime factorings $G = G_1 \times \cdots \times G_j$ and $H = G_{j+1} \times \cdots \times G_k$, so we have an involution φ of $G \times H = (G_1 \times \cdots \times G_j) \times (G_{j+1} \times \cdots \times G_k)$.

Utilizing Theorem 4.2.1, the involution φ permutes the prime factors of this product such that the permutation π satisfies $\pi^2 = \text{id}$. Using commutativity of $\times$, we can group together the prime factors G_i of G for which $1 < \pi(i) \leq j$, and call their product A. Note that $A = K_1^*$, where K_1^* is a single vertex with a loop, if no such factors G_i exist. The same applies for the graphs B and D defined below. Let B be the product of the remaining factors G_i of G. Also group together the prime factors G_i of H for which $j + 1 < \pi(i) \leq k$, and call their product D. The direct product of the remaining factors of H is then a graph isomorphic to B. The structure of φ under this scheme is as indicated below, where the arrows represent isomorphisms $\varphi_i : G_{\pi(i)} \to G_i$ between factors.



We have now coordinatized G and H as $G = A \times B$ and $H = B \times D$, and φ is an involution of $G \times H = (A \times B) \times (B \times D)$ for which $\varphi\big((a, b), (b', d)\big) = \big((\alpha(a), \beta(b')), (\gamma(b), \delta(d))\big)$, for automorphisms $\alpha : A \to A$, $\beta, \gamma : B \to B$ and $\delta : D \to D$. But because φ^2 is the identity, it must be that $\alpha^2 = \mathrm{id}$, $\gamma = \beta^{-1}$ and $\delta^2 = \mathrm{id}$. Thus we have involutions α and δ of A and D, respectively, and

$$\varphi\big((a,b),(b',d)\big) = \big((\alpha(a),\beta(b')),(\beta^{-1}(b),\delta(d))\big), \tag{4.1}$$

From (4.1) it is evident that the fixed points of φ (if any) are precisely

$$\big((a_0,\beta(b)),(b,d_0)\big) \quad \text{with } \alpha(a_0) = a_0,\ \delta(d_0) = d_0, \text{ and } b \in V(B). \tag{4.2}$$

Thus φ has a fixed point if and only if both α and δ have fixed points. Further, if φ has a fixed point, then it has exactly $|V(B)|$ of them.

Now suppose $G \times H$ is even palindromic. Let φ be an even palindromic involution of $G \times H$ (having no fixed point). From (4.2), at least one of α or δ has no fixed point, so suppose it is α. Then α is an even palindromic involution of A, so A is even palindromic. By the first part of the theorem, $G = A \times B$ is even palindromic. Similarly H is even palindromic if δ has no fixed points.

Suppose $G \times H$ is odd palindromic. Let φ be an odd palindromic involution whose sole fixed point is $\big((a_0,\beta(b_0)),(b_0,d_0)\big)$. The remark following (4.2) implies φ has at least $|V(B)|$ fixed points, so $B = K_1$. Thus we can drop B from our discussion, so $G = A$, $H = D$ and $\varphi(a,d) = \big(\alpha(a),\delta(d)\big)$. We now have involutions $\alpha : G \to G$ and $\delta : H \to H$ with fixed points a_0 and d_0, respectively. Also (a_0,d_0) is a fixed point of φ. If the involution α of G had a second fixed point a_1, then (a_0,d_0) and (a_1,d_0) would be two distinct fixed points of φ. Thus a_0 is the only fixed point of α, so α (hence also G) is odd palindromic. By the same reasoning H is odd palindromic. □

The following proposition is the desired result for connected and nonbipartite graphs.

Proposition 4.3.3. *Suppose G and H are connected and non-bipartite. Then:*

1. *G or H is even palindromic if and only if $G \times H$ is even palindromic.*

2. *G and H are odd palindromic if and only if $G \times H$ is odd palindromic.*

Proof. The forward direction is Proposition 4.3.1.

The converses of the two statements will be broken up into three parts. The first part codifies the structure of the involutions of $G \times H$.

Part I (Involution Structure) Let $\varphi : G \times H \to G \times H$ be an involution. By the remarks preceding this theorem, φ induces an automorphism $\widetilde{\varphi}$ of the R-thin graph $(G \times H)/R \cong G/R \times H/R$. Because φ is an involution, we have $\widetilde{\varphi}^2 = id$.

Take prime factorings $G/R = G_1 \times \ldots \times G_j$ and $H/R = G_{j+1} \times \ldots \times G_k$. Then $\widetilde{\varphi}$ is an automorphism (of order 2, or possibly of order 1, if φ fixes each R-class) of the graph $G/R \times H/R = (G_1 \times \ldots \times G_j) \times (G_{j+1} \times G_k)$. Now, $\widetilde{\varphi}$ permutes the prime factors of this product in the sense of Theorem 4.2.1, where the permutation π satisfies $\pi^2 = id$. As in the proof of Proposition 4.3.2, group together the prime factors G_i of G/R for which $1 < \pi(i) \leq j$, and call their product A. Let B be the product of the remaining factors of G/R. Also group together the prime factors G_i of H/R for which $j + 1 < \pi(i) \leq k$, and call their product D. The product of their remaining factors of H/R is then a graph isomorphic to B. Now we have $G/R = A \times B$ and $H/R = B \times D$. So $\widetilde{\varphi}$ is an automorphism of $G/R \times H/R = (A \times B) \times (B \times D)$ satisfying $\widetilde{\varphi}^2 = id$. As in the proof of Proposition 4.3.2, we have

$$\widetilde{\varphi}((a,b),(b',d)) = ((\alpha(a),\beta(b')),(\beta^{-1}(b),\delta(d))) \tag{4.3}$$

with automorphisms $\alpha : A \to A$, $\beta : B \to B$, and $\delta : D \to D$ for which $\alpha^2 = id$ and $\delta^2 = id$.

In Equation (4.3), the ordered pairs (a,b) and $(\alpha(a),\beta(b'))$ label vertices of G/R which are R-classes of G. Hence, they have cardinalities $|(a,b)|$ and $|(\alpha(a),\beta(b'))|$.

Likewise, (b', d) and $(\beta^{-1}(b), \delta(d))$ are R-classes of G which cardinalities $|(b', d)|$ and $|(\beta^{-1}(b), \delta(d))|$.

By the remarks preceding this theorem, the involution φ of $G \times H$ sends the R-class $(a, b) \times (b', d)$ bijectively to the R-class $(\alpha(a), \beta(b')) \times (\beta^{-1}(b), \delta(d))$, thus

$$|(a, b)| \cdot |(b', d)| = |(\alpha(a), \beta(b'))| \cdot |(\beta^{-1}(b), \delta(d))| \tag{4.4}$$

for all $a \in V(A)$, $b, b' \in V(B)$, and $d \in V(D)$. Putting $b' = \beta^{-1}(b)$ yields

$$|(a, b)| \cdot |(\beta^{-1}(b), d)| = |(\alpha(a), b))| \cdot |(\beta^{-1}(b), \delta(d))|. \tag{4.5}$$

In Equation (4.13), replace d with $\delta(d)$ and use $\delta^2 = id$ to get

$$|(a, b)| \cdot |(\beta^{-1}(b), \delta(d))| = |(\alpha(a), b))| \cdot |(\beta^{-1}(b), d)|. \tag{4.6}$$

Equations (4.13) and (4.6) imply that $|(a, b)| = |(\alpha(a), b)|$. Form an automorphism $\tilde{\alpha} : A \times B \to A \times B$ as $\tilde{\alpha}(a, b) = (\alpha(a), b))$. Then $\tilde{\alpha}^2 = id$, so we have an involution (if it is not the identity map) $\tilde{\alpha} : G/R \to G/R$ that maps each vertex (R-class) (a, b) to the vertex (R-class) $(\alpha(a), b)$ of the same cardinality.

Also, (4.13) and (4.6) imply that $|(\beta^{-1}(b), \delta(d))| = (\beta^{-1}(b), d)|$. This implies that $|(b, \delta(d))| = |(b, d)|$ for all $b \in V(B)$ and $d \in V(D)$. Form the automorphism $\tilde{\delta} : B \times D \to B \times D$ where $\tilde{\delta}(b, d) = (b, \delta(d))$. Then $\tilde{\delta}^2 = id$, so we have an involution (if not the identity map) $\tilde{\delta} : H/R \to H/R$ mapping each R-class (b, d) to the R-class $(b, \delta(d))$ of the same cardinality.

In summary, given an involution φ of $G \times H$, we have constructed automorphisms $\tilde{\alpha}$ and $\tilde{\delta}$ of G/R and H/R, respectively, satisfying $\tilde{\alpha}^2 = id$ and $\tilde{\delta}^2 = id$. Further $|\tilde{\alpha}((a, b))| = |(a, b)|$ for any R-class (a, b) of G. Thus we can lift $\tilde{\alpha}$ to an automorphism $\lambda : G \to G$ by declaring that λ restricts to a bijection $(a, b) \mapsto (\alpha(a), b)$ for each R-class (a, b) of G. Similarly, $|\tilde{\delta}((b, d))| = |(b, \delta(d))|$ for any R-class (b, d) of H, so we can lift

$\tilde{\delta}$ to an automorphism $\mu : H \to H$. In parts II and III of the proof, these lifts will be palindromic involutions.

To carry out this plan, we will need to consider R-classes of $G \times H$ that are fixed by φ (i.e. the R-classes whose vertices are permuted by φ). By Equation (4.3), the fixed points of $\widetilde{\varphi}$ (respectively, the fixed R-classes of φ) are

$$((a_0, \beta(b)), (b, d_0)) \text{ where } \alpha(a_0) = a_0, \delta(d_0) = d_0, \text{ and } b \in V(B) \tag{4.7}$$

$$(a_0, \beta(b)) \times (b, d_0) \text{ where } \alpha(a_0) = a_0, \delta(d_0) = d_0, \text{ and } b \in V(B) \tag{4.8}$$

We call an R-class **even** (**odd**) if it has even (odd) cardinality.

Part II (Converse of Statement (1))

Suppose $G \times H$ is even palindromic. Then there is an even palindromic involution φ of $G \times H$. We adopt the development and notation of Part I of the proof.

Our strategy is to show that one of $\tilde{\alpha} : G/R \to G/R$ or $\tilde{\delta} : H/R \to H/R$ has no odd fixed point (R-class). If one of the above, say $\tilde{\alpha}$, has no odd fixed point, then $\tilde{\alpha}$ can be lifted to an automorphism $\lambda : G \to G$ sending any R-class (a, b) bijectively to $(\alpha(a), b)$. Whenever $\tilde{\alpha}$ fixes an R-class (a, b), we can arrange for λ to restrict to a permutation (of order 2 with no fixed points) of the even set (a, b). Now, λ will be an even palindromic involution of G, so G is even palindromic.

Suppose to the contrary that $\tilde{\alpha}$ had an odd fixed point (a, b) and $\tilde{\delta}$ had an odd fixed point (b', d). (So $\alpha(a) = a$ and $\delta(d) = d$). By (4.11), we have that

$$|(a, b)| \cdot |(b', d)| = |(a, \beta(b'))| \cdot |(\beta^{-1}(b), d)|$$

where $|(a, b)|$ is odd and $|(b', d)|$ is odd, so $(a, \beta(b'))$ is odd. Thus, $(a, \beta(b')) \times (b', d)$ is an odd R-class of $G \times H$. However, the involution φ fixes this odd R-class, by (4.8). Thus φ fixes some point of this R-class, which contradicts that φ is even palindromic.

