2022

AUTOMATED CONJECTURING ON THE INDEPENDENCE NUMBER AND MINIMUM DEGREE OF DIAMETER-2-CRITICAL GRAPHS

Joshua R. Forkin

Automated Conjecturing on the independence number and minimum degree of Diameter-2-critical graphs

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

Joshua Forkin
Master of Science

Director: Dr. Craig Larson, Professor
Department of Mathematics and Applied Mathematics

Virginia Commonwealth University
Richmond, Virginia
May 2022

Acknowledgements

I first want to thank Dr. Larson for everything he has done in helping me with my research, and writing this thesis. I also want to thank my other professors at VCU for helping me significantly grow as a mathematician. All have been very willing to meet with me outside of class if I needed help with anything in my classes, as well as if I just wanted to talk. In addition, I want to thank my fellow VCU classmates who have always been friendly to me, and my family back home, who have always been my biggest supporters.

Table of Contents

List of Figures

Abstract

Abstract: A diameter-2-critical (D2C) graph is a graph with diameter two such that removing any edge increases the diameter or disconnects the graph. In this paper, we look at other lesser-studied properties of D2C graphs, focusing mainly on their independence number and minimum degree. We show that there exist D2C graphs with minimum degree strictly larger than their independence number, and that this gap can be arbitrarily large. We also exhibit D2C graphs with maximum number of common neighbors strictly greater than their independence number, and that this gap can be arbitrarily large. Furthermore, we exhibit a D2C graph whose number of distinct degrees in its degree sequence is strictly greater than its independence number. Additionally, we characterize D2C graphs with independence number 2 and show that all such graphs have independence number greater or equal to their minimum degree.

Chapter 1

Preliminaries

1.1 Basic Graph Theory Definitions

A **graph** G is a set of vertices and edges such that each edge has exactly two vertices as its endpoints. The set of vertices in G is called the **vertex set** of G, denoted $V(G)$. Likewise, the set of edges in G is called the **edge set** of G, denoted $E(G)$. The **complement** of a graph G, denoted G^c, is defined such that $V(G) = V(G^c)$ and for $u, v \in V(G)$, $uv \in E(G^c)$ if and only if $uv \notin E(G)$.

A **loop** is an edge whose endpoints are the same [26]. **Multiple edges** are two or more edges all having the same two endpoints. A **simple graph** is a graph that contains no loops or multiple edges. Unless otherwise stated, all graphs are assumed to be simple. The **order** of a graph G, denoted $n(G)$, is the number of vertices in G. The **size** of G is the number of edges in G. If an edge $e \in E(G)$ has endpoints u and v, the edge e is said to be **incident** with u and v, and u and v are said to be **adjacent** to each other. If two vertices u and v are adjacent, we say that they are **neighbors**. If a vertex w is adjacent to vertices u and v, then we say that u and v have w as a common neighbor. The **open neighborhood** of a vertex v, denoted $N(v)$, is the set of vertices adjacent to v. The **closed neighborhood** of v, denoted $N[v]$, is $N(v) \cup \{v\}$ (v together with its neighbors). A set of pairwise non-adjacent vertices form an **independent set**, and a set of pairwise adjacent vertices form a **clique**. The maximum cardinality of a clique in a graph G is called the **clique number** of G, denoted $\omega(G)$.

The **degree** of a vertex v, denoted $d(v)$, is the number of edges incident with v. A vertex v with $d(v) = 1$ is called a **leaf**, and an edge incident with a leaf is called a **pendant edge**. A vertex $v \in V(G)$ has **minimum degree** if, given any vertex $u \in V(G)$, $d(v) \leqslant d(u)$. The minimum degree of a graph G, denoted $\delta(G)$, is the cardinality of a vertex of minimum degree. Likewise, a vertex $v \in V(G)$ has **maximum degree** if, given any vertex $u \in V(G)$, $d(v) \geqslant d(u)$. The maximum degree of a graph G, denoted $\Delta(G)$, is the cardinality of a vertex of maximum degree. A **regular graph** is a graph where every vertex has the same degree.

A **walk** on a graph is an alternating set of vertices and edges $v_0, e_0, v_1, e_1, ..., v_{n-1}, e_{n-1}, v_n$ such that e_i has endpoints v_i and v_{i+1}. A **path** is a walk with no repeating edges or vertices. A path which begins at a vertex u and ends at a vertex v is referred to as a u, v-path. A **cycle** is a closed path. The length of the shortest cycle in G is called the **girth** of G. If G does not contain a cycle, then G is said to have infinite girth.

A **connected graph** is a graph G such that for all $u, v \in V(G)$, there exists a u, v-path. A **tree** is a connected graph that contains no cycles. A graph which is not connected is called a disconnected graph. A **component** H of a graph G is a maximal connected subgraph of G, meaning that adding any edge or vertex from $V(G)$ or $E(G)$ that is not part of $V(H)$ or $E(H)$ results in a disconnected graph. A **cut vertex** v in a graph G is a vertex whose removal from G increases the number of components in G. Likewise, a **cut edge** (or a bridge) e in a graph G is an edge whose removal from G increases the number of components in G. A **vertex cut** is a set $S \subset V(G)$ such that $G - S$ has more than one component or $G - S$ has only one vertex. The **vertex connectivity** of a graph, denoted $\kappa(G)$, is the minimum size of a vertex cut. If $\kappa(G) \geqslant k$, we say that G is k-connected.

There are several very basic classes of graphs. A **cycle graph**, denoted C_n, is a graph such that $V(C_n)$ induces a cycle of length n. A **path graph**, denoted P_n, is a graph such that $V(P_n)$ induces a path of length n. A **complete graph**, denoted K_n, is a graph such that if $u, v \in V(G)$ and $u \neq v$, then $uv \in E(G)$. A **bipartite graph** is a graph which every $v \in V(G)$ can be partitioned into one of two independent sets, called partite sets. A **complete bipartite graph**, denoted $K_{m,n}$, is a bipartite graph with partite sets X and Y such that $|X| = m$, $|Y| = n$, and if $x \in X$ and $y \in Y$, then $xy \in E(G)$. The complete bipartite graphs $K_{m,n}$ with $m = 1$ and/or $n = 1$ are called **stars**. The complete bipartite graphs $K_{m,n}$ where $m = n$ are called **balanced complete bipartite graphs**.

A **subgraph** H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $H \neq G$, then H is called a **proper subgraph** of G. If $V(H) = V(G)$, then H is called a **spanning subgraph** of G. A spanning subgraph that is also a tree is referred to as a **spanning tree** of G. If $S \subseteq V(G)$, then the subgraph induced by S, denoted $G[S]$, is the subgraph H formed by letting $V(H) = S$, and for $u, v \in S$, $uv \in E(H)$ if and only if $uv \in E(G)$.

A **dominating set** is a set $D \subset V(G)$ such that if $v \in V(G) - D$, then there exists a vertex $u \in D$ adjacent to v. If there exists $v \in V(G)$ such that $\{v\}$ is a dominating set, then v is called a **dominating vertex**. A **dominating edge** is an edge whose endpoints form a dominating set.

The **distance** between two vertices u and v, denoted $d(u, v)$, is the length of a shortest path between u and v. The **diameter** of a graph is the maximum distance between any two vertices in G. Given a vertex v, the distance to the vertex furthest away from v is called the **eccentricity** of v, denoted $\epsilon(v)$. More formally, $\epsilon(v)$ is the smallest integer k such that for all $u \in V(G)$, $d(u, v) \leqslant k$. If u has the smallest eccentricity of all the vertices in G, then the **radius** of G is the value of $\epsilon(u)$. More formally, the radius of G is the greatest integer k such that for all $u \in V(G)$, $\epsilon(u) \geqslant k$. (We can similarly define the diameter of G to be the smallest integer j such that for all $u \in V(G)$, $\epsilon(u) \leqslant k$.)

1.2 Independence Number

A maximal independent set is an independent set that is not contained in a larger independent set. More precisely, an independent set I is maximal if, given any vertex $v \in V(G) \setminus I$, $\{v\} \cup I$ is not an independent set. An independent set M is called maximum if, given another independent set S in G, $|S| \leqslant |M|$. The **independence number** of a graph G, denoted $\alpha(G)$, is the cardinality of a maximum independent set. The independence number is a widely studied graph invariant that is NP-hard to compute [10].

1.2.1 Basic Results on the Independence Number

Suppose that you partitioned $V(G)$ into independent sets $I_1, I_2, ..., I_n$. Letting each I_j represent a color, we call such a partition a **proper coloring** of G. The **chromatic number** of a graph, denoted $\chi(G)$, is the minimum number of colors needed in a proper coloring of G.

The chromatic number provides a well-known lower bound on the independence number of a graph.

Proposition 1. *For a graph* G, $\alpha(G)\chi(G) \geqslant n(G)$.

Proof. Let $\chi(G) = j$, let $\alpha(G) = k$ and let $X_1, X_2, ..., X_j$ be a proper coloring of G. We see that since a proper coloring partitions $V(G)$, $n(G) = |X_1| + |X_2| + ... + |X_j|$. Since each X_i is an independent set, $|X_1| + |X_2| + ... + |X_j| \leqslant jk = \alpha(G)\chi(G)$, as we wanted to show. □

Theorem 1. *[24] For any graph* G, $\chi(G) \leqslant \Delta(G) + 1$.

From Theorem 1 and Proposition 1, we get a lower bound of the independence number in terms of the maximum degree.

Corollary 1. *For any graph* G, $\alpha(G) \geqslant \frac{n}{\Delta(G)+1}$.

A **matching** in a graph is a set of edges such that no two edges share an endpoint. The **matching number** of a graph G (denoted $\alpha'(G)$) is the maximum size of a matching in G. The matching number of a graph gives both a lower and an upper bound on the independence number of a graph.

Proposition 2. *For any graph* G, $n(G) - 2\alpha'(G) \leqslant \alpha(G) \leqslant n(G) - \alpha'(G)$

Proof. Let M be a maximum matching in G. Since M is maximum, removing M from G leaves an independent set I of size $n(G) - 2\alpha'(G)$. Thus, $\alpha(G) \geqslant n(G) - 2\alpha'(G)$.

Now consider a maximum-sized independent set I. For each edge $e \in M$, the set I contains at most one endpoint from e. Since there are $\alpha'(G)$ edges in M, $\alpha(G) \leqslant n - \alpha'(G)$, as we wanted to show. □

Chapter 2

Diameter-2-Critical Graphs

2.1 Diameter 2 Graphs

Diameter 2 graphs are of particular interest because many graphs which are not widely studied have diameter 2. In fact, almost all graphs have diameter 2.

Theorem 2. *[1] Almost all graphs have diameter 2.*

Proof. For $u, v \in V(G)$, assume that the probability that u and v are adjacent in G, $P(uv \in E(G)) = \frac{1}{2}$. This clearly implies that $P(uv \notin E(G)) = \frac{1}{2}$. Next, let $w \in V(G)$. Since $P(uw \in E(G)) = \frac{1}{2}$ and $P(vw \in E(G)) = \frac{1}{2}$, the probability that both $uw, vw \in E(G)$ is $\frac{1}{4}$. This means that the probability that at least one of $uw, vw \notin E(G)$ is $\frac{3}{4}$.

