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Selected Problems in Graph Coloring

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DISSERTATION

SELECTED PROBLEMS IN GRAPH COLORING

A Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Virginia Commonwealth University.

by

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TABLE OF CONTENTS

Chapter | Page

LIST OF FIGURES

Abstract

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The Borodin–Kostochka Conjecture states that for a graph G, if $\Delta(G) \geq 9$ and $\omega(G) \leq \Delta(G) - 1$, then $\chi(G) \leq \Delta(G) - 1$. We prove the Borodin–Kostochka Conjecture for (P_5, gem)-free graphs, i.e., graphs with no induced P_5 and no induced $K_1 \vee P_4$.

For a graph G and $t, k \in \mathbb{Z}^+$ a *t-tone k-coloring* of G is a function $f : V(G) \to \binom{[k]}{t}$ such that $|f(v) \cap f(w)| < d(v, w)$ for all distinct $v, w \in V(G)$. The *t-tone chromatic number* of G, denoted $\tau_t(G)$, is the minimum k such that G is t-tone k-colorable. For small values of t, we prove sharp or nearly sharp upper bounds on the t-tone chromatic number of various classes of sparse graphs. In particular, we determine $\tau_2(G)$ exactly when $\text{mad}(G) < 12/5$ and also determine $\tau_2(G)$, up to a small additive constant, when G is outerplanar. Finally, we determine $\tau_t(C_n)$ exactly when $t \in \{3, 4, 5\}$.

CHAPTER 1

INTRODUCTION

Almost all definitions and notation needed for Chapter 2 and Chapter 3 follow [1]. Any that differ from [1], or are important to recall, will be explicitly mentioned. All graphs considered in this dissertation are assumed finite and simple.

We begin with key definitions and notations that we use frequently. We denote by $\Delta(G)$, $\omega(G)$, and $\chi(G)$ the *maximum degree*, *clique number*, and *chromatic number* of a graph G. A *vertex coloring*, or k*-coloring*, is a function that assigns a single color to each vertex so that adjacent vertices get distinct colors.

Common graphs for us include K_n the complete graph on n vertices; $K_{n_1,n_2,\ldots,n_k}$ the complete multipartite graph with k partite sets with size n_i for $1 \leq i \leq k$; E_n the edgeless graph on n vertices; P_n the path on n vertices; C_n the cycle on n vertices; and $P_n \square P_m$ the grid graph on $n \cdot m$ vertices where $n \geq m \geq 2$.

Chapter 2 proves the Borodin–Kostochka Conjecture for a certain class of graphs. We first present a relevant history of the conjecture and previous partial results. In Section 2.1 we discuss definitions and results needed for our new result. Finally, in Section 2.2 we prove the new result.

Chapter 3 introduces t-tone coloring and discusses its history. In Section 3.1 we review the history of 2-tone coloring, and in Section 3.2 we present our new results. Similarly, in Section 3.3 we review the history of t-tone coloring for $t \geq 2$, and we present our new results in Section 3.4.

CHAPTER 2

THE BORODIN–KOSTOCHKA CONJECTURE

Every graph G satisfies $\chi(G) \leq \Delta(G) + 1$. To see this, consider the vertices of G in any order, we color each with an arbitrary color in $\{1, \ldots, \Delta(G) + 1\}$ not already used on its neighborhood. This process always succeeds since $d(v) \leq \Delta(G)$ for all $v \in V(G)$. In 1941, Brooks [2] strengthened this bound to $\chi(G) \leq \Delta(G)$.

Brooks' Theorem. Let G be a graph. If $\Delta(G) \geq 3$ and $\omega(G) \leq \Delta(G)$, then $\chi(G) \leq \Delta(G)$.

In words, Brooks' Theorem states that a $\Delta(G)$-coloring exists for each connected G that is neither a complete graph nor an odd cycle; and the hypotheses $\omega(G) \leq \Delta(G)$ and $\Delta(G) \geq 3$ respectively prevent those scenarios. In 1977, Borodin and Kostochka [3] conjectured a further strengthening of Brooks' bound, with similar hypotheses.

Borodin–Kostochka Conjecture. Let G be a graph. If $\Delta(G) \geq 9$ and $\omega(G) \leq \Delta(G) - 1$, then $\chi(G) \leq \Delta(G) - 1$.

By Brooks' Theorem, each graph G with $\chi(G) > \Delta(G) \geq 9$ contains $K_{\Delta(G)+1}$. So the Borodin–Kostochka Conjecture asserts that each G with $\chi(G) = \Delta(G) \geq 9$ contains $K_{\Delta(G)}$. If true, the Borodin–Kostochka Conjecture is best possible in the following two ways. First, if the hypothesis $\Delta(G) \geq 9$ is weakened to $\Delta(G) \geq 8$ then the conjecture is false, as witnessed by G_1, on the left in Figure 1. Note that G_1 is a counterexample to this stronger version since $\Delta(G_1) = 8$ and $\omega(G_1) = 6 \leq 7 = \Delta(G_1) - 1$, but $\chi(G) = 8 > 7 = \Delta(G_1) - 1$. Second, if the hypothesis $\omega(G) \leq \Delta(G) - 1$

is strengthened to $\omega(G) \leq \Delta(G) - 2$, then the bound $\chi(G) \leq \Delta(G) - 1$ cannot be strengthened to $\chi(G) \leq \Delta(G) - 2$, as witnessed by G_2, on the right in Figure 1. For each $t \geq 9$, note that G_2 is a counterexample to this stronger version, since $\Delta(G) = t \geq 9$ and $\omega(G_2) = \Delta(G_2) - 2$, but $\chi(G_2) = \Delta(G_2) - 1$.

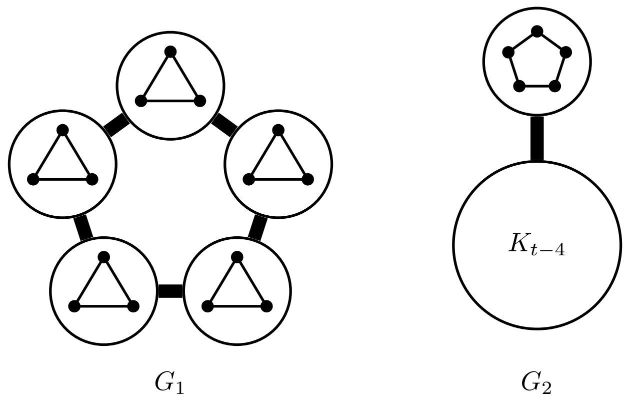

Fig. 1.: Each bold edge denotes a complete bipartite graph. We have $\Delta(G_1) = 8$, $\omega(G_1) = 6$, but $\chi(G_1) = 8$. And we have $\Delta(G_2) = t$, $\omega(G_2) = t-2$, but $\chi(G_2) = t-1$.

The strongest partial result toward the Borodin–Kostochka Conjecture is due to Reed [4]. In 1999, he proved this conjecture for every graph G with $\Delta(G)$ sufficiently large using probabilistic methods. Reed stated that a more careful analysis of his argument could reduce the lower bound on $\Delta(G)$ to about 10^3, but definitely not to 10^2.

Theorem 2.0.1. *[4] Every graph with $\chi(G) = \Delta(G) \geq 10^{14}$ contains $K_{\Delta(G)}$.*

The Borodin–Kostochka Conjecture has been proved for many interesting classes of graphs, particularly those defined by forbidden subgraphs. Rabern proved a stronger result for line graphs [5]. Dhurandhar [6] proved the conjecture for graphs that forbid as induced subgraphs $K_{1,3}$, $K_5 - e$, and another graph D that they define. This class of graphs is a superset of line graphs. Kierstead and Schmerl [7] proved the conjecture for graphs that forbid as induced subgraphs $K_{1,3}$ and $K_5 - e$.

Finally, Cranston and Rabern [8] proved it for graphs with no induced $K_{1,3}$. Note that *claw-free* means no $K_{1,3}$ is contained as an induced subgraph.

Theorem 2.0.2. *[8] Every claw-free graph with $\chi(G) \geq \Delta(G) \geq 9$ contains $K_{\Delta(G)}$.*

In 1998, Reed also conjectured [9] that $\chi(G) \leq \lceil(\omega(G) + \Delta(G) + 1)/2\rceil$. This conjecture, now called Reed's Conjecture, is weaker than the Borodin–Kostochka Conjecture when $\omega(G) \in \{\Delta(G) - 1, \Delta(G) - 2\}$, equivalent to it when $\omega(G) \in \{\Delta(G) - 3, \Delta(G) - 4\}$, and stronger otherwise. Chudnovsky, Karthick, Maceli, and Maffray [10] proved Reed's Conjecture for all (P_5, gem)-free graphs. To prove the Borodin–Kostochka Conjecture for the same class of graphs (Theorem 2.2.1) we use their structure theorem for (P_5, gem)-free graphs, as well as one of their key lemmas.

In Section 2.1 we mention all definitions and previous results needed to prove our result. Then in Section 2.2 we prove the result. We conclude this chapter with Section 2.3 containing future work.

2.1 Definitions and Lemmas

Most of our definitions follow [1], but we highlight a few terms. By *coloring* or *vertex coloring* we mean a proper vertex coloring. A graph is *k-colorable* (or has a *k-coloring*) if there exists a coloring $\varphi : V(G) \to \{1, \ldots, k\}$. Given two graphs S and T, their *join*, denoted $S \vee T$ is formed from their disjoint union by adding all edges with one endpoint in S and the other in T.

For graphs $H_1, \ldots, H_s$, a graph G is $(H_1, \ldots, H_s)$-free if G does not contain any of $H_1, \ldots, H_s$ as an induced subgraph. Thus, the class of (P_5, gem)-free graphs is all graphs that contain neither an induced P_5 nor an induced gem. Here *gem* is $K_1 \vee P_4$; see Figure 2.

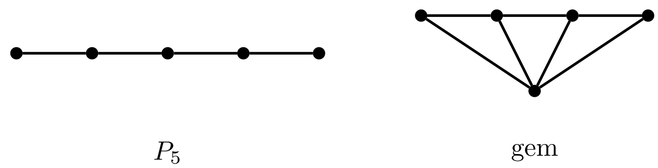

Fig. 2.: A P_5 and a gem.

A graph G is *perfect* if for every induced subgraph H of G we have $\omega(H) = \chi(H)$. An *odd hole* is an induced odd cycle of length at least 5, and an *odd antihole* is a complement of an odd hole. A well known characterization of perfect graphs is the Strong Perfect Graph Theorem.

Strong Perfect Graph Theorem. [11] A graph G is perfect if and only if G does not contain an odd hole or odd antihole.

Perfect graphs trivially satisfy Reed's Conjecture and the Borodin–Kostochka Conjecture. So when considering a counterexample G to either conjecture, we may assume G is not perfect. By the Strong Perfect Graph Theorem G either contains an odd hole or an odd antihole. Every odd hole of order 7 or greater contains a P_5 and every odd antihole of order 7 or greater contains a gem. So if we consider a counterexample G to either conjecture that is (P_5, gem)-free, by the Strong Perfect Graph Theorem, G must contain an odd hole or odd antihole of order exactly 5, and those are both isomorphic to C_5. So in a sense (P_5, gem)-free graphs are "close" to being perfect graphs.

We use the following notation from [10]. Let G be a graph and $X, Y \subseteq V(G)$. Let $[X, Y]$ denote the set of edges with one end in X and other end in Y. If every vertex in X is adjacent to every vertex in Y, then X is *complete* to Y or $[X, Y]$ is complete; if $[X, Y] = \emptyset$, then X is *anticomplete* to Y. A set X is a *homogeneous set* if every vertex with a neighbor in X is complete to X.

An *expansion* of a graph H is any graph G such that $V(G)$ can be partitioned into $|V(H)|$ non-empty sets Q_v, for each $v \in V(H)$, such that $[Q_u, Q_v]$ is complete if $uv \in E(H)$, and $[Q_u, Q_v]$ is anticomplete if $uv \notin E(H)$. An expansion of a graph is a *clique expansion* if each Q_v is a clique, and is a *P_4-free expansion* if each Q_v induces a P_4-free graph.

Chudnovsky, Karthick, Maceli, and Maffray [10] proved a structure theorem for (P_5, gem)-free graphs, here recorded as Theorem 2.1.1; however, we will need to review some definitions to fully understand this theorem.

Figure 3 shows the ten graphs $G_1, G_2, \ldots, G_{10}$. For each $i \in [10]$ let $\mathcal{G}_i$ be the class of graphs that are P_4-free expansions of G_i, and let $\mathcal{G}_i^*$ be the class of graphs that are clique expansions of G_i.

Let $\mathcal{H}$ be the class of connected (P_5, gem)-free graphs G such that $V(G)$ can be partitioned into seven non-empty sets $A_1, A_2, \ldots, A_7$ such that the following are true (the last graph in Figure 3 shows the behavior of A_i for each $i \in [7]$). Each A_i induces a P_4-free graph. The vertex-set of each component of $G[A_7]$ is a homogeneous set. Every edge with exactly one endpoint in $V(A_7)$ must have its other endpoint in $V(A_6)$. For all $i, j \in [6]$ if A_i and A_j are joined by a solid edge, then $[A_i, A_j]$ is complete, and if no edge is present then $[A_i, A_j] = \emptyset$. Let $\mathcal{H}^*$ be the class of graphs that are in $\mathcal{H}$ with $A_1, A_2, \ldots, A_5$ being cliques and also each component of A_7 being a clique.

We record the main structure theorem from [10] as well as a result from [12, 10]. These are, respectively, Theorem 2.1.1 and Theorem 2.1.2.

Theorem 2.1.1. *[10] If G is a connected (P_5, gem)-free graph that contains an induced C_5, then either $G \in \mathcal{H}$ or $G \in \mathcal{G}_i$, for some $i \in \{1, \ldots, 10\}$.*

Theorem 2.1.2. *[12, 10] Fix $i \in \{1, \ldots, 10\}$. For every $G \in \mathcal{G}_i$ (respectively $G \in \mathcal{H}$) there is $G^* \in \mathcal{G}_i^*$ (respectively $G^* \in \mathcal{H}^*$) such that $\omega(G) = \omega(G^*)$ and $\chi(G) = \chi(G^*)$.*

Further, G^* is an induced subgraph of G.

To prove our result in Section 2.2, namely Theorem 2.2.1, we assume it is false and choose G to be a counterexample that is *vertex-critical*; that is $\chi(G-v) < \chi(G)$ for each $v \in V(G)$. Ultimately, we reach a contradiction, by constructing a $(\Delta(G)-1)$-coloring of G. To construct this coloring, we repeatedly use Lemma 2.1.3.

Lemma 2.1.3. *Fix $k \in \mathbb{Z}^+$. Let G be a graph and let $I_1, \ldots, I_t$ be pairwise disjoint independent sets of G. If $G - \bigcup_{j=1}^{t} I_j$ is $(k-t-1)$-degenerate, then $\chi(G) \leq k$.*

Proof. Let G, k, and $I_1, \ldots, I_t$ satisfy the hypotheses. If $G - \bigcup_{j=1}^{t} I_j$ is $(k-t-1)$-degenerate, then it has a $(k-t)$-coloring φ'. We extend φ' to a k-coloring of G by giving each I_j its own color. □

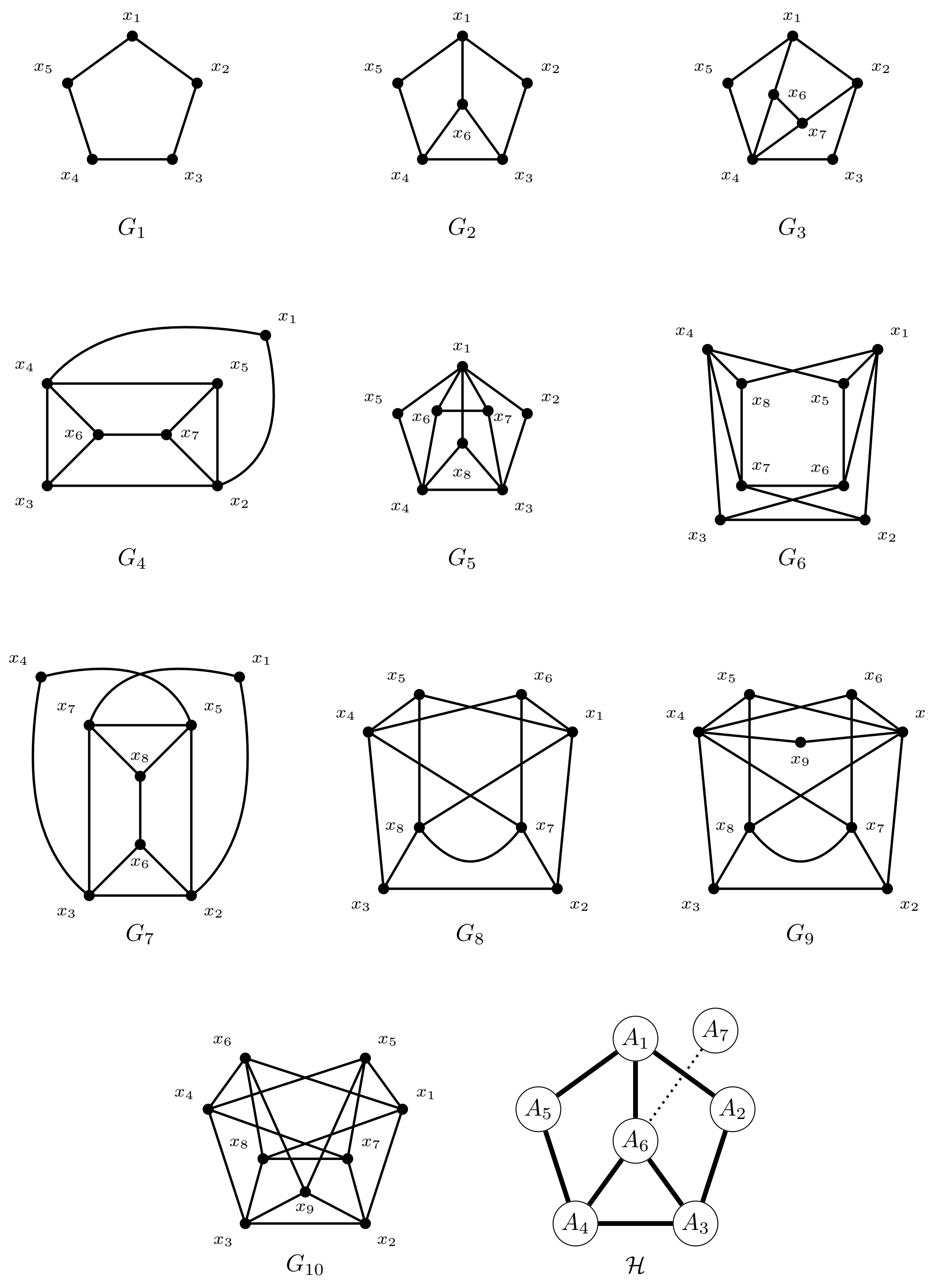

Fig. 3.: For $\mathcal{H}$, the dotted line between A_6 and A_7 indicates that $[A_6, A_7]$ is not necessarily complete even though $[A_6, A_7] \neq \emptyset$. Each edge with exactly one endpoint in $V(A_7)$ must have its other endpoint in $V(A_6)$.

We often use Lemma 2.1.3 to 8-color G, typically with $t = 2$. Let $G' := G - \bigcup_{j=1}^{2} I_j$. To prove that G' is 5-degenerate we give a vertex order $\sigma = (v_1, \ldots, v_n)$ such that each v_i has at most 5 neighbours earlier in σ. For a vertex partition $S_1 \uplus \cdots \uplus S_t$ of $V(G)$ we write $(S_1, \ldots, S_t)$ to denote a vertex order σ where all vertices in S_i come before all vertices in S_{i+1}, for each i, and vertices within each S_i are ordered arbitrarily.

Lemma 2.1.4. *Every vertex-critical counterexample G to the Borodin–Kostochka Conjecture has $\delta(G) \geq \Delta(G) - 1$. In particular, $\delta(G) \geq 8$ and $|d(v) - d(w)| \leq 1$ for all $v, w \in V(G)$.*

Proof. Assume that G is a vertex-critical counterexample to the the Borodin–Kostochka Conjecture. Suppose there exists $v \in V(G)$ such that $d(v) < \Delta(G) - 1$. Since G is vertex-critical, $G - v$ has a $(\Delta(G) - 1)$-coloring φ. To extend φ to v, we simply color v with a color unused on $N_G(v)$. $\square$

We want to reuse the idea in the proof of Lemma 2.1.4 to show that other induced subgraphs cannot appear in G. For this, we introduce a bit of notation. For a graph H, a d_1-*assignment* L gives to each $v \in V(H)$ a set (of allowable colors) $L(v)$ such that $|L(v)| = d_H(v) - 1$. A graph H is d_1-*choosable* if H has a proper coloring φ with $\varphi(v) \in L(v)$ for every d_1-assignment L.

Lemma 2.1.5. *If G is vertex-critical and $\chi(G) = \Delta(G)$, then G cannot contain any non-empty, d_1-choosable, induced subgraph H. So such a G contains neither of the following as induced subgraphs:*

- *$K_3 \vee 3K_2$ or*
- *$K_4 \vee H$, where $V(H)$ contains two disjoint pairs of nonadjacent vertices.*

Proof. We begin with the first statement. Suppose, to the contrary, that G contains an induced subgraph H, where G and H satisfy the hypotheses. Since G is vertex-critical, $G - H$ has a $(\Delta(G) - 1)$-coloring φ. For each $v \in V(H)$, let $L(v)$ denote the colors in $\{1, \ldots, \Delta(G) - 1\}$ that are unused by φ on $N_G(v)$. Thus, $|L(v)| \geq \Delta(G) - 1 - (d_G(v) - d_H(v)) \geq d_H(v) - 1$. Now we can extend φ to H precisely because H is d_1-choosable. This gives a $(\Delta(G) - 1)$-coloring of G, which is a contradiction; this proves the first statement.