Part III (Converse of Statement (2))

Suppose $G \times H$ is odd palindromic. Then, there is an odd palindromic involution φ of $G \times H$ with fixed point (x_0, y_0). Then φ fixes the R-class X that contains (x_0, y_0), which, by (4.8), has form $X = (a_0, \beta(b_0)) \times (b_0, d_0)$, where $\alpha(a_0) = a_0$ and $\delta(d_0) = d_0$. As the involution φ fixes exactly one vertex, which is in X, we know X has odd cardinality. Thus $(a_0, \beta(b_0))$ is an odd R-class of G/R, and (b_0, d_0) is an odd R-class of H/R. Note that $(a_0, \beta(b_0))$ is a fixed point of $\tilde{\alpha}$ and (b_0, d_0) is a fixed point of $\tilde{\delta}$.

Suppose $\tilde{\delta}$ had another fixed point, say (b_1, d_1). Then, $\delta(d_1) = d_1$, and by Equation (4.11), we have

$$|(a_0, \beta(b_0))| \cdot |(b_1, d_1)| = |(a_0, \beta(b_1))| \cdot |(b_0, d_1)|,$$

where $|(a_0, \beta(b_0))|$ is odd and $|(b_1, d_1)|$ is odd, so $|(a_0, \beta(b_1))|$ and $|(b_0, d_1)|$ are odd. Then, $(a_0, \beta(b_1)) \times (b_1, d_1)$ and $(a_0, \beta(b_0)) \times (b_0, d_1)$ are odd R-classes of $G \times H$ that are fixed by φ. However, $X = (a_0, \beta(b_0)) \times (b_0, d_0)$ is the only such R-class, so $\beta(b_1) = \beta(b_0)$ and $d_1 = d_0$, meaning $(b_1, d_1) = (b_0, d_0)$. So, we can conclude that (b_0, d_0) is the only odd R-class of H/R that is fixed by $\tilde{\delta}$. Therefore, we can lift $\tilde{\delta} : H/R \to H/R$ to an odd palindromic involution $\mu : H \to H$, sending each R-class (b, d) bijectively to $(b, \delta(d))$, having only one fixed vertex on the odd fixed class (b_0, d_0), and no fixed points on any other fixed (even) R-class. Thus, H is odd palindromic. Likewise, by a symmetric argument, G is also odd palindromic. $\square$

4.4 Major Theorems for One Nonbipartite Factor and One Bipartite Factor

Using the counterexample from Section 4.1 as our guide, we can adjust the case of one bipartite factor and one nonbipartite by including the possibility of 2-fold involutions in the factors. The following two proposition present sufficient conditions for a palindromic direct product when one factor is bipartite and the other is nonbipartite. We begin with sufficient conditions for an even palindromic direct product.

Proposition 4.4.1. *Let G and H be connected graphs where G is nonbipartite and H is bipartite. Then:*

1. *If G has an even 2-fold involution, then $G \times H$ is even palindromic.*
2. *If H is even palindromic, then $G \times H$ is even palindromic.*
3. *If H has an involution η that preserves partite sets where all fixed points of η are in the same partite set, and G has a 2-fold involution (α, β) where only one of α or β is fixed-point free, then $G \times H$ is even palindromic.*

Proof. **Proof of Statement (1)**: Suppose G has an even 2-fold involution, so G admits permutations (α, β) such that α and β are of order 2 and have no fixed points. Suppose H is bipartite with partite sets H_0 and H_1. Define $\varphi : G \times H \to G \times H$ such that

$$\varphi(g, h) = \begin{cases} (\alpha(g), h) & \text{if } h \in H_0 \\ (\beta(g), h) & \text{if } h \in H_1. \end{cases}$$

To see that φ is an automorphism, let $(g, h)(g', h') \in E(G \times H)$ for $h \in H_0$ and $h' \in H_1$. This implies that $gg' \in E(G)$, so $\alpha(g)\beta(g') \in E(G)$, and $hh' \in E(H)$. Thus $(\alpha(g), h)(\beta(g'), h') \in E(G \times H)$, so $\varphi(g, h)\varphi(g', h') \in E(G \times H)$. Moreover, φ is an even palindromic involution because both α and β are of order 2 with no fixed points. So $G \times H$ is even palindromic.

Proof of Statement (2): Suppose H is even palindromic, so there exists an even palindromic involution η of H. Then $(g, h) \mapsto (g, \eta(h))$ is an even palindromic involution of $G \times H$.

Proof of Statement (3): Suppose H has an even palindromic involution η that preserves the partite sets H_0 and H_1 of H. The restrictions of η to the two partite sets H_0 and H_1 give us two bijections $\eta_0 : H_0 \to H_0$ and $\eta_1 : H_1 \to H_1$. Let all fixed points of η be in one partite set, say H_0. So η_0 may have fixed points, and η_1 has no fixed points. Let (α, β) be a 2-fold involution of G such that only one of α or β is fixed point free. Without loss of generality, let α be fixed point free. Define $\varphi : G \times H \to G \times H$ such that

$$\varphi(g,h) = \begin{cases} (\alpha(g), \eta_0(h)) & \text{if } h \in H_0 \\ (\beta(g), \eta_1(h)) & \text{if } h \in H_1. \end{cases}$$

To see that φ is an automorphism, let $(g,h)(g',h') \in E(G \times H)$ for $h \in H_0$ and $h' \in H_1$. This implies that $gg' \in E(G)$ and $hh' \in E(H)$, so $\alpha(g)\beta(g') \in E(G)$ and $\eta_0(h)\eta_1(h') \in E(H)$. Thus $(\alpha(g), \eta_0(h))(\beta(g'), \eta_1(h')) \in E(G \times H)$, so $\varphi(g,h)\varphi(g',h') \in E(G \times H)$. Moreover, observe that $(g,h) \mapsto (\alpha(g), \eta_0(h))$ for $h \in H_0$ is a bijective map with no fixed points because α has no fixed points. Similarly, $(g,h) \mapsto (\beta(g), \eta_1(h))$ for $h \in H_1$ is a bijective map with no fixed points because η_1 has no fixed points. So φ is an automorphism with no fixed points. Because α, β, and η_1 have order 2 (η_0 may have order 1 or 2), φ is an involution with no fixed points. Thus $G \times H$ is even palindromic. □

The following proposition provides sufficient conditions for an odd palindromic direct product. Recall that an odd 2-fold involution is a 2-fold involution with permutations (α, β) such that at least one of α or β has exactly one fixed point.

Proposition 4.4.2. *Let G and H be connected where G is nonbipartite and H is bipartite. If G has an odd 2-fold involution and H is odd palindromic, then $G \times H$ is odd palindromic.*

Proof. Suppose G admits an odd 2-fold involution and H is odd palindromic. Then G admits permutations (α, β) where α and β are of order 2 and at least one of α or β has exactly one fixed point. Say α has exactly one fixed point at g_0. Also H has an involution η with exactly one fixed point h_0, so $\eta(h_0) = h_0$. Let H be bipartite with partite sets H_0 and H_1, and let the fixed point of η be in H_0.

Then $\varphi : G \times H \to G \times H$ such that

$$\varphi(g,h) = \begin{cases} (\alpha(g), \eta(h)) & \text{if } h \in H_0 \\ (\beta(g), \eta(h)) & \text{if } h \in H_1 \end{cases}$$

is an involution with a fixed point at (g_0, h_0). Suppose φ had another fixed point at, say, (g_1, h_1). Suppose $h_1 \in H_0$. Then α has another fixed point at g_1 and η has another fixed point at h_1, so it must be the case that $g_0 = g_1$ and $h_0 = h_1$ to preserve that α and η each have exactly one fixed point at g_0 and h_0, respectively. Similarly, if φ had another fixed point at (g_1, h_1) where $h_1 \in H_1$, then β would have a fixed point at g_1 and η would have another fixed point at h_1. This contradicts that η has exactly one fixed point at h_0. Thus φ is an odd palindromic involution with exactly one fixed point at (g_0, h_0), so $G \times H$ is odd palindromic. □

The following proposition addresses the sufficient and necessary conditions on a palindromic direct product where one factor is bipartite and the other is nonbipartite.

Proposition 4.4.3. *Let G and H be connected where G is nonbipartite and H is bipartite. Then:*

1. *$G \times H$ is even palindromic if and only if:*
   - *G has an even 2-fold involution, or*
   - *H is even palindromic, or*

- *H has an involution η that preserves partite sets where all fixed points of η are in the same partite set, and G has a 2-fold involution (α, β) where only one of α or β is fixed-point free*

2. *$G \times H$ is odd palindromic if and only if G has an odd 2-fold involution and H is odd palindromic*

Proof. The converse of Statement (1) is proven in Proposition 4.4.1, and the converse of Statement (2) is proven in Proposition 4.4.2.

Forward Directions of Statement (1) and (2): Suppose $G \times H$ has a palindromic involution φ. Observe that the direct product $G \times H$ will be bipartite given a nonbipartite G and bipartite H. Assuming that G and H are connected, any palindromic involution φ of $G \times H$ will either reverse the bipartition or preserve the bipartition. This splits the proof up into two cases.

Case 1: The palindromic involution φ reverses the bipartition of $G \times H$

In this case, we will show that H must be even palindromic. If φ is a palindromic involution that reverses the bipartition of $G \times H$, then φ must be an even palindromic involution because it has no fixed points. Then by Theorem 4.2.2, H admits an involution that reverses its partite sets, which is an even palindromic involution, so H is even palindromic. This completes the proof of Case 1.

Case 2: The palindromic involution φ preserves the bipartition of $G \times H$

This case will be broken up into three parts. The first part will codify the structure of involutions of $G \times H$. The remaining two parts will address Statements (1) and (2).

Part I: Involution Structure

Suppose G and H are connected, where G is nonbipartite, and H is bipartite with partite sets H_0 and H_1. Suppose the palindromic involution $\varphi : G \times H \to G \times$

H preserves the partite sets of $G \times H$, that is, $\varphi(V(G) \times H_0) = V(G) \times H_0$ and $\varphi((V(G) \times H_1) = V(G) \times H_1$. Further, let $\varphi(g,h) = (\varphi_G(g,h), \varphi_H(g,h))$. By the remarks preceding this theorem, φ induces an automorphism $\widetilde{\varphi}$ of the R-thin graph $(G \times H)/R \cong G/R \times H/R$. Because φ is an involution, we have $\widetilde{\varphi}^2 = id$.

Applying the Cartesian skeleton operator and using Proposition 1.6.2, this is an involution

$$\widetilde{\varphi} : S(G/R \times H/R) \to S(G/R \times H/R).$$

By Proposition 1.6.1, this is an involution

$$\widetilde{\varphi} : S(G/R) \,\square\, S(H/R) \to S(G/R) \,\square\, S(H/R).$$

Because H/R is connected and bipartite, by (Proposition 1.6.3), $S(H/R)$ has two connected components H_0' and H_1' whose respective vertex sets are the partite sets of H. Thus this is an involution

$$\widetilde{\varphi} : S(G/R) \,\square\, (H_0' + H_1') \to S(G/R) \,\square\, (H_0' + H_1').$$

By the distributive property of the Cartesian product, we have:

$$\widetilde{\varphi} : S(G/R) \,\square\, H_0' + S(G) \,\square\, H_1' \to S(G/R) \,\square\, H_0' + S(G) \,\square\, H_1'.$$

Now the restrictions of $\widetilde{\varphi}$ to the two components $S(G/R) \,\square\, H_0'$ and $S(G/R) \,\square\, H_1'$ give us two isomorphisms:

$$\widetilde{\varphi}_0 : S(G/R) \,\square\, H_0' \to S(G) \,\square\, H_0'$$
$$\widetilde{\varphi}_1 : S(G/R) \,\square\, H_1' \to S(G) \,\square\, H_1'.$$

Make note that as $V(G/R) = V(S(G/R))$ and $V(H) = V(H_0') + V(H_1')$, the map $\widetilde{\varphi}$ is an involution of both $G/R \times H/R$ and $S(G/R) \,\square\, S(H/R)$.