Thus, the probability that u and v are not adjacent and do not share a common neighbor is $\frac{1}{2}\left(\frac{3}{4}\right)^{n-2}$. Summing over all $\binom{n}{2}$ pairs of vertices, we have that the probability that G does not have diameter 2 is $\frac{1}{2}\binom{n}{2}\left(\frac{3}{4}\right)^{n-2}$, which goes to 0 as $n \to \infty$. This is what we wanted to show. $\square$

Proposition 3. *Let G have diameter 2, and let $v \in V(G)$. Then $N(v)$ is a dominating set*

Proof. Suppose there exists $w \in V(G)$ such that $w \notin N[v]$ and if $u \in N(v)$, u is not adjacent to w. We then see that $d(v, w) \geqslant 3$, contradicting the fact that G has diameter 2. Thus, we conclude that $N(v)$ forms a dominating set in G. $\square$

2.2 What is a Diameter-2-Critical Graph?

Often when trying to learn the different properties about a general class of graphs, it is useful to study their critical graphs.

We can think of a critical graph as a subset of a class of graphs $\mathcal{C}$ with property X. If $G \in \mathcal{C}$ and removing any edge from $\mathcal{C}$ changes property X, then we say that G is an X-critical graph.

For example, if we wanted to study the graphs with independence number 2, we may want to study a smaller class of independence number 2 graphs for which more is known. Independence number critical graphs (which are known as **alpha-critical** graphs) are graphs in which removing any edge increases the independence number of the graph.

There are also vertex critical graphs. In these graphs, the critical property X must change if any vertex is removed (but not necessarily when any edge is removed). These generally do not share the same properties as the class of edge critical graphs for a given property.

A **diameter-2-critical graph** (abbreviated D2C) is a diameter 2 graph G such that for any edge $e \in E(G)$ that is not a cut-edge, $G - e$ has diameter strictly larger than 2. If for an edge $e \in E(G)$, $G-e$ has strictly larger diameter than G, we call e a critical edge. Thus, we can alternatively define D2C graphs as diameter 2 graphs in which every edge is a critical edge. Examples of D2C graphs include P_3, C_4, C_5, all complete bipartite graphs, the Petersen Graph, the Clebsch Graph, the Wagner Graph, and the Hoffman-Singleton Graph.

D2C graphs are a subset of a large class of graphs known as diameter-n-critical graphs, diameter n graphs G such that for any edge $e \in E(G)$ that is not a cut-edge, $G - e$ has diameter strictly larger than n.

In general, D2C graphs do not have a lot of underlying structure. Indeed, it was proven in [11] that diameter-n-critical graphs cannot be constructed by finite extensions

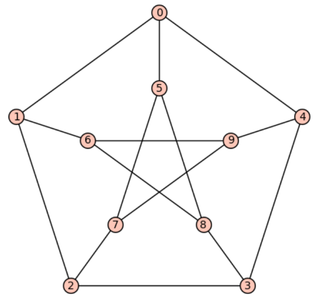


Figure 2.1: Petersen Graph

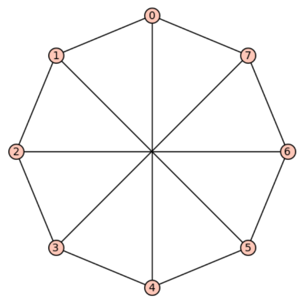


Figure 2.2: Wagner Graph

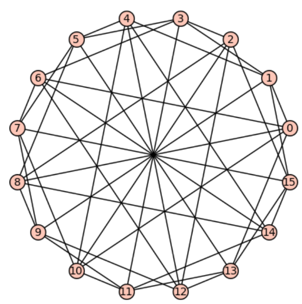


Figure 2.3: Clebsch Graph

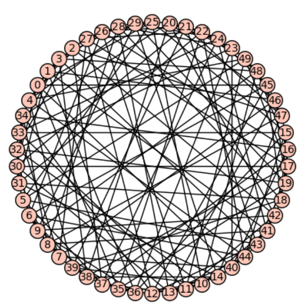


Figure 2.4: Hoffman-Singleton Graph

or characterized by forbidden subgraphs. However, there are methods for constructing D2C graphs from non-D2C graphs. In Section 2.6, we will show two such methods for constructing D2C graphs introduced by Loh and Ma (see [19]).

2.3 Properties of D2C Graphs

Proposition 4. *If* P *is a proper spanning subgraph of a D2C graph, then* P *is not D2C.*

Proof. If P is a proper spanning subgraph of a D2C graph G, then it is obtained from G by removing a non-empty set of edges. By the definition of a D2C graph, P will not have diameter 2 and thus cannot be a D2C graph. □

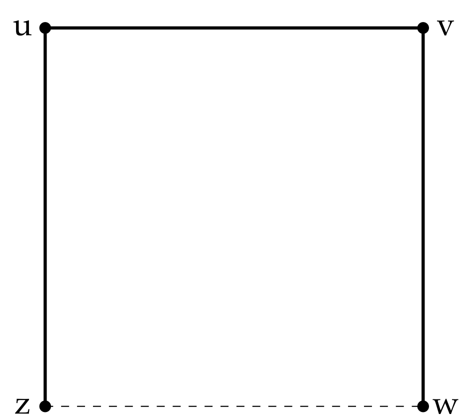


Figure 2.5: Removing edges from a D2C graph always increases its diameter.

Proposition 5. *If* G *contains a D2C proper spanning subgraph, then* G *is not D2C.*

Proof. If G contains a D2C graph as a proper spanning subgraph H, then G can be transformed into H by removing a non-empty set S of edges from G. But then the edges in S are not critical edges. Thus, G is not D2C. □

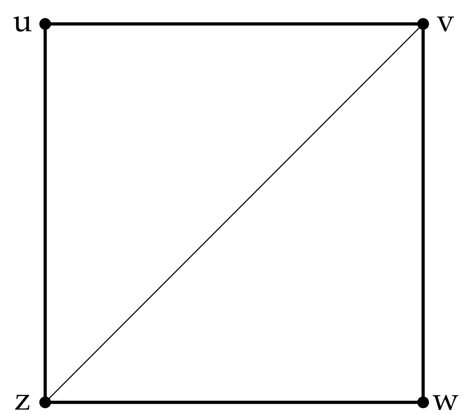


Figure 2.6: The edge vz can be removed to obtain a D2C graph C_4, meaning that vz is not a critical edge.

Theorem 3. *The only D2C bipartite graphs are complete bipartite graphs. Furthermore, all complete bipartite graphs are D2C.*

Proof. Every non-complete bipartite graph is a proper spanning subgraph of a larger complete bipartite graph. We will prove that all complete bipartite graphs are D2C, from which it will follow from Proposition 4 that any other bipartite graph is not D2C.

Let G be a complete bipartite graph, and let $xy \in E(G)$ having endpoints $x, y \in V(G)$. We first note that any two distinct vertices in the same partite set share at least 1 common neighbor in the other partite set (since G is a complete bipartite graph). Thus, G has diameter 2. We also note that removing xy from G leaves the vertices x and y at least distance three from each other, so $G - xy$ has at least diameter 3. Thus, G is D2C. □

All trees are bipartite graphs. The only complete bipartite graphs which are trees are stars, so we get another result directly from Theorem 3.

Corollary 2. *The only D2C trees are stars.*

We can also now classify D2C graphs with a dominating vertex.

Corollary 3. *The only D2C graphs with a dominating vertex are stars.*

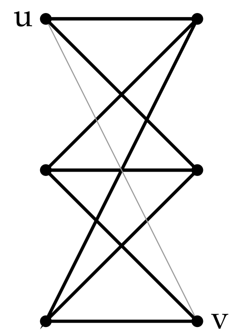


Figure 2.7: If an edge uv is removed from $K_{m,n}$, $d(u, v) = 3$.

Proof. Any graph G with a dominating vertex contains a star as a spanning subgraph H. If G = H, then G is a star. If G ≠ H, then by definition of a proper subgraph, it contains a star as a proper spanning subgraph. By Proposition 5, G is not D2C. □

Theorem 4. *The only D2C graphs with cut-edges are stars.*

Proof. Suppose that a D2C graph G contains a cut-edge uv. If u has a neighbor $y \neq v$, and v has a neighbor $z \neq u$, since uv is a cut-edge, we would have that $d(y, z) = 3$. We also see that if G is just the cut edge uv, then then diameter of G is 1. Thus, if G is a D2C graph with a cut-edge, then either $N(u) = \{v\}$ or $N(v) = \{u\}$, but not both.

Without loss of generality, suppose that $N(u) = \{v\}$. We now show that v is a dominating vertex. If v is not a dominating vertex, there exists some vertex $x \notin N(v)$. Since $N(u) = \{v\}$, it follows that $d(u, v) \geqslant 3$. This contradicts our assumption that G is D2C, and thus v is a dominating vertex. By Corollary 3, G can only be a star. Since every edge in a star is a cut-edge, we conclude that the only D2C graphs with cut-edges are stars. □

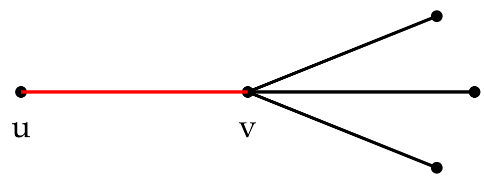


Figure 2.8: D2C graph with cut-edge uv

Since every pendant edge is a cut-edge, we see that the only D2C graphs with pendant edges are stars.

Corollary 4. *The only D2C graphs containing pendant edges are stars. Thus, if* G *is a D2C graph and* G *is not a star, then* $\delta(G) \geqslant 2$*.*

Theorem 5. *The only D2C graphs with cut vertices are stars.*

Proof. Suppose that a D2C graph G contains a cut-vertex v. We show that v is a dominating vertex. If v is not a dominating vertex, then there exists some vertex $x \notin N(v)$. Since v is a cut vertex, $G - v$ contains at least two components. At most one of the components will contain x, so let u be a vertex which is not in this component. We see that $d(u, x) \geqslant 3$, contradicting our assumption that G is D2C. Thus, v is a dominating vertex. By Corollary 3, G can only be a star. Since every edge in a star is a cut-edge, the only D2C graphs which contain cut-vertices are stars. □

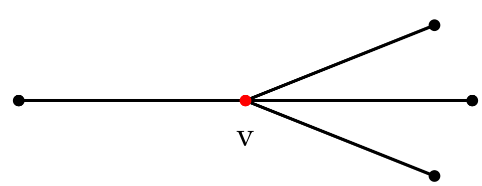


Figure 2.9: D2C graph with cut vertex v

Theorem 6. *Let* v *be a vertex of minimum degree in a D2C graph* G*, and let* S *be the set of vertices distance 2 from* v *in* G*. For all* $w \in N(v)$*,* w *must contain a neighbor in* S*.*

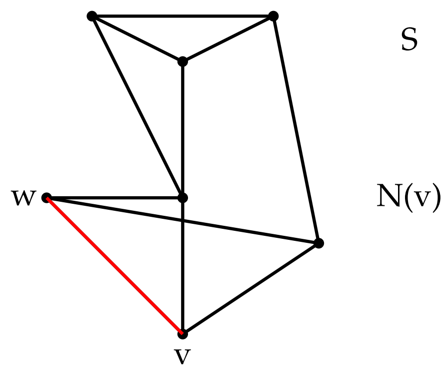


Figure 2.10: If w does not have a neighbor in S, then wv is not a critical edge.