The second statement follows from two results by Cranston and Rabern [13], who proved that the two subgraphs listed above are d_1-choosable. □

Lemma 2.1.6. *Let G be a vertex-critical counterexample to the Borodin–Kostochka Conjecture. If G contains nonempty, disjoint homogeneous sets A and B such that A and B are cliques and $N(A) \subseteq N(B)$, then $|A| > |B|$.*

Proof. Let G, A, and B satisfy the hypotheses. Suppose, for the sake of contradiction, that $|A| \leq |B|$. Since G is vertex-critical, $G - A$ has a $(\Delta(G) - 1)$-coloring φ. Since $|A| \leq |B|$, $N(A) \subseteq N(B)$, and B is a homogeneous set, we can extend φ to G by coloring A with colors used on B. Thus, G is not a counterexample to the Borodin–Kostochka Conjecture, a contradiction. □

A graph class $\mathcal{G}$ is *hereditary* if for each $G \in \mathcal{G}$ we have $H \in \mathcal{G}$ for each induced subgraph H of G. Every class of graphs characterized by a list of forbidden induced subgraphs is a hereditary class; in particular, the class of (P_5, gem)-free graphs is hereditary.

Theorem 2.1.7. *[14, 15] Let $\mathcal{G}$ be a hereditary class of graphs. If the Borodin–Kostochka Conjecture is false for some $G \in \mathcal{G}$, then it is false for some $G \in \mathcal{G}$ with $\Delta(G) = 9$.*

Proof. We assume the Borodin–Kostochka Conjecture is true for all $G \in \mathcal{G}$ with $\Delta(G) = 9$ and show that it is true for all $G \in \mathcal{G}$.

Suppose instead that some $G \in \mathcal{G}$ with $\Delta(G) > 9$ is a counterexample; in particular, $\omega(G) \leq \Delta(G) - 1$. Among all such G, choose one to minimize $\Delta(G)$. We want to find a maximum independent set I that intersects each clique of size $\Delta(G)-1$. If $\omega(G) < \Delta(G) - 1$, then any maximum independent set I suffices. Otherwise, I is guaranteed by a result of King [16]. Let $G' := G - I$. Note that $\omega(G') \leq \Delta(G) - 2$ and $\Delta(G') \leq \Delta(G) - 1$, since I is maximum.

If $\Delta(G') \leq \Delta(G) - 3$, then we can greedily color G' with at most $\Delta(G) - 2$ colors. By using a new color on I, we get a $(\Delta(G) - 1)$-coloring of G, a contradiction. If $\Delta(G') = \Delta(G) - 2$, then G' has a $(\Delta(G) - 2)$-coloring by Brooks' Theorem, since $\omega(G') \leq \Delta(G) - 2$. By using a new color on I, we again get a $(\Delta(G) - 1)$-coloring of G, a contradiction. So we must have $\Delta(G') = \Delta(G) - 1$ and $\omega(G') \leq \Delta(G) - 2$. Since $G' \in \mathcal{G}$ and $\Delta(G') < \Delta(G)$, we know that G' is not a counterexample to the Borodin–Kostochka Conjecture. In particular G' has a $(\Delta(G') - 1)$-coloring. By using a new color on I, we extend this coloring of G' to a $(\Delta(G) - 1)$-coloring of G, a contradiction. □

2.2 New Result for the Borodin–Kostochka Conjecture

Our result is the following theorem.

Theorem 2.2.1. *Let G be a (P_5, gem)-free graph. If $\Delta(G) \geq 9$ and $\omega(G) \leq \Delta - 1$, then $\chi(G) \leq \Delta(G) - 1$.*

Proof. Suppose the theorem is false. Let G be a counterexample that minimizes $\Delta(G)$. Further, we can choose G to be vertex-critical. By Theorem 2.1.7, we can assume that $\Delta(G) = 9$. Next we show that we can also assume that either $G \in \mathcal{G}_i^*$

for some $i \in \{1, \ldots, 10\}$ or $G \in \mathcal{H}^*$.

If G is perfect, then $\chi(G) = \omega(G) \leq \Delta(G) - 1$, a contradiction. Since G is not perfect, the Strong Perfect Graph Theorem [11] implies that G must contain an odd hole or odd anti-hole (that is, either G or its complement contains an induced odd cycle of length at least 5). Every odd hole of length at least 7 contains a P_5 as an induced subgraph. Similarly, every odd antihole of length at least 7 contains a gem as an induced subgraph. Since G is (P_5, gem)-free, G must contain a hole or antihole of length 5, both of which are congruent to C_5. So, by Theorem 2.1.1, either $G \in G_i$ for some $i \in \{1, \ldots, 10\}$ or $G \in \mathcal{H}$. Finally, by Theorem 2.1.2, we can assume that either $G \in \mathcal{G}_i^*$ or $G \in \mathcal{H}^*$. This is because G is vertex-critical. Since $\chi(G^*) = \chi(G)$, we conclude that $G^* \cong G$. For each Q_i, we write x_i, x_i', x_i'', and x_i''' to denote arbitrary distinct vertices in Q_i (provided that such vertices exist). For independent sets I_1 and I_2 and $G - (I_1 \cup I_2)$, we write $(Q'_{j_1}, Q'_{j_2}, Q'_{j_3}, \ldots)$ to denote the vertex order $(Q_{j_1}, Q_{j_2}, Q_{j_3}, \ldots)$ restricted to $V(G) \setminus (I_1 \cup I_2)$. Now we consider these 11 cases in succession. Each case is independent of all others.

Case 1: $G \in \mathcal{G}_1^*$. Since $\Delta(G) = 9$, there exists $i \in \{1, \ldots, 5\}$ such that $|Q_i| \geq 4$. By symmetry, assume $i = 1$. If $|Q_5| \geq 2$ and $|Q_2| \geq 2$, then

$$G[\{x_1, x_1', x_1'', x_1''', x_2, x_2', x_5, x_5'\}] \cong K_4 \vee H,$$

where H contains two disjoint pairs of nonadjacent vertices; this subgraph is d_1-choosable, which contradicts Lemma 2.1.5. Assume instead, by symmetry, that $|Q_5| = 1$. Lemma 2.1.4 implies that $1 \geq d(x_2) - d(x_1) = |N[x_2]| - |N[x_1]| = |Q_3| - |Q_5|$. Thus, $|Q_3| \leq 2$. Since $8 \leq d(x_4) = |Q_3| + |Q_4| - 1 + |Q_5|$, we know $|Q_4| \geq 6$. But now $d(x_5) = |Q_4| + (|Q_5| - 1) + |Q_1| \geq 6 + (1 - 1) + 4 = 10$. This contradicts that $\Delta(G) = 9$.

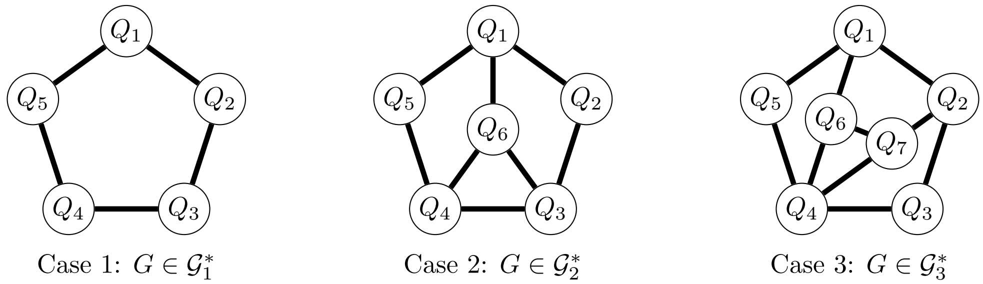

Fig. 4.: Cases 1–3: $G \in \mathcal{G}_1^*, G \in \mathcal{G}_2^*$, or $G \in \mathcal{G}_3^*$.

Case 2: $\boldsymbol{G \in \mathcal{G}_2^*}$. By Lemma 2.1.6, $|Q_2| > |Q_6|$ and $|Q_5| > |Q_6|$; in particular, $|Q_2|, |Q_5| \geq 2$ since each Q_i is nonempty. If $|Q_6| \geq 2$, then let $I_1 = \{x_2, x_5, x_6\}$ and $I_2 = \{x_2', x_5', x_6'\}$. Now $G - (I_1 \cup I_2)$ is 5-degenerate with vertex order $(Q_1', Q_4', Q_3', Q_5', Q_2', Q_6')$; by Lemma 2.1.3, G is 8-colorable, a contradiction. Assume instead that $|Q_6| = 1$. If $|Q_1| \geq 2$, then let $I_1 = \{x_2, x_5, x_6\}$. Now $G - I_1$ is 6-degenerate with vertex order $(Q_1', Q_5', Q_2', Q_4', Q_3', Q_6')$; by Lemma 2.1.3, G is 8-colorable, a contradiction. Assume instead that $|Q_1| = 1$. Now Lemma 2.1.4 implies that $1 \geq d(x_3) - d(x_2) = |N[x_3]| - |N[x_2]| = |Q_4| + |Q_6| - |Q_1| = |Q_4|$. By symmetry, $|Q_3| \leq 1$. Thus, $d(x_6) = |Q_1| + |Q_3| + |Q_4| + |Q_6| - 1 = 3$, which contradicts that $\delta(G) \geq 8$.

Case 3: $\boldsymbol{G \in \mathcal{G}_3^*}$. By Lemma 2.1.6, $|Q_5| > |Q_6|$ and $|Q_3| > |Q_7|$; in particular, $|Q_5|, |Q_3| \geq 2$, since each Q_i is nonempty. If $|Q_4| \geq 2$, then let $I_1 = \{x_2, x_5, x_6\}$ and $I_2 = \{x_1, x_3, x_7\}$. Now $G - (I_1 \cup I_2)$ is 5-degenerate with vertex order $(Q_4', Q_3', Q_5', Q_2', Q_1', Q_6', Q_7')$; by Lemma 2.1.3, G is 8-colorable, a contradiction. Assume instead that $|Q_4| = 1$. Lemma 2.1.4 implies that $1 \geq d(x_2) - d(x_3) = |N[x_2]| - |N[x_3]| = |Q_1| + |Q_7| - |Q_4| = |Q_1| + |Q_7| - 1$. Thus, $|Q_1| = |Q_7| = 1$. By symmetry, $|Q_2| = |Q_6| = 1$. So $d(x_6) = |Q_1| + |Q_7| + |Q_4| + |Q_6| - 1 = 3$, which contradicts that $\delta(G) \geq 8$.

Case 4: $\boldsymbol{G \in \mathcal{G}_4^*}$. By Lemma 2.1.6, $|Q_1| > |Q_5|$; so $|Q_1| \geq 2$ since each Q_i

is nonempty. Let $I_1 := \{x_1, x_5, x_7\}$ and $I_2 := \{x'_1, x_3, x_6\}$. Now $G - (I_1 \cup I_2)$ is 5-degenerate with vertex order $(Q'_4, Q'_2, Q'_1, Q'_5, Q'_3, Q'_7, Q'_6)$; by Lemma 2.1.3, G is 8-colorable, a contradiction.

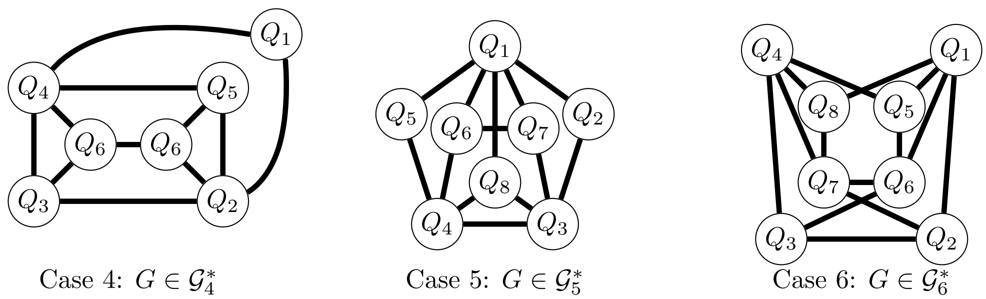

Fig. 5.: Cases 4–6: $G \in \mathcal{G}_4^*, G \in \mathcal{G}_5^*$, or $G \in \mathcal{G}_6^*$.

Case 5: $G \in \mathcal{G}_5^*$. By Lemma 2.1.6, $|Q_5| > |Q_6|$ and $|Q_2| > |Q_7|$; so $|Q_5|, |Q_2| \geq 2$ since each Q_i is nonempty. If $|Q_1| \geq 3$, then $G[\{x_1, x'_1, x''_1, x_2, x'_2, x_5, x'_5, x_6, x_7\}] \cong K_3 \vee 3K_2$; this subgraph is d_1-choosable, contradicting Lemma 2.1.5. Assume instead that $|Q_1| \leq 2$. If $|Q_3| \geq 4$, then $G[\{x_3, x'_3, x''_3, x'''_3, x_2, x_4, x_7, x_8\}] \cong K_4 \vee H$ where H contains two disjoint pairs of nonadjacent vertices; this subgraph is d_1-choosable, contradicting Lemma 2.1.3. So $|Q_3| \leq 3$. By symmetry, $|Q_4| \leq 3$. Thus, $|Q_2|, |Q_5| \geq 4$, since $d(x_2), d(x_5) \geq 8$. However, now $d(x_1) = (|Q_1| - 1) + |Q_2| + |Q_5| + |Q_6| + |Q_7| \geq (1 - 1) + 4 + 4 + 1 + 1 = 10$, which contradicts that $\Delta(G) = 9$.

Case 6: $G \in \mathcal{G}_6^*$. Let $I_1 := \{x_3, x_5, x_7\}$ and $I_2 := \{x_2, x_6, x_8\}$. Now $G - (I_1 \cup I_2)$ is 5-degenerate with vertex order $(Q'_4, Q'_1, Q'_8, Q'_5, Q'_7, Q'_6, Q'_3, Q'_2)$. By Lemma 2.1.3, G is 8-colorable, a contradiction.

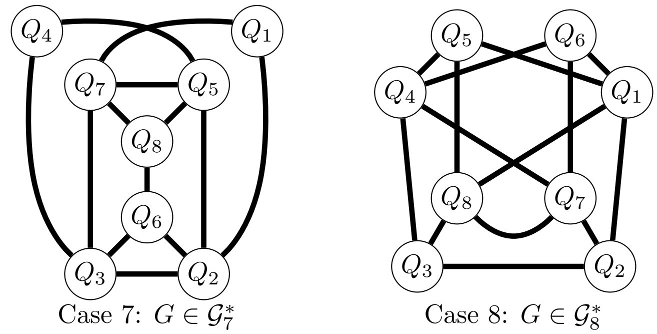

Fig. 6.: Cases 7–8: $G \in \mathcal{G}_7^*$ or $G \in G_8^*$.

Case 7: $\boldsymbol{G \in \mathcal{G}_7^*}$. By Lemma 2.1.6, $|Q_4| > |Q_7|$; so $|Q_4| \geq 2$ since each Q_i is nonempty. If $|Q_1| \geq 7$, then $d(x_2) = |Q_3| + (|Q_2| - 1) + |Q_6| + |Q_5| + |Q_1| \geq 10$, contradicting that $\Delta(G) = 9$. Thus, $|Q_1| \leq 6$. Let $I_1 := \{x_4, x_6, x_7\}$ and $I_2 := \{x_2, x_4', x_8\}$. Now $G - (I_1 \cup I_2)$ is 5-degenerate with vertex order $(Q_5', Q_3', Q_4', Q_7', Q_2', Q_8', Q_6', Q_1')$. By Lemma 2.1.3, G is 8-colorable, a contradiction.

Case 8: $\boldsymbol{G \in \mathcal{G}_8^*}$. Let $I_1 := \{x_3, x_5, x_7\}$ and $I_2 := \{x_2, x_6, x_8\}$. Now $G - (I_1 \cup I_2)$ is 5-degenerate with order $(Q_4', Q_1', Q_3', Q_2', Q_7', Q_8', Q_6', Q_5')$; by Lemma 2.1.3, G is 8-colorable, a contradiction.

Case 9: $\boldsymbol{G \in \mathcal{G}_9^*}$. By the Lemma 2.1.6 $|Q_9| > |Q_5|$; so $|Q_9| \geq 2$ since each Q_i is nonempty. Let $I_1 := \{x_3, x_5, x_7, x_9\}$ and $I_2 := \{x_2, x_6, x_8, x_9'\}$. Now $G - (I_1 \cup I_2)$ is 5-degenerate with order $(Q_4', Q_1', Q_3', Q_2', Q_7', Q_8', Q_9', Q_6', Q_5')$. By Lemma 2.1.3, G is 8-colorable, a contradiction.

Case 10: $\boldsymbol{G \in \mathcal{G}_{10}^*}$. Let $I_1 := \{x_3, x_5, x_7\}$ and $I_2 := \{x_2, x_6, x_8\}$. Now $G - (I_1 \cup I_2)$ is 5-degenerate with order $(Q_9', Q_4', Q_1', Q_3', Q_2', Q_7', Q_8', Q_6', Q_5')$. By Lemma 2.1.3, G is 8-colorable, a contradiction.

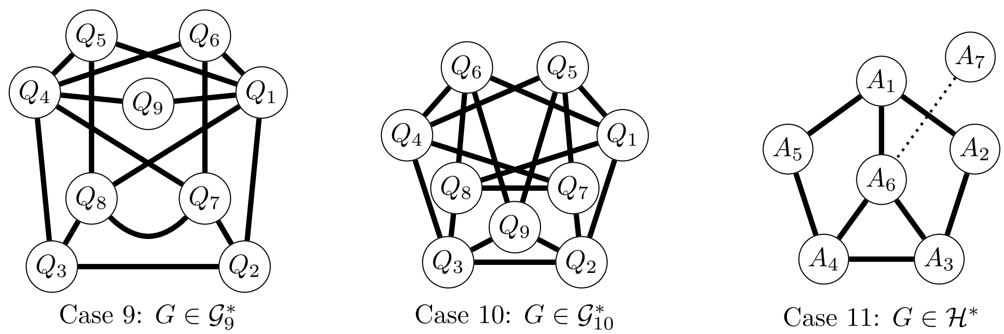

Fig. 7.: Cases 9–11: $G \in \mathcal{G}_9^*$, $G \in \mathcal{G}_{10}^*$, or $G \in \mathcal{H}^*$.

Case 11: $G \in \mathcal{H}^*$. As we defined in Section 2.1, let $\mathcal{H}$ be the class of connected (P_5, gem)-free graphs G for which $V(G)$ can be partitioned into non-empty sets $A_1, \ldots, A_7$ such that the following properties all hold: each A_i induces a P_4-free graph, the vertex-set of each component of $G[A_7]$ is a homogeneous set, each edge with exactly one endpoint in $V(A_7)$ has the other endpoint in $V(A_6)$, and for all distinct $i, j \in \{1, \ldots, 6\}$, if a solid edge appears between A_i and A_j in the rightmost graph in Figure 7, then $[A_i, A_j]$ is complete, but if no edge appears then $[A_i, A_j] = \emptyset$. Let $\mathcal{H}^*$ be the class of graphs that are in $\mathcal{H}$ with $A_1, \ldots, A_5$ being cliques and also each component of A_7 being a clique. By Theorem 2.1.2, we know that if $G \in \mathcal{H}$, then in fact $G \in \mathcal{H}^*$.

If $|A_5| \leq \omega(A_6)$, then by criticality $G - A_5$ has an 8-coloring, call it φ. To extend φ to G, we color A_5 with colors used on a maximum clique in A_6. This yields a proper 8-coloring of G, a contradiction. Thus, $|A_5| > \omega(A_6)$. By symmetry, $|A_2| > \omega(A_6)$. Since A_6 is nonempty by definition, $|A_2|, |A_5| \geq 2$. Since $d(a_6) \leq 9$ and each of A_1, A_3, and A_4 is nonempty, each component of A_7 has size at most 6. If $|A_6| \geq 2$, then let $I_1 = \{a_2, a_5, a_6\}$ and $I_2 = \{a_2', a_5', a_6'\}$. Now $G - (I_1 \cup I_2)$ is 5-degenerate with order $A_1', A_5', A_2', A_4', A_3', A_6', A_7'$. By Lemma 2.1.3, G is 8-colorable, a contradiction. So assume instead that $|A_6| = 1$.

By criticality, there exists an 8-coloring φ of $G' := G - V(A_7)$. Each component of A_7 is a clique of size at most 6 that is adjacent to only a_6 in G'. Thus we can color each component of A_7 with 8 colors so that φ extends to G, which contradicts that $\chi(G) \geq \Delta(G) = 9$. □

2.3 Future Work

As mentioned in Section 2.1, iweconsider a minimal counterexample G to the Borodin–Kostochka Conjecture, we may assume G is not perfect; thus, by the Strong Perfect Graph Theorem, G either contains an odd hole or odd antihole. By forbidding a set of induced subgraphs, like $\{P_5, \text{gem}\}$, we can eliminate possible odd hole and odd antiholes. In fact without even forbidding any induced subgraphs we can eliminate most odd antiholes. Cranston and Rabern [13] proved that $E_2 \vee P_4$ is d_1-choosable, and all antiholes of order at least 8 contain an induced $E_2 \vee P_4$. So by Lemma 2.1.5 we know odd antiholes of order at least 9 cannot be contained within a minimal counterexample to the Borodin–Kostochka Conjecture. So a minimal counterexample to the Borodin–Kostochka Conjecture must contain an odd antihole of order 7 or an odd hole since an odd antihole of order 5 is exactly an odd hole of order 5. We could try to eliminate all but the odd anthole of order 7 and an odd hole of order 5 by forbidding say P_5. However, the case of determining if P_5-free graphs containing an induced odd hole of order 5 do satisfy the Borodin–Kostochka Conjecture does not seem to be easy. Of course we could forbid a set of two distinct graphs that give a bit more structure in these cases left over, just as forbidding the set $\{P_5, \text{gem}\}$ did. For example, Gupta and Pradhan [17] proved the Borodin–Kostochka Conjecture for (P_5, C_4)-free graphs, citing our result from Section 2.2 and using a similar method to prove their result (C_4 is an induced subgraph of all odd antiholes of order at least 7).