Prime factor $S(G/R) = G_1 \square \cdots \square G_j$ and $H_0' = H_{01} \square \cdots \square H_{0k}$. Then $\widetilde{\varphi}_0$ is an automorphism (of order 2, or possibly of order 1, if φ fixes each R-class) of the graph $S(G/R) \square H_0' = (G_1 \square \cdots \square G_j) \square (H_{01} \square \cdots \square H_{0k})$. Similarly, prime factor $H_1' = H_{11} \square \cdots \square H_{1\ell}$. Then $\widetilde{\varphi}_1$ is an automorphism (of order 2, or possibly of order 1, if φ fixes each R-class) of the graph $S(G/R) \square H_1' = (G_1 \square \cdots \square G_j) \square (H_{11} \square \cdots \square H_{1\ell})$.

Now by Theorem 1.5.1, $\widetilde{\varphi}$ permutes the prime factors of $S(G/R) \square H_0'$ and $S(G/R) \square H_1'$ in the following way: There is a permutation π_0 (of order 2) that permutes the prime factors of $S(G/R) \square H_0'$, and there is a permutation π_1 (of order 2) that permutes the prime factors of $S(G/R) \square H_1'$. Now π_0 may send some prime factors of $S(G/R)$ to H_0'. If π_0 sends prime factors of $S(G/R)$ to H_0', let X be the product of such factors. If π_0 sends prime factors of $S(G/R)$ to itself, let G_X be the product of such factors. If π_0 sends prime factors of H_0' to itself, let H_X be the product of such factors. Similarly, π_1 may send some prime factors of $S(G/R)$ to H_1'. If π_1 sends prime factors of $S(G/R)$ to H_1', let Y be the product of such factors. If π_1 sends prime factors of $S(G/R)$ to itself, let G_Y be the product of such factors. If π_1 sends prime factors of H_1' to itself, let H_Y be the product of such factors. Figure 5 provides a visual of the above, where the factors have been re-ordered and grouped by product.

For $\alpha : G_X \to G_X$, $\chi_x, \gamma_x : X \to X$, and $\eta_x : H_X \to H_X$, we can express $\widetilde{\varphi}_0$ as follows:

$$\widetilde{\varphi}_0((g_x, x), (x', h_x)) = ((\alpha(g_x), \chi_x(x')), (\gamma_x(x), \eta_x(h_x))).$$

Because φ is an involution, we have that $\alpha^2 = id$, $\eta_x^2 = id$, and $\chi_x = \gamma_x^{-1}$, so we can update the expression of $\widetilde{\varphi}_0$:

$$\widetilde{\varphi}_0((g_x, x), (x', h_x)) = ((\alpha(g_x), \chi_x(x')), (\chi_x^{-1}(x), \eta_x(h_x))). \tag{4.9}$$

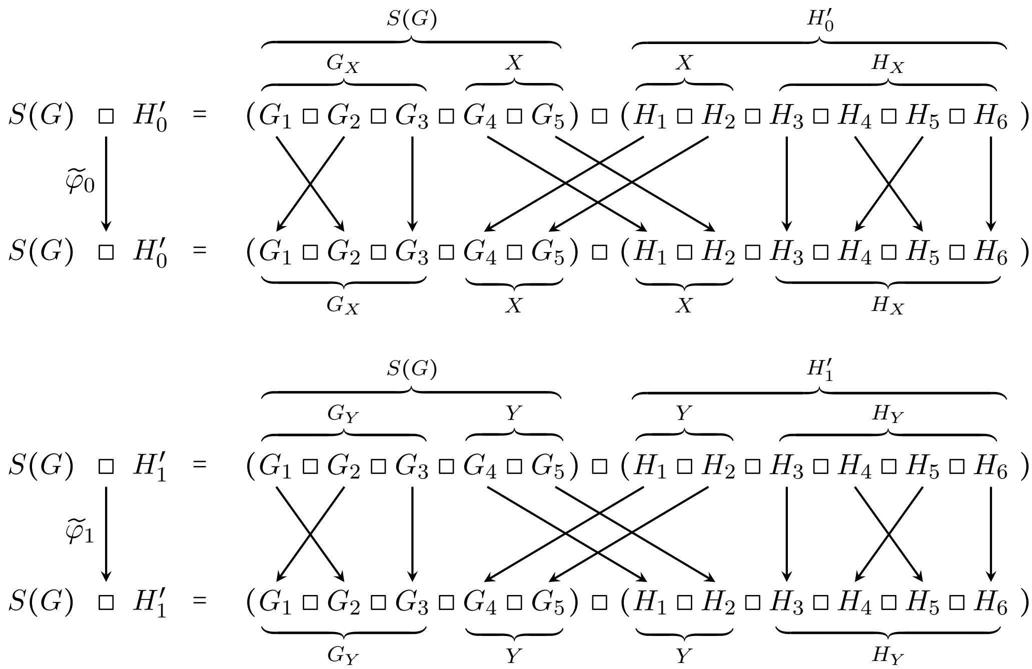


Fig. 5. Permutation of prime factors in the Cartesian skeleton

Similarly, we have

$$\widetilde{\varphi}_1((g_y, y), (y', h_y)) = ((\beta(g_y), \chi_y(y')), (\chi_y^{-1}(y), \eta_y(h_y))) \tag{4.10}$$

for $\beta : G_Y \to G_Y$, $\chi_y : Y \to Y$, and $\eta_y : H_Y \to H_Y$.

Recall that $\widetilde{\varphi}_0$ and $\widetilde{\varphi}_1$ are simply restrictions of the involution $\widetilde{\varphi}$ and have order 1 or 2, so $\alpha^2 = id$, $\beta^2 = id$, $\eta_x^2 = id$, and $\eta_y^2 = id$.

In Equation (4.9), the ordered pairs (g_x, x) and $(\alpha(g_x), \chi_x(x'))$ label vertices of $S(G/R)$ which are R-classes of $S(G)$. Hence, they have cardinalities $|(g_x, x)|$ and $|(\alpha(g_x), \chi_x(x'))|$. Likewise, (x', h_x) and $(\chi_x^{-1}(x), \eta_x(h_x))$ are R-classes of H'_0 with cardinalities $|(x', h_x)|$ and $|(\chi_x^{-1}(x), \eta_x(h_x))|$.

In Equation (4.10), the ordered pairs (g_y, y) and $(\beta(g_y), \chi_y(y'))$ label vertices of $S(G/R)$ which are R-classes of $S(G)$ with cardinalities $|(g_y, y)|$ and $|(\beta(g_y), \chi_y(y'))|$. Likewise, (y', h_y) and $(\chi_y^{-1}(y), \eta_y(h_y))$ are R-classes of H'_1 with cardinalities $|(y', h_y)|$

and $|(\chi_y^{-1}(y), \eta_y(h_y))|$.

By the remarks preceding this theorem, φ sends the R-class $(g_x, x) \times (x', h_x)$ bijectively to the R-class $(\alpha(g_x), \chi_x(x')) \times (\chi_x^{-1}(x), \eta_x(h_x))$, thus

$$|(g_x, x)| \cdot |(x', h_x)| = |(\alpha(g_x), \chi_x(x'))| \cdot |(\chi_x^{-1}(x), \eta_x(h_x))| \qquad (4.11)$$

for all $g_x \in V(G_X)$, $x, x' \in V(X)$, and $h_x \in V(H_X)$.

Putting $x' = \chi_x^{-1}(x)$ yields

$$|(g_x, x)| \cdot |(\chi_x^{-1}(x), h_x)| = |(\alpha(g_x), x)| \cdot |(\chi_x^{-1}(x), \eta_x(h_x))|. \qquad (4.12)$$

Replacing h_x with $\eta_x(h_x)$ and using $\eta_x^2 = id$ yields

$$|(g_x, x)| \cdot |(\chi_x^{-1}(x), \eta_x(h_x))| = |(\alpha(g_x), x)| \cdot |(\chi_x^{-1}(x), h_x)|. \qquad (4.13)$$

We will show that $|(g_x, x)| = |(\alpha(g_x), x)|$. Dividing the left side of Equation (4.12) by the right side of Equation (4.13) and the right side of Equation (4.12) by the left side of Equation (4.13), then simplifying by clearing fractions, we get

$$|(g_x, x)| \cdot |(g_x, x)| = |(\alpha(g_x), x)| \cdot |(\alpha(g_x), x)|.$$

which implies that

$$|(g_x, x)| = |(\alpha(g_x), x)|. \qquad (4.14)$$

Define $\widetilde{\alpha} : V(G_X \,\square\, X) \to V(G_X \,\square\, X)$ as $\widetilde{\alpha}(g_x, x) = (\alpha(g_x), x)$. Then $\widetilde{\alpha}^2 = id$, so we have an order-2 permutation (if it is not the identity map) $\widetilde{\alpha} : S(G/R) \to S(G/R)$ that maps each vertex (R-class) (g_x, x) to the vertex (R-class) $(\alpha(g_x), x)$ of the same cardinality.

Similarly, φ sends the R-class $(g_y, y) \times (y', h_x)$ bijectively to the R-class

$(\beta(g_y), \chi_y(y')) \times (\chi_y^{-1}(y), \eta_y(h_y))$. Following similar reasoning, we have

$$|(g_y, y)| = |(\beta(g_y), y)|. \tag{4.15}$$

Define $\widetilde{\beta} : V(G_Y \square Y) \to V(G_Y \square Y)$ as $\widetilde{\beta}(g_y, y) = (\beta(g_y), y)$. Then $\widetilde{\beta}^2 = id$, so we have an order-2 permutation (if it is not the identity map) $\widetilde{\beta} : S(G/R) \to S(G/R)$ that maps each vertex (R-class) (g_y, y) to the vertex (R-class) $(\beta(g_y), y)$ of the same cardinality.

We will now show that $(\widetilde{\alpha}, \widetilde{\beta})$ is a 2-fold automorphism of G/R, that is, given $(g, z)(g', z') \in E(G/R)$, we will show $\widetilde{\alpha}(g, z)\widetilde{\beta}(g', z') \in E(G/R)$. Select an edge $(w, h)(w', h') \in E(H/R)$ where $h \in H_0'$ and $h' \in H_1'$. Then $((g, z), (w, h))((g', z'), (w', h'))$ $\in E(G/R \times H/R)$ where $((g, z), (w, h)) \in V(G_X \square X \square X \square H_X)$ and $((g', z'), (w', h')) \in$ $V(G_Y \square Y \square Y \square H_Y)$.

Apply $\widetilde{\varphi}$:

$((\alpha(g), \chi_x(w)), (\chi_x^{-1}(z), \eta_x(h)))((\beta(g'), \chi_y(w')), (\chi_y^{-1}(z'), \eta_y(h'))) \in E(G/R \times H/R)$.

This implies that $(\chi_x^{-1}(z), \eta_x(h))(\chi_y^{-1}(z'), \eta_y(h')) \in E(H/R)$. Because $(g, z)(g', z') \in$ $E(G/R)$, we have $((g, z), (\chi_x^{-1}(z), \eta_x(h)))((g', z'), (\chi_y^{-1}(z'), \eta_y(h'))) \in E(G/R \times H/R)$.

Apply $\widetilde{\varphi}$ again:

$((\alpha(g), \chi_x(\chi_x^{-1}(z)), (\chi_x^{-1}(z), \eta_x^2(h)))((\beta(g'), \chi_y(\chi_y^{-1}(z'))), (\chi_y^{-1}(z'), \eta_y^2(h'))) \in E(G/R$ $\times H/R)$. This implies that $(\alpha(g), z)(\beta(g'), z') \in E(G/R)$, so $\widetilde{\alpha}(g, z)\widetilde{\beta}(g', z') \in E(G/R)$. Thus G/R admits a 2-fold automorphism. Now because $\widetilde{\alpha}^2 = id$ and $\widetilde{\beta}^2 = id$, $(\widetilde{\alpha}, \widetilde{\beta})$ is a 2-fold involution of G/R.