Proof. Suppose that w has no neighbor in S. This means that the possible neighbors of w are v and vertices in $N(v)$. We see that $|\{v\} \cup N(v) - \{w\}| = |N(v)| = \delta(G)$, so w can have at most $\delta(G)$ neighbors. Therefore, it must have exactly $\delta(G)$ neighbors, meaning that $N(w) = N(v) - \{w\} + \{v\}$.

We show that the edge wv is not a critical edge. In $G - wv$, w is still adjacent to all the other vertices in $N(v)$. Also, since w does not have a neighbor in S, the distance from v to a vertex in S does not change. Likewise, the distance between a vertex in $N(v)$ and S will not be affected by the removal of wv. Thus, we see that the diameter of $G - wv$ is still 2. This contradicts our assumption that G was D2C, and so we conclude that w must contain a neighbor in S. $\square$

Theorem 6 gives some structure which is common to all D2C graphs, but not all diameter 2 graphs. This will be used in Section 4.4 to classify all D2C graphs with $\alpha = 2$.

2.4 The Murty-Simon Conjecture

Much of the research done on D2C graphs centers on finding the maximum number edges in an n-vertex D2C graph. This extremal problem has been studied since at least the 1960's, when Murty and Simon conjectured that the upper edge bound for D2C graph is $\lfloor \frac{n^2}{4} \rfloor$.

Conjecture 1. *[3] If G is a D2C graph with e edges, then $e \leqslant \lfloor \frac{n^2}{4} \rfloor$, with equality if and only if G is a complete bipartite graph with balanced bipartite sets.*

Although the conjecture has not been fully proven, it has been proven for small D2C graphs ($n \leqslant 24$, see [7]) and very large D2C graphs ($n \geqslant 2 \uparrow\uparrow 10^{14}$, see [9] and [17]).

The best upper bound on the number of edges in any D2C graph was proven in 1987 by Fan (see [7]).

Theorem 7. *[7] If G is a D2C graph, then the maximum number of edges in G is $.2532n^2$.*

Many recent results have focused on proving the Murty-Simon Conjecture for different classes of graphs. Notably, it has been shown that the upper bound on the number of edges for non-bipartite D2C graphs with a dominating edge is strictly less than the Murty-Simon Conjecture bound.

Theorem 8. *[3] If G is a non-bipartite D2C graph with a dominating edge and e edges, then apart from the graph H_5 (see [3]), $e \leqslant \lfloor \frac{n^2}{4} \rfloor - 2$.*

2.5 Regular D2C Graphs

The edge bounds of D2C graphs can be very useful in narrowing down the possible number of regular D2C graphs. For instance, we can classify regular D2C graphs with a dominating edge using Theorem 8.

Theorem 9. *The only regular D2C graphs with a dominating edge are complete bipartite graphs with balanced partite sets.*

Proof. Let G be a k-regular non-bipartite D2C graph with order n, e edges, and a dominating edge. We see that the degree sum of G is kn. By the Handshaking Lemma (see [26]), $e = \frac{kn}{2}$. By Theorem 8, $e \leqslant \frac{n^2}{4} - 2$, and so $\frac{kn}{2} \leqslant \frac{n^2}{4} - 2$. Multiplying by two, we have that $kn \leqslant \frac{n^2}{2} - 4$. Dividing by n, we have that $k \leqslant \frac{n}{2} - \frac{4}{n} < \frac{n}{2}$. Thus,

$2k < n$. However, since G has a dominating edge e_d with endpoints u and v, we have that $n(G) = |N[u]-\{v\}|+|N[v]-\{u\}| = k+k = 2k$. By this contradiction, we conclude that if G is a regular D2C graphs with a dominating edge, then it cannot be non-bipartite.

Thus, if G contains a dominating edge, then it must be bipartite. From Theorem 3, we see that the only D2C bipartite graphs are complete bipartite graphs. Indeed, we see that every edge in a complete bipartite graph is a dominating edge. Since the only regular complete bipartite graphs are $K_{n,n}$, we conclude that the only regular D2C graphs with a dominating edge are complete bipartite graphs with balanced partite sets. □

Using the calculations done in the proof of Theorem 9, Theorem 7 shows that for regular D2C graphs, $k \leqslant .5064n$. An upper bound on k can be found by using the fact that k-regular diameter 2 graphs have at most $1 + k + k^2$ vertices. Thus, $n \leqslant 1 + k + k^2$. This makes it possible to generate all of the k-regular D2C graphs when k is small.

2.6 Constructing Diameter-2-Critical Graphs

2.6.1 Using Constructions by Loh and Ma

Infinite families of D2C graphs can be constructed using a method introduced by Po-Shen Loh and Jie Ma in [19].

To do this, begin with a diameter 2 graph G whose complement also has diameter 2. If $n(G) = k$, begin with a copy of G such that its k vertices are labeled $v_1, ..., v_k$. These vertices will be called vertex set A, and this graph will be referred to as G.

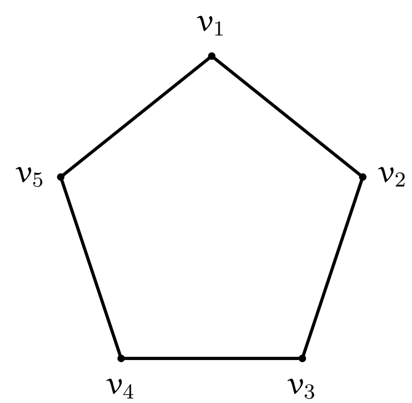


Figure 2.11: C_5 can be used as G since diam(C_5) = 2 and it is self-complementary.

Next, add k more vertices labeled $v'_1, ..., v'_k$, which will be referred to as vertex set B. Then, add edge $v'_i v'_j$ $(i \neq j)$ to B if and only if $v_i v_j \notin E(G)$. (So in this step, we add a copy of the complement of G to the vertex set B.) The edges in the vertex set B must be added exactly the way that was done in this step to assure that the final graph is D2C.

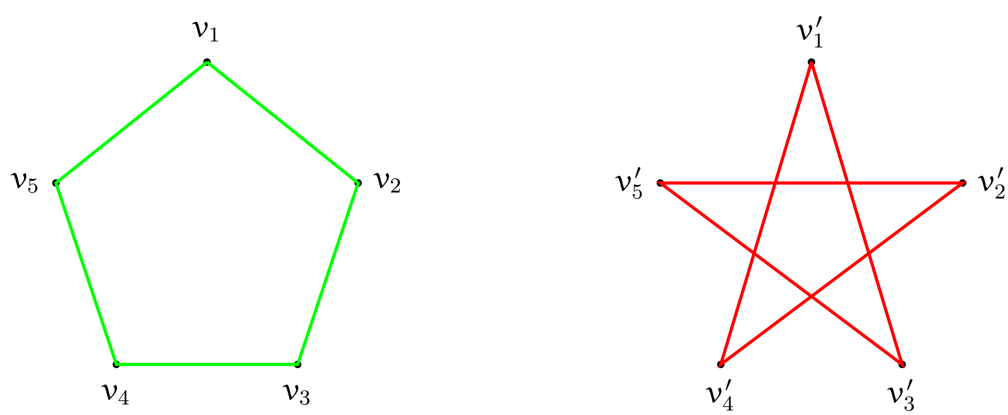


Figure 2.12: On left: Vertex set A with copy of G. On right: vertex set B with copy of G^c.

Finally, we add the edge $v_i v'_i$ for all $1 \leqslant i \leqslant k$. This adds a perfect matching between the vertex sets A and B. The resulting graph, which we denote G', is a D2C graph.

To prove that G' is D2C, the authors first argue that G' has diameter 2. First, looking at $v_i \in A$ and $v'_k \in B$, by construction, either $v_i v_k \in E(G')$ or $v'_i v'_k \in E(G')$. If $v_i v_k \in E(G')$, using this edge and the edge $v_k v'_k$ in the perfect matching, we see that v_i and v'_k

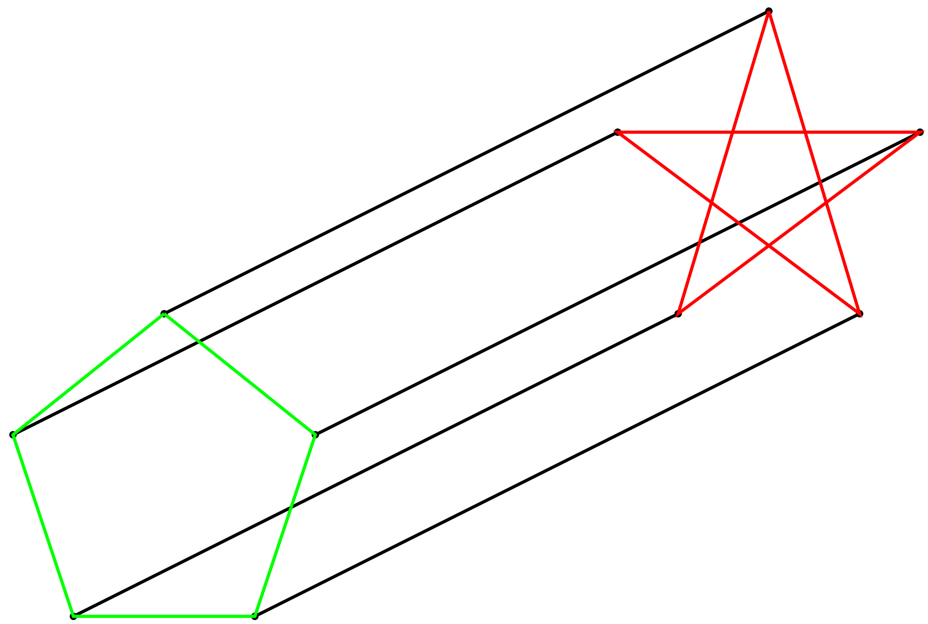


Figure 2.13: Final construction of D2C graph G', which is isomorphic to the Petersen graph.

are distance two apart. On the other hand, if $v_i'v_k' \in E(G')$, using this edge and the edge v_iv_i' in the perfect matching, we again see that v_i and v_k' are distance 2 apart.

Since G and its complement have diameter at most 2, any pair of vertices in the same vertex set are at most distance 2 apart. Finally, vertices $v_i \in A$ and $v_i' \in B$ are distance one apart (adjacent by the perfect matching), and so we conclude that G' has diameter 2.

Next we need to show that every edge in G' is critical. If the edge v_iv_i' is removed, given that if $v_iv_k \in E(G')$, then $v_i'v_k' \notin E(G')$, we have that $d(v_i, v_i') \geqslant 3$. Next, if any edge v_iv_k is removed from A, given that if $v_iv_k \in E(G')$, then $v_i'v_k' \notin E(G')$, we have that $d(v_i, v_k') \geqslant 3$. Likewise, if any edge $v_i'v_k'$ is removed from B, given that if $v_i'v_k' \in E(G')$, then $v_iv_k \notin E(G')$, we have that $d(v_i', v_k) \geqslant 3$. Thus, every edge is critical, and G' is D2C. We will refer to this construction as Loh-Ma's Construction 2.1 (as it is called in [19]).

Given such a graph G', we have another construction using G' to create an additional infinite family of D2C graphs. To do this, simply join a set of disjoint vertices C with all of the vertices in A. This new graph G'' will also be D2C.