There are also versions of the Borodin–Kostochka Conjecture using generalizations of the chromatic number. One example is the *list chromatic number*, denoted

χ_ℓ, and defined as follows. Let G be a graph. A *k-assignment* L of G is a function that assigns each vertex v in G a list $L(v)$ of colors where $|L(v)| = k$. An *L-coloring* of G is a proper coloring f such that $f(v) \in L(v)$ for each $v \in V(G)$. If G has an L-coloring for every k-assignment L, then G is *k-choosable*. The list-chromatic number $\chi_\ell(G)$ is the smallest k for which G is k-choosable.

Borodin–Kostochka Conjecture (list-chromatic version). Let G be a graph. If $\Delta(G) \geq 9$ and $\omega(G) \leq \Delta(G) - 1$, then $\chi_\ell(G) \leq \Delta(G) - 1$.

Conjecture 2.3 already has a result similar to Reed's result in Theorem 2.0.1 for the Borodin-Kostochka Conjecture. It was proved by Choi, Kierstead, and Rabern [18].

Theorem 2.3.1. *[18] Every graph with $\chi_\ell(G) \geq \Delta(G) \geq 10^{20}$ contains $K_{\Delta(G)}$.*

Cranston and Rabern also have a similar result to their claw-free result from Theorem 2.0.2 for the Borodin-Kostochka Conjecture.

Theorem 2.3.2. *[19] If G is a claw-free graph with $\Delta(G) > \omega(G)$ and $\Delta(G) \geq 69$, then $\chi_\ell(G) \leq \Delta(G) - 1$.*

The tactics we outlined above for standard coloring cannot be used for the list-chromatic version of the Borodin–Kostochka Conjecture since there is no Strong Perfect Graph Theorem for χ_ℓ. However, we can use *chordal graphs*, which are graphs with no induced cycles other than C_3. Chordal graphs are perfect graphs, but more importantly Tuza and Voigt [20] proved chordal graphs are *chromatic-choosable*, meaning $\chi = \chi_\ell$. So a counterexample to the list-chromatic version of the Borodin–Kostochka Conjecture cannot be chordal; thus it must have an induced cycle of length at least 4. So it could be possible to apply techniques similar to those presented in Section 2.1 and Section 2.2. For example, if a (P_5, gem)-free graph is a counterexample to the list-chromatic version of the Borodin–Kostochka Conjecture it must not be chordal, thus have an induced C_4 or C_5 since any larger induced cycle would contain an induced P_5. In the case that an induced C_5 is present, we can still apply Theorem 2.1.1.

CHAPTER 3

T-TONE COLORING

We write $[k]$ to denote $\{1, \ldots, k\}$ and write $\binom{[k]}{t}$ to denote the collection of all subsets of $[k]$ of size t. The elements of $\binom{[k]}{t}$ are referred to as *t-sets*. For a graph G and $v, w \in V(G)$, we write $d_G(v, w)$ for the distance (length of the shortest path) between v and w in the graph G, and when context is clear we simply write $d(v, w)$; we also write $N_G(v)$ for the set of vertices at distance 1 from v (*the neighbours of* v) and $N_G^2(v)$ for the set of vertices at distance 2 from v (*the second neighbours of* v) and when context is clear we simply write $N(v)$ and $N^2(v)$ respectively.

In 2009, Ping Zhang led Fonger, Goss, Phillips, and Segroves [21] in developing a new generalization of proper vertex coloring called *t-tone coloring* (Fonger et al. also mention that Gary Chartrand helped develop this definition). Here *t-tone coloring* is recorded as Definition 3.0.1.

Definition 3.0.1. *For a graph G and $t, k \in \mathbb{Z}^+$ a* ***t-tone k-coloring*** *of G is a function $f : V(G) \to \binom{[k]}{t}$ such that $|f(v) \cap f(w)| < d_G(v, w)$ for all distinct $v, w \in V(G)$.*

Every t-tone k-coloring f of a graph G assigns to each vertex $v \in V(G)$ a t-set; the t-set assigned to v is denoted $f(v)$. A t-set can also be assigned to other vertices in G as long as f satisfies the condition in Definition 3.0.1 (see Figure 8 for some examples of t-tone k-colorings). In the literature of t-tone coloring "assigning a t-set to v" is also called "giving a *label* to v". So the elements of an assigned t-set (or given label) are called the colors assigned (or given) to v or the colors on v. An equivalent formulation of Definition 3.0.1 is "a t-tone k-coloring of a graph assigns

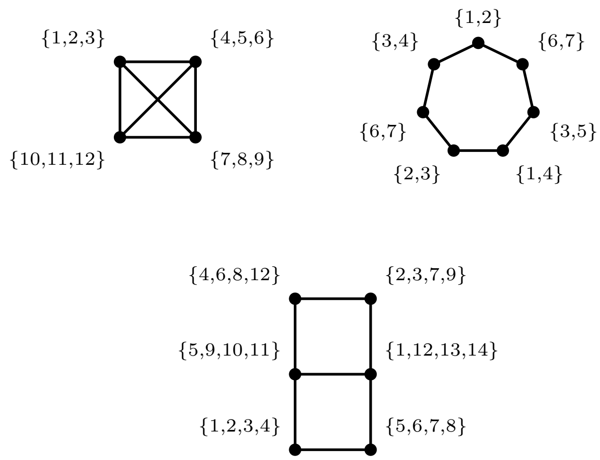

Fig. 8.: A 3-tone 12-coloring of K_4; a 2-tone 7-coloring of C_7; and a 4-tone 14-coloring of $P_2 \square P_3$.

t-sets to vertices so that distinct vertices at distance d have their corresponding t-sets sharing at most $d-1$ colors". In general, when coloring substructures of a graph, most commonly the vertices or the edges, it is natural to ask for the minimum number of colors needed to color a particular graph. This approach motivated Definition 3.0.2.

Definition 3.0.2. *A graph G that has a t-tone k-coloring is **t-tone k-colorable**, and the **t-tone chromatic number** of G, denoted $\boldsymbol{\tau_t(G)}$, is the minimum k such that G is t-tone k-colorable.*

From Definition 3.0.1 and Definition 3.0.2, we can easily check that every graph G satisfies $\tau_1(G) = \chi(G)$, and thus τ_t is a generalization of χ. Some other immediate properties of τ_t are recorded here as Proposition 3.0.3 and Proposition 3.0.4.

Proposition 3.0.3. *[21] If H is a subgraph of G, then every t-tone coloring of G induces a t-tone coloring of H. In particular $\tau_t(H) \leq \tau_t(G)$; equivalently τ_t is monotone under taking subgraphs.*

Proof. Let G be a graph, H be a subgraph of G, and f be a t-tone k-coloring of G. Note that $\tau_t(G) \leq k$. Define a t-tone k-coloring f' for H by defining $f'(v) := f(v)$

for all $v \in V(H)$. For any distinct $v, w \in V(H)$ we have that $d_H(v, w) \geq d_G(v, w)$, which implies

$$|f'(v) \cap f'(w)| = |f(v) \cap f(w)| < d_G(v, w) \leq d_H(v, w).$$

Thus f' is a t-tone k-coloring of H, which implies $\tau_t(H) \leq k$. The result follows by letting $k = \tau_t(G)$. □

Figure 9 shows an example of Proposition 3.0.3.

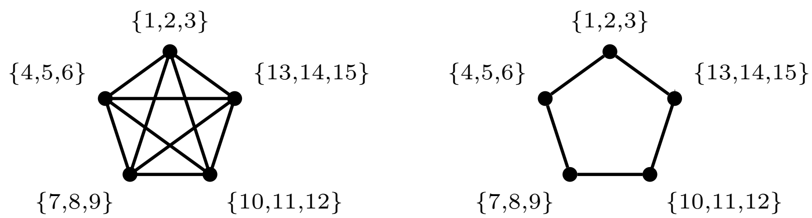

Fig. 9.: A 3-tone 15-coloring of K_5 that is used on a C_5 subgraph.

Proposition 3.0.4. *[22] If G is a graph and $t \in \mathbb{Z}^+$, then $\tau_{t-1}(G) \leq \tau_t(G)$.*

Proof. Let G be a graph, $t \in \mathbb{Z}^+$, and f be a t-tone k-coloring of G. Note that $\tau_t(G) \leq k$. Form a $(t-1)$-tone k-coloring f' of G by, for each $v \in V(G)$, deleting from $f(v)$ any single element calling the resulting set $f'(v)$. Now for any distinct $v, w \in V(G)$ we have $|f'(v) \cap f'(w)| \leq |f(v) \cap f(w)|$, which implies

$$|f'(v) \cap f'(w)| \leq |f(v) \cap f(w)| < d_G(v, w).$$

Thus f' is a $(t-1)$-tone k-coloring of G, which implies $\tau_{t-1}(G) \leq k$. The result follows by letting $k = \tau_t(G)$. □

Note that f' in Proposition 3.0.4 might use fewer than k colors among its assigned t-sets, but it is still a $(t-1)$-tone k-coloring as there is no requirement we use all k colors among the t-sets (see Figure 10 for an example). Of course if colors go unused among the assigned t-sets then a t-color k'-coloring easily exists where k' is the size of the union of all assigned t-sets. For example in Figure 10 the 3-tone 14-coloring is also a 3-tone 12-coloring since only 12 distinct colors appear on all assigned 3-sets (1 and 11 do not appear).

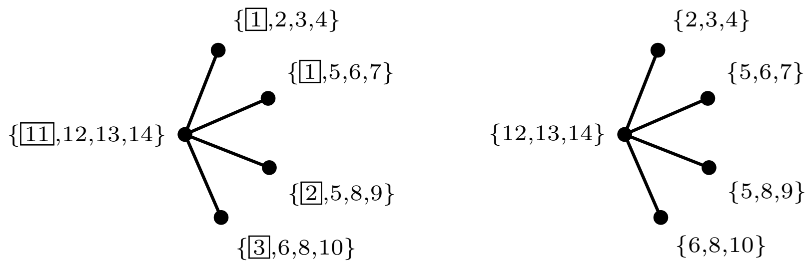

Fig. 10.: A 4-tone 14-coloring f of $K_{1,4}$ that is used to create a 3-tone 14-coloring of $K_{1,4}$ by deleting a member from each t-set assigned by f.

The most widely studied case of t-tone coloring is the case $t = 2$ and many papers focus solely on this case. However, bounds and exacts values of τ_t, for $t \geq 2$, have been studied for various graph classes [23, 24, 25, 26, 27, 28, 29] with several papers investigating graph products [30, 31, 32] and one studying τ_t of random graphs [33].

In Section 3.1 we will only consider 2-tone colorings of graphs. We will review graph classes where τ_2 is known and best known bounds for τ_2. We will also mention any interesting open problems. In Section 3.2 we will present new results on τ_2.

In Section 3.3 we will do the same as in Section 3.1 except that we will consider t-tone colorings of graphs for general $t \geq 2$. In Section 3.4 we will present new results on τ_t when $t \geq 2$.

3.1 Background and Results for τ_2

For any graph G the following is true, which is a combination of three trivial bounds on $\chi(G)$:

$$\max\left\{\omega(G), \frac{|G|}{\alpha(G)}\right\} \leq \chi(G) \leq |G|.$$

This statement is also true for τ_1 since $\chi = \tau_1$, and it can be generalized to τ_t for $t \geq 2$; see Proposition 3.1.2. It will be necessary for the proof of Proposition 3.1.2 to have Proposition 3.1.1.

Proposition 3.1.1. *[21]* $\tau_t(K_n) = tn$

Proof. Let G be a complete graph on n vertices. By definition any two distinct vertices in G are adjacent, so any t-tone k-coloring f of G must assign a t-set to each vertex that is disjoint from all other assigned t-sets. This implies $\tau_2(G) \geq tn$. It is trivial to create a coloring with disjoint assigned t-sets when $k = tn$. So $\tau_2(G) = tn$. □

Proposition 3.1.2. *[21, 22] For any graph G and each $t \geq 1$, we have*

$$\max\left\{t\omega(G), \frac{t|G|}{\alpha(G)}\right\} \leq \tau_t(G) \leq t|G|.$$

Proof. Let G be a graph and $t \geq 1$. By assigning each vertex of G a t-set such that all assigned t-sets are pairwise disjoint we obtain a t-tone $t|G|$-coloring of G, so $\tau_t(G) \leq t|G|$. And by Proposition 3.1.1 and Proposition 3.0.3 we have that $t\omega(G) = \tau_t(K_{\omega(G)}) \leq \tau_t(G)$. It now suffices to prove $\frac{t|G|}{\alpha(G)} \leq \tau_t(G)$.

Let f be a t-tone k-coloring of G. Each color appears on an independent set of vertices; thus at most $\alpha(G)$ vertices. Since the total number of colors, not necessarily distinct, used by vertices of G is $t|G|$ we have

$$k \cdot \alpha(G) \geq \text{ “number of colors used by vertices of } G\text{”} = t|G|.$$

If we let $k = \tau_t(G)$, this implies $\frac{t|G|}{\alpha(G)} \leq \tau_t(G)$. □

Recall that $\chi = \tau_1$. Just like for τ_1 these bounds are trivial for all $t \geq 2$, but are still sharp for all n as witnessed by K_n. For other graphs and graph classes these bounds are far from the correct value of τ_t, even when $t = 1$. However, this is where the similarities of τ_1 and τ_t for $t \geq 2$ start to diverge. The first major example is the behavior of τ_1 with trees compared to τ_t when $t \geq 2$. It will be shown in Section 3.3 that for any tree T and fixed t we have for some constant c

$$\left\lceil 0.5 + t + \sqrt{(t^2 - t)\Delta(T) + 0.25} \right\rceil \leq \tau_t(T) \leq c\sqrt{\Delta(T)}.$$

Thus, even for $T = K_{1,n}$, we have the bounds in Proposition 3.1.2 giving

$$\max\left\{2t, t + \frac{1}{n}\right\} \leq \tau_t(K_{1,n}) \leq t(n+1),$$

but we know $\tau_t(K_{1,n}) \approx c\sqrt{\Delta(K_{1,n})} = c\sqrt{n}$ (see Figure 11 for an example). So for each $t \geq 2$ as n grows $\frac{\tau_t(K_{1,n})}{\max\left\{2t, \frac{t(n+1)}{n}\right\}}$ and $\frac{t(n+1)}{\tau_t(K_{1,n})}$ are arbitrarily large.

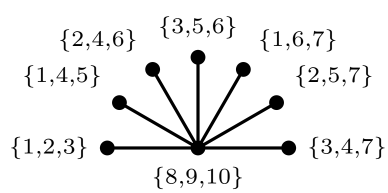

Fig. 11.: $\tau_3(K_{1,6}) = 10$, however, $\max\left\{6, 3 + \frac{1}{6}\right\} = 6 < \tau_3(K_{1,6}) < 21 = 3(6+1)$.

Moving forward in this section we will investigate τ_2 only and defer larger values of t to Section 3.3. We will first investigate τ_2 for fundamental graph classes. These results will help us to later solidify better bounds for τ_2 than those given by Proposition 3.1.2. Then selected problems and results are listed. We will assume for the rest of this section that graphs have $\Delta \geq 2$ since τ_2 is trivial to compute for graphs with $\Delta \leq 1$. The reason is that the only graphs with $\Delta \leq 1$ are E_n and $aK_1 + bK_2$, but $\tau_2(E_n) = 2$ for all $n \geq 1$ and $\tau_2(aK_1 + bK_2) = 4$ for all $a, b \in \mathbb{Z}^+$ (see Figure 12).

Fig. 12.: A 2-tone 2-coloring of E_3 and a 2-tone 4-coloring of $3K_1 + 2K_2$.

It is helpful for the remaining part of this section to exchange our definition of 2-tone k-colorings for the following equivalent version: we assign a 2-set to each vertex so that adjacent vertices receive disjoint 2-sets and vertices at distance two receive distinct 2-sets. This allows for a quick checklist when trying to assign a 2-set to a vertex in a graph.

Two of the first results regarding τ_2 were due to Fonger et al. [21], calculating τ_2 for all stars and trees. We record these here as Proposition 3.1.3 and Proposition 3.1.4, respectively. It is worthwhile to present these proofs, since many ideas from these proofs spawned techniques that were used in many later results. These techniques will be reviewed in detail later in Section 3.2.

Proposition 3.1.3. *[21] For all $n \geq 1$ we have $\tau_2(K_{1,n}) = \left\lceil \sqrt{2n + 0.25} + 2.5 \right\rceil$.*

Proof. Assume $n \geq 1$. Let x be the vertex of degree n in $K_{1,n}$ and fix $k \in \mathbb{Z}^+$ such that there exists a 2-tone k-coloring f of $K_{1,n}$. It must be that x was assigned a unique 2-set since x is adjacent to every other vertex. Without loss of generality, we may assume that the 2-set assigned to x was $\{k, k-1\}$. Thus f assigned 2-sets to the leaves of $K_{1,n}$ from $[k-2]$. For some leaf v, say we removed the assigned 2-set from v, so f is now a partial 2-tone k-coloring. The total number of colors in assigned 2-sets on neighbours of v is 2, since v has only a single neighbor x. Also, the total number of assigned 2-sets on the second neighbours of v is $n - 1$ because each leaf of $K_{1,n}$ is distance two from every other leaf, so the leaves must all have distinct assigned

2-sets. So the removed assigned 2-set of v cannot have used either of the 2 colors on x, nor can v have duplicated any assigned 2-set on the other $n-1$ leaves. However, it must be true that some 2-set from $[k-2]$ could be assigned to v since f was a 2-tone k-coloring of $K_{1,n}$. Thus the following inequality must hold

$$\text{“total number of 2-sets”} = \binom{k-2}{2} > n-1 \geq \text{“number of 2-sets forbidden for } v\text{”}.$$

This is equivalent to $k^2-5k+(6-2n) \geq 0$, which simplifies to $k \geq \left\lceil \sqrt{2n+0.25}+2.5 \right\rceil$ since $k \in \mathbb{Z}^+$. This inequality is true if and only if f is a 2-tone k-coloring of $K_{1,n}$. So we have

$$\tau_2(K_{1,n}) = \left\lceil \sqrt{2n+0.25}+2.5 \right\rceil,$$

since τ_2 is the minimum number of colors needed to 2-tone color a graph. □

Proposition 3.1.4. *[21] If T is a tree, then $\tau_2(T) = \left\lceil \sqrt{2\Delta(T)+0.25}+2.5 \right\rceil$.*

Proof. Let T be a tree. We may assume $\Delta(T) \geq 2$. Using Proposition 3.0.3 and Proposition 3.1.3 we have

$$\left\lceil \sqrt{2\Delta(T)+0.25}+2.5 \right\rceil = \tau_2(K_{1,\Delta(T)}) \leq \tau_2(T).$$

So it suffices to prove $\tau_2(T) \leq \left\lceil \sqrt{2\Delta(T)+0.25}+2.5 \right\rceil$.

Let T be a minimal counterexample with respect to $|T|$. Pick a leaf v of T and let $T' := T - v$. By minimality and since $\Delta(T') \leq \Delta(T)$ we have

$$\tau_2(T') \leq \left\lceil \sqrt{2\Delta(T')+0.25}+2.5 \right\rceil \leq \left\lceil \sqrt{2\Delta(T)+0.25}+2.5 \right\rceil.$$

Let f' be a 2-tone k-coloring of T' where $k := \left\lceil \sqrt{2\Delta(T)+0.25}+2.5 \right\rceil$. We will extend f' to T by first defining a partial 2-tone k-coloring f where $f(u) := f'(u)$ for all $u \in V(T') = V(T) - v$. To extend f to v we have to consider the neighbours of v in T and the second neighbours of v in T since v must have a disjoint 2-set from its neighbours and a distinct 2-set from its second neighbours. Thus we need to choose a set of two colors from the remaining $k-2$ colors that do not appear on the 2-set assigned to the single neighbour of v. We must also choose this set of two colors such that it does not appear as an assigned 2-set on a second neighbour of v. There are at most $\Delta(T)-1$ second neighbours of v, each of which has a distinct assigned 2-set among the other second neighbours as they are all pairwise distance 2 from one another. So we can find a set of two colors to assign to v as long as the following inequality holds

$$\binom{k-2}{2} > \Delta(T) - 1.$$

This is always satisfied, which is seen by manipulating the inequality as follows

$$\binom{k-2}{2} \geq \Delta(T) \iff (k-5/2)^2 \geq 2\Delta(T)-6+25/4 \iff k \geq \sqrt{2\Delta(T)+0.25}+2.5.$$

Thus f can be extended to v and $\tau_2(T) \leq \left\lceil 2.5 + \sqrt{2\Delta(T)+0.25} \right\rceil$. □

As a corollary of Proposition 3.1.4, we have $\tau_2(P_n) = 5$ for all $n \geq 3$. We can also adapt the proof of Proposition 3.1.3, as Fonger et al. did, to show

$$\tau_2(K_{n_1,n_2,\ldots,n_k}) = \sum_{i=1}^{k} \left\lceil \sqrt{2n_i + 0.25} + 0.5 \right\rceil,$$

since all vertices within any one of the partite sets are pairwise distance two and must have disjoint labels from all other parts, so it is exactly like coloring the leaves of a star for each partite set. Of note is that Proposition 3.1.4 uses minimality explicitly, and this form of proof is very precarious when considering τ_2 (even more so when considering τ_t for $t > 2$). The issue stems from the fact that when deleting structures from a graph we can increase the distance between vertices. Thus a coloring of the smaller graph by induction might not yield a 2-tone coloring that can be used on the unmodified graph. The assigned 2-sets of the smaller graph will have more restrictions when considered on the unmodified graph where vertices are a shorter distance from one another. In the proof of Proposition 3.1.4 we avoid this issue by deleting a leaf, so distances between all other vertices are preserved when the leaf is removed. Thus, proofs by induction or minimality should be handled with care. Bickle and Phillips [22] captured the idea that when computing τ_2 for any graph G, the leaves can largely be ignored (see Proposition 3.1.5).