Also, because we know that $|(g_x, x)| = |(\alpha(g_x), x)|$, equation (4.13) quickly implies that $|(\chi_x^{-1}(x), \eta_x(h_x))| = (\chi_x^{-1}(x), h_x)|$. This implies that

$$|(x, \eta_x(h_x))| = |(x, h_x)|, \tag{4.16}$$

for all $x \in V(X)$ and $h_x \in V(H_X)$. Define the permutation $\widetilde{\eta}_x : V(X \square H_X) \to$

$V(X \square H_X)$ such that $\widetilde{\eta}_x(x, h_x) = (x, \eta_x(x))$. Similarly, we can conclude that

$$|(y, \eta_y(h_y))| = |(y, h_y)|. \tag{4.17}$$

Define the permutation $\widetilde{\eta}_y : V(Y \square H_Y) \to V(Y \square H_Y)$ such that $\widetilde{\eta}_y(y, h_y) = (y, \eta_y(y))$.

Now define the partition-preserving permutation $\widetilde{\eta} : H/R \to H/R$ such that

$$\widetilde{\eta}(z,h) = \begin{cases} \widetilde{\eta}_x(z,h) = (z, \eta_x(h)) & \text{if } (z,h) \in V(X \square H_X) \\ \widetilde{\eta}_y(z,h) = (z, \eta_y(h)) & \text{if } (z,h) \in V(Y \square H_Y). \end{cases}$$

We need to show that $\widetilde{\eta}$ is an automorphism, that is, given $(z,h)(z',h') \in E(H/R)$, we need to show that $\widetilde{\eta}(z,h)\widetilde{\eta}(z',h') \in E(H/R)$. Let $h \in H'_0$ and $h' \in H'_1$. Select an edge $(g,w)(g',w') \in E(G/R)$. Then $((g,w),(z,h))((g',w'),(z',h')) \in E(G/R \times H/R)$ where $((g,w),(z,h)) \in V(G_X \square X \square X \square H_X)$ and $((g',w'),(z',h')) \in V(G_Y \square Y \square Y \square H_Y)$.

Apply $\widetilde{\varphi}$:

$((\alpha(g), \chi_x(z)), (\chi_x^{-1}(w), \eta_x(h)))((\beta(g'), \chi_y(z')), (\chi_y^{-1}(w'), \eta_y(h'))) \in E(G/R \times H/R)$.

This implies that $(\alpha(g), \chi_x(z))(\beta(g'), \chi_y(z')) \in E(G/R)$. Because $(z,h)(z',h') \in E(H/R)$, we have $((\alpha(g), \chi_x(z)), (z,h))((\beta(g'), \chi_y(z')), (z',h')) \in E(G/R \times H/R)$.

Apply $\widetilde{\varphi}$ again:

$((\alpha^2(g), \chi_x(z)), (\chi_x^{-1}(\chi_x(z)), \eta_x(h)))((\beta^2(g'), \chi_y(z')), (\chi_y^{-1}(\chi_y(z')), \eta_y(h'))) \in E(G/R \times H/R)$. This implies that $(z, \eta_x(h))(z', \eta_y(h')) \in E(H/R)$, so $\widetilde{\eta}(z,h)\widetilde{\eta}(z',h') \in E(H/R)$. Thus $\widetilde{\eta}$ is an automorphism and $\widetilde{\eta}^2 = id$, so we have a partition-preserving involution (if not the identity map) $\widetilde{\eta} : H/R \to H/R$ mapping each R-class $(x, h_x) \in H'_0$ to the R-class $(x, \eta_x(h_x)) \in H'_0$ of the same cardinality and mapping each R-class $(y, h_y) \in H'_1$ to the R-class $(y, \eta_y(h_y)) \in H'_1$ of the same cardinality.

Now consider the fixed points of $\widetilde{\varphi}$. These will be fixed R-classes (whose vertices are permuted) of φ. By equations (4.9) and (4.10), the fixed points of $\widetilde{\varphi}$ are:

$$((g_{x_0}, \chi_x(x)), (x, h_{x_0})) \text{ where } \alpha(g_{x_0}) = g_{x_0}, \eta_x(h_{x_0}) = h_{x_0}, \text{ and } x \in V(X) \tag{4.18}$$

$$((g_{y_0}, \chi_y(y)), (y, h_{y_0})) \text{ where } \beta(g_{y_0}) = g_{y_0}, \eta_y(h_{y_0}) = h_{y_0}, \text{ and } y \in V(Y) \tag{4.19}$$

By the remarks preceding this theorem, (4.18) and (4.19) can be represented as:

$$(g_{x_0}, \chi_x(x)) \times (x, h_{x_0}) \text{ where } \alpha(g_{x_0}) = g_{x_0}, \eta_x(h_{x_0}) = h_{x_0}, \text{ and } x \in V(X) \tag{4.20}$$

$$(g_{y_0}, \chi_y(y)) \times (y, h_{y_0}) \text{ where } \beta(g_{y_0}) = g_{y_0}, \eta_y(h_{y_0}) = h_{y_0}, \text{ and } y \in V(Y) \tag{4.21}$$

We call an R-class **even** (**odd**) if it has even (odd) cardinality.

Part II: Proof of Statement (1): Suppose G is nonbipartite and H is bipartite. If $G \times H$ is even palindromic with even palindromic partition preserving φ then:

- **G has an even 2-fold involution, or**
- **H is even palindromic, or**
- **H has an involution η that preserves partite sets where all fixed points of η are in the same partite set, and G has a 2-fold involution (α, β) where only one of α or β is fixed-point free.**

Suppose $G \times H$ is even palindromic with even palindromic involution φ with no fixed points. By the remarks in Part I, φ induces an automorphism $\widetilde{\varphi}$ of $(G \,\square\, H)/R$ that restricts to involutions $\widetilde{\varphi}_0$ and $\widetilde{\varphi}_1$ of $S(G/R) \,\square\, H_0'$ and $S(G/R) \,\square\, H_1'$, respectively. Because φ is even palindromic, any R-class fixed by φ must be even.

Consider $(\widetilde{\alpha}, \widetilde{\beta}) : G/R \to G/R$ and $\widetilde{\eta} : H/R \to H/R$ from Part I. These functions may have fixed points (R-classes), and if so, the cardinalities of those fixed points will either be even or odd. In what follows, we will lift the 2-fold involution of G/R to a

2-fold involution of G, and we will lift the involution of H/R to an involution of H. As we will see shortly, if any of the fixed R-classes of $\widetilde{\alpha}$, $\widetilde{\beta}$, or $\widetilde{\eta}$ are odd, the 2-fold involution of G or the involution of H may contain fixed points. We will consider several possible cases where fixed R-classes, if there are any, may be even or odd, and we will conclude from those cases that G admits an even 2-fold involution, or H is even palindromic, or G admits a 2-fold involution $(\widetilde{\alpha}', \widetilde{\beta}')$ where only one of $\widetilde{\alpha}'$ or $\widetilde{\beta}'$ has no fixed point and H admits an involution with fixed points in only one partite set.

First, we can lift $\widetilde{\alpha}$ to $\widetilde{\alpha}' : G \to G$ sending any R-class (g_x, x) bijectively to $(\alpha(g_x), x)$ (by Equation (4.14)) where $\widetilde{\alpha}'$ is an order-2 permutation. Similarly, $\widetilde{\beta}$ can be lifted to an order-2 permutation $\widetilde{\beta}' : G \to G$ sending any R-class (g_y, y) bijectively to $(\beta(g_y), y)$ (by Equation (4.15)). By Proposition 1.5.1, $(\widetilde{\alpha}', \widetilde{\beta}')$ is a 2-fold automorphism of G. Because $\widetilde{\alpha}'^2 = id$ and $\widetilde{\beta}'^2 = id$, $(\widetilde{\alpha}', \widetilde{\beta}')$ is a 2-fold involution of G, by definition.

Next, we can lift $\widetilde{\eta}_x$ to an order-2 permutation $\widetilde{\eta}'_x : H_0 \to H_0$, sending each R-class (x, h_x) bijectively to $(x, \eta_x(h_x))$ (by Equation (4.16)). We can also lift $\widetilde{\eta}_y$ to an order-2 permutation $\widetilde{\eta}'_y : H_1 \to H_1$, sending each R-class (y, h_y) bijectively to $(y, \eta_y(h_y))$ (by Equation (4.17)). Now we can define $\widetilde{\eta}' : H \to H$ such that $\widetilde{\eta}'(z, h) = \widetilde{\eta}'_x(z, h)$ for $(z, h) \in H_0$ and $\widetilde{\eta}'(z, h) = \widetilde{\eta}'_y(z, h)$ for $(z, h) \in H_1$. By Proposition 1.5.1, $\widetilde{\eta}'$ is an automorphism of H. Because $\widetilde{\eta}'_x$ and $\widetilde{\eta}'_y$ each have order 2, we can arrange for $\widetilde{\eta}'$ to be an involution of H.

Now we will consider several cases of the parity of fixed R-classes of $\widetilde{\alpha}$, $\widetilde{\beta}$, and $\widetilde{\eta}$ (if there are any fixed R-classes).

Case 1: G has no odd R-classes.

Given that $(\widetilde{\alpha}, \widetilde{\beta})$ is a 2-fold involution of G/R, this implies that $\widetilde{\alpha}$ and $\widetilde{\beta}$ fix only even R-classes, if they fix any at all. If $\widetilde{\alpha}$ or $\widetilde{\beta}$ (say $\widetilde{\alpha}$) fixes an even R-class

(g_x, x), we can arrange for $\widetilde{\alpha}'$ (defined above) to restrict to an order-2 fixed point free permutation of (g_x, x) that swaps vertices of the even class pairwise. Similarly, we can arrange for $\widetilde{\beta}'$ to be an order-2 fixed point free permutation, swapping vertices of any even fixed R-class of $\widetilde{\beta}$. Thus $(\widetilde{\alpha}', \widetilde{\beta}')$ is an even 2-fold involution of G.

Case 2: H has no odd R-classes.

Given that $\widetilde{\eta}$ is an involution of H/R, this implies $\widetilde{\eta}$ fixes only even R-classes, if it fixes any at all. If $\widetilde{\eta}$ fixes an even R-class, say $(x, h_x) \in H_0$, we can arrange for $\widetilde{\eta}'_x$ (defined above) to restrict to an order-2 fixed point free permutation of the even set (x, h_x). The same is true if $\widetilde{\eta}$ fixes an even class in H_1, so we can arrange for $\widetilde{\eta}'_y$ (defined above) to restrict to an order-2 fixed point free permutation of any fixed even set of $\widetilde{\eta}_y$. Now $\widetilde{\eta}'$ (defined above) is an even palindromic involution of H, so H is even palindromic.

Case 3: G and H both have at least one odd R-class.

Given that $(\widetilde{\alpha}, \widetilde{\beta})$ is a 2-fold involution of G/R, it is either the case that neither $\widetilde{\alpha}$ nor $\widetilde{\beta}$ fixes any odd R-class (and swaps them pairwise with other odd R-classes of the same cardinality) or at least one of $\widetilde{\alpha}$ or $\widetilde{\beta}$ fixes an odd R-class. Similarly, given that $\widetilde{\eta}$ is an involution of H/R, it is either the case that $\widetilde{\eta}$ does not fix any odd R-class or at least one of $\widetilde{\eta}_x$ or $\widetilde{\eta}_y$ fixes an odd R-class.