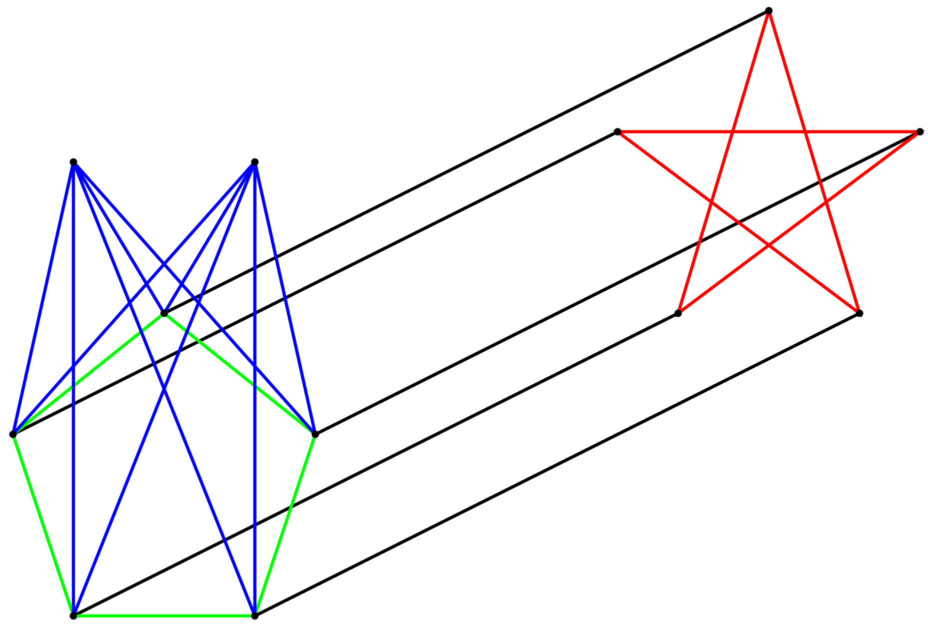


Figure 2.14: D2C graph G'', where two new vertices and the blue edges have been added to G'.

To prove this, consider $c \in C$. We see that if we remove the edge cv_i, then since the only neighbor of v_i' in A is v_i, $d(c, v_i') \geqslant 3$. Thus, every new edge added is a critical edge. Also, using what we already know about G' above, we again see that every edge between two vertices in the G' subgraph is a critical edge. Thus, we conclude that G'' is D2C.

This new construction of joining vertices to the vertex set A will be referred to as Loh-Ma's Construction 2.2 (as it is referred to in [19]).

2.6.2 Using Random Graphs

Another method of obtaining D2C graphs can be done using a mathematical computer software such as Sage (see [23]). Begin by generating a random graph of any size G. We can generate code so that Sage will check if a given edge in a graph is critical. If it is not, remove the edge from the graph. Do this until every edge in the graph is critical.

Since no critical edges are removed, the diameter of the resulting graph must still be two. Also, since all non-critical edges are removed, all edges of the resulting graph must be critical. Thus, this resulting graph must be D2C.

Chapter 3

Automated Conjecturing For Diameter-2-Critical Graphs

3.1 Methods

In this section, we give a number of conjectures related to the independence number and minimum degree of D2C graphs. These conjectures were found using the mathematics software Sage.

To do this, we first inputted the G6 strings of a collection of known diameter-2-critical graphs into Sage. (A list of the G6 strings for all D2C graphs with $n \leqslant 9$ can be found in [20].) We then used the computer program Conjecturing (see [18]) to generate a number of conjectures about the independence number and minimum degree of D2C graphs. The program Conjecturing takes the list of D2C graphs which we inputted and compares the independence number/minimum degree of the graphs to a multitude of other graph invariants. Conjecturing then outputs the invariants for which the independence number/minimum degree are lower/upper bounds for all the inputted D2C graphs. If Conjecturing outputs a conjecture which is already known to be true, then that conjecture can be added to a list called 'theory'. This will prevent the conjecture from being outputted again as a conjecture, and can help the program make better conjectures.

In this section, we show a select list of the conjectures which Conjecturing generated on D2C graphs. In the following conjectures, independence_number(x) refers to the independence number and min_degree(x) refers to the minimum degree.

3.2 Lower Bounds on the Independence Number of D2C Graphs

The parameter max_common_neighbors(x) refers to the maximum number of common neighbors between any pair of vertices in a graph G.

Conjecture 2. *For all D2C graphs, independence_number(x) $\geqslant$ max_common_neighbors(x).*

Conejcture 2 is false. A construction of an infinite family of graphs can be found in Section 5.1.

The parameter diameter(x) simply refers to the diameter of a graph G.

Proposition 6. *For all D2C graphs, independence_number(x) $\geqslant$ diameter(x).*

Proof. All D2C graphs have diameter 2 by definition. For a graph G to have diameter 2, there must exist at least two non-adjacent vertices u and v in $V(G)$. Thus, G contains the independent set $\{u, v\}$, from which it follows that $\alpha(G) \geqslant 2$. □

The parameter radius(x) refers to the radius of a graph G. By definition, the radius of a graph is less than or equal to the diameter of a graph. Thus, we have the following corollary to Proposition 6.

Proposition 7. *For all D2C graphs, independence_number(x) $\geqslant$ radius(x).*

The degree sequence of a graph G is a listing of the degree of each vertex in $V(G)$, typically in non-ascending order. The parameter distinct_degrees(x) refers to the number of distinct integers in the degree sequence of G.

Conjecture 3. *For all D2C graphs, independence_number(x) $\geqslant$ distinct_degrees(x).*

Conjecture 3 is false; the construction of a counterexample can be found in Section 5.2. This has led us to conjecture the following:

Conjecture 4. *For all natural numbers* k*, there exists a D2C graphs such that independence_number(x) $<$ distinct_degrees(x)* + k*.*

The parameter median_degree(x) refers to the median of the degree sequence of a graph G.

Conjecture 5. *For all D2C graphs, independence_number(x) $\geqslant$ median_degree(x).*

We trivially have that min_degree(x) $\leqslant$ median_degree(x). Given our construction of D2C graphs in Section 5.1 such that min_degree(x) $>$ independence_number(x), we conclude that Conjecture 5 is false.

The Havel-Hakimi algorithm is used to determine whether a given non-increasing sequence of non-negative integers comes from a graph (see [12] and [13]). If we repeatedly apply the Havel-Hakimi algorithm to a degree sequence which represents a graph (a graphical sequence), then the Havel-Hakimi algorithm will produce a finite number of zeros. The number of zeroes in the degree sequence after applying the Havel-Hakimi algorithm is called the residue of a graph G. The parameter residue(x) refers to the residue of G.

The residue of a graph provides a lower bound for the independence number of a graph (regardless of whether the graph is D2C or not).

Theorem 10. *[8] For any graph* G*, independence_number(x) $\geqslant$ residue(x).*

The parameter average_degree(x) refers to the median of the degree sequence of a graph G. We trivially have that for all graphs, min_degree(x) $\leqslant$ average_degree(x).

Proposition 8. *For all graphs, min_degree(x) $\leqslant$ average_degree(x).*

3.3 Upper Bounds on the Independence Number of D2C Graphs

Given a degree sequence $d_1, d_2, ..., d_n$ in non-increasing order, the parameter barrus_q(x) is the maximum value of k for which $k \leqslant d_k$ (see [4]). If a graph G has order $n(G)$, then the parameter barrus_bound(x) is equal to $n(G)-$barrus_q(x). The parameter barrus_bound(x) is an upper bound on the independence number of a graph.

Theorem 11. *[4] For all D2C graphs, independence_number(x) $\leqslant$ barrus_bound(x).*

The parameter size(x) simply refers to the size of a graph G.

Proposition 9. *For all D2C graphs, independence_number(x) $\leqslant$ size(x).*

Proof. Since all D2C graphs G are connected, it must contain at least $n-1$ edges. Since D2C graphs are connected graphs, G is not a disjoint set of n vertices. Thus, $\alpha(G) \leqslant n-1$. $\square$

Given a degree sequence $d_1, d_2, ..., d_n$ in non-decreasing order, the annihilation number is the largest index k such that $\sum_{i=1}^{k} d_i \leqslant \sum_{i=k+1}^{n} d_i$. The parameter annihilation_number(x) denotes the annihilation number of a graph G. Pepper proved that the annihilation number is an upper bound on the independence number of a graph in [21].

Theorem 12. *For any graph, independence_number(x) $\leqslant$ annihilation_number(x).*

The parameter girth(x) simply refers to the girth of a graph.

Conjecture 6. *For all D2C graphs, independence_number(x) $\leqslant$* $\mathtt{diameter}(x)^{\mathtt{girth}(x)}$

We see that the girth of any complete bipartite graph is 4. Thus, for any complete bipartite graph, $\mathtt{diameter}(x)^{\mathtt{girth}(x)} = 16$. However, for $n \geqslant 17$, the complete graph $K_{n,n}$ has independence number strictly greater than 16. Thus, there are infinitely many D2C graphs for which Conjecture 6 is false.

3.4 Vertex Connectivity of D2C Graphs

One conjecture regarding the vertex connectivity of D2C graphs was made in hopes that it would help prove Conjecture 8.

Conjecture 7. *For all D2C graphs* G, $\delta(G) = \kappa(G)$.

Graphs in which $\delta(G) = \kappa(G)$ are known as **maximally connected graphs**. Thus, Conjecture 7 asks if every D2C graph is maximally connected. This conjecture has not yet been proven or disproven. It is, however, true for any D2C graph that could be constructed using Loh-Ma's Construction 2.1.

Theorem 13. *For all D2C graphs* G *formed using Loh-Ma's Construction 2.1,* $\delta(G) = \kappa(G)$.

Proof. Let G' be a D2C graph constructed by Loh-Ma's Construction 2.1 with vertex sets A and B, where $G'[A]$ is the complement of $G'[B]$. We first argue that in any cut set $S \subset V(G')$ leaving $G'[A]$ or $G'[B]$ connected, $|S| \geqslant \delta$. We first note that removing S must disconnect $G'[A]$ or $G'[B]$ to disconnect G' (since A and B are connected by a perfect matching). Therefore, without loss of generality, suppose removing S disconnects $G'[A]$ but leaves $G'[B]$ connected. Let $S_A = S \cap V(G'[A])$ and $S_B = S \cap V(G'[B])$, and let v be a vertex in some component C of $G'[A] - S_A$. Denote the set of vertices in $G'[B]$ adjacent to the vertices in C as C_B. In order to disconnect the component that v is in, we must remove all of the vertices in C_B (since S leaves $G'[B]$ connected). Thus, any cut set of G' must have cardinality at least $|S| \geqslant |C_B| + |S_A| = |C| + |S_A| \geqslant |N(v) \cap C| + |N(v) \cap C_B| + |N(v) \cap S_A| \geqslant d(v)$. Thus, we conclude for this case that $\kappa(G') \geqslant \delta(G')$. Since removing the neighbors of a minimum degree vertex form a cut set of size $\kappa(G')$, we conclude that $\kappa(G') = \delta(G')$ for this case.