Proposition 3.1.5. *[22] Let G be a graph. If H is the largest induced subgraph of G where $\delta(H) \geq 2$ and B is the subgraph of G induced by the edges not contained within H, then* $\tau_2(G) = \max\left\{\left\lceil \sqrt{2\Delta(B) + 0.25} + 2.5 \right\rceil, \tau_2(H)\right\}$.

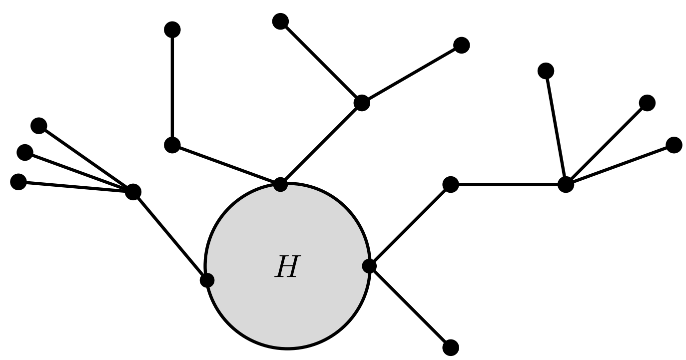

Fig. 13.: Example of Proposition 3.1.5: a graph G with H in grey and B in black.

Bickle and Phillips [22] also determined how τ_2 can change when removing any single edge from the graph; see Proposition 3.1.6 and Proposition 3.1.7.

Proposition 3.1.6. *[22] Let G be a connected graph with a cut-edge $e = uv$. Let C_1 and C_2 be the components of $G - e$ containing u and v, respectively. If $H_1 := G[V(C_1) \cup v]$ and $H_2 := G[V(C_2) \cup u]$, then $\tau_2(G) = \max\{\tau_2(H_1), \tau_2(H_2)\}$.*

Proposition 3.1.6 and Proposition 3.1.5 together imply that when computing τ_2 of a graph we may consider the graph to be 2-edge connected.

Proposition 3.1.7. *[22] Let G be a graph containing an edge $e = uv$. Then $\tau_2(G) - \tau_2(G - e) \leq 1$.*

For Proposition 3.1.7 the proof had to be careful when removing an edge, since just like when removing a vertex, the distances between vertices can grow. So a coloring of the smaller graph might have two vertices with too many shared colors among their 2-sets relative to their distance in the original graph. Here in Proposition 3.1.7 they sidestepped this issue by allowing the use of one extra color to repair labels on vertices that share too many colors.

Of course some results do not rely on induction or minimality, and others do apply induction by removing structures other than a single leaf or edge. For example, Proposition 3.1.8 is a direct proof of τ_2 for cycles.

Proposition 3.1.8. *[22, 21]*

$$\tau_2(C_n) = \begin{cases} 6, & \text{if } n = 3, 4, 7 \\ 5, & \text{otherwise} \end{cases}$$

Proof. We know $\tau_2(C_3) = \tau_2(K_3) = 6$ by Proposition 3.1.1 and $\tau_2(C_4) = \tau_2(K_{2,2}) = 6$ from the discussion right after Proposition 3.1.3. For C_7, with some case work it can be shown that at least 6 colors are needed. Briefly, if we try to color C_7 with only 5 colors at least 4 of those 5 colors must appear in three assigned 2-sets. This is because $\alpha(C_7) = 3$ and 14 colors, not necessary distinct, appear among all 2-sets. However those 4 colors themselves cannot be placed without a repeated 2-set at distance 2. Thus $\tau_2(C_7) = 6$ as witnessed by the 2-tone 6-coloring of C_7 in Figure 14.

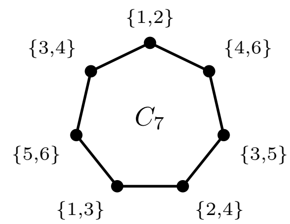

Fig. 14.: A 2-tone 6-coloring of C_7.

Now using the fact that $\tau_2(C_n) \geq \tau_2(P_3) = 5$ and the following 2-tone 5-colorings of C_m for $m \in \{5, 6, 8, 9\}$ provided in Figure 15, we can create a 2-tone 5-coloring of any cycle of length at least 10. Fix $n \geq 10$. Note that n can be written as a linear combination of $5, 6, 8, 9$ since $5+5 = 10, 5+6 = 11, 6+6 = 12, 5+8 = 13, 5+9 = 14$, and any larger number can be formed by adding a some number of 5's to one of those five numbers listed.

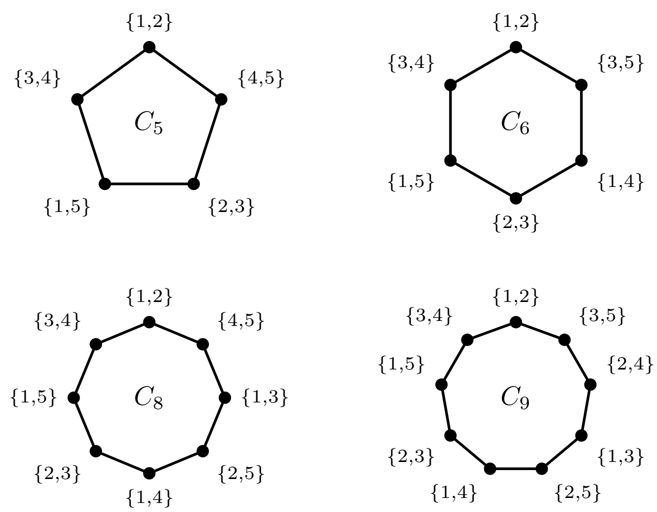

Fig. 15.: A 2-tone 5-coloring of C_5, C_6, C_8, and C_9.

By deleting the edge from the colorings from each labeled cycle in Figure 15 with labels $\{1, 2\}$ and either $\{3, 5\}$ or $\{4, 5\}$, we get a 2-tone 5-coloring f_m of P_m for some $m \in \{5, 6, 8, 9\}$ where the first three labels are the same, namely $\{1, 2\}, \{3, 4\}, \{1, 5\}$. To create a 2-tone 5-coloring of C_n we write n as a linear combination of $5, 6, 8, 9$.

We then concatenate copies of P_m, labeled by f_m, so that the vertex labelled $\{1,2\}$ in each copy is immediately preceded by a vertex labelled $\{3,5\}$ or $\{4,5\}$ in another cycle copy. □

Figure 16 shows an example of using Proposition 3.1.8. The proof idea behind Proposition 3.1.8 is formally extended for all $t \geq 2$ in Section 3.4 and used to compute τ_t for $t > 2$.

Using Proposition 3.0.3 and Proposition 3.1.7 we get the following corollary.

Corollary 3.1.9. *[22] Let G be a graph and H be a subgraph of G with at least* 1 *edge. Then*

$$\tau_2(H) \leq \tau_2(G) \leq \tau_2(H) + (||G|| - ||H||).$$

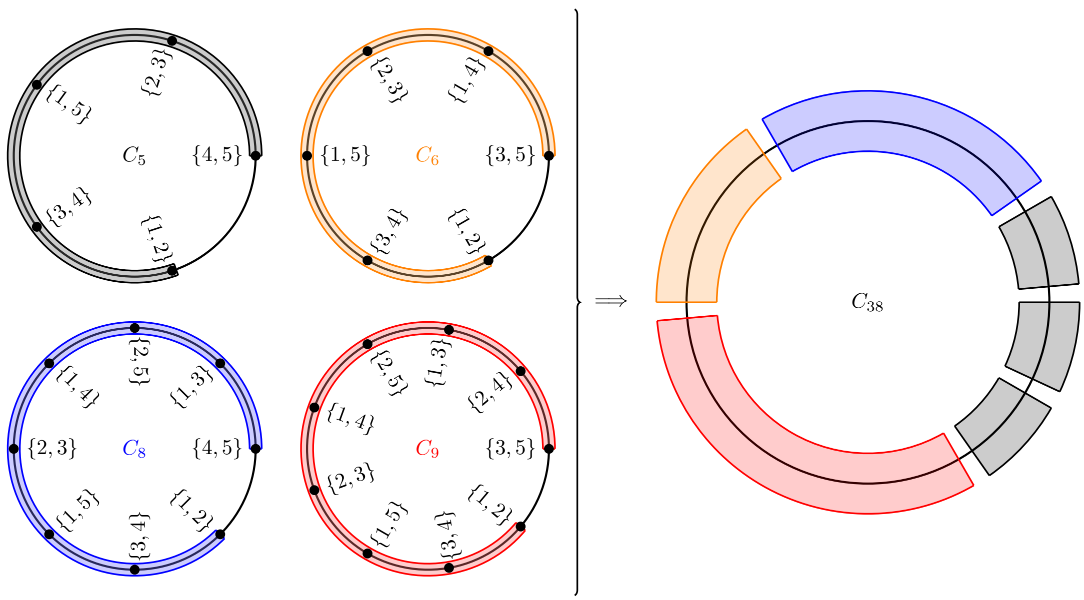

Fig. 16.: Using 2-tone 5-colorings of C_n for each $n \in \{5, 6, 8, 9\}$ to get a 2-tone 5-coloring of C_{38}, since $38 = 3(5) + 1(6) + 1(8) + 1(9)$.

Corollary 3.1.9 gives some insights into which subgraphs of a graph might be a good measure of τ_2 for the entire graph. Specifically, large dense subgraphs will "control" τ_2 for the whole graph. Of course, this might not be completely true, but it gives some intuition about τ_2. Theorem 3.1.10 and Theorem 3.1.11 were both proved by Bal, Bennett, Dudek, and Frieze [33] and they give similar insights into how τ_2 is determined by the density of the graph. Theorem 3.1.10 handles the "dense" case and Theorem 3.1.11 handles the "sparse" case. Recall that "with high probability" we are assuming that n is tending towards infinity and $G_{n,p}$ is the Erdős-Rényi-Gilbert random graph on n vertices in which each potential edge in $\binom{[n]}{2}$ is present with probability p. The "dense" case is referred to as such since the authors show that $G_{n,p}$ has diameter 2 with high probability. The "sparse" case is referred to as such since the authors show that $G_{n,p}$ is a forest with high probability.

Theorem 3.1.10. *[33] Let $p := p(n)$ satisfy $Cn^{-1/4}(\ln(n))^{9/4} \leq p < \epsilon < 1$, where C is a sufficiently large constant and ϵ is any constant strictly less than 1. Then with high probability,*

$$\tau_2(G_{n,p}) = (2 + o(1))\chi(G_{n,p}).$$

Theorem 3.1.11. *[33] Let C be a constant and $p := C/n$. Then with high probability,*

$$\tau_2(G_{n,p}) = \left\lceil \sqrt{2\Delta(G_{n,p}) + 0.25} + 2.5 \right\rceil.$$

These results do not give guarantees, but they indicate that sparse graphs might need only roughly $\sqrt{\Delta}$ colors to 2-tone color, and dense graphs might "usually" need roughly 2Δ colors to 2-tone color (using Brooks' Theorem 2). We confirm these intuitions with a series of results. The first is a corollary of Proposition 3.1.3, using Proposition 3.0.3, to prove a lower bound for τ_2. We recorded it as Lemma 3.1.12, as we will reference it several times.

Lemma 3.1.12. *[21] Let G be a graph. Then* $\tau_2(G) \geq \left\lceil \sqrt{2\Delta(G) + 0.25} + 2.5 \right\rceil$.

Proof. Using Proposition 3.0.3 and Proposition 3.1.4 we have

$$\tau_2(G) \geq \tau_2(K_{1,\Delta(G)}) = \left\lceil \sqrt{2\Delta(G) + 0.25} + 2.5 \right\rceil .$$

□

This bound is sharp for several sparse graph classes including trees with at least one edge and cycles with at least 8 edges. Now we consider upper bounds on τ_2.

The *square* of a graph G, denoted G^2, has $V(G^2) := V(G)$ and $E(G^2) := \{vw : v, w \in V(G^2) \text{ and } d_G(v, w) \leq 2\}$. Fonger et al [21] showed that a 2-tone k-coloring f can be produced from a vertex coloring f_1 of G and a vertex coloring f_2 of G^2, when f_1 and f_2 use completely disjoint color sets. Thus, we let $f(v) := \{f_1(v), f_2(v)\}$. This shows $\tau_2(G) \leq \chi(G) + \chi(G^2)$. With Brooks' Theorem 2 we have $\tau_2(G) \leq \Delta(G) + 1 + \Delta(G^2) + 1 \leq \Delta(G)^2 + \Delta(G) + 2$. Bickle and Phillips in [22] improved this to $\chi(G) \leq \Delta(G)^2 + \Delta(G)$ by analyzing the cases when G is a clique or odd cycle or when G^2 is a clique. This enabled them to use the second statement of Brooks' Theorem 2. Next, Cranston, Kim, and Kinnersley [23] made a significant improvement, here recorded as Theorem 3.1.13, to just $O(\Delta(G))$.

Theorem 3.1.13. *[23] For every graph G we have* $\tau_2(G) \leq (2 + \sqrt{2})\Delta(G)$.

Proof. Order the vertices arbitrarily and color each vertex in order greedily. Each uncolored vertex v has at most $\Delta(G)$ neighbours, and thus sees at most $2\Delta(G)$ colors within assigned 2-sets on those neighbours. And v has at most $\Delta(G)(\Delta(G) - 1)$ second neighbours. The label for v differ from each each of these so it suffices to check that $\binom{\sqrt{2}\Delta(G)}{2} > \Delta(G)(\Delta(G) - 1)$. This inequality holds, which implies the result. □

Greedily coloring is a common proof technique that appears in several papers on t-tone coloring (thus far, we have seen it in some slightly different forms). Bickle [30] improved the bound in Theorem 3.1.13 by a small additive constant by a choosing a specific vertex order and using Proposition 3.1.7. In either case, neither upper bound is known to be sharp.

The conjectured best general upper bound was originally posed by Bickle and Phillips [22] and posted to Doug West's webpage [34].

Conjecture 3.1.14. *[22, 34] Let G be a graph. Then $\tau_2(G) \leq 2\Delta(G) + 2$ with equality if and only if $w(G) = \Delta(G) + 1$ or $\Delta(G) = 2$ and $G \in \{C_4, C_7\}$.*

Proposition 3.1.8 and Proposition 3.1.4, show that Conjecture 3.1.14 holds for graphs G with $\Delta(G) = 2$, as G is either a cycle or path and $\tau_2(G) \leq 6$ with equality if and only if $G \in \{K_3, C_4, C_7\}$. Some further progress has been made towards this conjecture. Cranston et al. [23] proved $\tau_2(G) \leq 8$ if $\Delta(G) \leq 3$ and Dong [24] proved $\tau_2(G) \leq 12$ if $\Delta(G) \leq 4$. Neither paper proved the sharpness condition, and the latter is still off by 2.

Many authors studied τ_2 for specific graph classes. Some other graph classes where τ_2 is known are theta graphs, Mobius ladders, wheels, fans, Sierpinski triangle graphs, and Hanoi graphs [30, 22, 28]. Loe, Middelbrooks, Morris, and Wash [31] as well as Cooper and Wash [32] and Bickle [30] studied τ_2 for graph products as well, specifically the direct product, Cartesian product, and strong product. Loe et al. [31] proved the best known bounds on the direct product of two graphs:

$$\max\left\{\left\lceil\frac{5+\sqrt{1+8\Delta(G)\Delta(H)}}{2}\right\rceil, \tau_2(K_{\omega(G)} \times K_{\omega(H)})\right\} \leq \tau_2(G \times H) \leq \chi\left(G^2\right) + \chi\left(H^2\right).$$

Here the lower bound comes from Proposition 3.0.3 and Lemma 3.1.12. They proved an exact formula for $\tau_2(K_{\omega(G)} \times K_{\omega(H)})$ as well as the upper bound, which can be a better bound than Theorem 3.1.13 as the authors noted with $P_3 \times P_4$. They also proved the best known upper bounds for the Cartesian product of two graphs:

$$\tau_2(G \square H) \leq \begin{cases} 2\chi(G^2) & \text{if } \chi(G^2) \text{ is odd,} \\ 2\left(\chi(G^2)+1\right) & \text{otherwise.} \end{cases}$$

As noted by the authors this bound is sharp for $P_3 \square P_3$. For the strong product of two graphs:

$$\tau_2(G \boxtimes H) \leq \min\left\{\tau_2(G)\chi(H^2), \chi(G^2)\tau_2(H)\right\},$$

As noted by the authors this bound is sharp for $K_3 \boxtimes K_3$. Cranston et al. proved, using techniques similar to those in the proof of Theorem 3.1.13, the best known upper bounds for bipartite graphs with $\tau_2(G) \leq 2\left\lceil\sqrt{2}\Delta(G)\right\rceil$, which is sharp when $\Delta = 2$. They also have the best known upper bounds for chordal graphs with $\tau_2(G) \leq \left\lceil(1+\sqrt{6}/2)\Delta(G)\right\rceil + 1$. In addition they prove, for all $\epsilon > 0$, that there exists r_0 such that all chordal graphs G with $\Delta(G) > r_0$ have $\tau_2(G) \leq (2+\epsilon)\Delta(G)$. Cranston et al. [23] bounded τ_t for graphs that are k-degenerate. Their result will be discussed in more detail in Section 3.3 since Bickle [30] improved their general result when $t = 2$.

Proposition 3.1.15. *[30] Let G be a k-degenerate graph. Then*

$$\tau_2(G) \leq 2k + \left\lceil \frac{1+\sqrt{9+8(2k\Delta(G)-\Delta(G)-k^2)}}{2} \right\rceil.$$

Since planar graphs are 5-degenerate[1], we get as a corollary of Proposition 3.1.15 that $\tau_2(G) \leq \left\lceil\sqrt{18\Delta(G)-47.75}+0.5\right\rceil + 10$ for all planar graphs G. This gives some partial results towards a problem posed by West: "Determine bounds on $\tau_t(G)$

[1]This is implied by Euler's Formula, $f - e + n = 2$, as follows: $3f \leq 2e \implies 2+e-n \leq \frac{2}{3}e \implies e \leq 3n-6$, so $\sum_{v\in V(G)} d(v) = 2e \leq 6n-12$.

in terms of $\Delta(G)$ when G is planar" [34].

As mentioned before, the proof for the upper bound in Proposition 3.1.15 uses a standard technique in this area. We create a vertex order and count how many colors appear on previous colored neighbours in the vertex order, then count how many distinct labels appear on previous colored second neighbours in the vertex order. We then ensure that there are enough colors to create a label that the vertex can use that avoids the colors on its neighbours and the labels on its second neighbours. This technique will be consolidated and formally presented in a single lemma in Section 3.2.

In Section 3.2 we will prove new results for τ_2. In particular, we prove a sharp bound on τ_2 for a class graphs which includes planar graphs with girth at least 12, and a nearly sharp bound on τ_2 for all outerplanar graphs. We also prove a new upper bound on τ_2 for all planar graphs; that is sharp up to a factor of $2/\sqrt{3} \approx 1.155$. We conclude the section with some challenging conjectures and open problems. These results in Section 3.2 make significant progress towards the question posed by West [34] about t-tone coloring of general planar graphs.

As a last note for this section, there have been two variations presented on the definition of t-tone coloring in the literature. The first of which is a *pair k-coloring*, which was introduced by Bickle [30] to try to capture some behavior of certain 2-tone colorings of graphs. A *pair k-coloring* of a graph G is a 2-tone k-coloring in which every assigned 2-set is distinct; just like τ_2 we have the *pair chromatic number* of G, denoted $\mathrm{pc}(G)$. Clearly $\tau_2(G) \leq \mathrm{pc}(G)$ for all graphs G. The second and last variation is a *tight t-tone k-coloring* and it was introduced by Yang [26]. A *tight t-tone k-coloring* f of a graph G is a t-tone k-coloring of G where $|f(u) \cap f(v)| = d_G(u, v) - 1$ for all distinct $v, u \in V(G)$. If a graph has a tight t-tone k-coloring, then it is a coloring that cannot be improved any further, as every pair of distinct vertices shares as many colors as possible.

3.2 New Results for τ_2

The *average degree* of a graph G is the average of the degrees of the vertices of G. The *maximum average degree* of a graph G, denoted $\mathrm{mad}(G)$, is the maximum of the average degree of all the subgraphs of G.

In Theorem 3.2.6 we determine $\tau_2(G)$ for all outerplanar graphs, up to a small additive constant. And in Theorem 3.2.8 we determine $\tau_2(G)$ for all graphs G with $\mathrm{mad}(G) < 12/5$ and $\Delta(G) \geq 11$. This includes planar graphs with girth at least 12. We will start with Theorem 3.2.5 which bounds $\tau_2(G)$ for all planar graphs; as $\Delta(G)$ grows, our bound is sharp asymptotically up to a factor of $2/\sqrt{3} \approx 1.155$.