Suppose first that neither $\widetilde{\alpha}$ nor $\widetilde{\beta}$ fix any odd R-classes. We can lift $\widetilde{\alpha}$ to $\widetilde{\alpha}'$ (defined above), and if $\widetilde{\alpha}$ fixes any even R-classes, we can arrange for $\widetilde{\alpha}'$ to restrict to an order-2 fixed point free permutation of the even R-class. We can also lift $\widetilde{\beta}$ to $\widetilde{\beta}'$ (defined above). If $\widetilde{\beta}$ fixes any even R-classes, we can arrange for $\widetilde{\beta}'$ to restrict to an order-2 fixed point free permutation of the even R-class. Now $(\widetilde{\alpha}', \widetilde{\beta}')$ is an even 2-fold involution of G. Similarly, suppose $\widetilde{\eta}$ does not fix any odd R-classes. We can lift $\widetilde{\eta}_y$ to $\widetilde{\eta}'_y$ (defined above), and if $\widetilde{\eta}_y$ fixes any even R-classes, we can arrange for $\widetilde{\eta}'_y$ to restrict to an order-2 fixed point free permutation of the even R-class. We can also lift $\widetilde{\eta}_x$ to $\widetilde{\eta}'_x$ (defined above). If $\widetilde{\eta}_x$ fixes any even R-classes, we can arrange for $\widetilde{\eta}'_x$ to restrict to an order-2 fixed point free permutation of the even R-class. Now $\widetilde{\eta}'$ (as defined above) is an even palindromic involution of H.

Now suppose that at least one of $\widetilde{\alpha}$ and $\widetilde{\beta}$ fix an odd R-class, and suppose $\widetilde{\eta}$ fixes an odd R-class. We will first show by contradiction that $\widetilde{\alpha}$ and $\widetilde{\beta}$ cannot both fix odd R-classes. Suppose to the contrary that $\widetilde{\alpha}$ fixes the odd R-class (g_x, x) and $\widetilde{\beta}$ fixes the odd R-class (g_y, y). We have assumed $\widetilde{\eta}$ fixes an odd R-class, say (z, h_z), and suppose (z, h_z) is in H'_1, so $\widetilde{\eta}_y$ fixes (z, h_z). By the remarks in Part I, we have $\beta(g_y) = g_y$, $\widetilde{\eta}_y(h_z) = h_z$. By Equation (4.11), we have

$$|(g_y, y)| \cdot |(z, h_z)| = |(g_y, \chi_y(z))| \cdot |(y, h_z)|. \tag{4.22}$$

Because $|(g_y, y)|$ and $|(z, h_z)|$ are odd, we get that $|(g_y, \chi_y(z))|$ is odd, so $(g_y, \chi_y(z)) \times (z, h_z)$ is an odd R-class of $G \times H$. However, φ fixes this odd R-class, which implies that φ has a fixed point, which contradicts that φ is even palindromic. Because $\widetilde{\alpha}$ also fixes an odd R-class, we would reach the same conclusion if (z, h_z) had been in H'_0. Thus it cannot be the case that both $\widetilde{\alpha}$ and $\widetilde{\beta}$ fix odd R-classes, so only one of $\widetilde{\alpha}$ or $\widetilde{\beta}$ fixes odd R-classes, say $\widetilde{\beta}$ with a fixed odd R-class (g_y, y). Because we have assumed that $\widetilde{\eta}$ fixes an odd R-class, such an R-class will either be fixed by $\widetilde{\eta}_x$ or $\widetilde{\eta}_y$. Suppose towards a contradiction that $\widetilde{\eta}_y$ fixes an odd R-class (z, h_z). As we have previously concluded using Equation (4.22), this would imply that $(g_y, \chi_y(z)) \times (z, h_z)$ is a fixed odd R-class of $G \times H$ which implies that there is a fixed point in φ, a contradiction. Thus an R-class fixed by $\widetilde{\eta}_y$ (if there are any) must be even. Further, it must be that $\widetilde{\eta}_x$ fixes an odd R-class, so $\widetilde{\eta}$ fixes odd R-classes only in H'_0. Now $\widetilde{\alpha}$ and $\widetilde{\eta}_y$ fix no odd R-classes while $\widetilde{\beta}$ and $\widetilde{\eta}_x$ each fix at least one odd R-class.

We can lift $\widetilde{\alpha}$ to $\widetilde{\alpha}'$ (defined above), and if $\widetilde{\alpha}$ fixes any even R-classes, we can arrange for $\widetilde{\alpha}'$ to restrict to an order-2 fixed point free permutation of the even R-

class. We can also lift $\widetilde{\beta}$ to $\widetilde{\beta}'$ (defined above). If $\widetilde{\beta}$ fixes any even R-classes, we can arrange for $\widetilde{\beta}'$ to restrict to an order-2 fixed point free permutation of the even R-class. When $\widetilde{\beta}$ fixes an odd R-class, we can arrange for $\widetilde{\beta}'$ to have a fixed point on the odd R-class. We can lift $\widetilde{\eta}_y$ to $\widetilde{\eta}'_y$ (defined above), and if $\widetilde{\eta}_y$ fixes any even R-classes, we can arrange for $\widetilde{\eta}'_y$ to restrict to an order-2 fixed point free permutation of the even R-class. We can also lift $\widetilde{\eta}_x$ to $\widetilde{\eta}'_x$ (defined above). If $\widetilde{\eta}_x$ fixes any even R-classes, we can arrange for $\widetilde{\eta}'_x$ to restrict to an order-2 fixed point free permutation of the even R-class. When $\widetilde{\eta}'_x$ fixes an odd R-class, we can arrange for $\widetilde{\eta}'_x$ to have a fixed point on the odd R-class. Now $\widetilde{\eta}'$ (as defined above) is an involution of H with fixed points only in the partite set H_0, and $(\widetilde{\alpha}', \widetilde{\beta}')$ is a 2-fold involution of G such that only $\widetilde{\beta}'$ has fixed points.

Part III: Proof of Statement (2): Suppose G is nonbipartite and H is bipartite. If $G \times H$ is odd palindromic, then G admits an odd 2-fold involution and H is odd palindromic.

Suppose $G \times H$ is odd palindromic. Then there is an odd palindromic involution φ of $G \times H$ with fixed point (x_0, y_0). Without loss of generality, let $y_0 \in V(X \,\square\, H_X)$. Then φ fixes the R-class Z that contains (x_0, y_0), which, by Equation (4.20), has form $Z = (g_{x_0}, \chi_x(x_0)) \times (x_0, h_{x_0})$, where $\alpha(g_{x_0}) = g_{x_0}$ and $\eta_x(h_{x_0}) = h_{x_0}$. Because φ fixes Z, we know that Z has odd cardinality. Thus $(g_{x_0}, \chi_x(x_0))$ is an odd R-class of G/R, and (x_0, h_{x_0}) is an odd R-class of H/R. Note that $(g_{x_0}, \chi_x(x_0))$ is a fixed point of $\widetilde{\alpha}$ and (x_0, h_{x_0}) is a fixed point of $\widetilde{\eta}$.

Suppose $\widetilde{\eta}$ fixes another odd R-class, say (x_1, h_{x_1}), without loss of generality. Then $\widetilde{\eta}_x$ has a fixed point at (x_1, h_{x_1}), meaning that $\eta_x(h_{x_1}) = h_{x_1}$, and by Equation (4.11), we have

$$|(g_{x_0}, \chi_x(x_0))| \cdot |(x_1, h_{x_1})| = |(g_{x_0}, \chi_x(x_1))| \cdot |(x_0, h_{x_1})|,$$

where $|(g_{x_0}, \chi_x(x_0))|$ is odd and $|(x_1, h_{x_1})|$ is odd, so $|(g_{x_0}, \chi_x(x_1))|$ and $|(x_0, h_{x_1})|$ are odd. Then, $(g_{x_0}, \chi_x(x_1)) \times (x_1, h_{x_1})$ and $(g_{x_0}, \chi_x(x_0)) \times (x_0, h_{x_1})$ are odd R-classes of $G \times H$ that are fixed by φ. However, $Z = (g_{x_0}, \chi_x(x_0)) \times (x_0, h_{x_0})$ is the only such R-class, so $\chi_x(x_1) = \chi_x(x_0)$ and $h_{x_1} = h_{x_0}$, meaning $(x_1, h_{x_1}) = (x_0, h_{x_0})$. So, we can conclude that (x_0, h_{x_0}) is the only odd R-class of H/R that is fixed by $\widetilde{\eta}_x$.

We will show that $(g_{x_0}, \chi_x(x_0))$ is the only odd R-class fixed by $\widetilde{\alpha}$ (note that this would be $\widetilde{\beta}$ if y_0 was in $V(Y \square H_Y)$). Suppose $\widetilde{\alpha}$ had another odd fixed point at $(g_{x_1}, \chi_x(x_1))$, which implies that $\alpha(g_{x_1}) = g_{x_1}$. Then, by Equation (4.11), we have

$$|(g_{x_1}, \chi_x(x_1))| \cdot |(x_0, h_{x_0})| = |(g_{x_1}, \chi_x(x_0))| \cdot |(x_1, h_{x_0})|$$

where $|(g_{x_1}, \chi_x(x_1))|$ and $|(x_0, h_{x_0})|$ are odd, so $|(g_{x_1}, \chi_x(x_0))|$ and $|(x_1, h_{x_0})|$ are odd. Then $(g_{x_1}, \chi_x(x_1)) \times (x_1, h_{x_0})$ is odd and $(g_{x_1}, \chi_x(x_0)) \times (x_0, h_{x_0})$ is odd and are fixed by φ. However, $(g_{x_0}, \chi_x(x_0)) \times (x_0, h_{x_0})$ is the only odd R-class fixed by φ, so $g_{x_0} = g_{x_1}$ and $\chi_x(x_0) = \chi_x(x_1)$, thus $(g_{x_1}, \chi_x(x_1)) = (g_{x_0}, \chi_x(x_0))$. This implies that $(g_{x_0}, \chi_x(x_0))$ is the only odd R-class fixed by $\widetilde{\alpha}$.

Therefore, we can lift $\widetilde{\alpha}$ to a permutation $\widetilde{\alpha}' : G \to G$ sending any R-class (g_x, x) bijectively to $(\alpha(g_x), x)$ (by Equation (4.14)). When $\widetilde{\alpha}$ fixes the odd R-class $(g_{x_0}, \chi_x(x_0))$, we can arrange for the order-2 permutation $\widetilde{\alpha}'$ to fix exactly one vertex on the odd R-class $(g_{x_0}, \chi_x(x_0))$. Likewise, we can lift $\widetilde{\beta}$ to a permutation $\widetilde{\beta}' : G \to G$ sending any R-class (g_y, y) bijectively to $(\beta(g_y), y)$ (by Equation (4.15)). Because $\widetilde{\beta}$ is an order 2 permutation of an odd cardinality set of R-classes, $\widetilde{\beta}$ will fix one or more odd R-classes. When $\widetilde{\beta}$ fixes one or more odd R-class $(g_{y_0}, \chi_y(y_0))$, we can arrange for the order-2 permutation $\widetilde{\beta}'$ to fix a vertex on the odd R-class $(g_{y_0}, \chi_y(y_0))$. As we showed in Part II, $(\widetilde{\alpha}', \widetilde{\beta}')$ is a 2-fold involution, and because $\widetilde{\alpha}'$ and $\widetilde{\beta}'$ each have exactly one fixed point, $(\widetilde{\alpha}', \widetilde{\beta}')$ is an odd 2-fold involution of G.