Now suppose that any cut set S of G' disconnects both $G'[A]$ and $G'[B]$. Then if G was the original graph that was used in construction 2.1, $\kappa(G') \geqslant \kappa(G) + \kappa(G^c)$, where G^c is the complement of G. Since for all graphs, $\kappa(G) + \kappa(G^c) \geqslant \min\{\delta(G), \delta(G^c)\} + 1 = \delta(G')$

(see [16]). Thus, in this case, $\kappa(G') \geqslant \delta(G')$. Since removing the vertices of a minimum degree vertex form a cut set of size $\kappa(G')$, $\kappa(G') \leqslant \delta(G')$. Hence, $\kappa(G') = \delta(G')$ for this case. $\square$

Chapter 4

Diameter-2-Critical Graphs where $\delta \leqslant \alpha$

4.1 Original Conjecture of Study

As mentioned, we originally studied D2C graphs in order to find properties which extended to general diameter 2 graphs. We first used the computer program Conjecturing (see [18]) to find and prove conjectures specific to D2C graphs. One of these conjectures became of great interest, as it was not true for general diameter 2 graphs, but was true for tens of thousands of D2C graphs that were tested in Sage (see [23]).

Conjecture 8. *For all D2C graphs,* $\delta(G) \leqslant \alpha(G)$.

Since the independence number of a graph is NP-hard to compute, we cannot use brute force calculations to find the independence number of D2C graphs. Thus, finding the relationship between the minimum degree and the independence number can be quite difficult for a class of graph as broad as D2C graphs.

This led to the creation of a number of different proof techniques, which we will discuss in Section 4.2.

4.2 Proving that $\delta \leqslant \alpha$

There are several useful proof methods for proving that $\delta \leqslant \alpha$ for a D2C graph. These include finding a stable vertex, as well as assuming that $\delta > \alpha$ for a class of D2C graphs, and getting a contradiction.

4.2.1 Stable Vertex Argument

A stable vertex is a vertex v such that the subgraph induced by $N(v)$ forms an independent set S. If v is a stable vertex, then we see that $\delta(G) \leqslant d(v) = |S| \leqslant \alpha(G)$. Therefore, if a graph contains a stable vertex, then $\delta \leqslant \alpha$.

Proposition 10. *If G contains a stable vertex, then $\delta(G) \leqslant \alpha(G)$.*

Since all the vertices of a triangle-free graph are stable vertices, we immediately get the following lemma.

Lemma 1. *If G is triangle-free, then $\delta(G) \leqslant \alpha(G)$.*

We originally conjectured that every D2C graph contains a stable vertex. A subgraph of the Cameron Graph (see [2]), shown in Figure 4.1, was the first counterexample found (for its G6 String, see Appendix A).

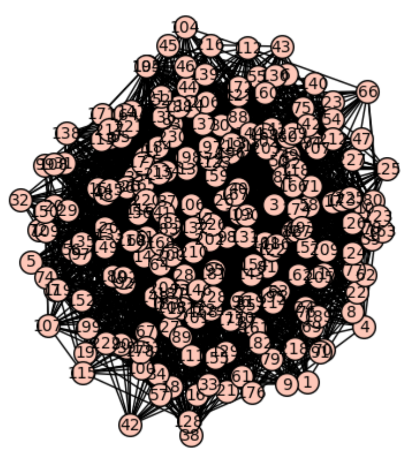


Figure 4.1: First D2C graph found with no stable vertex

Given that that there are infinitely many counterexamples to Conjecture 8 (see Section 5.1), there are also infinitely many D2C graphs that do not contain a stable vertex.

We also note that D2C graphs with no stable vertex can also have the property that $\alpha(G) \geqslant \delta(G)$ (such as the D2C Cameron subgraph in Figure 4.1).

4.2.2 Looking at Subsets of D2C Graphs

For proving any conjecture about D2C graphs, we can first try to prove the conjecture for smaller classes of graphs. This originally led to the proof that all D2C graphs with $\alpha = 2$ have the property that $\alpha \geqslant \delta$ (see Theorem 14), and allowed us to classify all D2C graphs with $\alpha = 2$ (see Section 4.4).

Another class of graphs for which some results on D2C graphs are known are regular D2C graphs (see Section 2.5). It is known that for all 2- and 3-regular D2C graphs, $\alpha(G) \geqslant \delta(G)$ (see Section 4.4). We conjecture that $\alpha(G) \geqslant \delta(G)$ for all 4-regular D2C graphs.

4.3 Proving that $\delta \leqslant \alpha$ for all D2C Graphs with $\alpha = 2$

We already know that all complete bipartite graphs are D2C. Since all complete bipartite graphs are triangle-free, by Lemma 1, we have that $\delta \leqslant \alpha$ for all complete bipartite graphs.

A more interesting class of D2C graphs where $\delta \leqslant \alpha$ are $\alpha = 2$ graphs.

Theorem 14. *If G is a D2C critical where $\alpha(G) = 2$, then $\alpha(G) \geqslant \delta(G)$.*

To prove this, we will first prove two lemmas from which the result will directly follow.

Lemma 2. *If G is a diameter 2 graph with $\alpha(G) = 2$, then for any $v \in V(G)$, the set of vertices S distance 2 from v forms a clique.*

Proof. Suppose S is not a clique. Then there exists two non-adjacent vertices $s_1, s_2 \in S$. We then see that v, s_1, s_2, form an independent set of size 3. This contradicts our assumption that $\alpha(G) = 2$, and so we conclude that S must be a clique. $\square$

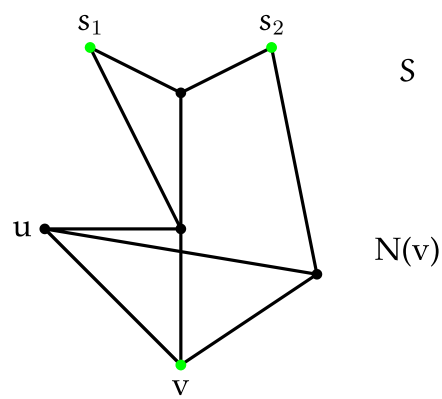


Figure 4.2: If S is not a clique, then the green vertices form an independent set of size 3

Lemma 3. *If* G *is a D2C graph with* $\alpha(G) = 2$*, then for any* $v \in V(G)$ *where* $d(v) = \delta$*,* $N(v)$ *is an independent set.*

Proof. Let $v \in V(G)$ have minimum degree, and suppose that for $u, w \in N(v)$, $uw \in E(G)$. We argue that uw is not a critical edge, contradicting our assumption that G is D2C. We first see that since G has diameter 2, removing uw cannot increase the distance between v and any of the vertices in S. We also see that the vertices in $N(v)$ all share v as a neighbor, and so vertices in $N(v)$ will still be at most distance 2 apart in $G - uw$. Since (by Lemma 2) S forms a clique, these vertices are all distance 1 apart.

Now consider $s \in S$. Since v has minimum degree, it follows from Theorem 6 that u contains a neighbor $r \in S$. Since S is a clique, s is adjacent to (or is) r. Thus, u and s are at most distance 2 apart in $G - uw$. A similar argument works to show that the vertices w and s are at most distance 2 apart. Having considered all pairs of vertices in G, we conclude that $G - uw$ has diameter 2. This contradicts our assumption that G is D2C, and so it must be true that v is a stable vertex. $\square$

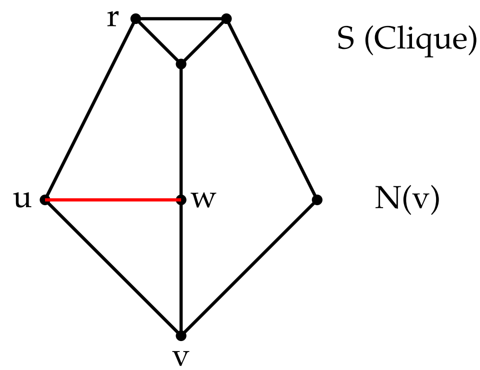


Figure 4.3: Any edge in $N(v)$, such as the red edge uw, is not critical

Thus, any vertex of minimum degree in a D2C graph with $\alpha = 2$ is a stable vertex. So we conclude that $\alpha \geqslant \delta$ for all D2C graphs with $\alpha = 2$.

4.4 Characterizing D2C Graphs with $\alpha = 2$

We can use the result from Theorem 14 to characterize all D2C graphs with $\alpha = 2$. To do so, we will argue that if v is a minimum degree vertex, then $|S| \leqslant 2$, where S denotes the set of vertices distance 2 from v.

We will again prove two lemmas, from which the result will follow.

Lemma 4. *There does not exist a D2C graph* G *with the following properties: (i)* $\alpha = 2$, *(ii) if* $v \in V(G)$ *has minimum degree, then* $|S| \geqslant 3$, *and (iii)* $d(v) = 1$.

Proof. Let $u \in N(v)$. If $d(v) = 1$, then in order for G to have diameter 2, u must be a dominating vertex. However, if $|S| \geqslant 3$, and S is a clique, then G is not a star. From Corollary 3, G is not D2C. □

Lemma 5. *There does not exist a D2C graph* G *with the following properties: (i)* $\alpha = 2$, *(ii) if* $v \in V(G)$ *has minimum degree, then* $|S| \geqslant 3$, *and (iii) if* $u, w \in N(v)$, *then* u *and* w *share a common neighbor* $x \in S$.

Proof. We first note that by Theorem 14 and Lemma 4, v must have exactly two neighbors u and w. Suppose that u and w share a common neighbor $x \in S$. We will argue that either ux or wx is not a critical edge. Since G is D2C and has $\alpha = 2$, u and w must form an independent set (Lemma 3), and S must be a clique (Lemma 2).

Since $|S| \geqslant 3$, and since $N(v)$ must form a dominating set, at least one of u or w has at least two neighbors in S. Without loss of generality, suppose that u has at least two neighbors in S.

We now argue that ux is not a critical edge. Since $wx \in E(G)$, $d(v, x) = 2$. Since S is a clique, and both u and w have a neighbor in S, for all $s \in S$, $d(u, s) \leqslant 2$ and $d(w, s) \leqslant 2$. Furthermore, we see that if in $G - ux$, v is distance 3 (or more) from a vertex $s \in S$, $s \neq x$, then v needed to be distance 3 from s in G. Since G has diameter 2, we conclude that this is not possible. Finally, since u and w share a common neighbor v, they are still distance 2 apart in $G - ux$. Thus, we conclude that $G - ux$ has diameter 2.

Thus, the edge ux is not critical, and so we conclude that such a graph G does not exist. □

Lemma 6. *There does not exist a D2C graph* G *with the following properties: (i)* $\alpha = 2$*, (ii) if* $v \in V(G)$ *has minimum degree, then* $|S| \geqslant 3$*.*

Proof. Let $u, w \in N(V)$ as in the proof of Lemma 5. Since $|S| \geqslant 3$, and since $N(v)$ must form a dominating set, at least one of u or w has at least two neighbors in S. Without loss of generality, suppose that u has at least two neighbors in S. Suppose that u has neighbors $x, y \in S$.

We now argue that xy is not a critical edge. We first see that since S is a clique, removing xy will not increase the distance between any two vertices in S above 2. We also see that $N(v)$ still forms a dominating set, and so v is still distance 2 from vertices in $N(v)$ and in S. We also see that since the vertices in $N(v)$ all still share a common neighbor v, these vertices are still at most distance 2 from each other.