All our proofs in this section proceed by minimal counterexample. As mentioned in Section 3.1, this approach requires extra care, since a 2-tone coloring of a subgraph H of G might fail to induce a 2-tone coloring of $G[V(H)]$. Specifically, if we delete a vertex v to form a subgraph H, we allow the possibility that neighbors of v in G will receive identical labels in H; of course, this is forbidden in a 2-tone coloring of G. To avoid this difficulty, rather than deleting vertices, we often instead contract edges, which never increases distances. However, this adds the potential issue of increasing the maximum degree. To avoid this pitfall, we typically contract an edge with one endpoint of degree at most 2. We will use the local structure of planar and outerplanar graphs presented in Lemma 3.2.1 and Lemma 3.2.2 respectively, in order to find an edge to contract.

Lemma 3.2.1. *[35] For every planar graph G there exists $v \in V(G)$ such that $d(v) \leq 5$ and v has at most two neighbors with degree at least 11.*

Lemma 3.2.2. *[36, 37] For every outerplanar graph G there exists $xy \in E(G)$ with either (i) $d(x) = 1$ or (ii) $d(x) = 2$ and $d(y) \leq 4$.*

Lemma 3.2.3 constructs a planar graph that improves the lower bound of τ_2 in

Lemma 3.1.12 (Figure 17 illustrates this construction).

Lemma 3.2.3. *For each $t \geq 1$, we form H_t from K_3 by replacing each edge $vw \in E(K_3)$ with a copy of $K_{2,t}$, identifying the high degree vertices with v and w. For all t we have $\left\lceil \sqrt{3\Delta(H_t) + 0.25} + 0.5 \right\rceil \leq \tau_2(H_t) \leq \left\lceil \sqrt{3\Delta(H_t) + 30.25} + 0.5 \right\rceil$. (When $t \geq 27$ these two bounds differ by at most 1.)*

Proof. Fix a positive integer k to be determined later. We consider a 2-tone k-coloring of H_t. It is easy to check that $\tau_2(C_6) = 5$, so assume $t \geq 2$. Let x, y, and z denote the vertices of degree at least 4. For the lower bound, note that all $3\Delta(H_t)/2 = 3t$ vertices excluding x, y, z must get distinct 2-element subsets of $[k]$. The inequality $\binom{k}{2} \geq 3\Delta(H_t)/2$ is equivalent to the lower bound.

Now we prove the upper bound. Color x with $\{1, 2\}$; color y with $\{3, 4\}$; and color z with $\{5, 6\}$. Now we assign each remaining vertex of H_t a distinct element of $\binom{[k]}{2} \setminus \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$. This requires that no vertex of degree 2 receive a label from $\binom{[6]}{2}$. Thus, we need $\binom{k}{2} - \binom{6}{2} \geq 3t$. This inequality is equivalent to the upper bound. We must also ensure that the coloring is proper, i.e., all labels including 1 or 2 (other than $\{1, 2\}$) be used on vertices non-adjacent to x, and similarly for $\{3, 4\}$ with y and for $\{5, 6\}$ with z. However, this is easy to ensure.

Finally, we show that the bounds differ by at most 1 when $t \geq 27$. For this conclusion, it suffices that $\sqrt{3\Delta(H_t) + 30.25} - \sqrt{3\Delta(H_t) + 0.25} \leq 1$. This inequality holds when $\Delta(H_t) \geq 70$, i.e., when $t \geq 35$. And it easy to check the remaining eight cases by hand. □

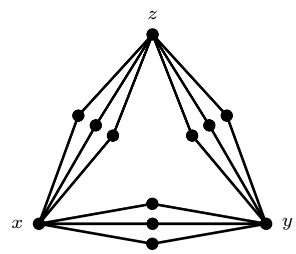

Fig. 17.: The graph H_3.

As mentioned in Section 3.1, a common technique is to consider the neighbors and second neighbours of an uncolored vertex in order to extend a partial 2-tone coloring of a graph G. Lemma 3.2.4 formalizes this approach.

Lemma 3.2.4. *Let G be a graph and f be a partial 2-tone k-coloring of G. For any uncolored vertex $v \in V(G)$, if $\binom{k-2|N(v)|}{2} > |N^2(v)|$, then f can be extended to v.*

Proof. Let G, f, and v be as in the lemma. To extend φ to v, we must avoid all colors that appear in 2-sets assigned to vertices in $N(v)$, which forbids at most $2|N(v)|$ colors. We must also avoid all assigned 2-sets used on vertices in $N^2(v)$, which forbids at most $|N^2(v)|$ 2-sets. Thus, it suffices to have $\binom{k-2|N(v)|}{2} > |N^2(v)|$ (see Figure 18). □

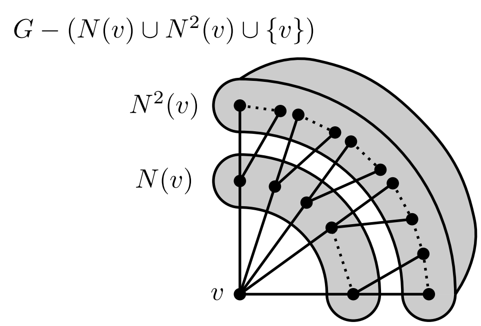

Fig. 18.: A vertex v with $N(v)$, which has at most $\Delta(G)$ vertices within it, and $N^2(v)$ which has at most $\Delta(G)(\Delta(G) - 1)$ vertices within it.

We first prove an upper bound on $\tau_2(G)$ for every planar graph G, and then show how to strengthen it for two classes of "sparse" planar graphs. For a general planar graph G (with maximum degree $\Delta(G)$), our next upper bound differs from the lower bound in Lemma 3.1.12 by a factor of approximately $\sqrt{2}$. However, for our construction H_t in Lemma 3.2.3, the present upper bound differs from the lower bound by only a factor of $2/\sqrt{3} \approx 1.155$.

Theorem 3.2.5. *Let G be a planar graph. Then $\tau_2(G) \leq \left\lfloor \sqrt{4\Delta(G) + 50.25} + 31.1 \right\rfloor \leq \left\lfloor \sqrt{4\Delta(G)} + 36.5 \right\rfloor$. Furthermore, $\tau_2(G) \leq \max\left\{41, \left\lfloor \sqrt{4\Delta(G) + 50.25} + 11.5 \right\rfloor\right\}$.*

Proof. In the first statement, the second inequality is easy to verify, so we focus on the first. The second statement is clearly stronger when $\Delta(G)$ is sufficiently large, but we include the first to give a better bound when $\Delta(G)$ is small. We prove both statements simultaneously.

Suppose the theorem is false and let G be a counterexample that minimizes $|V(G)|$. If $\Delta(G) \leq 12$, then Theorem 3.1.13 gives

$$\tau_2(G) \leq \left\lceil (2 + \sqrt{2})\Delta(G) \right\rceil \leq \lfloor \sqrt{4\Delta(G) + 50.25} + 31.1 \rfloor \leq 41.$$

So we assume that $\Delta(G) \geq 13$. By Lemma 3.2.1, there exists $v \in V(G)$ such that $d(v) \leq 5$ and v has at most two neighbors of degree at least 11. If $d(v) \geq 3$, then pick $w \in N(v)$ with $d(w) \leq 10$; otherwise let w be an arbitrary neighbor of v. Form H from G by contracting vw. Since $|V(H)| < |V(G)|$ by induction $\tau_2(H) \leq \max\left\{41, \left\lfloor \sqrt{4\Delta(H) + 50.25} + 11.5 \right\rfloor\right\}$, and since $\Delta(H) \leq \max\{\Delta(G), 5 + 10 - 2\} = \Delta(G)$ we have $\tau_2(H) \leq \max\left\{41, \left\lfloor \sqrt{4\Delta(G) + 50.25} + 11.5 \right\rfloor\right\}$. Similarly,

$$\tau_2(H) \leq \left\lfloor \sqrt{4\Delta(H) + 50.25} + 31.1 \right\rfloor \leq \left\lfloor \sqrt{4\Delta(G) + 50.25} + 31.1 \right\rfloor.$$

This 2-tone coloring of H induces a partial 2-tone coloring of G, with v uncolored. Now $N_G(v)$ forbids at most $2|N_G(v)| \leq 10$ colors from use on v. Further, vertices in $N_G^2(v)$ forbid at most $2(\Delta(G) - 1) + 3(9) = 2\Delta(G) + 25$ distinct 2-sets from use on v.

By Lemma 3.2.4 we can extend any partial 2-tone k-coloring of G (with v uncolored) to a 2-tone k-coloring of G whenever $\Delta(G) \geq 13$ and

$$\binom{k-10}{2} > 2\Delta(G) + 25.$$

This inequality is easy to verify when $k = \left\lfloor \sqrt{4\Delta(G) + 50.25} + 11.5 \right\rfloor$, which completes the proof of both statements. $\square$

In the next two theorems, we consider special classes of "sparse" planar graphs that are in a sense "tree-like". For these graphs, we improve the leading coefficient in the bound of Theorem 3.2.5 by a factor of approximately $\sqrt{2}$, so that it matches that in the lower bound given by Lemma 3.1.12.

Theorem 3.2.6. *If G is outerplanar, then* $\tau_2(G) \leq \left\lfloor \sqrt{2\Delta(G) + 4.25} + 5.5 \right\rfloor \leq \left\lfloor \sqrt{2\Delta(G)} + 6.6 \right\rfloor$.

Proof. The second inequality is easily verified by algebra, so we focus on the first. Suppose the theorem is false and let G be a counterexample minimizing $|V(G)|$. Note that the class of outerplanar graphs is closed under edge contraction.

By Lemma 3.2.2 there exists $vw \in E(G)$ such that $d(v) = 1$, or $d(v) = 2$ and $d(w) \leq 4$. In either case, form H by contracting vw (restricting to the underlying simple graph if we create a pair of parallel edges). Note that $|H| < |G|$ and $\Delta(H) \leq \Delta(G)$. By the minimality of G,

$$\tau_2(H) \leq \left\lfloor \sqrt{2\Delta(H) + 4.25} + 5.5 \right\rfloor \leq \left\lfloor \sqrt{2\Delta(G) + 4.25} + 5.5 \right\rfloor.$$

The vertices in $N_G(v)$ forbid at most $2|N(v)| \leq 4$ colors from use on v. Further, the vertices in $N_G^2(v)$ forbid at most $\Delta(G) - 1 + (4 - 1) = \Delta(G) + 2$ distinct 2-sets from use on v. By Lemma 3.2.4 we can extend any 2-tone k-coloring of H to G when

$$\binom{k-4}{2} > \Delta(G) + 2.$$

This inequality is easy to verify when $k = \left\lfloor \sqrt{2\Delta(G) + 4.25} + 5.5 \right\rfloor$. □

Lemma 3.2.7 is a structural result that we will use to prove Theorem 3.2.8. As a special case, that theorem will exactly determine τ_2 for planar graphs with sufficiently large girth and max degree.

We will also need some new definitions. A *d^+-vertex*, *d^--vertex*, or *d-vertex* is, respectively, a vertex of degree at least d, at most d, and exactly d. An *ℓ-thread* in a graph G is a trail of length $\ell + 1$ in G whose ℓ internal vertices have degree 2 in G. We refer to the non-internal vertices of an ℓ-thread as *endpoints*. So an ℓ-thread has two endpoints, not necessarily distinct.

For Lemma 3.2.7 and Theorem 3.2.8 we present the proofs as if each ℓ-thread has two distinct endpoints, but all arguments remain valid if the endpoints are not distinct.

Lemma 3.2.7. *Let G be a graph with $\delta(G) \geq 2$. If mad$(G) < 12/5$, then G contains at least one of the following:*

(a) a 4-thread,

(b) a 3-thread with a 5^--vertex as an endpoint, or

(c) a 2-thread with a 3^--vertex and a 5^--vertex as endpoints.

Proof. Let G be a graph with $\delta(G) \geq 2$ and $\text{mad}(G) < 12/5$. Assume for contradiction that G has no threads of type (a), (b), and (c). If G contains a 2-regular component, then it contains an instance of (c); so assume no component of G is 2-regular. Thus, every 2-vertex appears in a unique maximal thread, and the endpoints of that thread are 3^+-vertices. We give each vertex v initial charge $d(v)$. To redistribute charge, each maximal thread takes charge $1 - 12/(5d(v))$ from each of its endpoints. Each thread redistributes its charge equally to its internal vertices. Below we show that each vertex ends with charge at least $12/5$, contradicting that $\text{mad}(G) < 12/5$.

Since G has no 4-thread, each maximal thread has at most 3 internal vertices. If a thread t has a vertex v as an endpoint, then the charge that t receives from v is: $1 - 12/(3(5)) = 1/5$ if $d(v) = 3$; and at least $1 - 12/(4(5)) = 2/5$ if $d(v) \geq 4$; and at least $1 - 12/(6(5)) = 3/5$ if $d(v) \geq 6$.

Each 1-thread gains at least $1/5$ from each endpoint, so finishes with at least $12/5$.

Each 2-thread cannot be an instance of (c), so either (i) both of its endpoints are 4^+-vertices or (ii) it has a 6^+-vertex as an endpoint. So a 2-thread gains either (i) at least $2/5$ from each endpoint or (ii)at least $3/5$ from the endpoint that is a 6^+-vertex and at least $1/5$ from the other endpoint. Thus, each 2-thread finishes with at least $2(2) + 4/5 = 2(12/5)$.

Each 3-thread has a 6^+-vertex for each endpoint, otherwise G contains (b). So a 3-thread gains at least $3/5$ from each endpoint. Thus, each 3-thread finishes with at least $3(2) + 6/5 = 3(12/5)$. If v is an endpoint of a thread, then v sees at most $d(v)$ threads. Thus, v has final charge $d(v) - d(v)(1 - 12/(5d(v)) = 12/5$. This implies that $\overline{d}(G) \geq 12/5$; which contradicts the hypothesis $\text{mad}(G) < 12/5$. □

Theorem 3.2.8. *Let G be a graph with $mad(G) < 12/5$. Then*

$$\tau_2(G) \leq \max\left\{7, \left\lceil \sqrt{2\Delta(G) + 0.25} + 2.5 \right\rceil \right\}.$$

Furthermore, if G is planar with girth at least 12 and $\Delta(G) \geq 7$, then

$$\tau_2(G) = \left\lceil \sqrt{2\Delta(G) + 0.25} + 2.5 \right\rceil.$$

Proof. The second statement follows from the first since a planar graph G with girth at least 12 has $\text{mad}(G) < 2(12)/(12-2) = 12/5$ and Lemma 3.1.12 implies that if $\Delta(G) \geq 7$, then $\tau_2(G) \geq 7$. We now prove the first statement.

Suppose the theorem is false and let G be a counterexample minimizing $|V(G)|$. If there exists v with $d(v) \leq 1$, then by minimality

$$\tau_2(G-v) \leq \max\left\{7, \left\lceil \sqrt{2\Delta(G)+0.25}+2.5 \right\rceil \right\}.$$

And by Lemma 3.2.4 we get $\tau_2(G) \leq \max\left\{7, \left\lceil \sqrt{2\Delta(G)+0.25}+2.5 \right\rceil \right\}$. Thus, we assume $\delta(G) \geq 2$.

By Lemma 3.2.7 we know G contains configuration (a), (b), or (c) in that lemma. We will show that none of these configurations can appear in our minimal counterexample G. To do so, we form a subgraph H by deleting some vertices of G, color H by minimality, and extend our coloring of H to the deleted vertices of G, to contradict that G was a counterexample. Let $k_G = \max\left\{7, \left\lceil \sqrt{2\Delta(G)+0.25}+2.5 \right\rceil \right\}$. For an arbitrary subgraph H of G (which will be clear from context), let $k_H = \max\left\{7, \left\lceil \sqrt{2\Delta(H)+0.25}+2.5 \right\rceil \right\}$.

Case 1: G contains a 4-thread, as shown in Figure 19. Form H from G by deleting v_2 and v_3. Note that $|H| < |G|$ and $\Delta(H) \leq \Delta(G)$. By the minimality of G, we have $\tau_2(H) \leq k_H \leq k_G$. Let φ be a 2-tone k_G-coloring of H. By Lemma 3.2.4, since $k_G \geq 7$ we can extend φ to v_2 followed by v_3, a contradiction.

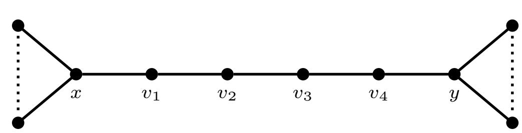

Fig. 19.: A 4-thread with endpoints x and y.

Case 2: G contains a 3-thread, as shown in Figure 20. Form H by deleting v_2 and v_3. Note that $|H| < |G|$ and $\Delta(H) \leq \Delta(G)$. By the minimality of G, we have $\tau_2(H) \leq k_H \leq k_G$. Let φ be a 2-tone k_G-coloring of H. By Lemma 3.2.4, since $k_G \geq 7$ we can extend φ to v_3 followed by v_2, a contradiction. In particular, since y forbids 2 colors from use on v_3 and the vertices at distance 2 (in G) from v_3 forbid at most 5 distinct 2-sets from use on v_3, since $k_G \geq 7$ we have at least $\binom{5}{2} - 5 = 5$ remaining 2-sets available for v_3. Afterwards, it is easy to color v_2. This finishes the extension of φ to a 2-tone k_G-coloring of G, which is a contradiction.

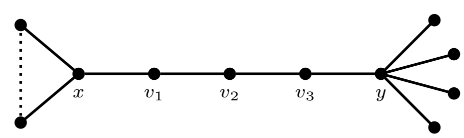

Fig. 20.: A 3-thread with endpoints x and y, where $d(y) \leq 5$.

Case 3: *G* contains a 2-thread, as shown in Figure 21. Form H from G by deleting v_1. Note that $|H| < |G|$ and $\Delta(H) = \Delta(G)$. By the minimality of G, we have $\tau_2(H) \leq k_H \leq k_G$. Let φ be a 2-tone k_G-coloring of H. Note that φ might fail to induce a partial 2-tone coloring of G since it is possible that $\varphi(v_2) = \varphi(x)$, which creates a problem since $d(v_2, x) = 2$. To avoid this issue we can simply recolor v_2, since $d(v_2) = 2$. In this case, v_2 is a leaf of H, so its neighbor forbids 2 colors from use on v_1; furthermore, the vertices at distance 2 from v_2 (in G) forbid at most 5 distinct 2-sets from use on v_1. So we can recolor w with another 2-set, since $\binom{7-2}{2} > 4 + 1$; in fact, we have at least 5 choices of label for v_2. Thus, we assume that φ induces a proper 2-tone coloring of G. Finally, we consider coloring v_1. Its two neighbors forbid at most $2(2) = 4$ colors. And the three vertices at distance two forbid at most an additional three 2-sets. If $k_G \geq 8$, then we have a 2-set available to use on v_1. So assume instead that $k_G = 7$. If no 2-sets are available to use on v_1, then the two 2-sets used on its neighbors are disjoint. Further, the three 2-sets used on vertices at distance two are distinct, and they are all disjoint from the set of colors used on its neighbors. But now to escape this situation we can recolor v_2 with one of the other 4 possible 2-sets we had to choose from. Afterward, we can extend the 2-tone 7-coloring to G, a contradiction.

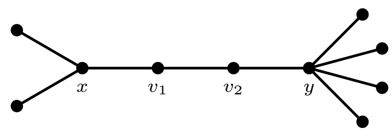

Fig. 21.: A 2-thread with endpoints x and y, where $d(x) = 3$ and $d(y) \leq 5$.

□

We conclude this section with a few conjectures.

Conjecture 3.2.9. *There exists a constant C such that all planar G satisfy $\tau_2(G) \leq \sqrt{3\Delta(G)} + C$.*

Perhaps the following stronger statement holds. It is essentially best possible, due to Lemma 3.2.3.

Conjecture 3.2.10. *If G is planar with $\Delta(G)$ sufficiently large, then*

$$\tau_2(G) \leq \left\lceil \sqrt{3\Delta(G) + 30.25} + 0.25 \right\rceil.$$

We also believe that for planar graphs the girth requirement in Theorem 3.2.8 can be significantly weakened.

Conjecture 3.2.11. *There exists a constant C such that every planar graph G with girth at least 5 satisfies $\tau_2(G) \leq \sqrt{2\Delta(G)} + C$.*

It is interesting to note the following. For every integer $t \geq 2$ there exists a girth g_t and a maximum degree Δ_t such the maximum value of $\tau_t(G)$, taken over all planar graphs G with girth at least g_t and $\Delta(G) \geq \Delta_t$, is achieved by a tree. Cranston et al. [23] showed that this maximum (for trees) is bounded by $c_t\sqrt{\Delta(G)}$ for some constant c_t; and this is asymptotically sharp. We briefly sketch the extension to planar graphs with sufficiently large girth and maximum degree. Following an approach similar to (but simpler than) the proof of Lemma 3.2.7, we can prove that if G has sufficiently low maximum average degree, then it contains either a 1^--vertex or a $3t$-thread. Every 1^--vertex can be handled inductively (by coloring greedily). For a $3t$-thread, we delete the middle t vertices and color the smaller graph by induction. We choose Δ_t large enough that $\tau_t(K_{1,\Delta_t}) \geq 3\tau_t(P_t)$. (Recall that $\tau_t(K_{1,\Delta_t}) \geq \tau_2(K_{1,\Delta_t}) \geq \sqrt{2\Delta_t}$, by Lemma 3.1.12.) Now the number of colors forbidden on all of the uncolored vertices (taken together) is at most $2\tau_t(P_t)$. Thus, we have at least $\tau_t(P_t)$ colors that are available for use on all of the uncolored vertices. So we can extend the coloring.