Similarly, we can lift $\widetilde{\eta}_x$ to an order-2 permutation $\widetilde{\eta}_x' : H_0 \to H_0$, sending each

R-class (x, h_x) bijectively to $(x, \eta_x(h_x))$ (by Equation (4.16)). We can also lift $\widetilde{\eta}_y$ to an order-2 permutation $\widetilde{\eta}'_y : H_1 \to H_1$, sending each R-class (y, h_y) bijectively to $(y, \eta_y(h_y))$ (by Equation (4.17)). We can define $\widetilde{\eta}' : H \to H$ such that $\widetilde{\eta}'(z, h) = \widetilde{\eta}'_x(z, h)$ for $(z, h) \in H_0$ and $\widetilde{\eta}'(z, h) = \widetilde{\eta}'_y(z, h)$ for $(z, h) \in H_1$. We can let $\widetilde{\eta}'$ have only one fixed vertex on the odd fixed class (x_0, h_{x_0}), and no fixed points on any other fixed (even) R-class. As we showed in Part II, $\widetilde{\eta}'$ is an involution, and because it has only one fixed point, $\widetilde{\eta}'$ is an odd palindromic involution. Thus H is odd palindromic.

Thus G has an odd 2-fold involution $(\widetilde{\alpha}', \widetilde{\beta}')$ and H is odd palindromic. □

4.5 Discussion on the Case of Two Bipartite Factors

Although this dissertation focuses only on connected products, it is worth briefly noting that there are some pre-existing theorems that may help us make sense of the complexity of a palindromic direct product of two bipartite factors, which results in a disconnected product with exactly two components. These components may either be isomorphic or nonisomorphic, so there are two cases to consider here.

The following theorem by Hammack characterizes conditions under which the direct product of two bipartite graphs results in two isomorphic components of the product.

Theorem 4.5.1 ([6], Hammack). *Suppose G and H are connected bipartite graphs. The two components of $G \times H$ are isomorphic if and only if at least one of G or H admits an automorphism that interchanges its partite sets.*

This theorem implies that if a direct palindromic product is disconnected with two isomorphic components, meaning that there is an even palindromic involution that interchanges the components, then at least one of G or H admits an automorphism that interchanges its partite sets. In an ideal scenario, this automorphism would be an involution, which would mean that one of G or H is even palindromic. However, we are not guaranteed that the automorphism will have order two, so in general, it is not necessarily the case that an even palindromic direct product with two isomorphic components requires that at least one of its factors is even palindromic. Any conjecture related to a direct palindromic product of two bipartite factors will need to reflect this situation.

The above scenario detailed complications with a direct palindromic product that has two isomorphic components. The other possible scenario is that the components are nonisomorphic, so a palindromic involution of the product will preserve the components. Each component will be bipartite, so a palindromic involution will either preserve the partitions of each component, reverse the partitions of each component, or preserve the partition of one component and reverse the partition of the other component. Because the product is bipartite, this problem is reminiscent of the challenges presented in the case where one factor is bipartite and the other factor is nonbipartite. In that case, we were presented with a scenario where the partite sets in the product were preserved by the palindromic involution, restricted to each partite set, and we ultimately showed that a 2-fold involution existed in the nonbipartite factor. This occurred because the product was bipartite, giving the product's partite sets their own freedom to behave in their own separate ways, and we showed that those two separate behaviors formed a 2-fold involution in the nonbipartite factor.

With this in mind, returning to the case where both factors are bipartite, not only do we have a bipartite product, but we have two components, each of which is bipartite. If we could conclude that the partitions of each component are preserved, then we may be in a place to reasonably conjecture that at least one of the factors admits a 2-fold involution, but it could be more complex than that, especially given the differences between even versus odd palindromic involutions in the product and what that could mean in the factors. Moreover, this ideal scenario would require elimination of the cases where the partite sets of both components are reversed, or one component's partite sets are preserved while the other's is reversed, which may be difficult, involved, and not necessarily possible.

At this time, we leave the case where both factors are bipartite to conjecture.

Conjecture 4.5.1. *Suppose G and H are connected and bipartite. Then:*

1. *$G \times H$ is even palindromic if and only if*
   - *G or H admits an automorphism that interchanges its partite sets or*
   - *G or H admits an even 2-fold involution*
2. *$G \times H$ is odd palindromic if and only if G and H admit odd 2-fold involutions.*

CHAPTER 5

THE LEXICOGRAPHIC PRODUCT OF PALINDROMIC GRAPHS

5.1 Summary of Project

We are left with just one more standard associative graph product: the lexicographic product.

The following two theorems are the main results of the paper.

Theorem 5.1.1. *$G \circ H$ is even palindromic if and only if G or H is even palindromic.*

Theorem 5.1.2. *$G \circ H$ is odd palindromic if and only if G and H are odd palindromic.*

This project was included in a paper submitted in *The Art of Discrete and Applied Mathematics* [9].

5.2 Preliminary Information

The structure of the lexicographic product is unusual, and as a result, the lexicographic product has fewer desirable theorems for us to utilize. Because the definition of the edge set of the lexicographic product resembles only a partial definition of the edge set of the strong product, the lexicographic product is not commutative nor does it follow the standard factoring methods that are presented in the Cartesian, strong, and direct products. As a result, we do not have a theorem analogous to Theorem 1.5.1, Theorem 1.5.2, and Theorem 4.2.1.

With this in mind, we will need to take a different approach. We can utilize a specific structure within the lexicographic product called an ***S*- or *R*-tower**. If X is an S-class, then the subgraph $\langle X \rangle \circ H$ of $G \circ H$ is called the ***S*-tower over *X***, or simply an ***S*-tower**. Note that an S-tower $\langle X\rangle \circ H$ is isomorphic to $K_{|X|} \circ H$. If X is an R-class, then the subgraph $\langle X\rangle \circ H$ of $G \circ H$ is called the ***R*-tower over *X***, or simply an ***R*-tower**. Note that an R-tower $\langle X\rangle \circ H$ is isomorphic to $D_{|X|} \circ H$.

Observe that S-classes and R-classes are externally related in G, because any two vertices in an S-class must have the same closed neighborhood, and any two vertices in an R-class must have the same open neighborhood, so any vertex adjacent to a vertex in an S- or R-class will be adjacent to all vertices in an S- or R-class. Similarly, S-towers and R-towers are externally related in $G \circ H$ because if $xx' \in E(G)$ where x and x' are in different S- or R-classes, then by definition of the lexicographic product, $(x,y)(x',y') \in E(G\circ H)$ for every $y, y' \in V(H)$. We show that S-towers (R-towers) are mapped bijectively to S-towers (R-towers) if H is connected (disconnected), which has a similar theme to Theorem 1.5.1, Theorem 1.5.2, and Theorem 4.2.1. After showing that towers map to towers, we can very carefully proceed with a proof that is similar to those presented in prior chapters.

5.3 Major Theorems

Before proving our main result, we need the following lemma. The proof is an immediate consequence of the definitions.

Lemma 5.3.1. *If $\alpha \in \mathrm{Aut}(G)$ and $\beta \in \mathrm{Aut}(H)$, then the map $(\alpha,\beta) : (x,y) \mapsto (\alpha(x),\beta(y))$ is an automorphism of $G \circ H$.*

Lemma 5.3.1 and the lexicographic product seen in Figure 3 suggest that if both factors are odd palindromic, then the product itself will be odd palindromic. Indeed, we have the following result.

Proposition 5.3.1. *If G or H is even palindromic, then $G \circ H$ is even palindromic. If G and H are odd palindromic, then $G \circ H$ is odd palindromic.*

Proof. Let G be even palindromic. Then there exists an even palindromic involution $\alpha : G \to G$. Then $(x, y) \mapsto (\alpha(x), y)$ is an involution of $G \circ H$, by Lemma 5.3.1, and it has no fixed points. So $G \circ H$ is even palindromic.

Now let H be even palindromic. Then there exists an even palindromic involution $\beta : H \to H$. Then $(x, y) \mapsto (x, \beta(y))$ is an involution of $G \circ H$, by Lemma 5.3.1, and it has no fixed points. So $G \circ H$ is even palindromic.

Let G and H be odd palindromic. Then G has an involution α with exactly one fixed point, $x_0 = \alpha(x_0)$, and H has an involution β with exactly one fixed point, $y_0 = \beta(y_0)$. So $(x, y) \mapsto (\alpha(x), \beta(y))$ is an involution of $G \circ H$, by Lemma 5.3.1, that has exactly one fixed point at (x_0, y_0). So $G \circ H$ is odd palindromic. $\square$

The following proposition shows that S-towers are mapped to S-towers under any automorphism of $G \circ H$ when H is connected.

Proposition 5.3.2. *Suppose G is arbitrary and H is connected. If φ is an automorphism of $G \circ H$, then φ maps S-towers to S-towers.*

Proof. Let φ be an automorphism of $G \circ H$ for an arbitrary G and a connected H. To show that φ maps S-towers to S-towers, it suffices to show that each S-tower is connected and that two adjacent vertices in the same S-tower $\langle X \rangle \circ H$ map to two adjacent vertices in the same S-tower $\langle X' \rangle \circ H$.

First, we will show that S-towers are connected. Let X be an S-class of G. If $|X| = 1$, then $\langle X \rangle \circ H$ is isomorphic to H, where H is connected, so $\langle X \rangle \circ H$ is connected. Suppose $|X| \geq 2$. Now $\langle X \rangle \circ H = K_{|X|} \circ H$ is connected by definition of the lexicographic product. Thus all S-towers are connected.

Let (x, y) and (x', y') be adjacent vertices in the same S-tower. Suppose $\varphi(x, y) = (x'', y'')$ and $\varphi(x', y') = (x''', y''')$. See Figure 3. Note that, as mentioned earlier, $\langle X \rangle \circ H$ is externally related, so $\varphi(\langle X \rangle \circ H)$ is externally related as well.

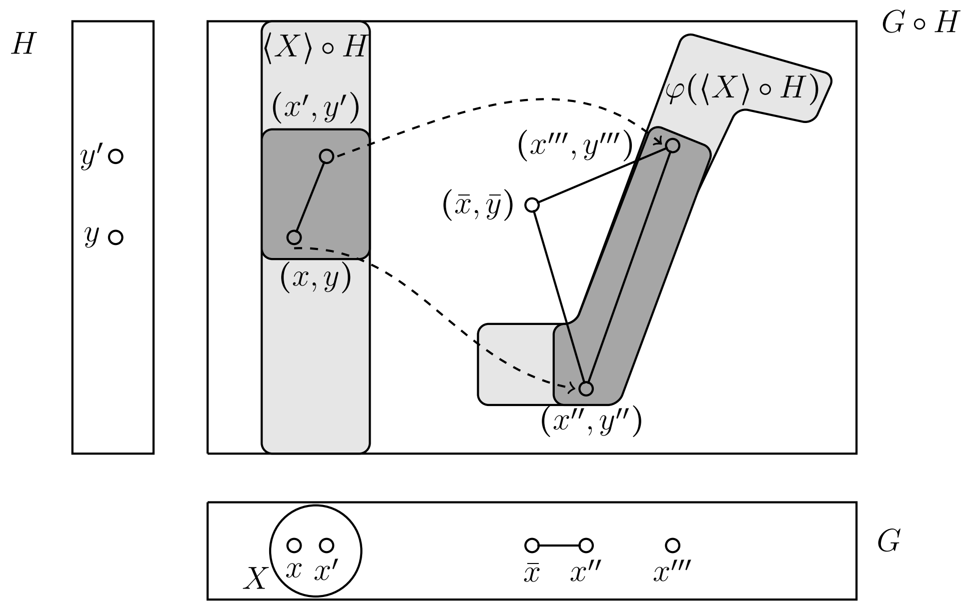


Fig. 6. Visual Aid for Proof of Proposition 5.3.2 – the image of an S-tower is an S-tower.