Next, since S is a clique in G and u is adjacent to x and y, removing xy from G will still leave u at most distance 2 from every vertex. Finally, by Lemma 5, w shares no common neighbors with u, and by Theorem 6, it contains some other neighbor $z \in S$. Since S was originally a clique, z will still be distance 1 from all of the vertices in S after removing xy. Thus, w will still be distance at most 2 from every vertex in S. Thus, the edge xy is not critical, and so we conclude that such a graph G does not exist. $\square$

Thus, if G is a D2C graph with $\alpha = 2$, then given a minimum degree vertex $v \in V(G)$, $|S| \leqslant 2$. Since it follows from Theorem 14 that $\delta(G) \leqslant 2$, we see that the only D2C graphs with $\alpha(G) = 2$ are P_3, C_4, and C_5.

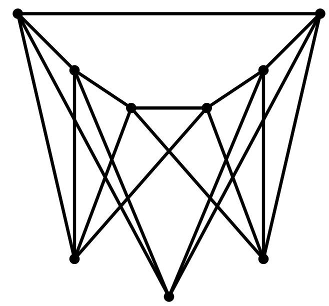
Chapter 5

Counter-examples to Select D2C Graph Conjectures

5.1 Diameter-2-Critical Graphs Where $\delta > \alpha$

Using the construction of D2C graphs outlined in [19], we can show that Conjecture 8 is not true for all D2C graphs. The construction of a counterexample relies on the fact that Loh-Ma's Construction 2.1 requires a graph with diameter 2 whose complement has diameter 2, but neither graph needs to be D2C. We note that for diameter 2 graphs, it is possible for $\delta > \alpha$. In fact, the difference can be quite stark. Consider an n-vertex complete graph G with exactly one edge removed. Then G has diameter 2, but $\delta(G) = n - 2$ and $\alpha(G) = 2$.

Finding a diameter 2 graph H whose complement H^c has diameter 2, while also having that $\delta(H) > \alpha(H)$ and $\delta(H^c) > \alpha(H^c)$ is more difficult. This is because graphs with larger minimum degree and smaller independence numbers generally have larger clique sizes. However, larger graphs have more room to space out edges in order to have a large minimum degree and both a small independence number and clique size. An example of a fairly small diameter 2 graph G with $\delta(H) > \alpha(H)$ and $\delta(H^c) > \alpha(H^c)$ is shown below:



We see that we cannot create a graph with $\delta > \alpha$ simply using Loh-Ma's Constructions 2.1 or 2.2 with this graph G. We can, however, use G to create a larger diameter 2 graph H where the minimum degree is much larger than the independence number in both H and H^c. This lead to the construction of the graph in Figure 5.1, a D2C graph with minimum degree 9 and independence number 8.

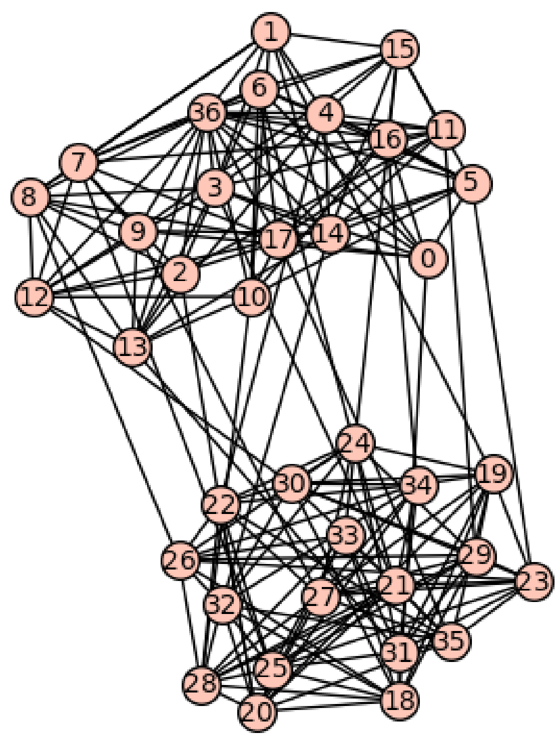


Figure 5.1: First Counterexample Constructed

To find an infinite family of graphs for which $\alpha < \delta$, it is helpful to look at self-complementary graphs. These are graphs G that are isomorphic to their complement graph G^c. This makes computing the independence number and minimum degree easy when using Loh-Ma's Construction 2.1 or 2.2. It also assures that, provided that G has diameter 2, G^c has diameter 2, allowing it to be used in Loh-Ma's Construction 2.1.

One such class of self-complementary graphs is known as Paley graphs. Paley graphs G are strongly regular graphs with q vertices such that q is a prime power, $q \equiv 1 \mod 4$, and each vertex has a distinct label from 0 to $q-1$ such that given two distinct vertices labeled r and s, r and s are adjacent if and only if $r-s$ is a square number in $\mathbb{Z}_q$.

By their construction, Paley graphs with q vertices are regular graphs, where each vertex has degree $\frac{q-1}{2}$ [25]. It is also known that the clique number of these graphs are at most $\sqrt{q}$ [25]. Since Paley graphs are self-complementary, for any Paley graph G and its complement G^c, $\omega(G) = \alpha(G) = \omega(G^c) = \alpha(G^c)$. Therefore, if we use Loh-Ma's Construction 2.1 with a Paley graph G, we would have that $\delta(G) = \frac{q-1}{2} + 1 = \frac{q+1}{2}$ and $\alpha(G) \leqslant 2\sqrt{q}$. Since $2\sqrt{q} < \frac{q+1}{2}$ when $q \geqslant 14$, any Paley graph with at least 14 vertices can be used in Loh-Ma's Construction 2.1 to create a D2C graph with $\delta > \alpha$.

In fact, since $\frac{d}{dq}(2\sqrt{q}) < \frac{d}{dq}(\frac{q+1}{2})$ when $q \geqslant 14$, the gap between δ and α can be arbitrarily large.

Proposition 11. *Given any natural number k, there exists a D2C graph such that $\delta > \alpha + k$.*

The smallest Paley graph G for which Loh-Ma's Construction 2.1 can be used is the Paley graph of order 13, where the graph G' has order 26, independence number 6 and minimum degree 7. This is the smallest known D2C graph (by its order) for which $\alpha < \delta$.

It is also known that the maximum common neighbors between a pair of vertices in a Paley graph is $\frac{q-1}{4}$ [25]. Given that in Loh-Ma's Construction 2.1, the independence number in the D2C graph G' is at most $2\sqrt{q}$, it will be true that the maximum common neighbors in G' is greater than the independence number of G' whenever $\frac{q-1}{4} > 2\sqrt{q}$. This is true when $q \geqslant 66$.

Since $\frac{d}{dq}(2\sqrt{q}) < \frac{d}{dq}(\frac{q-1}{4})$ when $q \geqslant 66$, we get a proposition for the maximum common neighbors analogous to Proposition 12.

Proposition 12. *Let* c *be the maximum common neighbors of any pair of vertices in* G*. Given any natural number* k*, there exists a D2C graph such that* $c > \alpha + k$*.*

5.2 Diameter-2-Critical Graphs Where distinct_degrees(G)> $\alpha(G)$

The parameter distinct_degrees(G) refers to the number of distinct integers in the degree sequence of G. For all D2C graphs where $n \leqslant 11$, distinct_degrees(G) $\leqslant \alpha(G)$. However, we can construct a larger D2C graph such that distinct_degrees(G) $> \alpha(G)$.

Begin with the Paley graph P with order 13. Label the vertices of P as done in Figure 5.2 (using the same labels $v_0, v_1, ..., v_{12}$).

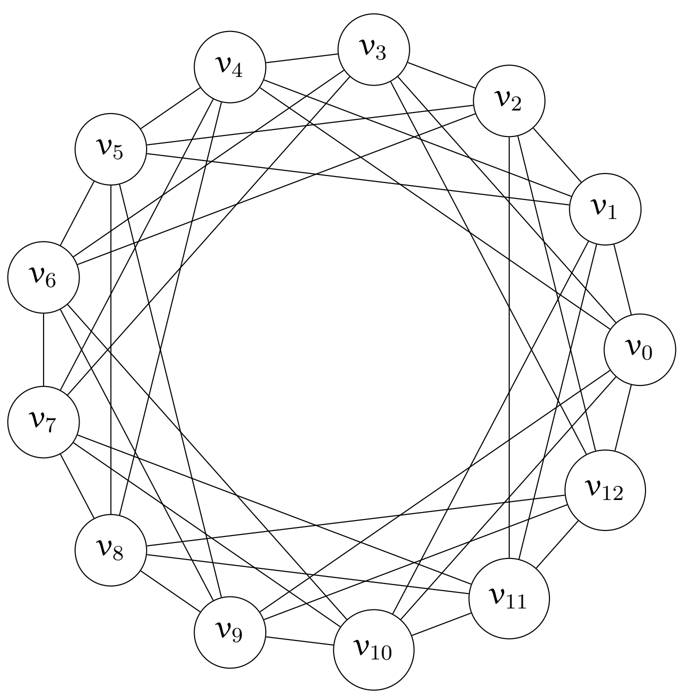


Figure 5.2: Paley Graph on 13 vertices

Create a new graph G by adding the edges v_1v_3 and v_3v_9 from P, and then removing the edges v_7v_8 and v_7v_{11}. This new graph G has five distinct degrees: $d(v_3) = 8$, $d(v_1) = d(v_9) = 7$, $d(v_8) = d(v_{11}) = 5$, $d(v_7) = 4$, and all other vertices have degree 6. Its complement G^c also has the same five distinct degrees.

We can then use Loh-Ma's Construction 2.1 on G to create a D2C graph G' which will again have 5 distinct degrees (5,6,7,8,9). Label the vertices of G' containing a copy of G^c as $v'_0, ..., v'_{12}$ so that v_i and v'_j are endpoints of the perfect matching if and only if $i = j$. Although $\alpha(G) = 4$ and $\alpha(G^c) = 4$, no two maximum independent sets can be used from G and G^c to form a maximum independent set in G'. Thus, $\alpha(G') \leqslant 7$. Given that the vertices $v_0, v_1, v_7, v_8, v'_3, v'_9, v'_{12}$ form an independent set, we have that $\alpha(G') = 7$.

Finally, we use Loh-Ma's Construction 2.2 on G' to create a D2C graph with three more distinct degrees and the same independence number as G'. To do this, join two vertices x and y to the vertices making up the copy of G in G'. Call this new graph G''.

Since $\alpha(G^c) = 4$, we cannot have that $\alpha(G'') > \alpha(G')$. Thus, $\alpha(G'') = 7$. We also see now that $d(v_1) = d(v_9) = 10$, $d(v_3) = 11$, and $d(x) = d(y) = 13$, so three new distinct degrees have been added to G'. Therefore, G'' has eight distinct degrees and independence number 7, making it a counter-example to Conjecture 3. (For the G6 string of G'', see Appendix A).

We conjecture that the gap between the number of distinct degree in G and the independence number of G can be arbitrarily large.