3.3 Background and Results for τ_t when $t \geq 2$

We assume for the remainder of this section that all graphs have $\Delta \geq 2$; otherwise, just as in Section 3.1, we can compute τ_t easily (see Figure 22).

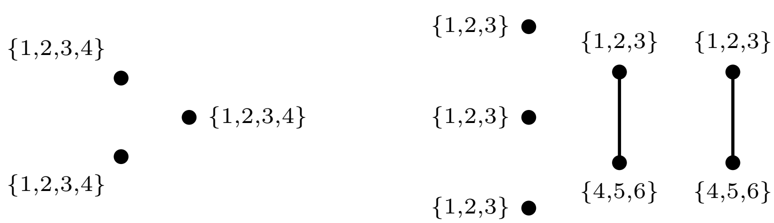

Fig. 22.: A 4-tone 4-coloring of E_3 and a 3-tone 6-coloring of $3K_1 + 2K_2$.

While τ_2 is the most widely studied case of t-tone coloring, several papers have also studied τ_t in the general case when $t \geq 2$. In this section we assume that $t \geq 2$.

Similar to Section 3.1, we have Theorem 3.3.1 and Theorem 3.3.2 which were both proved by Bal, Bennett, Dudek, and Frieze [33]. Theorem 3.3.1 is the "dense" case and Theorem 3.3.2 is the "sparse" case.

Theorem 3.3.1. *[33] Let $p := p(n)$ satisfy $Cn^{-1/4}(\ln(n))^{9/4} \leq p < \epsilon < 1$ where C*

is a sufficiently large constant and ϵ is any constant strictly less than 1*. Then with high probability,*

$$\tau_t(G_{n,p}) = (t + o(1))\chi(G_{n,p}).$$

Theorem 3.3.2. *[33] Let c be a constant and $p := c/n$ and $t \geq 3$. Then there exists constants c_1 and c_2 such that with high probability,*

$$c_1\sqrt{\Delta(G_{n,p})} \leq \tau_t(G_{n,p}) \leq c_2\sqrt{\Delta(G_{n,p})}.$$

These results do not give guarantees, but they indicate that sparse graphs might need only roughly $\sqrt{\Delta}$ colors to t-tone color and dense graphs might "usually" need roughly $t\Delta$ colors to t-tone color (using Brooks' Theorem 2). Using information from earlier sections, we can provide a lower bound for τ_t with Proposition 3.3.3, which follows directly from Lemma 3.1.12 and Proposition 3.0.4. In Section 3.4 we will improve this lower bound. This bound matches the lower bound from Theorem 3.3.2.

Proposition 3.3.3. *[23] Let G be a graph and $t \geq 2$. Then*

$$\tau_t(G) \geq \left\lceil \sqrt{2\Delta(G) + 0.25} + 2.5 \right\rceil.$$

For almost all results regarding τ_t we have that t is arbitrary and fixed, but the graph is variable (usually in some graph class). Proposition 3.3.4 shows what happens to τ_t for a fixed graph, but t is variable. It gives a new lower bound for τ_t; however, for "small" values of t it is almost always worse than Proposition 3.3.3 (sometimes even giving a negative value). For a connected graph G, the *Wiener Index of G*, denoted $W(G)$, is the sum of the distances of all pair of vertices of G.

Proposition 3.3.4. *[25] Let G be a connected graph. Then*

$$\tau_t(G) \geq t|G| - W(G) + \binom{|G|}{2}$$

with equality if and only if $t \geq M$ where $M := \max_{v \in V(G)} M(v)$ and $M(v) := \sum_{v,u \in V(G), v \neq u}(d_G(u,v) - 1)$.

There are some known upper bounds. For a connected graph G, Yang [26] provided an alternate proof for Proposition 3.3.4 by forming from G an auxiliary multigraph G^* and fixing a specific orientation $\vec{G^*}$. Using $\vec{G^*}$, Yang created an algorithm that extends a partial t-tone coloring of G. Yang also proved an upper bound for τ_t using the same auxiliary graph G^*. The bound is better than the trivial upper bound Proposition 3.1.2 by an additive constant.

Fonger et al. [21] proved an upper bound for τ_t akin to their upper bound of $\tau_2(G) \leq \chi(G) + \chi(G^2)$. The *kth power* of a graph G, denoted G^k, is formed by $V(G^k) := V(G)$ and $E(G^k) := \{vw : v, w \in V(G^k) \text{ and } 1 \leq d_G(v,w) \leq k\}$. Fonger et al [21] proved $\tau_t(G) \leq \sum_{i=1}^{t} \chi(G^i)$, and with Brooks' Theorem 2 we have $\tau_t(G) \leq O(\Delta(G)^t)$.

The best known general upper bound for τ_t was proved by Cranston et al. [23]; recorded here as Proposition 3.3.5. A sketch of the proof for Proposition 3.3.5 is provided here, omitting some calculations. Note that $O(\Delta(G))$ matches the asymptotics of the bounds for $\tau_t(G_{n,p})$ from Theorem 3.3.1.

Proposition 3.3.5. *[23] Let G be a graph. Then $\tau_t(G) \leq (t^2 + t)\Delta(G)$.*

Sketch of Proof. Suppose f is a partial t-tone k-coloring of G and let $v \in V(G)$ be an uncolored vertex. Clearly, at most $t\Delta(G)$ colors used on $N(v)$, since $|N(v)| \leq \Delta(G)$. Then for vertices that have assigned t-sets that are distance d from v, by definition, can share at most $d - 1$ colors with v. So the number t-sets forbidden from use on

v due to each vertex at distance d is at most $\binom{t}{d}\binom{(k-t\Delta(G))-d}{t-d}$, and there are at most $\Delta(G)(\Delta(G)-1)^{d-1}$ such vertices. Let $k' := k - t\Delta(G)$. Now if the following inequality holds, then we can extend the coloring to v:

$$\sum_{d=2}^{t} \binom{t}{d}\binom{k'-d}{t-d}\Delta(G)(\Delta(G)-1)^{d-1} < \binom{k'}{t}.$$

And this inequality does hold when $k' = t^2\Delta(G)$. $\square$

We have good bounds for τ_t on fewer graph classes than we do for τ_2. Proposition 3.1.1 computes τ_t for cliques and Proposition 3.3.6 computes τ_t for paths.

Proposition 3.3.6. *[22] For all $t, n \geq 1$ we have $\tau_t(P_n) = \sum_{i=0}^{n-1} \max\left\{0, t - \binom{i}{2}\right\}$.*

Proof. Let P_n be the path on n vertices and label the vertices $v_1, \ldots, v_n$. Color the vertices in order of increasing subscript. When vertex v_i is being colored, for each $j \in [i-1]$ there are j colors in the t-set assigned to v_{i-j-1} that do not appear in any of the assigned t-sets of v_{i-j} to v_{i-1}. We use these colors on v_i until either (a) v_i has t colors or (b) we run out of vertices. In the latter case, we have used $\sum_{j=0}^{i-1} j = \binom{i}{2}$ colors from previous vertices, and need $t - \binom{i}{2}$ new colors. When $\binom{i}{2} \geq t$, no more new colors are needed. $\square$

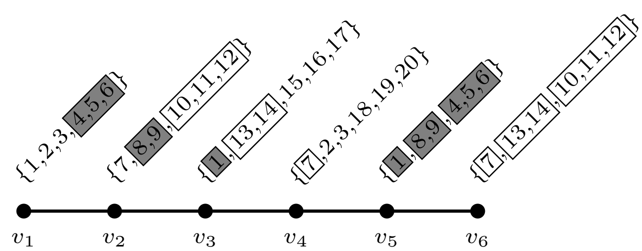

Fig. 23.: Using the algorithm outlined in Proposition 3.3.6 to 6-tone 20-color P_6.

For some partial results, Wu [27] computed $\tau_t(C_n)$ for all $t \geq 3$ and $4 \leq n \leq 7$, recorded here as Lemma 3.3.7.

Lemma 3.3.7. *[27] For all $t \geq 3$ we have $\tau_t(C_4) = 4t - 2$ and $\tau_t(C_5) = 5t - 5$. For all $t \geq 4$ we have $\tau_t(C_6) = 6t - 12$ with $\tau_3(C_6) = 8$. For all $t \geq 6$ we have $\tau_t(C_7) = 7t - 21$ with $\tau_5(C_7) = 17$, $\tau_4(C_7) = 13$, and $\tau_3(C_7) = 9$.*

We recall two bounds on τ_t for $t \geq 2$ for some graph classes that were mentioned in Section 3.1, both proved by Cranston et al. [23]. The first is an upper bound for $t \geq 2$ of τ_t for all trees; recorded here as Propositions 3.3.8. This bound is asymptotically tight due to Proposition 3.3.3.

Proposition 3.3.8. *[23] For every $t \geq 2$ there exists a constant $c := c(t)$ such that $\tau_t(T) \leq c\sqrt{\Delta(T)}$ whenever T is a tree.*

The second is an upper bound for all k-degenerate graphs. For example, this result implies a bound on τ_t for all planar graphs, since they are 5-degenerate. Bickle [30] made an improvement for $t = 2$, in Proposition 3.1.15.

Proposition 3.3.9. *[23] Let G be a graph and $k \geq 2$. Then for each $t \geq 2$ we have $\tau_t(G) \leq kt + kt^2\Delta(G)^{1-1/t}$.*

Only two papers have studied τ_t for specific values of $t > 2$. Cooper and Wash [32] proved $10 \leq \tau_3(P_n \Box P_m) \leq 12$ for all $n \geq m \geq 2$; and Dong [29] proved $\tau_3(G) \leq 21$ for graphs G with $\Delta(G) \leq 3$.

In Section 3.4 we will improve the lower bound on τ_t in Proposition 3.3.3 using results from design theory and studying $\tau_t(K_{1,n})$. In addition, we determine $\tau_t(C_n)$ exactly, for all $t \in \{3, 4, 5\}$ and all $n \geq 3$ and we determine $P_m \Box P_n$ exactly for τ_3 and τ_4 and bound τ_5 for all $n \geq m \geq 2$. Finally we consider graphs classes with *polynomial expansion*, graphs where there exists a polynomial p where the total number of vertices at distance d from any single vertex is at most $p(d)$. We prove a general upped bound for graph classes that have polynomial expansion and as an example prove a new upper bound on τ_t for grid graphs.

3.4 New Results for τ_t when $t \geq 2$

We first will prove a best known general lower bound for τ_t for $t > 2$, by determining a new lower bound on $\tau_t(K_{1,n})$. Due to Proposition 3.3.8, we can increase the lower bound in Proposition 3.3.3 by at most a multiplicative constant.

We first notice that t-tone k-coloring of $K_{1,n}$ is equivalent to finding a family of $(k-t)$-sets from $\binom{[k-t]}{t}$ of size n that pairwise intersect in at most 1 element. This is because the non-leaf vertex of $K_{1,n}$ must be a t-set disjoint from all others. Thus, all leaves must be assigned t-sets from $[k-t]$. Finding a t-tone k-coloring is thus equivalent to finding n edge-disjoint copies of K_t within a K_{k-t}. Each copy of K_t corresponds to a t-set assigned to a leaf of $K_{1,n}$. Since these K_t's are pairwise edge-disjoint, this will imply the assigned t-sets pairwise share at most 1 color (as all leaves in $K_{1,n}$ are distance two from one another). Figure 24 shows an example.

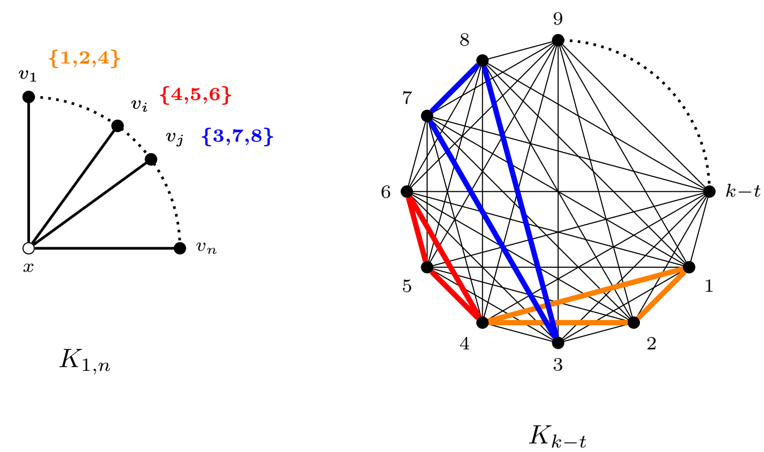

Fig. 24.: An example of 3-tone $(k-t)$-coloring the leaves of $K_{1,n}$, by looking at edge disjoint K_3's in a K_{k-t}.

A $q-(k,t,\lambda)$ *packing design* is a family of t-sets from $\binom{[k]}{t}$ where any q-set from $\binom{[k]}{q}$ is in at most λ member of the family. The *packing number* $D_\lambda(k,t,q)$ is the maximum size of a $q-(k,t,\lambda)$ packing design. The Johnson-Schönheim Bound [38, 39] gives an upper bound for the packing number:

$$D_\lambda(k,t,q) \le \left\lfloor \frac{k}{t} \left\lfloor \frac{k-1}{t-1} \cdots \left\lfloor \frac{\lambda(k-q+1)}{t-q+1} \right\rfloor \cdots \right\rfloor \right\rfloor.$$

So a $2-(k,t,1)$ packing design is equivalent to finding a family of edge-disjoint copies of K_t in a K_k. Thus

$$D_1(k,t,2) \le \left\lfloor \frac{k}{t} \left\lfloor \frac{k-1}{t-1} \right\rfloor \right\rfloor.$$

Lemma 3.4.1. *Fix $t \ge 2$. Then $\tau_t(K_{1,n}) \ge \sqrt{(t^2-t)n + 0.25} + t + 0.5$.*

Proof. Let $t \ge 2$, $n \ge 1$, and x be the non-leaf vertex of $K_{1,n}$. Let f be a t-tone $(k+t)$-coloring of $K_{1,n}$. Note that x must receive a t-set disjoint from all others since x is adjacent to all other vertices. So we assume that f assigns t-sets from $\binom{[k]}{t}$ to all n leaves of $K_{1,n}$. Thus there exists n edge disjoint copies of K_t in a K_k. The Johnson-Schönheim Bound implies

$$n \le \left\lfloor \frac{k}{t} \left\lfloor \frac{k-1}{t-1} \right\rfloor \right\rfloor \le \frac{k(k-1)}{t(t-1)}.$$

Thus $0.5 + \sqrt{(t^2-t)n + 0.25} \le k$. And the result follows. □

For a graph G, Proposition 3.0.3 bounds $\tau_t(G)$ below using any subgraph of G. If we take that subgraph to be $K_{1,\Delta(G)}$ then we can compare the bounds on $\tau_t(K_{1,\Delta(G)})$ given by Proposition 3.3.4 and Lemma 3.4.1 to determine when one is more preferable as a lower bound on $\tau_t(G)$. Proposition 3.3.4 gives $\tau_t(K_{1,\Delta(G)}) = t(\Delta(G)+1) + \frac{\Delta(G)}{2} - \frac{\Delta(G)^2}{2}$ whenever $t \ge 2\Delta(G) - 1$. This is because $W(K_{1,n}) = n^2$ and

$$\max_{v \in V(K_{1,n})} \left\{ \sum_{u \in V(K_{1,n})-v} d(u,v) - 1 \right\} = \max\{2n-1, n-1\} = 2n-1.$$

Thus Lemma 3.4.1 can only improve the bound on $\tau_t(K_{1,\Delta(G)})$ when $t < 2\Delta(G) - 1$, however, this will happen. If $\Delta(G)$ is fixed and t is variable, then Proposition 3.3.4

will eventually become the better bound as t grows. If t is fixed and $\Delta(G)$ is variable, then Lemma 3.4.1 will eventually become the better bound.

The problem of packing edge disjoint complete graphs in a larger complete graph has been studied in detail. Bailey and Burgess [40] investigated a more generalized version of the problem and cite Schönheim [39] and Brouwer [41] as being responsible for solving the base packing problem for K_3 and K_4, respectively. These complete solutions give us $\tau_3(K_{1,n})$ and $\tau_4(K_{1,n})$. Bailey and Burgess [40] show these complete solutions in two different tables. Both tables are recorded in Proposition 3.4.2.

Proposition 3.4.2. *For each $t \in \{3, 4\}$, a t-tone $(k + t)$-coloring of $K_{1,n}$ exists if and only if k and n satisfy the relevant inequality below:*

$t = 3$	$k \equiv 1, 3 \pmod 6$	$n \leq k(k-1)/6$
$t = 3$	$k \equiv 0, 2 \pmod 6$	$n \leq k(k-2)/6$
$t = 3$	$k \equiv 4 \pmod 6$	$n \leq (k^2 - 2k - 2)/6$
$t = 3$	$k \equiv 5 \pmod 6$	$n \leq (k^2 - k - 8)/6$
$t = 4$	$k \equiv 7, 10 \pmod{12}, k \neq 10, 19$	$n \leq \left\lfloor \frac{k}{4} \left\lfloor \frac{k-1}{3} \right\rfloor \right\rfloor - 1$
$t = 4$	$k \equiv 9, 17 \pmod{12}$	$n \leq \left\lfloor \frac{k}{4} \left\lfloor \frac{k-1}{3} \right\rfloor \right\rfloor - 1$
$t = 4$	$k = 8, 10, 11$	$n \leq \left\lfloor \frac{k}{4} \left\lfloor \frac{k-1}{3} \right\rfloor \right\rfloor - 2$
$t = 4$	$k = 19$	$n \leq \left\lfloor \frac{k}{4} \left\lfloor \frac{k-1}{3} \right\rfloor \right\rfloor - 3$
$t = 4$	*otherwise*	$n \leq \left\lfloor \frac{k}{4} \left\lfloor \frac{k-1}{3} \right\rfloor \right\rfloor$

3.4.1 $\tau_t(C_n)$ when $t \in \{3, 4, 5\}$

Now we will consider τ_t for cycles when $t \in \{3, 4, 5\}$. We can easily prove that $\tau_t(C_n) = O(t^{3/2})$, as follows. Let $f(t) := \tau_t(P_t)$. By Lemma 3.3.6, there exists a constant c such that $\tau_t(P_t) \leq ct^{3/2}$ for all t. Further, $\tau_t(P_n) = \tau_t(P_t)$ for all $n \geq t$.

Whenever $n \geq 2t + 2$, to prove $\tau_t(C_n) \leq 2f(t)$ we simply color the first $t + 1$ vertices with one set of $f(t)$ colors and the remaining vertices with a disjoint set of $f(t)$ colors. But is it true that $\tau_t(C_n) = \tau_t(P_n)$ for all n sufficiently large (as a function of t)? Proposition 3.1.8 showed that $\tau_2(C_n) = 6$ when $n \in \{3, 4, 7\}$ and otherwise $\tau_2(C_n) = \tau_2(P_n) = 5$. We generalize their approach to prove analogous results for τ_3, τ_4, and τ_5. Lemma 3.4.3 is a generalization to the proof method of Proposition 3.1.8.

Lemma 3.4.3. *Fix $t, k, n \in \mathbb{Z}^+$. Let $\mathcal{C}$ be a set of positive integers, each at least t. If n can be written as an integer linear combination of elements in $\mathcal{C}$ (with nonnegative coefficients), then $\tau_t(C_n) \leq k$ provided that the following two properties hold:*

(1) For each $\ell \in \mathcal{C}$, there exist a t-tone k-coloring φ_ℓ of C_ℓ; and

(2) For each ordered pair $(\ell_1, \ell_2) \in \mathcal{C} \times \mathcal{C}$ (allowing $\ell_1 = \ell_2$), we get a t-tone k-coloring of C_{2t} if we color its first t vertices as vertices $\ell_1 - t + 1, \ldots, \ell_1$ of C_{ℓ_1} under φ_{ℓ_1} and we color its last t vertices as vertices $1, \ldots, t$ of C_{ℓ_2} under φ_{ℓ_2}.

Proof. Fix t, k, and $\mathcal{C}$ satisfying the hypotheses. We prove the stronger statement that if n satisfies the hypotheses, then C_n has a t-tone k-coloring in which its vertices are partitioned into copies of P_{ℓ_i}, with each $\ell_i \in \mathcal{C}$, and each copy of P_{ℓ_i} colored by φ_{ℓ_i}. Our proof is by induction on the sum of the coefficients in the integer linear combination representation of n.

Assume, by symmetry, that ℓ_1 has a positive coefficient, and let $n' := n - \ell_1$. By hypothesis, we have the desired t-tone k-coloring $\varphi_{n'}$ of $C_{n'}$. We insert a path on ℓ_1 vertices between the“first” and “last” vertex of the cycle $C_{n'}$ to get C_n. Note that $\varphi_{n'}$ induces a partial t-tone k-coloring of C_n, with these ℓ_1 successive vertices uncolored. To extend this partial coloring, we color the uncolored vertices using φ_{ℓ_1}. By properties (1) and (2), this yields a t-tone k-coloring of C_n, as desired. □

Note that Property (2) holds trivially if each t-tone coloring φ_{ℓ_i} agrees on (is identical on) its first t vertices. For example, in Figure 25, we can use Lemma 3.4.3 with $\mathcal{C} = \{4, 5\}$ to show $\tau_3(C_{13}) \leq 10$ since $13 = 2(4)+1(5)$ and the 3-tone 10-colorings of C_4 and C_5 agree in the first 3 vertices.

