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DISSERTATION
INVESTIGATIONS IN THE SEMI-STRONG PRODUCT OF GRAPHS AND
BOOTSTRAP PERCOLATION

A Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor
of Philosophy at Virginia Commonwealth University.

by
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Abstract

The semi-strong product of graphs G and H , denoted $G \overline{\times} H$, is a way of forming a new graph from the graphs G and H . The vertex set of the semi-strong product is the Cartesian product of the vertex sets of G and H , $V(G) \times V(H)$. The edges of $G \overline{\times} H$ determined as follows: $(g_1, h_1)(g_2, h_2) \in E(G \overline{\times} H)$ whenever $g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$ or $g_1 = g_2$ and $h_1 h_2 \in E(H)$.

A natural subject for investigation is to determine properties of the semi-strong product in terms of those properties of its factors. We investigate distance, independence, matching, and domination in the semi-strong product

Bootstrap Percolation is a process defined on a graph. We begin with an initial set of infected vertices. In each subsequent round, uninfected vertices become infected if they are adjacent to at least r infected vertices. Once infected, vertices remain infected. The parameter r is called the percolation threshold. When G is finite, the infection either stops at a proper subset of G or all of $V(G)$ becomes infected. If all of $V(G)$ eventually becomes infected, then we say that the infection percolates and we call the initial set of infected vertices a percolating set.

The cardinality of a minimum percolating set of G with percolation threshold r is denoted $m(G, r)$. We determine $m(G, r)$ for certain Kneser graphs and bipartite Kneser graphs.

Chapter 1

Introduction

In this dissertation, we will investigate two topics: the semi-strong product of graphs and bootstrap percolation. Before each section we will provide appropriate definitions and background. All graphs are simple and finite. Standard graph theory terminology is used, for instance, as found in [24]. The semi-strong product was first introduced (as the strong tensor product) by Garman, Ringeisen, and White in 1976 [31]. The three authors presented some basic properties and also investigated the product in the context of graph imbeddings on orientable surfaces. Ghidewon Abay-Asmerom investigated the semi-strong product in 1990 in his dissertation [2] and in subsequent papers [3, 4, 5], one of which includes a new proof of the genus of Q_n [6]. Brooks studied the semi-strong product in his master's thesis in 2015 [20]. Zhao et. al. have studied book imbeddings of the semi-strong product [90]. The semi-strong product may also have been studied under the name “semi-Cartesian Kronecker product” in two papers in Russian [44, 66].

The works cited so far have studied the semi-strong product from a topological perspective. However, this product has also been investigated in other contexts. Petrosyan studied interval edge colorings of product graphs, including the semi-strong product [73]. Li and Zhang have studied nowhere-zero 3-flows and the group connectivity number in the semi-strong product [61, 62]. Pisanski et. al. provided a sufficient condition for a 1-factor in the semi-strong product [74]. Jaradat also studied 1-factorization of the semi-strong product, in the context of edge coloring [51]. Zhou has studied Hamiltonian decomposition of cycles in the semi-strong product of digraphs [91]. A series of papers have also examined the basis number of the semi-strong product of various graphs [8, 9, 49, 50, 52]. In addition, the semi-strong product has been investigated with regards to various coloring problems and signed graphs [45, 84, 85]. In this dissertation, we continue research on non-topological properties of the semi-strong product, such as distance, domination, independence, Eulerian circuits,

Hamiltonian cycles, and matchings.

In section 2.1, we determine the distance between two vertices in the semi-strong product in terms of distance in the factors. Using our knowledge of distance in the product, we find the eccentricities of the vertices of the semi-strong product. This enables us to determine the radius, diameter, center, and periphery of the semi-strong product. Section 2.2 continues the study of distance in the semi-strong product. We prove formulas for the Wiener index of the semi-strong product of a complete graph and a connected graph and a complete bipartite graph and a connected graph. In section 2.3, we present Garman, Ringeisen, and White's formula for the chromatic number of the semi-strong product and find a similar formula for the clique number of the semi-strong product. In section 2.4, we use Garman, Ringeisen, and White's result on the degree of a vertex of the semi-strong product along with Euler and Hierholzer's well-known theorem to characterize which semi-strong products contain Eulerian circuits and Eulerian trails. We also provide a sufficient condition to guarantee a Hamiltonian cycle in the semi-strong product.