We need to show that (x'', y'') and (x''', y''') are in the same S-tower, which means we need to show that $N_G[x''] = N_G[x''']$. This is trivially true if $x'' = x'''$, so for the remainder of the proof, we can assume that $x'' \neq x'''$. First, we will show that $N_G[x''] \subseteq N_G[x''']$. Let $\bar{x} \in N_G[x'']$. We will show that $\bar{x} \in N_G[x''']$.

If $\bar{x} = x'''$, then clearly $\bar{x} \in N_G[x''']$, so assume $\bar{x} \in N_G[x'']$ and $\bar{x} \neq x'''$.

Because $\bar{x}x'' \in E(G)$, it follows that $(\bar{x}, \bar{y})(x'', y'') \in E(G \circ H)$ for some $\bar{y} \in V(H)$. Consider the case where $(\bar{x}, \bar{y}) \notin \varphi(\langle X \rangle \circ H)$. Because $\varphi(\langle X \rangle \circ H)$ is externally related, $(\bar{x}, \bar{y})(x'', y'') \in E(G \circ H)$ implies that $(\bar{x}, \bar{y})(x''', y''') \in E(G \circ H)$. It follows that $\bar{x}x''' \in E(G)$, so $\bar{x} \in N_G[x''']$.

Thus we must consider the case where $(\bar{x}, \bar{y}) \in \varphi(\langle X \rangle \circ H)$ for all $\bar{y} \in V(H)$. We claim there exists a $\bar{y} \in V(H)$ for which the pre-image $(\bar{x}', \bar{y}')$ of $(\bar{x}, \bar{y})$ does not have the form $(x', \bar{y}')$, that is, $\bar{x}' \neq x'$. Suppose towards a contradiction that for all $\bar{y} \in V(H)$, $(\bar{x}, \bar{y})$ has a pre-image of the form $(x', \bar{y}')$. Note that the pre-image of

(x''', y''') is (x', y'). Then there are $|V(H)|+1$ vertices whose pre-image is of the form $(x', \bar{y}')$ which is a contradiction because there are only $|V(H)|$ such vertices. Thus there must exist a $\bar{y} \in V(H)$ for which the pre-image $(\bar{x}', \bar{y}')$ of $(\bar{x}, \bar{y})$ does not have the form $(x', \bar{y}')$, that is, $\bar{x}' \neq x'$. Now $\bar{x}' \in X$ where $\bar{x}'x' \in E(G)$, which implies that $(\bar{x}', \bar{y}')(x', y') \in E(G \circ H)$. Thus $(\bar{x}, \bar{y})(x''', y''') \in E(G \circ H)$ where $\bar{x} \neq x'''$, so $\bar{x}x''' \in E(G)$, which implies that $\bar{x} \in N_G[x''']$, so $N_G[x''] \subseteq N_G[x''']$. Similarly, $N_G[x'''] \subseteq N_G[x'']$, so $N_G[x''] = N_G[x''']$. Now $N_G[x''] = N_G[x''']$ implies that x'' and x''' are in the same S-class, so (x'', y'') and (x''', y''') are in the same S-tower. Thus the image of an S-tower under φ is itself an S-tower. $\square$

The following proposition shows that R-towers are mapped to R-towers under any automorphism of $G \circ H$ where H is disconnected.

Proposition 5.3.3. *Suppose G is arbitrary and H is disconnected. If φ is an automorphism of $G \circ H$, then φ maps R-towers to R-towers.*

Proof. Let φ be an automorphism of $G \circ H$ for an arbitrary G and a disconnected H. To show that φ maps R-towers to R-towers, it suffices to show that each R-tower is disconnected and that two nonadjacent vertices in the same R-tower $\langle X \rangle \circ H$ map to two nonadjacent vertices in the same R-tower $\langle X' \rangle \circ H$. This is sufficient for the following reason. Suppose some vertex (x, y) is in $\langle X \rangle \circ H$ and $\varphi(x, y) \in \langle X' \rangle \circ H$. Let (x', y') be any other vertex of $\langle X \rangle \circ H$. If (x', y') is in a different component of $\langle X \rangle \circ H$ than (x, y), then (x, y) and (x', y') are nonadjacent and both must map to the same R-tower $\langle X' \rangle \circ H$. However, if (x, y) and (x', y') are in the same component of $\langle X \rangle \circ H$, then there must be a vertex (x'', y'') in a different component of $\langle X \rangle \circ H$. Then (x, y) and (x'', y'') must map to $\langle X' \rangle \circ H$ while (x', y') and (x'', y'') must map to $\langle X' \rangle \circ H$, so (x, y) and (x', y') must map to $\langle X' \rangle \circ H$.

First we will show that R-towers are disconnected. Let X be an R-class of G.

If $|X| = 1$, then $\langle X\rangle \circ H$ is isomorphic to H, where H is disconnected, so $\langle X\rangle \circ H$ is disconnected. Suppose $|X| \geq 2$. Then $\langle X\rangle \circ H = D_{|X|} \circ H$ is $|X|$ disjoint copies of the form $\{v\} \circ H$ for each $v \in X$, so $\langle X\rangle \circ H$ is disconnected. Thus all R-towers are disconnected.

Let (x, y) and (x', y') be nonadjacent vertices in $G \circ H$ in the same R-tower $\langle X\rangle \circ H$ in $G \circ H$. Suppose $\varphi(x, y) = (x'', y'')$ and $\varphi(x', y') = (x''', y''')$. We need to show that (x'', y'') and (x''', y''') are in the same R-tower, so we need to show that $N_G(x'') = N_G(x''')$. First, we will show that $N_G(x'') \subseteq N_G(x''')$. Let $\bar{x} \in N_G(x'')$. We need to show that $\bar{x} \in N_G(x''')$. Because $\bar{x}x'' \in E(G)$, then $(\bar{x}, \bar{y})(x'', y'') \in E(G \circ H)$ for some $\bar{y} \in V(H)$. It is either the case that $(\bar{x}, \bar{y}) \in \varphi(\langle X\rangle \circ H)$ or $(\bar{x}, \bar{y}) \notin \varphi(\langle X\rangle \circ H)$. We will show it is impossible that $(\bar{x}, \bar{y}) \in \varphi(\langle X\rangle \circ H)$. First, we need the following claim.

Claim: If $(\bar{x}, \bar{y}) \in \varphi(\langle X\rangle \circ H)$, then for all $\tilde{y} \in V(H)$, $(\bar{x}, \tilde{y}) \in \varphi(\langle X\rangle \circ H)$.

Suppose towards a contradiction that some $(\bar{x}, \tilde{y}) \notin \varphi(\langle X\rangle \circ H)$. Because $\bar{x}x'' \in E(G)$, it follows that $(\bar{x}, \tilde{y})(x'', y'') \in E(G \circ H)$. Because the R-tower $\langle X\rangle \circ H$ is externally related, $\varphi(\langle X\rangle \circ H)$ is externally related as well, so it follows that $(\bar{x}, \tilde{y})(\bar{x}, \bar{y}) \in E(G \circ H)$. This implies that $\tilde{y}\bar{y} \in E(H)$ for all $\bar{y} \in V(H)$, so H is connected, which is a contradiction. Thus it must be that for all $\tilde{y} \in V(H)$, $(\bar{x}, \tilde{y}) \in \varphi(\langle X\rangle \circ H)$.

Because $(\bar{x}, \tilde{y}) \in \varphi(\langle X\rangle \circ H)$ for all $\tilde{y}$, the pre-images of such are in $\langle X\rangle \circ H$. Consider the pre-image $(\bar{x}', \bar{y}')$ of $(\bar{x}, \bar{y})$. Because $\bar{x}' \in X$ and $(\bar{x}', \bar{y}')(x, y) \in E(G \circ H)$, then $\bar{x}'x \in E(G)$ or $\bar{x}' = x$. If $\bar{x}'x \in E(G)$, this contradicts that $\langle X\rangle$ is the edgeless graph $D_{|X|}$, so it must be that $\bar{x}' = x$. Similarly, it must be that for all $\tilde{y} \in V(H)$, the pre-image of $(\bar{x}, \tilde{y})$ is of the form $(x, \tilde{y})$. Now because $(\bar{x}, \tilde{y})(x'', y'') \in E(G \circ H)$ for all $\tilde{y} \in V(H)$, we have that $(x, \tilde{y}')(x, y) \in E(G \circ H)$ for all $\tilde{y}' \in V(H)$. This implies that $\tilde{y}'y \in E(H)$ for all $\tilde{y}' \in V(H)$ which implies that H is connected, a contradiction.

Thus it is impossible that $(\bar{x}, \bar{y}) \in \varphi(\langle X \rangle \circ H)$.

Now it is only possible that $(\bar{x}, \bar{y}) \notin \varphi(\langle X \rangle \circ H)$. Since $\varphi(\langle X \rangle \circ H)$ is externally related, $(\bar{x}, \bar{y})(x'', y'') \in E(G \circ H)$ implies that $(\bar{x}, \bar{y})(x''', y''') \in E(G \circ H)$, so $\bar{x}x''' \in E(G)$. Thus $\bar{x} \in N_G(x''')$, so $N_G(x'') \subseteq N_G(x''')$. Similarly, $N_G(x''') \subseteq N_G(x'')$, so $N_G(x'') = N_G(x''')$.

Now $N_G(x'') = N_G(x''')$ means that x'' and x''' are in the same R-class, so (x'', y'') and (x''', y''') are in the same R-tower. Thus the image of the R-tower over X in $G \circ H$ is itself an R-tower. □

The following two propositions, corollary, and two lemmas will be used in proving the main theorem. Recall that D_n is the edgeless graph on n vertices.

Proposition 5.3.4. *If $D_n \circ H$ is odd palindromic where H is arbitrary, then D_n and H are odd palindromic.*

Proof. Suppose $D_n \circ H$ is odd palindromic, and let φ be an odd palindromic involution of $D_n \circ H$.

First, observe that $D_n \circ H$ must have an odd number of vertices, so D_n and H must have an odd number of vertices. Now, because D_n has an odd number of vertices, any arbitrary involution of $V(D_n)$ with exactly one fixed point is an odd palindromic involution, so D_n is odd palindromic.

Next, we want to show that H is odd palindromic. Observe that $D_n \circ H$ is the disjoint union of n copies of H, each of the form $\{v\} \circ H$, for a vertex v of D_n. Let (x_0, y_0) be the fixed point of φ. Let C be the component of $G \circ H$ that contains (x_0, y_0). Then φ restricts to an odd palindromic involution of C. We claim that the number k of components of H that are isomorphic to C is odd. Observe that $D_n \circ H$ (with n odd) has kn components isomorphic to C, and φ fixes one of these components and interchanges the others in pairs. Hence kn is odd, so k is odd. Let

$C' \ncong C$ be a component of H that is not even palindromic. We claim that H has an even number j of components isomorphic to C'. Observe that $D_n \circ H$ has jn components isomorphic to C', and φ interchanges these components in pairs. Hence jn is even, so j must be even. Thus we can construct an odd palindromic involution β of H as follows: β restricts to an odd palindromic involution on one copy of C and interchanges the others in pairs. If H has any even palindromic components, then β restricts to an even palindromic involution on each one of them. Finally, β interchanges the remaining components in pairs. Thus H is odd palindromic. □

The following two propositions, corollary, and two lemmas will be used in proving the main theorem. Recall that D_n is the edgeless graph on n vertices.

Proposition 5.3.5. *If $D_n \circ H$ is even palindromic where H is arbitrary, then D_n or H is even palindromic.*

Proof. Suppose $D_n \circ H$ is even palindromic, and let φ be an even palindromic involution of $D_n \circ H$.

Observe that $D_n \circ H$ must have an even number of vertices, so D_n or H (or both) must have an even number of vertices. If D_n has an even number of vertices, then D_n is even palindromic.