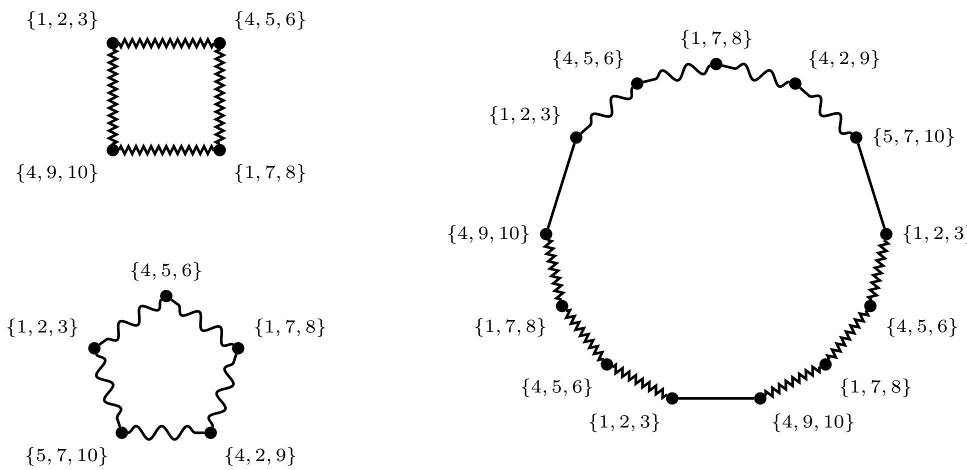

Fig. 25.: Using 3-tone 10-colorings of C_4 and C_5 to show $\tau_3(C_{13}) \leq 10$.

We use Lemma 3.4.3 to prove the next three theorems, which show that $\tau_t(C_n) = \tau_t(P_n)$ for all $t \in \{3, 4, 5\}$, for all but a small (finite) number of values of n.

Theorem 3.4.4.

$$\tau_3(C_n) = \begin{cases} 10 & \textit{if } n \in \{4, 5\} \\ 9 & \textit{if } n \in \{3, 7, 10, 13\} \\ 8, & \textit{otherwise} \end{cases}$$

Proof. It is easy to check that $\tau_3(P_3) = 8$. So $\tau_3(C_n) \geq \tau_3(P_3) = 8$ for all $n \geq 3$. Lemma 3.3.7 shows that $\tau_3(C_n) = 9$ when $n \in \{3, 7\}$ and that $\tau_3(C_n) = 10$ when $n \in \{4, 5\}$. So we assume below that $n = 6$ or $n \geq 8$. The case $n \in \{10, 13\}$ is exceptional, so we defer it briefly to handle the general case. In Lemma 3.4.3, we let $\mathcal{C} = \{6, 8, 9, 11\}$ and take φ_k as described below.

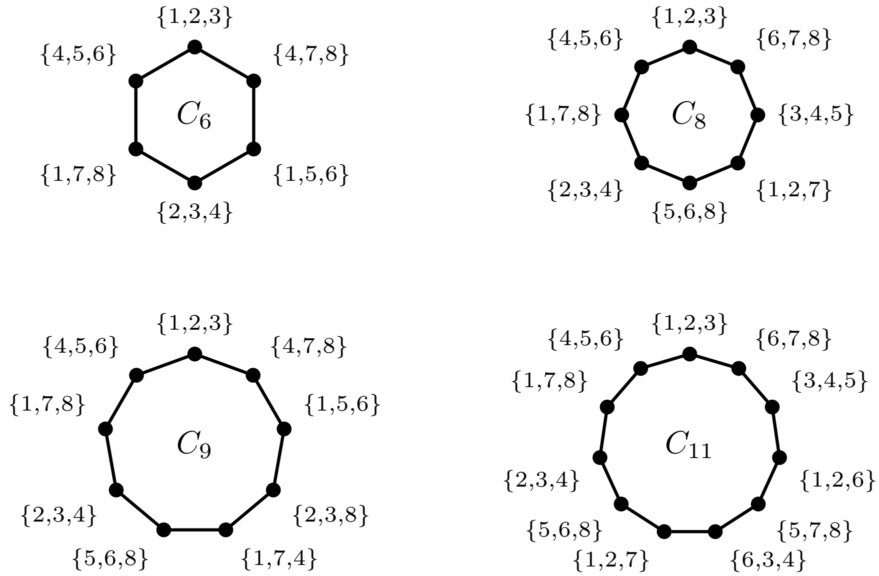

Fig. 26.: A 3-tone 8-coloring of C_n for each $n \in \{6, 8, 9, 11\}$.

So it remains to show that n can be written as an integer linear combination of elements of $\mathcal{C}$ whenever $n \geq 3$ and $n \notin \{3, 4, 5, 7, 10, 13\}$. To see this, we consider the integer linear combinations, $6, 8, 9, 11, 6+6, 6+8, 6+9, 8+8, 8+9, 9+9, 8+11$ and note that every larger integer can be written as one of the final 6, plus some multiple of 6.

Now assume $n \in \{10, 13\}$. To see that $\tau_3(C_n) \leq 9$, consider the two following 3-tone 9-colorings.

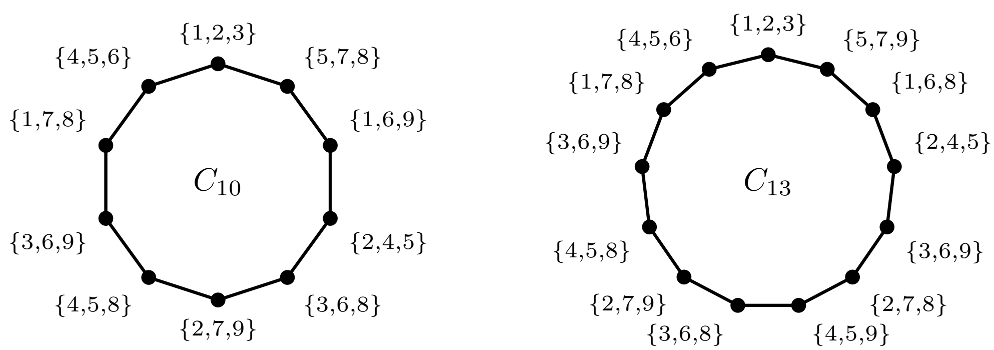

Fig. 27.: A 3-tone 9-coloring of C_n for each $n \in \{10, 13\}$.

Finally, we show for each $n \in \{10, 13\}$ that $\tau_3(C_n) > 8$. Assume the contrary, let φ be a 3-tone 8-coloring of C_n, and let c_i denote the number of vertices receiving color i under φ for each $i \in [8]$. Let $s := (n-1)/3$. It is straightforward to check that, for at least $(c_i - s)2$ pairs of vertices at distance 2, both vertices receive color i. Note that $\sum_{i=1}^{8} c_i = 3n = 9s+3$. Further, $\sum_{i=1}^{8}(c_i - s)2 = 18s + 6 - 16s = 2s + 6$. Observe that C_n has precisely $n = 3s + 1$ pairs of vertices at distance 2. Since $n \in \{10, 13\}$, we have $s \in \{3, 4\}$, so $2s + 6 > 3s + 1$. Thus, by pigeonhole some pair of vertices at distance 2 receive two common colors under φ, a contradiction. $\square$

Theorem 3.4.5.

$$\tau_4(C_n) = \begin{cases} 15 & \textit{if } n = 5 \\ 14 & \textit{if } n = 4 \\ 13 & \textit{if } n = 7 \\ 12, & \textit{otherwise} \end{cases}$$

Proof. We have $\tau_4(C_n) \geq \tau_4(P_n) = 12$. By Lemma 3.3.7 we have $\tau_4(C_3) = 12$, $\tau_4(C_4) = 14$, $\tau_4(C_5) = 15$, and $\tau_4(C_7) = 13$. We let $\mathcal{C} = \{6, 8, 9, 10, 11, 12, 13\}$ and take φ_k as described below.

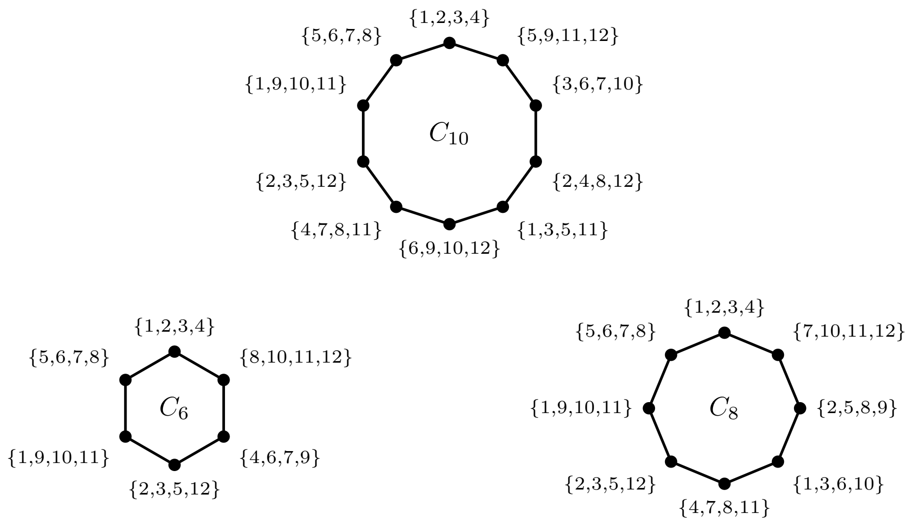

Fig. 28.: A 4-tone 12-coloring of C_n for each $n \in \{8, 9, 10\}$.

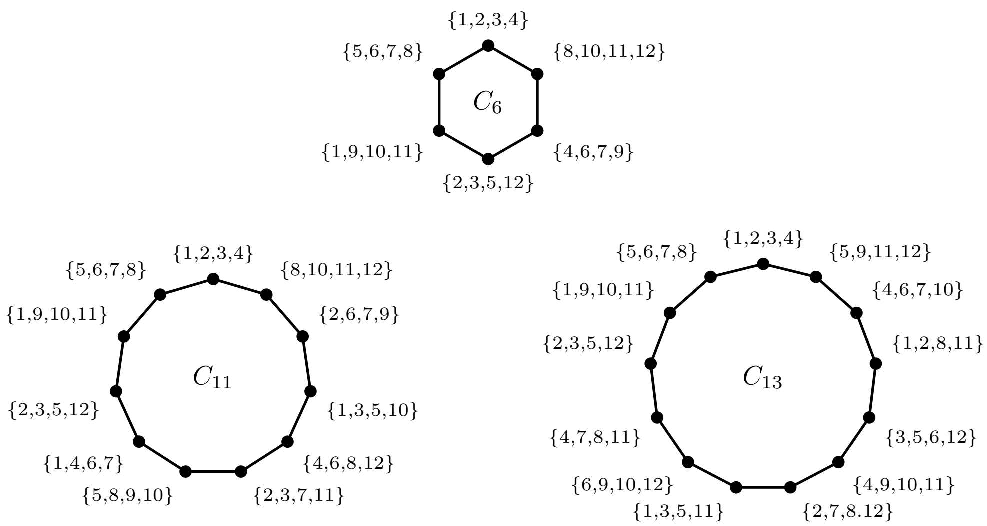

Fig. 29.: A 4-tone 12-coloring of C_n for each $n \in \{6, 11, 13\}$.

So it remains to show that n can be written as an integer linear combination of elements of $\mathcal{C}$ whenever $n \geq 3$ and $n \notin \{3, 4, 5, 7\}$. To see this, we consider the integer linear combinations, $6, 8, 9, 10, 11, 6 + 6, 13$ and note that every larger integer can be written as one of the final 6, plus some multiple of 6. $\square$

Theorem 3.4.6.

$$\tau_5(C_n) = \begin{cases} 20 & \textit{if } n = 5 \\ 18 & \textit{if } n \in \{4, 6\} \\ 17 & \textit{if } n \in \{7, 9\} \\ 15 & \textit{if } n = 3 \\ 16, & \textit{otherwise} \end{cases}$$

Proof. We have $\tau_5(C_n) \geq \tau_5(P_n) = 16$ when $n \geq 4$. Using Lemma 3.3.7, we have $\tau_5(C_3) = 15$, $\tau_5(C_4) = 18$, $\tau_5(C_5) = 20$, $\tau_5(C_6) = 18$, and $\tau_5(C_7) = 17$. We let $\mathcal{C} = \{8, 10, 11, 12, 13, 14, 15, 17\}$ and take φ_k as described below.

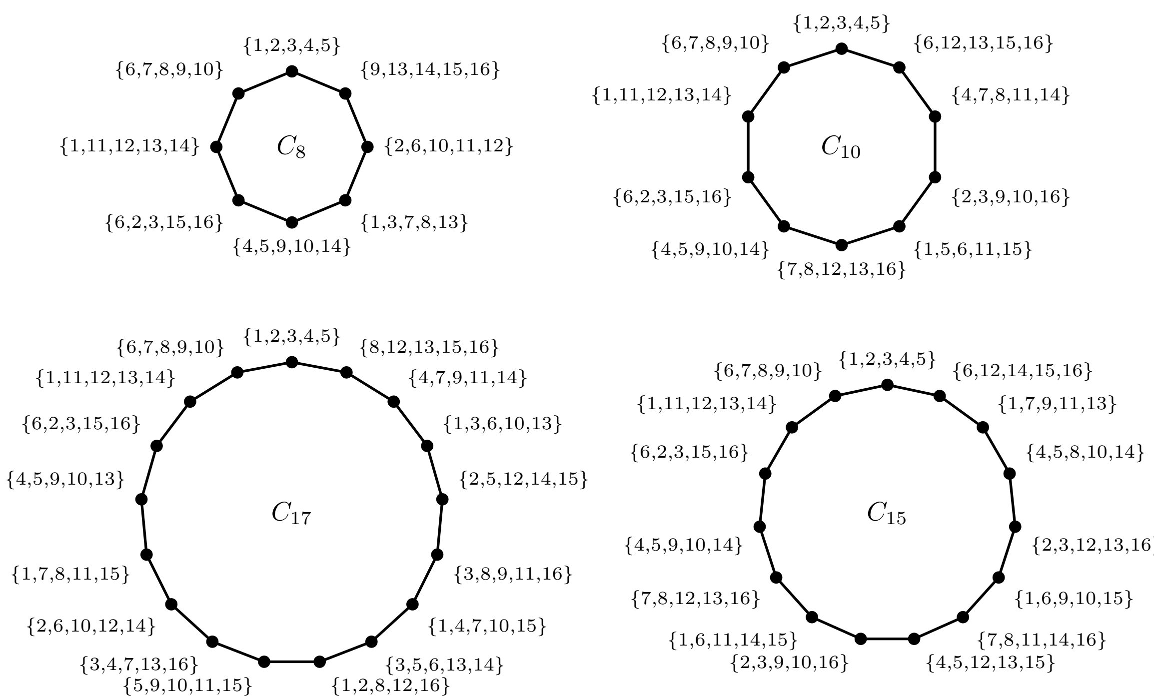

Fig. 30.: A 5-tone 16-colorings of C_n for each $n \in \{8, 10, 15, 17\}$.

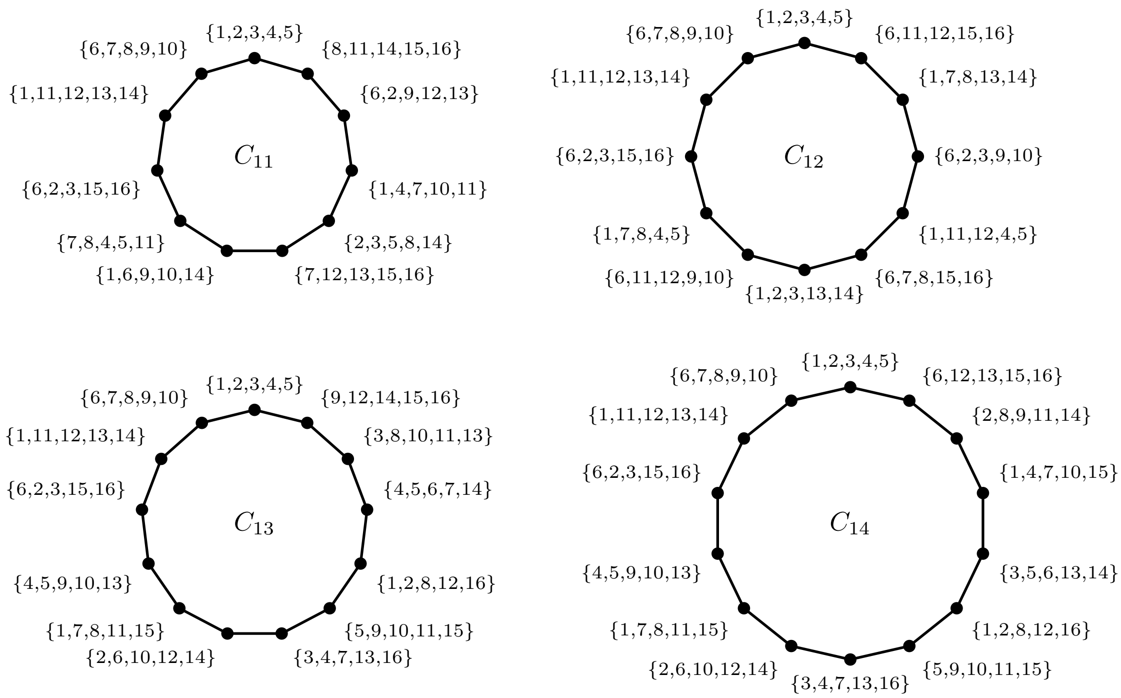

Fig. 31.: A 5-tone 16-colorings of C_n for each $n \in \{11, 12, 13, 14\}$.

So it remains to show that n can be written as an integer linear combination of elements of $\mathcal{C}$ whenever $n \geq 3$ and $n \neq 9$. To see this, we consider the integer linear combinations, $8, 10, 11, 12, 13, 14, 15, 8+8, 17, 8+10, 8+11, 10+10, 10+11, 11+11, 8+15, 8+8+8, 10+15$ and note that every larger integer can be written as one of the final 8, plus some multiple of 8.

Now assume that $n = 9$. To see that $\tau_5(C_9) \leq 17$, consider the following 5-tone 17-coloring.

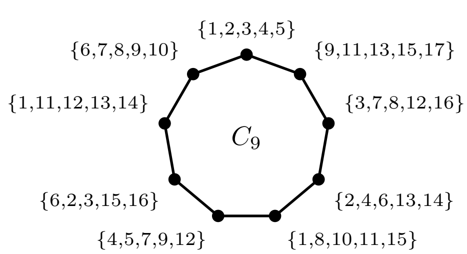

Fig. 32.: A 5-tone 17-coloring of C_9.

Finally, we will prove that $\tau_5(C_9) \geq 17$. Assume, to the contrary, that C_9 has a 5-tone 16-coloring. Note that each color appears on at most 4 vertices. Each color must appear on at least one vertex, since $\tau_5(C_9) \geq \tau_5(P_4) = 16$. For each $i \in [4]$, let s_i denote the number of colors used on exactly i vertices. So we have $\sum_{i=1}^{4} s_i = 16$ and $\sum_{i=1}^{4} is_i = 9(5) = 45$. Further, let s_3' denote the number of colors used on exactly 3 vertices, where some pair is at distance 2, and let s_3'' denote the number of colors used on exactly 3 vertices, where each pair is distance 3. Note that each color used on 4 vertices is used on 3 pairs of vertices at distance 2. Since C_9 has 9 pairs of vertices at distance 2, and each pair can share at most 1 common color, we get $3s_4 + s_3' \leq 9$. Similarly, by considering vertex pairs with a common color that are at distance 3, we get $s_3' + 3s_3'' \leq 18$. Multiplying the first inequality by 2, adding it to the second inequality, and dividing by 3 (recalling $s_3' + s_3'' = s_3$) gives

$$2s_4 + s_3 \leq 12. \tag{$*$}$$

Recall that $\sum_{i=1}^{4} s_i = 16$ and $\sum_{i=1}^{4} is_i = 9(5) = 45$. Multiplying the first equation by 3 and subtracting the second gives $2s_1 + s_2 - s_4 = 3$. Adding this to $(*)$ gives $2s_1 + s_2 + s_3 + s_4 \leq 12 + 3 = 15$. Since $s_1 \geq 0$, this contradicts the first equation, and this contradiction finishes the proof. $\square$

We end this discussion on τ_t for cycles with the following conjecture.

Conjecture 3.4.7. *For each $t \geq 2$ there exists $N \in \mathbb{N}$ such that $\tau_t(C_n) = \tau_t(P_n)$ for all $n \geq N$.*

3.4.2 $\tau_t(P_n \square P_m)$ when $t \in \{3, 4, 5\}$

Now we consider τ_t for grid graphs for each $t \in \{3, 4, 5\}$. Bickle [30] (also Cooper and Wash [32]) showed that $\tau_2(P_n \square P_m) = 6$ for all $n, m \geq 2$.

It is useful in their proof, and in the following three theorems, to imagine the grid graph as being drawn in the first quadrant of the xy-plane with vertices as integer points. Now their proof can be viewed as coloring lines of slope 1 by cycling through the colors 1, 2, 3 and coloring lines of slope -1 by cycling through the colors 4, 5, 6. Each vertex v needs two colors; it takes one color from the line through it of slope 1 and takes the other color from the line through it of slope -1.

For Theorem 3.4.8, the proof can be viewed as coloring the lines of slope 1 and slope -1 as above, but also coloring lines of slope 2. This theorem improves a result in [32]. For Theorem 3.4.9, the proof can be viewed as coloring the lines of slope 1, slope -1, and slope 2 as in Theorem 3.4.8, but further coloring lines of slope $-\frac{1}{2}$. Finally, for Theorem 3.4.10, the proof can also be viewed as coloring the lines of slope 1, slope -1, slope 2, and slope $-\frac{1}{2}$ as in Theorem 3.4.9, but adding colors to lines of slope 1.