Conjecture 9. *Let* D *be the number of distinct integers in the degree sequence of* G*. Given any natural number* k*, there exists a D2C graph such that* $D > \alpha + k$*.*

5.3 Conclusion

There are still many open questions with regards to the independence number of D2C graphs. While we know that Conjecture 8 is false for infinitely many graphs, it is unclear what the smallest order of a counterexample is. It is also unclear what the smallest value of α is in a counterexample to Conjecture 8. Indeed, there are also still many classes of D2C graphs for which it does appear that Conjecture 8 is true. These include 4-regular, 5-regular, and $\alpha = 3$ graphs. Similar questions remain for Conjectures 2 and 3. Perhaps most notably, it remains unknown if Conjecture 7 is true for all D2C graphs. The result of Theorem 13 provides evidence that it could be.

Bibliography

[1] B. Bollobás, Random graphs, Cambridge Studies in Advanced Mathematics, Series Numbers 73, 2nd Edition, Cambridge University Press, Cambridge, 2001.

[2] "Cameron Graph", https://en.wikipedia.org/wiki/Cameron_graph.

[3] A. Dailly, F. Foucaud, A. Hansberg, *Strengthening the Murty-Simon conjecture on diameter 2 critical graphs*, Discrete Mathematics. Elsevier, **342**, no. 11 (2019), 3142–3159. ⟨10.1016/j.disc.2019.06.023⟩

[4] M. D. Barrus, *Havel–Hakimi residues of unigraphs*, Information Processing Letters. **112**, no. 1 (2012), 44–48.

[5] R. Eggleton, J. Macdougall, *Triangle-free and Triangle-saturated Graphs*, Journal of Combinatorial Mathematics and Combinatorial Computing. **25** (1997), 3–21.

[6] R. Eggleton, J. MacDougall, *Minimally triangle-saturated graphs: adjoining a single vertex.* Australas. J. Combin. **25** (2002), 263—278.

[7] G. Fan, *On Diameter 2-critical graphs*, Discrete Mathematics **67** (1987), 235–240.

[8] O. Favaron, M. Mahéo, and J-F. Saclé. *On the residue of a graph,* Journal of Graph Theory. **15**, no. 1 (1991), 39–64.

[9] Z. Füredi, *The maximum number of edges in a minimal graph of diameter 2*, J. Graph Theory, **16**, no. 1 (1992), 81-–98.

[10] M. R. Garey, D. S. Johnson, Computers and intractability. A guide to the theory of NP-completeness. A Series of Books in the Mathematical Sciences. W. H. Freeman and Co., San Francisco, Calif., 1979.

[11] F. Gliviak, *On the impossibility to construct diametrically critical graphs by extensions*, Archivum Mathematicum, **11**, no. 3 (1975), 131–137.

[12] S. Hakimi, *On the realizability of a set of integers as degree sequences of the vertices of a graph.* SIAM Appl. Math. **10** (1962), 496-–506.

[13] V. Havel, *A remark on the existence of finite graphs.* Časopis Pěst Mat. **80** (1955), 477-–480.

[14] T. W. Haynes, M. A. Henning. *A characterization of P5-free, diameter-2-critical graphs*, Discret. Appl. Math. **169** (2014), 135–139.

[15] T. W. Haynes, M. A. Henning, L. C. van der Merwe, A. Yeo, *Progress on the Murty–Simon Conjecture on diameter-2 critical graphs: a survey.* Journal of Combinatorial Optimization **30** (2015), 579–595.

[16] A. Hellwig, L. Volkmann, *The connectivity of a graph and its complement*, Discrete Applied Mathematics, **156**, no. 17 (2008), 3325–3328

[17] D. E. Knuth, *Mathematics and Computer Science: Coping with Finiteness*. Science. **194**, no. 4271 (1976): 1235–1242.

[18] C. E. Larson, N. Van Cleemput, *Automated conjecturing I: Fajtlowicz's Dalmatian heuristic revisited*, Artificial Intelligence, **231** (2016), 17-–38.

[19] P. Loh, J. Ma, *Diameter critical graphs*, Journal of Combinatorial Theory, Series B, **117** (2016), 34–58. https://doi.org/10.1016/j.jctb.2015.11.004.

[20] J. Radosavljević, M. Živković, *The List of Diameter-2-Critical Graphs with at Most 10 Nodes*, IPSI Transactions on Advanced Research. **16**, no. 1 (2020), 34–38.

[21] R. Pepper, ”On the Annihilation Number of a Graph, Proceedings of the 15th American Conference on Applied Mathematics. World Scientific and Engineering Academy and Society.” (2009), 217–220.

[22] M. Rosenfeld, *Independent sets in regular graphs*, Israel J. Math. **2** (1964), 262-–272. https://doi.org/10.1007/BF02759743

[23] SageMath, the Sage Mathematics Software System (Version 9.5), The Sage Developers, 2022, https://www.sagemath.org.

[24] G. Szekeres, H. Wilf, *An inequality for the chromatic number of a graph*. J. Combinatorial Theory. **4** (1968), 1-–3.

[25] E.W. Weisstein, ”Paley Graphs.” https://mathworld.wolfram.com/PaleyGraph.html.

[26] D. B. West, Introduction to graph theory. 2nd edition, Prentice Hall, Inc., Upper Saddle River, NJ, 2001.

[27] W. Willis, ”Bounds for the independence number of a graph.” Masters Thesis, (Virginia Commonwealth University, 2011)

Appendix A

G6 Strings for select D2C graphs

- Counter-example to conjecture 3:

```
'[?G?PGqMPxbQBqH|A]V??s?FG?MG?}C?v@?LoG@m?_Ew@?Ko@Clo?eEw?H_m?@kL'
```

- Cameron D2C subgraph:

```
"~?Bf???OPOAGPIWhFhw^\_??@?????????????EA???_??g??GP???a??DK?c@GwGCRoB?
No?????????A???\_C?????@?C????O???H?????U@?G?AHAa???r???A?B?????AF??CA`
V?@O??@o?????\_?\_?C?????????O??????????????aA@??O????????CC??s??????cH
_??????B_GK??gC\_`??\_???OA\_?Q???OPo??A?????@???????GE????@\_???C??\_?
????@W??????O?`????A????O??G????CAH??????c_@K????@??GCZ@_@E?W?_OOFGA_QD
AA?CF?`@@@e?CdB_????@??O?????_??????????G??AD?Q???????????C??O?O???????
AD?CG?C?oAE??A????G???a??W??????_??COR????K?g??C?m??\_????P_?SE?OI?@???
??CGI????@?????????????O????P?????CCA?G?A`??W?O@@\_?@G_?oCC???CC???????
?_???a??O?????H_CGCA??????GAa@GO?AOA???GOCB?CA?GW@????@?sGC?`??I@@??@@o
??????`c?G???C@???@????????AE????A???????B?CO????AA_?_?A@???????O????@_
`CC_?CAa@@O@O???@@E????@?_AA?COO@??CG???????????Ara?C???????A??`BoCK?c?
CK_@??OE?M??PACC??__??????????????OcOgR???????????????????_J?CB?AC@????
```

??CO?H@C????????AWE?pH_?????gO?C@??_C?A@?AOD????C?KC??gG_???@_?????GA?
??cC@??O??CKI_?C??G???????I?OA`?I@????????G?????G?G???D@??????@?C?AG_??
??C??G??????CA?o???_Bc?E?GOCOC??iC???S?G`_??G?G??A`?@A?HOKC`???`O??O_\
_C??IGM?@C??C?C???C???GPCGG?@?H??G???CC????DE??c????G?????AA?@_G_O?_???
?cA?C?AP?gA@a?I??g??_G??@?GO?O?@?OO????G@??????_??cCG?C?????????????APg
\}_?OOW????G@?G?CQCD?R??_@G???A?????ACH?A_????OOC??Go??????@@a_GK?H????
GA????????C????@??????O?@_G??_?gK_I?A???CGAO???@????W_??CGK?G??A_???A??
?COo??OD?????S??Ca??O??`G?A@GO???O?HO_?B???@Eg_C_CD_@?CAA??CO?A?cO?x?SH
A?I?GS?H?K@??C@@?W?_Q?@OCG?iC??OQH?`@_E?o?J?B?C_@?O????CO?@D????_??I?O@
CA???D@CO??C@?P?W@?G???`????cFC@???GA_E?O_H_H?G?_?OCO?G??@G?G?AH_???c_C
??@?SA???o?@??@@O?ED???IA___G??A_AA??A?G?C?G???SG?????Ao@O????O??OO?
?KGac?G_?F?A@CK???OC??C?????????ACo?G????OoP_???A`????????????@?G_OC?g
AA????????@_??__??CG??????Of?aGG??E_?GA_OAQG??E??G?H?G????C??@GOIG??
????????D????Q_?????????GC?@G??GO??????G?C@??K??_??I???A_????O?@_?_????
@???O_?@C???OOK??W?@????????G?@C??CB??????_?@@?_A?O?????G???WDG???ACCA?
?@P??????IG????ALP???GQCG?OgaA??@B???B?@_?????P?C????O?????G??oP????G??
Z??G???OO??A_??O????_?s???B?????????eG?A__?????AOA?Cg?????AA?OO@?AE????
??????YeCK@B?????P_OE?G?_CA?GGA?YCe?C@OG@??A@??GGD??CC????S__CGAO_?@O?@
_?@??Gc?_???@H`OC@?????????CG???_?W?CO?I?AC?????G_??i?H????????C???P_?`
_????AE???O@A@O??`??????@G??G?O?????C_????A????C?O??[?????A?A???__?????
??P@I@??I???????CA??G_?AE??@?O?C_Y?GkCA??????CP??CC??C???A_KAO?p_????@?
C?_?A_OI@?_?C??gDGQO??AB?AAOCa??G`_G?A???AAK??Z?C??@@??A??AA???AE??@???
H??HsG?EA?OC????????????C??Q??T?cB?????CG?????@??OI?_??O?gg?_?????A???
??A???A?_G????KE@??OC?C??????AD??????@???Q??W?KH?H?A?g???c?C_g??AAG???K
?@OG_?GA?GDC???`cP@CQ?H?AC??????????G??????O@O??W??A_A??Cg_??@?F???????
?SA??????G?_POP_??A?_AOB?AGHA?__A_?????D?Og?????g???@B?????????GO??A??F
???CgC_E?W_??????A??G??@?O??G??????K?A?`o?????g`AA???Se???OH@C?@_??oc_@

```
?_???Q_DCOcPC?@G@?S?CGG???_K?_GG?@A_?S@????@O?K???C_@?G??I?M?pC???G_???
??????A?_AODHOC_O_AKACO??G?G?AC_??????@?Q?W?AG@KR@?WAQ@OAB@P???GOG@A???
??_?BAq???HWO?HA@?_?CCCA@O?C?@?G_???KA??Co?Ao????@_C????B?I@??bAGE?E?O_
C??C??????DA‘G??@PID@?A?E????AA????‘?SO_a?C??@a?QOoOo?g?S???????????OOG
O_Og?????g??C?@@??QCC@??@COOA?OO???KAA?c??D??A???@???@O@??jC?OI?????HO?
?OC@_o???E@_?T?@??HA?g?OG????@C????A‘B?AO?O??AOg??@?A‘OW@@?__K?AK??IGgG
??CO???oC?????A__C___DAO@??_C???__??@ACCc???E?Q?G????CAO@OGG?E?_ACb??@O
????GgGOG??GA?GGC_?a?GGO@_C_??????@_E???g??G???@AA_?Ga?@A?@A???_aCG?Gh?
?_?G??????_OoGG?qA?_?O?_A@???@g?OO_gAA?????A??O_CGOO_?_ABO??@?D?G??_A?G
O????SC@@_G?@C??gGaC_??I@G??_Ga?_??@A?AG?OGO????D??Gc???????a?S?O?@?_OP
???DA????GIE??A???CO?????G@B?C_???ACC‘OA????CC???QA??c???‘?OHC?@O????\_
Go?_?G?gCG_a??C_?@OL??@?G???A???????K@A????iG?@H?OA??OGIG@????AGIGOPB?D
A??CA?O???ABO????@@?G?Oa?C_C_@O???H@G_G_??_??@???????A?F?@O_??D?W@?H?Q?
?@S?@??SG??CAa??gC_GA_@??a?GCAODG?QEA?KC????A??C???cBS?C?A??S_A?A?‘????
????Cg???????IC_??F@?C?_GCc?@c?AOA@A?????_?????O@?AAAaOBCdC??_@?SO\_\_?
??\_\_SoOoSGA???@aS???A?@??O?A????????H???ICA_aF?C?AGKGO?@???_?S?S???@@
?O@QG????????C_aE_Q?o?OG??@GG??@??CCOOACaI_?CA_?Sd?P_??Q?????c?A?@?C??C
??????CO@G?E_?_???I?A@??‘??OAa_?O?Q__S?C??CIAE??G_A??@P???Oa??I??O?WG?O
P?QCW@I?G@AO?G?@?@???C_c?D???_G????i??_????oC??aGW???‘Ac?AC_AA?C@C?_?I?
_??@E?OK???@‘@Q?????AK@CO?A?@?A@???C_?GSO???_G?A@CoQ???C?CO??@P??qO@??O
‘@?O?????????O??C?gIC??CCg_Q?@K?A?_GO???G???????????????CA?O??__??K?GsH
D?GO_PG??O@_?OC?‘_??g???@_@????_D_H?d?G??C??G@‘PA@O?OCg??????????BCg?C_
?BD??@@G‘a??G????CDA?_@AGG?CQ???A_G????GkO??SC?_??_??????_??@O?O_QE?_?G
AA?EC??@F??aI???C????A_??GG??????cIO?ObOKG???S?OO?_?_?w??"
```