Now suppose D_n is not even palindromic, so n is odd. We want to show that H is even palindromic. Again, $D_n \circ H$ is the disjoint union of n copies of H. Suppose φ fixes some component C in $D_n \circ H$. Because φ is fixed point free, C must be even palindromic, so C in H is even palindromic. Suppose there exists a component C' in H that is not even palindromic and H has k components isomorphic to C'. We claim that k is even. Observe that $D_n \circ H$ (with n odd) has kn components isomorphic to C', and φ interchanges these in pairs. Hence kn is even, so k must be even. Thus H is composed of components that are either even palindromic or can be interchanged in pairs by an involution, so H is even palindromic. $\square$

Corollary 5.3.1. *If $K_n \circ H$ is odd palindromic, then K_n and H are odd palindromic. If $K_n \circ H$ is even palindromic, then K_n or H is even palindromic.*

Proof. Let $K_n \circ H$ be odd palindromic. Then $\overline{K_n \circ H} = \overline{K_n} \circ \overline{H} = D_n \circ \overline{H}$ is odd palindromic. By Proposition 5.3.4, D_n and $\overline{H}$ are odd palindromic. Then $\overline{D_n} = K_n$ and $\overline{\overline{H}} = H$ are odd palindromic.

Let $K_n \circ H$ be even palindromic. Then $\overline{K_n \circ H} = \overline{K_n} \circ \overline{H} = D_n \circ \overline{H}$ is even palindromic. By Proposition 5.3.5, D_n or $\overline{H}$ is even palindromic. If D_n is even palindromic, then $\overline{D_n} = K_n$ is even palindromic. If $\overline{H}$ is even palindromic, then $\overline{\overline{H}} = H$ is even palindromic. Thus K_n or H is even palindromic. $\square$

Lemma 5.3.2. *Let H be connected. Any involution φ of $G \circ H$ induces an automorphism α' of G/S defined as $\alpha'(X) = X'$, where $\varphi(\langle X \rangle \circ H) = \langle X' \rangle \circ H$. Hence $|\alpha'(X)| = |X|$ for each $X \in G/S$. Moreover, any permutation α of $V(G)$ with the property that $\alpha(X) = \alpha'(X)$ for each X in G/S is an automorphism of G.*

Proof. Because H is connected, φ permutes S-towers by Proposition 5.3.2, so it follows that φ induces a corresponding permutation α' of S-classes which is clearly bijective. Suppose α is a permutation of $V(G)$ with the property that $\alpha(X) = \alpha'(X)$ for each X in G/S, that is, $\alpha(x) = x'$ for $x \in X$ and $x' \in \alpha'(X)$, so α is bijective. Let $xx' \in E(G)$. Suppose x and x' are in the same S-class X. Then $\alpha(x)$ and $\alpha(x')$ are in the same S-class of their own, and because $\langle X \rangle$ is a complete graph, it must be that $\alpha(x)\alpha(x') \in E(G)$. Now suppose x and x' are in different S-classes, say X and X' respectively. Then in $G \circ H$, there is an edge running from $\langle X \rangle \circ H$ to $\langle X' \rangle \circ H$, thus an edge running from $\varphi(\langle X \rangle \circ H)$ to $\varphi(\langle X' \rangle \circ H)$, so there is an edge running from $\alpha'(X)$ to $\alpha'(X')$. Now because S-classes are externally related, the existence of one edge between $\alpha'(X)$ and $\alpha'(X')$ implies that each vertex in $\alpha'(X)$ is adjacent to every vertex in $\alpha'(X')$. Thus $\alpha(x)\alpha(x') \in E(G)$ which means that α is a homomorphism, and further, an automorphism. $\square$

Lemma 5.3.3. *Let H be disconnected. Any involution φ of $G \circ H$ induces an automorphism α' of G/R defined as $\alpha'(X) = X'$, where $\varphi(\langle X \rangle \circ H) = \langle X' \rangle \circ H$. Hence $|\alpha'(X)| = |X|$ for each $X \in G/R$. Moreover, any permutation α of $V(G)$ with the property that $\alpha(X) = \alpha'(X)$ for each X in G/R is an automorphism of G.*

Proof. If we invoke Proposition 5.3.3 (instead of Proposition 5.3.2) along with the observation that R-classes are externally related, then this proof is the same as the proof of Lemma 5.3.2. $\square$

The following two theorems are the main results of the paper.

Theorem 5.3.1. *$G \circ H$ is odd palindromic if and only if G and H are odd palindromic.*

Proof. The converse direction of this statement is Proposition 5.3.1.

Let $G \circ H$ be odd palindromic with odd palindromic involution φ. We will consider two cases: the case where H is connected and the case where H is disconnected.

Case 1: H is connected.

We will first show that H is odd palindromic. Let (x_0, y_0) be the fixed point of φ, and suppose (x_0, y_0) is in the S-tower over X_0. By Proposition 5.3.2, $\varphi(\langle X_0 \rangle \circ H) = \langle X_0 \rangle \circ H$. Then φ restricts to an automorphism of $\langle X_0 \rangle \circ H \cong K_{|X_0|} \circ H$, where $|X_0|$ is odd, so $K_{|X_0|} \circ H$ is odd palindromic. By Corollary 5.3.1, $K_{|X_0|}$ and H are odd palindromic. Thus H is odd palindromic.

Now we will show that G is odd palindromic using Lemma 5.3.2 to construct an odd palindromic involution of G. Observe that φ will fix the S-tower $\langle X_0 \rangle \circ H$ that contains the fixed point (x_0, y_0). Any remaining odd-ordered S-towers must be interchanged in pairs by φ, and even-ordered S-towers are either fixed or interchanged in pairs by φ. By Lemma 5.3.2, φ induces an automorphism α' of G/S (recall: $\alpha'(X) = X'$, where $\varphi(\langle X\rangle \circ H) = \langle X'\rangle \circ H$). Since φ fixes the odd-ordered S-tower $\langle X_0\rangle \circ H$, $\alpha'(X_0) = X_0$. If φ interchanges odd-ordered S-towers $\langle X_1\rangle \circ H$ and $\langle X_2\rangle \circ H$, then $\alpha'(X_1) = X_2$ and $\alpha'(X_2) = X_1$. If φ fixes an even-ordered S-tower $\langle X\rangle \circ H$, then α' will also fix X. In what follows, we will construct an involution α of $V(G)$ such that $\alpha(X) = \alpha'(X)$ for each $X \in G/S$. Since α' fixes the odd-ordered S-class X_0, we can let $\alpha(x_0) = x_0$ for exactly one $x_0 \in X_0$, and let α interchange the remaining vertices of X_0 in pairs. If α' interchanges S-classes X and X', we can let α interchange vertices pairwise between X and X'. If α' fixes an even-ordered S-class X, we can let α interchange vertices of X in pairs. Now observe that, by Lemma 5.3.2, α is an involution of G with exactly one fixed point, so G is odd palindromic.

Case 2: H is disconnected.

We will first show that H is odd palindromic. Let (x_0, y_0) be the fixed point of φ, and suppose (x_0, y_0) is in the R-tower over X_0. By Proposition 5.3.3, $\varphi(\langle X_0\rangle \circ H) = \langle X_0\rangle \circ H$. Then φ restricts to an automorphism of $\langle X_0\rangle \circ H \cong D_{|X_0|} \circ H$, so $D_{|X_0|} \circ H$ is odd palindromic. By Proposition 5.3.5, $D_{|X_0|}$ and H are odd palindromic. Thus H is odd palindromic.

Now G is also odd palindromic using the same proof as in Case 1 but invoking Lemma 5.3.3 in place of Lemma 5.3.2. □

Theorem 5.3.2. *$G \circ H$ is even palindromic if and only if G or H is even palindromic.*

Proof. The converse direction of this statement is Proposition 5.3.1.

Let $G \circ H$ be even palindromic with even palindromic involution φ. We will consider two cases: the case where H is connected and the case where H is disconnected.

Case 1: H is connected.

Suppose φ fixes some S-tower $\langle X\rangle \circ H$ where $|X|$ is odd. Then $\langle X\rangle \circ H \cong K_n \circ H$

is even palindromic, where K_n is not even palindromic because n is odd, so H must be even palindromic by Corallary 5.3.1.

Suppose φ fixes no S-tower $\langle X \rangle \circ H$ where $|X|$ is odd. Observe that φ must interchange odd-ordered S-towers in pairs. By Lemma 5.3.2, φ induces an automorphism α' of G/S. Moreover, because φ interchanges odd S-towers $\langle X_1 \rangle \circ H$ and $\langle X_2 \rangle \circ H$, $\alpha'(X_1) = X_2$ and $\alpha'(X_2) = X_1$. If φ fixes any even S-tower $\langle X \rangle \circ H$, then α' will also fix X. In what follows, we will construct an involution α of $V(G)$ such that $\alpha(X) = \alpha'(X)$ for each X in G/S. Since α' interchanges odd-ordered S-classes X and X' in pairs, we can let α interchange vertices pairwise between X and X'. If α' fixes an even S-class X, we can let α interchange vertices of X in pairs. Now observe that, by Lemma 5.3.3, α is an involution of G with no fixed points, so G is even palindromic.

Case 2: H is disconnected.

Suppose φ fixes some R-tower $\langle X \rangle \circ H$ where $|X|$ is odd. Then $\langle X \rangle \circ H \cong D_n \circ H$ is even palindromic, where D_n is not even palindromic because n is odd, so H must be even palindromic by Proposition 5.3.5.

Suppose φ fixes no R-tower $\langle X \rangle \circ H$ where $|X|$ is odd. Using the same proof as in Case 1, invoking Lemma 5.3.3 instead of Lemma 5.3.2, it can be concluded that G is even palindromic.

$\square$

CHAPTER 6

CONCLUDING REMARKS

This dissertation set out to prove that palindromic factors result in palindromic products and vice versa. The results were quite clean for the Cartesian, strong, and lexicographic products, and although the results for the direct product were not analogous in the way we had hoped, we were still able to establish meaningful conditions.

While the conditions on palindromic products have been fully fleshed out, the restrictions that arose during the exploration of the direct product added an unexpected twist. An overarching goal of this dissertation was to provide evidence that automorphisms of the direct product transfer to the factors of the product, and while this was not true in general in the case of transferring a palindromic involution from the direct product to its factor, it was true when both factors were nonbipartite. It would be interesting to investigate and establish suitable conditions for transferring an automorphism of, say, order 3 from the direct product to its factors, starting with two nonbipartite factors and adding appropriate restrictions when we move to one or two bipartite factors. As we established in Chapter 4, transferring an automorphism of order 2 can be tricky, is surprisingly nontrivial, and does not always work. It is reasonable to believe that transferring automorphisms of any order other than 2 will require much more care for the case of two nonbipartite factors, and it may not hold for one or two bipartite factors.

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VITA

Jamie Lynn Shive, born and raised in Allentown, Pennsylvania on March 9, 1994, moved to Richmond, Virginia to obtain her Bachelor of Science in Applied Mathematics from Virginia Commonwealth University where she graduated Magna Cum Laude with University Honors. She was inspired to pursue a Ph.D. after participating in undergraduate research on topics in graph theory. Jamie is also an instructor, course author, and curriculum developer for the Graph Theory, Combinatorics, and Algorithms courses for Johns Hopkins Center for Talented Youth. Although her current research interests lie in product graph theory, graph invariants, combinatorial games, and graph algorithms, she maintains unwavering passion, enthusiasm, and excitement for all types of mathematics, statistics, and operations research problems. After graduating, Jamie is delighted to take her passion, enthusiasm, excitement, husband, stepson, and cats to Seattle, Washington where she will work on interesting, complex problems as an Applied Mathematician at Boeing.