For the following three theorems we consider the vertices of $P_m \square P_n$ as integer points on the xy-plane where a vertex (x_i, y_j) is denoted by (i, j) with $1 \leq i \leq m$ and $1 \leq j \leq n$. For all vertices (i_1, j_1) and (i_2, j_2) in $V(P_m \square P_n)$, note that the distance between them is exactly $|i_1 - i_2| + |j_1 - j_2|$. For the following theorems, note that $\tau_t(C_4) = 4t - 2$, since each pair of nonadjacent vertices can share 1 color.

Theorem 3.4.8. *$\tau_3(P_m \square P_n) = 10$ for all integers m and n with $2 \leq m \leq n$.*

Proof. We know $10 = \tau_3(C_4) \leq \tau_3(P_m \square P_n)$. So it suffices to construct a 3-tone 10-coloring of $P_m \square P_n$. Let $f : V(P_m \square P_n) \to \binom{[10]}{3}$ where we write $f((i, j))$ as $f(i, j)$ and we let $f(i,j) := \{f_1(i,j), f_2(i,j), f_3(i,j)\}$, where

$$f_1(i,j) := (i-j) \bmod 3$$
$$f_2(i,j) := ((i+j) \bmod 3) + 3$$
$$f_3(i,j) := ((2i+j) \bmod 4) + 6.$$

Denote v by (i_1, j_1) and w by (i_2, j_2). If $d(v,w) = 1$, then clearly $f(v) \cap f(w) = \emptyset$. It suffices to prove the following three claims.

Claim 1: If $|f(v) \cap f(w)| = 3$, then $d(v,w) \geq 4$.

If $|f(v) \cap f(w)| = 3$, then $f_i(v) = f_i(w)$ for all $i \in [3]$. So $(i_1 - j_1) \equiv (i_2 - j_2) \mod 3$ and $(i_1 + j_1) \equiv (i_2 + j_2) \mod 3$. Thus $i_1 \equiv i_2 \mod 3$ and $j_1 \equiv j_2 \mod 3$. If $d(v,w) \leq 3$ and $v \neq w$, then $i_1 \equiv i_2 \pm 3$ and $j_1 = j_2$ or else $i_1 = i_2$ and $j_1 = j_2 \pm 3$. But now $(2i_1 + j_1) \not\equiv (2i_2 + j_2) \mod 4$.

Claim 2: If $|f(v) \cap f(w)| = 2$, then $d(v,w) \geq 3$. Assume $|f(v) \cap f(w)| = 2$. If $\{f_1(v), f_2(v)\} = \{f_1(w), f_2(w)\}$, then the argument in Claim 1 still holds. Instead we assume $f_3(v) = f_3(w)$ and $d(v,w) \leq 2$. Thus $i_1 = i_2 \pm 2$ and $j_1 = j_2$, but now $f_1(v) \neq f_2(v)$ and $f_2(v) \neq f_2(w)$, a contradiction.

Claim 3: If $|f(v) \cap f(w)| = 1$, then $d(v,w) \geq 2$. Assume that $d(v,w) = 1$. So either $i_1 = i_2$ and $j_1 - j_2 = \pm 1$ or else $j_1 = j_2$ and $i_1 - i_2 = \pm 1$. Now clearly $f_i(v) \neq f_i(w)$ for all $i \in [3]$, a contradiction. $\square$

Theorem 3.4.9. *$\tau_4(P_m \square P_n) = 14$ for integers m and n with $2 \leq m \leq n$.*

Proof. We know $14 = \tau_4(C_4) \leq \tau_4(P_m \square P_n)$. So it suffices to construct a 4-tone 14-coloring of $P_m \square P_n$. Let $f : V(P_m \square P_n) \to \binom{[14]}{4}$, where we write $f((i,j))$ as $f(i,j)$

and we let $f(i, j) := \{f_1(i, j), f_2(i, j), f_3(i, j), f_4(i, j)\}$, where

$$\begin{aligned} f_1(i, j) &:= (i - j) \bmod 3 \\ f_2(i, j) &:= ((i + j) \bmod 3) + 3 \\ f_3(i, j) &:= ((2i + j) \bmod 4) + 6 \\ f_4(i, j) &:= ((i + 2j) \bmod 4) + 10. \end{aligned} \tag{$**$}$$

Denote v by (i_1, j_1) and w by (i_2, j_2). Assume $d(v, w) = 1$. It suffices to prove the following four claims.

Claim 1: If $|f(v) \cap f(w)| = 4$, then $d(v, w) \geq 5$. Assume $|f(v) \cap f(w)| = 4$. So $f_i(v) = f_i(w)$ for all $i \in [4]$. Claim 1 in Theorem 3.4.8 implies $d(v, w) \geq 4$. Suppose $d(v, w) = 4$. Since $f_4(v) = f_4(w)$ we have $i_1 - i_2 \equiv 0 \mod 4$ and $j_1 = j_2$, or $j_1 - j_2 \equiv 0 \mod 4$ and $i_1 = i_2$. In either case this implies $f_k(v) \neq f_k(w)$ for each $k \in \{1, 2\}$, a contradiction.

Claim 2: If $|f(v) \cap f(w)| = 3$, then $d(v, w) \geq 4$. Assume $|f(v) \cap f(w)| = 3$. Claim 1 in Theorem 3.4.8 implies $f_4(v) = f_4(w)$; and Claim 2 in Theorem 3.4.8 implies $d(v, w) \geq 3$. Suppose $d(v, w) = 3$. If $f_3(v) \neq f_3(w)$, then $i_1 \equiv i_2 \mod 3$ and $j_1 \equiv j_2 \mod 3$, but then $f_4(v) \neq f_4(w)$, a contradiction. If $f_3(v) = f_3(w)$, then $i_1 - i_2 \equiv j_1 - j_2 \mod 4$, which implies $f_1(v) \neq f_1(w)$ and $f_2(v) \neq f_2(w)$, contradicting $|f(v) \cap f(w)| = 3$.

Claim 3: If $|f(v) \cap f(w)| = 2$, then $d(v, w) \geq 3$. Assume $|f(v) \cap f(w)| = 2$. If $f_4(v) \neq f_4(w)$, then by Claim 2 in Theorem 3.4.8 we know $d(v, w) \geq 3$. So we may assume $f_4(v) = f_4(w)$ and $f_k(v) = f_k(w)$ for some single $k \in [3]$. From Claim 3 in Theorem 3.4.8 we have that $d(v, w) \geq 2$. Suppose $d(v, w) = 2$. Since $f_4(v) = f_4(w)$ it must be that $i_1 = i_2$. So $j_1 - j_2 \equiv 2 \mod 4$; but now $f_k(v) \neq f_k(w)$ for all $k \in \{1, 2\}$, a contradiction.

Claim 4: If $|f(v) \cap f(w)| = 1$, then $d(v, w) \geq 2$. Assume $|f(v) \cap f(w)| = 1$. If

$f_4(v) \neq f_4(w)$, then Claim 3 in Theorem 3.4.8 implies $d(v,w) \geq 2$. So $f_4(v) = f_4(w)$, which implies $d(v,w) \geq 2$. $\square$

Theorem 3.4.10. $20 \leq \tau_5(P_m \Box P_n) \leq 22$ *for all* $2 \leq m < n$.

Proof. We know $\tau_t(P_2 \Box P_3) = 6t - 10$; in fact, an optimal t-tone coloring f of $P_2 \Box P_3$ is unique up to relabelling. This implies $20 = \tau_5(P_2 \Box P_3) \leq \tau_5(P_m \Box P_n)$.

It now suffices to construct a 5-tone 22-coloring of $P_m \Box P_n$. Let $f : V(P_m \Box P_n) \to \binom{[22]}{5}$ where we will denote $f((i,j))$ as $f(i,j)$ and define
$f(i,j) := \{f_1(i,j), f_2(i,j), f_3(i,j), f_4(i,j), f_5(i,j)\}$ where

$$f_1(i,j) := (i-j) \bmod 3$$

$$f_2(i,j) := ((i+j) \bmod 3) + 3$$

$$f_3(i,j) := ((2i+j) \bmod 4) + 6$$

$$f_4(i,j) := ((i+2j) \bmod 4) + 10$$

$$f_5(i,j) := ((i+3j) \bmod 8) + 14.$$

Let $v = (i_1, j_1)$, $w = (i_2, j_2)$, and $q = |f(v) \cap f(w)|$. If $q \in \{0, \ldots, 4\}$ and $f_5(v) \neq f_5(v)$, then $(**)$ and the claims in Theorem 3.4.9 imply $d(v,w) \geq q+1$. So we assume $f_5(v) = f_5(v)$. This implies $d(v,w) \geq 4$ since otherwise $((i_1 - i_2) + 3(j_1 - j_2)) \bmod 8 \neq 0$. So it suffices to prove the following two claims.

Claim 1: If $|f(v) \cap f(w)| = 4$, then $d(v,w) \geq 5$. Assume $|f(v) \cap f(w)| = 4$. Suppose $d(v,w) = 4$. Since $f_5(v) = f_5(w)$, either: $i_1 - i_2 = \pm 1$ and $j_1 - j_2 = \mp 3$; or $i_1 - i_2 = \pm 2$ and $j_1 - j_2 = \pm 2$; or $i_1 - i_2 = \pm 3$ and $j_1 - j_2 = \mp 1$. In all cases $f_2(v) \neq f_2(w)$ and $f_3(v) \neq f_3(w)$, a contradiction to $|f(v) \cap f(w)| = 4$.

Claim 2: If $|f(v) \cap f(w)| = 5$, then $d(v,w) \geq 6$. Assume $|f(v) \cap f(w)| = 5$. Claim 1 implies $d(v,w) \geq 5$. So $|i_1 - i_2| + |j_1 - j_2| = 5$. But now $f_5(v) \neq f_5(w)$, a contradiction. $\square$

3.4.3 τ_t on Graphs with Polynomial Expansion

We now focus on graphs with polynomial expansion. Recall that a graph with polynomial expansion means exists a polynomial p where the total number of vertices at distance d from any single vertex is at most $p(d)$. Proposition 3.3.5, which is $\tau_t(G) \leq (t^2+t)\Delta(G)$, gives an upper bound for τ_t that is linear in Δ and quadratic in t. The proof roughly relies on a counting argument similar to the idea in Lemma 3.2.4, where we bound from above the number of t-sets that are forbidden to be assigned to an uncolored vertex v, and then we ensure we have enough colors that some t-set is still available to v. Theorem 3.4.11 proves an upper bound on τ_t that is sub-quadratic in t for any class of graphs where no vertex has too many vertices at any distance. As an example after proving Theorem 3.4.11 we will prove the best known upper bound for grid graphs.

Theorem 3.4.11. *Let $\mathcal{G}$ be a family of graphs and $p(x)$ a polynomial such that for all $G \in \mathcal{G}$ each vertex has at most $p(d)$ vertices at distance d, for each $d \in [|V(G)|-1]$. If $\epsilon = \frac{1}{2\deg(p(x))+3}$, then there exist constants c and t_0 such that $\tau_t(G) \leq ct^{2-\epsilon}$ for all $G \in \mathcal{G}$ and all $t \geq t_0$.*

Proof. Let $\mathcal{G}$, $p(x)$, and ϵ be as satisfy the hypotheses. If $k_2 = \deg(p(x))$, then $p(x) \leq k_1x^{k_2}$ for some sufficiently large constant k_1; so we assume that $p(x) = k_1x^{k_2}$. Fix $G \in \mathcal{G}$, and let f be a partial t-tone k-coloring of G. To prove the bound we consider an arbitrary uncolored vertex v and show how to extend f to v.

Fix $a, b \in \mathbb{Z}^+$, which we will specify later, with $a \geq 2$. And let $r := at^{2\epsilon} + b$. When extending f to v, we will simply forbid all colors used on vertices at distance from v no more than $\lfloor r \rfloor$. Recall that left Riemann sums underestimate the area under a monotonically increasing function. Thus, the set T of such vertices satisfies

$$|T| \leq \sum_{i=1}^{\lfloor r \rfloor} p(i) \leq c_0 \lfloor r \rfloor^{k_2+1} \leq c_0 r^{k_2+1}$$

for some sufficiently large constant c_0.

The number of colors forbidden by vertices in T is at most $t|T|$. Recall that $k_2 + 1 = \frac{1}{2}\left(\frac{1}{\epsilon} - 1\right)$. So, using another sufficiently large constant c_1, and letting $\hat{c} := c_0 c_1 a^{k_2+1}$, we get

$$t|T| \leq t\left(c_0 r^{k_2+1}\right) = t\left(c_0(at^{2\epsilon} + b)^{k_2+1}\right) \leq t\left(c_0 c_1 a^{k_2+1} t^{1-\epsilon}\right) = \hat{c} t^{2-\epsilon}.$$

Each vertex at distance $\lfloor r \rfloor + 1$ to t from v forbids at most $\binom{t}{d}\binom{|S|-d}{t-d}$ t-sets from appearing on v. To ensure that a t-set remains available for v, we need $|S|$ large enough such that

$$\sum_{d>r}^{t} \frac{\binom{t}{d}\binom{|S|-d}{t-d} p(d)}{\binom{|S|}{t}} < 1.$$

To ensure this inequality holds, we choose S large enough that the dth term in the sum is less than $\frac{1}{2^d}$ whenever $r < d \leq t$. We first use the following inequality

$$\begin{aligned} r - \left\lfloor \frac{(a-1)r+b}{a} \right\rfloor &< r - \frac{(a-1)r+b}{a} + 1 = t^{\frac{1-\epsilon}{k_2+1}} - 1 \\ &< \frac{r}{2} - 1 < \left\lfloor \frac{\lfloor r \rfloor + 1}{2} \right\rfloor \leq \left\lfloor \frac{d}{2} \right\rfloor = d - \left\lceil \frac{d}{2} \right\rceil \end{aligned}$$

to get that

$$\begin{aligned}
\frac{\binom{t}{d}\cdot\binom{|S|-d}{t-d}\cdot p(d)}{\binom{|S|}{t}} &= \frac{p(d)}{d!}\cdot\left(\frac{t!}{(t-d)!}\right)^2\cdot\frac{(|S|-d)!}{|S|!} \\
&= \frac{p(d)}{d!}\cdot\frac{t^2\cdot(t-1)^2\cdots(t-d+1)^2}{|S|\cdot(|S|-1)\cdots(|S|-d+1)} \\
&< \frac{p(d)}{d!}\cdot\frac{t^{2d}}{(|S|-t)^d} < \frac{p(d)}{d(d-1)\cdots\left(d-\lceil\frac{d}{2}\rceil\right)}\cdot\frac{t^{2d}}{(|S|-t)^d} \\
&< \frac{p(d)}{\left(d-\lceil\frac{d}{2}\rceil\right)^{\lceil\frac{d}{2}\rceil+1}}\cdot\frac{t^{2d}}{(|S|-t)^d} < \frac{p(d)}{\left(d-\lceil\frac{d}{2}\rceil\right)^{\frac{d}{2}}}\cdot\frac{t^{2d}}{(|S|-t)^d} \\
&< \frac{p(d)}{\left(r-\left\lfloor\frac{(a-1)r+b}{a}\right\rfloor\right)^{\frac{d}{2}}}\cdot\frac{t^{2d}}{(|S|-t)^d} \le \frac{p(d)}{\left(r-\frac{(a-1)r+b}{a}\right)^{\frac{d}{2}}}\cdot\frac{t^{2d}}{(|S|-t)^d} \\
&= \frac{p(d)}{(t^{2\epsilon})^{\frac{d}{2}}}\cdot\frac{t^{2d}}{(|S|-t)^d} = \frac{p(d)t^{d(2-\epsilon)}}{(|S|-t)^d}.
\end{aligned}$$

Next we need to bound $p(d)^{\frac{1}{d}}$ for all $d > r$. Since $\lim_{x\to\infty} p(x)^{\frac{1}{x}} = 1$ we have there exists $\delta \in \mathbb{R}$ such that $p(x)^{\frac{1}{x}} \le 2$ for all $x > \delta$. We have $r = at^{2\epsilon} + b \ge \delta$ for all $t \ge \left(\frac{\delta-b}{a}\right)^{\frac{1}{2\epsilon}}$. Thus for sufficiently large t we have $p(d)^{\frac{1}{d}} \le 2$ for all $d > r$.

Finally we have $|S| \ge 4t^{2-\epsilon} + t + 1$ will suffice since

$$\begin{aligned}
4t^{2-\epsilon} + t < |S| &\implies 2\cdot 2\cdot t^{2-\epsilon} < |S| - t \implies 2\cdot p(d)^{\frac{1}{d}}\cdot t^{2-\epsilon} < |S| - t \\
&\implies 2^d\cdot p(d)\cdot t^{d(2-\epsilon)} < (|S|-t)^d \\
&\implies \frac{p(d)t^{d(2-\epsilon)}}{(|S|-t)^d} < \frac{1}{2^d}.
\end{aligned}$$

The result follows with $c := \hat{c} + 4 + 1$

$$\tau_t(G) \le k = |T| + |S| \le \left(\hat{c}t^{2-\epsilon}\right) + \left(4t^{2-\epsilon} + t + 1\right) \le ct^{2-\epsilon}.$$

□

Corollary 3.4.12 proves a sub-quadratic bound in t for grid graphs and in addition Theorem 3.4.13 will provide a best known quadratic bound in t, which is useful for some smaller values of t.

Corollary 3.4.12. *For all integers $t \geq 3$ and $2 \leq m \leq n$ we have $\tau_t(P_m \Box P_n) \leq 23t^{\frac{9}{5}}$.*

Proof. Let $G := P_m \Box P_n$ with $n \geq m \geq 2$. For any $v \in V(G)$, and for all $d \in [|V(G)| - 1]$, vertex v has at most $4d$ vertices with distance exactly d from v. In particular, grid graphs satisfy the requirements for Theorem 3.4.11 where $p(x) = 4x$ and $\epsilon = \frac{1}{5}$. In particular, $\tau_t(G) \leq ct^{\frac{9}{5}}$ for some constant c and for all $t \geq 3$.

We can determine $c \leq 22.4$ by letting $r = 2t^{2\epsilon} + 1$. So there are $2\lfloor r \rfloor^2 + 2\lfloor r \rfloor$ vertices at distances 1 up to $\lfloor r \rfloor$. Since $t \geq 3$ we have $|T| \leq t(2r^2 + 2r) = 8t^{9/5} + 12t^{7/5} + 4t \leq 17.4t^{9/5}$ and $r \geq 4$. The latter implies $p(d)^{\frac{1}{d}} \leq 2$ for all $r < d \leq t$. Thus $\tau_t(P_m \Box P_n) \leq (17.4 + 4 + 1)t^{9/5} \leq 23t^{9/5}$. □

Theorem 3.4.13. $\tau_t(P_m \Box P_n) \leq 2t + \sqrt{5}t^2$ *for all integers $2 \leq m \leq n$.*

Proof. Let $t \geq 1$ and $G = P_m \Box P_n$ with $n \geq m \geq 2$. Order the vertices in increasing lexicographical order, i.e., for $v_i = (x_i, y_i)$ and $v_j = (x_j, y_j)$ we have $(x_i, y_i) < (x_j, y_j)$ if $y_i < y_j$ or $y_i = y_j$ and $x_i < x_j$. For each v_i, if we only consider vertices that appear before v_i in the vertex ordering, there is at most one vertex of distance d for each of the d columns to the left of v_i, for each of the d columns to the right of v, and one vertex for the current column of v_i. So v_i has at most $2d + 1$ neighbors at distance d for $1 \leq d \leq t$. Note that given $1 \leq q \leq d - 1$ we have $(t - q)^2 \leq t^2 - \frac{q}{\alpha}$ when $\frac{q+(1/\alpha)}{2} \leq t$; in particular, when $\alpha = \sqrt{5}$ since $\frac{q+(1/\sqrt{5})}{2} \leq \frac{d-1+(1/\sqrt{5})}{2} \leq d \leq t$. So if v_i is uncolored it has at most $t|N(v)| \leq 2t$ colors forbidden by its neighbors before it in the vertex ordering. So if we have $c := k - 2t$ colors remaining to color v, it suffices to show that

$$\sum_{d=2}^{t} \frac{\binom{t}{d}\binom{c-d}{t-d}(2d+1)}{\binom{c}{t}} < 1.$$

Let $c = \sqrt{5}t^2$. Then we just need to show that the every term in the sum is less than $\frac{1}{d!}$ for $2 \leq d \leq t$.

$$\begin{aligned}
\frac{\binom{t}{d}\binom{c-d}{t-d}(2d+1)}{\binom{c}{t}} &= \left(\frac{2d+1}{d!}\right)\left(\frac{t^2(t-1)^2(t-2)^2\cdots(t-(d-1))^2}{c(c-1)(c-2)\cdots(c-(d-1))}\right) \\
&= \left(\frac{2d+1}{d!}\right)\left(\frac{t^2(t-1)^2(t-2)^2\cdots(t-(d-1))^2}{\sqrt{5}t^2(\sqrt{5}t^2-1)(\sqrt{5}t^2-2)\cdots(\sqrt{5}t^2-(d-1))}\right) \\
&= \left(\frac{2d+1}{d!\cdot\sqrt{5}^d}\right)\left(\frac{t^2(t-1)^2(t-2)^2\cdots(t-(d-1))^2}{t^2(t^2-\frac{1}{\sqrt{5}})(t^2-\frac{2}{\sqrt{5}})\cdots(t^2-\frac{(d-1)}{\sqrt{5}})}\right) \\
&\leq \frac{2d+1}{d!\cdot\sqrt{5}^d} \leq \frac{1}{d!}
\end{aligned}$$

□

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