- Graphs with order 6 to 11 with $\delta \geqslant 3$ and $\delta = \alpha$:


```
"GCR‘vo", "GCrb‘o", "GCZJdo", "GCY^B_", "HCOfeW{", "HCQb‘rK",
```

"HCQbepl", "HCQeR‘}", "HCQeRg}", "HCRbdQY", "HCp‘eji", "HCpdehi",
"I?B~vrw}?", "I?rFf_{N?", "I?rDrqs\?","ICQ‘fIku_", "J?‘FE_{~@}?",
"J?‘FEc{NFw?", "J?‘F?~oNFw?", "J?‘DQnoNFw?", "J?bFF‘wN?{?",
"J?bBeLwNDs?", "J?bDMpw]@m?"

- D2C graphs with $7 \geqslant n \geqslant 10$ and $\delta \geqslant 3$:

’I???F~}~_’, ’I??E@~{~?’, ’I??F?~{~?’, ’I??F~z{~?’, ’I?AAFo}~_’,
’I?AEDp{^o’, ’I?AEFo}^?’, ’I?AEDx{]o’, ’I?AEDt{^?’, ’I?ABCt{~_’,
’I?ABEt{n_’, ’I?ABC~w^?’, ’I?ACNp{]_’, ’I?AFCp{^o’, ’I?AFEp{No’,
’I?AFCx{]o’, ’I?AFCt{^?’, ’I?AF?~w^?’, ’I?ABeT{n_’, ’I?ABc^w^?’,
’I?AELp{]o’, ’I?AFfP{Fo’, ’I?AFeX{Mo’, ’I?AFvp{^?’, ’I?BEDp{]o’,
’I?BEDo}]O’, ’I?BEFo}}?’, ’I?BEDxy^?’, ’I?BDEpyNO’, ’I?BD@xY}?’,
’I?BD@x{u_’, ’I?BD@xyu_’, ’I?BD?~w^_’, ’I?B@‘zI~_’, ’I?B@‘ziv_’,
’I?B@‘zM}?’, ’I?B@c^w^_’, ’I?B@d\\{u_’, ’I?B@d\\]}?’, ’I?BF@zWNg’,
’I?BF@zWMw’, ’I?BF@w{u_’, ’I?BDePyNO’, ’I?BDdPZ^?’, ’I?BDeS{^?’,
’I?BDeS{]_’, ’I?BDbS{u_’, ’I?BDbS{uG’, ’I?B@vC{v_’, ’I?B@vE{V_’,
’I?B@uK{}?’, ’I?B@tNwV_’, ’I?BDMp{Mo’, ’I?B@mZwMo’, ’I?B@lZwUo’,
’I?B@hzI}?’, ’I?BFHzWMg’, ’I?BDjZWMw’, ’I?B@~bgFW’, ’I?B@~bgEw’,
’I?B@|pwpW’, ’I?B@|pwow’, ’I?B@xzw}?’, ’I?B@~rw}?’, ’I?Bcu_{Yo’,
’I?Bcsza^?’, ’I?B~vrw}?’, ’I?‘@F_]~_’, ’I?‘@FrUN_’, ’I?‘DFa{V_’,
’I?‘DBo{t_’, ’I?‘DBrsV_’, ’I?‘DBr[\\g’, ’I?‘DEhw^o’, ’I?‘@f_{r_’,
’I?‘@famV_’, ’I?‘@fC{v_’, ’I?‘@fE]^?’, ’I?‘@eUu^?’, ’I?‘CV_{~_’,
’I?‘CV‘{|_’, ’I?‘FAqsNo’, ’I?‘FAo}lO’, ’I?‘FAovn?’, ’I?‘FBqsF_’,
’I?‘FBo{d_’, ’I?‘F@w{v_’, ’I?‘FEc{~?’, ’I?‘F?~o~?’, ’I?‘Db_{ro’,
’I?‘DeHw^o’, ’I?‘DeHyNo’, ’I?‘Dbo{p_’, ’I?‘DfHwVo’, ’I?‘DeXw\\o’,
’I?‘DbVSNg’, ’I?‘DbS{t_’, ’I?‘DbLwvO’, ’I?‘Da^on?’, ’I?‘Da\\w|?’,

’I?‘EVasN_’, ’I?‘DQno~?’, ’I?‘FfbKNG’, ’I?‘FdZoVO’, ’I?‘F‘zo~?’,
’I?bBF_{u_’, ’I?bBBo{m_’, ’I?bBDhYm_’, ’I?bBDc]^?’, ’I?b@aTong’,
’I?b@aTwl_’, ’I?b@aUq^g’, ’I?b@aUy\\g’, ’I?b@ddM^O’, ’I?b@eVoN_’,
’I?b@bTYl_’, ’I?bAV_{]_’, ’I?bAV_{Kw’, ’I?bATqs[w’, ’I?bATrs[o’,
’I?bBbQseo’, ’I?bBbQYlO’, ’I?bBbO]lo’, ’I?bB‘rI|O’, ’I?bBeUsm_’,
’I?bBeUsmO’, ’I?bBeUsMg’, ’I?bBeTsmG’, ’I?bB‘\\[kg’, ’I?bBQuw\\G’,
’I?bDMpsMw’, ’I?bDJpwd_’, ’I?bF‘xw{?’, ’I?bF‘xw]?’, ’I?bBrqw\\G’,
’I?bDjpw]?’, ’I?bDjpwl?’, ’I?‘bF‘wf_’, ’I?‘bEg{}_’, ’I?‘bEi{u_’,
’I?‘ad_}ro’, ’I?‘aeQuNo’, ’I?‘acgwzg’, ’I?‘acg{y_’, ’I?‘acii~?’,
’I?‘acgyz_’, ’I?‘acim}?’, ’I?‘afQqfo’, ’I?‘afQUNo’, ’I?‘adjINg’,
’I?‘adgyr_’, ’I?‘adg}qo’, ’I?‘aczoy_’, ’I?‘acza]_’, ’I?‘acyqy_’,
’I?‘adK]}?’, ’I?‘fBpwf_’, ’I?‘fCxwd_’, ’I?‘ebQ[ko’, ’I?‘ebQqfO’,
’I?‘eaqsJo’, ’I?‘eaqkLw’, ’I?‘eaoujW’, ’I?‘eaomlO’, ’I?‘eaquNO’,
’I?‘e‘qeVO’, ’I?‘e‘r[wo’, ’I?‘eeW{ko’, ’I?‘efD[n?’, ’I?‘ebVSn?’,
’I?‘eTjSN_’, ’I?‘eIqpN_’, ’I?‘cn@WNW’, ’I?‘cn@wFO’, ’I?‘cn@[MO’,
’I?‘cm‘wjO’, ’I?‘cm_xzG’, ’I?‘cjbgfW’, ’I?‘cjbWjW’, ’I?‘cmPpn?’,
’I?‘ckppzO’, ’I?‘ciroJO’, ’I?‘ciqszO’, ’I?‘ciox~?’, ’I?‘cioxz_’,
’I?‘cjpw~?’, ’I?‘cmhwjG’, ’I?‘cmhin?’, ’I?‘cjfgFW’, ’I?‘cmLwn?’,
’I?‘aljgfg’, ’I?‘aljING’, ’I?‘aliYzG’, ’I?‘ejpwn?’, ’I?‘c~‘w{?’,
’I?bfAoxd_’, ’I?bebOxl_’, ’I?bebpwN_’, ’I?bLbpw_’, ’I?rFf_{N?’,
’I?rDrqs\\?’, ’I?qa‘bu}O’, ’I?qa‘ro|_’, ’I?qa‘rSx_’, ’I?qa‘qe|?’,
’I?qa‘ped_’, ’I?qa‘ph|_’, ’I?qa‘iYZw’, ’I?qa‘i]Zo’, ’I?qadpUw_’,
’I?qabi[j_’, ’I?qabYYLw’, ’I?qabWuew’, ’I?qa‘nE}O’, ’I?qdRaYtO’,
’I?q‘qjo{?’, ’I?q‘qjoY_’, ’I?q‘qiw\\?’, ’I?q‘qiww_’, ’I?q‘uhsi_’,
’I?otQxef_’, ’I?opuJoBw’, ’I?opvJcv?’, ’ICOfBa[z?’, ’ICOfBa[Zo’,
’ICOf@pSb?’, ’ICOf@o]v?’, ’ICOf?zwpo’, ’ICOfEqsZW’, ’ICOfCxk{G’,
’ICOedO[|_’, ’ICOedRcN_’, ’ICOedO{x_’, ’ICOedPen?’, ’ICOedOm~?’,

```
'ICOebIgno', 'ICQ`fPUF_', 'ICQ`fIku_', 'ICQf@rWZG', 'ICQf@Xpv?',
'ICQe`pkmg', 'ICQe`qi^?', 'ICQb`rGzG', 'ICQb`rG^G', 'ICQb`rgV?',
'ICQb`rKr?', 'ICQeR_{\\_', 'ICQeR_{Xg', 'ICQeJ`wbO', 'ICQeHpwbo',
'ICQeHqk[o', 'ICQeHpqbo'
```