Our major work on the semi-strong product is in sections 2.5 and 2.6. In section 2.5, we find formulas for both the independence and vertex covering numbers of the semi-strong product in terms of the order of the first factor and the independence or vertex covering number of the second factor. We also determine the matching number and the edge covering number for certain graph classes of the semi-strong product. Although formulas for some graph invariants, such as the chromatic number, are known for all semi-strong products, other graph invariants, such as the domination number, prove more difficult to determine. In section 2.6, we determine the domination number of the semi-strong product of each of four standard graph classes with a connected graph. These four classes are complete graphs, complete bipartite graphs, paths, and cycles. We also provide a general upper bound for the domination number of the semi-strong product, which is attained by some semi-strong products. In this section, we also show that previous research on Vizing's conjecture in the direct product falsifies a natural conjectured lower bound. Subsection 2.6.1 contains no new results, but describes a connection between domination in the semi-strong product and a classic chessboard problem, which can be rephrased in terms of graph domination.

Bootstrap percolation is a process defined on a graph, G . The process begins by selecting an initial set of infected vertices, $A_0 \subseteq V(G)$. In each subsequent round, an uninfected vertex, v , becomes infected if v is adjacent to at least r previously infected vertices. We use A_t to

denote the set of vertices infected as of round t . Symbolically,

$$A_t = A_{t-1} \cup \{v \in V(G) : |N(v) \cap A_{t-1}| \geq r\}$$

The parameter r is called the percolation threshold and bootstrap percolation with fixed r is sometimes referred to as r -neighbor bootstrap percolation or r -bootstrap percolation. If G is a finite graph, then after a fixed number of rounds, either all vertices of G become infected or the infection stops at some proper subset of $V(G)$. The set of infected vertices after the percolation process finishes is called the closure of A_0 , denoted $\langle A_0 \rangle$. If $\langle A_0 \rangle = V(G)$, then we say that A_0 is contagious or A_0 percolates. Figure 1.1 shows two graphs containing percolating sets of size 2 when $r = 2$.

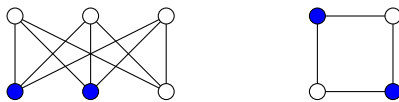


Figure 1.1: Percolating sets of size 2 when $r = 2$ in $K_{3,3}$ and C_4

Bootstrap percolation was introduced in 1979 by Chalupa, Leath, and Reich [22]. One model that has received much attention is when the vertices of A_0 are selected randomly: each vertex is selected independently and every vertex of G has probability p of being initially selected. After this, the infection proceeds deterministically. This model has been studied extensively, for instance, in [10, 11, 13, 14, 15, 43].

Another area of study is extremal problems. The minimum size of a percolating set in a graph G with percolation threshold r is denoted $m(G, r)$. Observe that if $|V(G)|$ is at least r , then $m(G, r) \geq r$. In this dissertation, we investigate minimum percolating sets in the Kneser graph and the bipartite Kneser graph. Previous research on bootstrap percolation in Kneser graphs has found $m(G, 2)$ for all Kneser graphs except those of the form $K(2k + 1, k)$. In Chapter 3, we find a sufficient condition for when $m(G, r) = r$ in Kneser graphs and bipartite Kneser graphs.

Chapter 2

The semi-strong product

We begin this chapter by introducing the semi-strong product. In this dissertation, we will use the tilde to denote adjacency: if $xy \in E(G)$, then we write $x \sim y$. The semi-strong product of G and H , denoted $G \overline{\times} H$, is a graph with vertex set $V(G) \times V(H)$ (the Cartesian product of $V(G)$ and $V(H)$). The edge set of the semi-strong product is defined as follows: $(g_1, h_1)(g_2, h_2) \in E(G \overline{\times} H)$ if $g_1 = g_2$ and $h_1 \sim h_2$ in H or $g_1 \sim g_2$ in G and $h_1 \sim h_2$ in H . Figure 2.1 shows $P_1 \overline{\times} P_2$.

Another way to describe the semi-strong product is in relation to other graph products. The Cartesian product, direct product, and strong product all have vertex set $V(G) \times V(H)$. The Cartesian product is denoted $G \square H$, with edge set defined as follows: $(g_1, h_1)(g_2, h_2) \in E(G \square H)$ if $g_1 = g_2$ and $h_1 \sim h_2$ in H or $h_1 = h_2$ and $g_1 \sim g_2$ in G . The direct product of G and H (also known as the tensor product), denoted $G \times H$, has the following edge set: $(g_1, h_1)(g_2, h_2) \in E(G \times H)$ if $h_1 \sim h_2$ in H and $g_1 \sim g_2$ in G . The strong product of G and H is denoted $G \boxtimes H$ and its edge set is the union of the edge sets of both the Cartesian and direct product. Figure 2.2 shows these three products with $G = H = P_2$. Having defined these products, we now see that the semi-strong product contains the direct product edges along with half of the Cartesian edges. Hence, the semi-strong product can be considered half of a strong product.

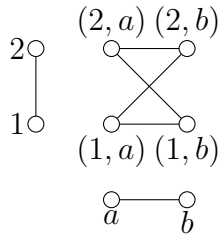


Figure 2.1: $P_2 \overline{\times} P_2$

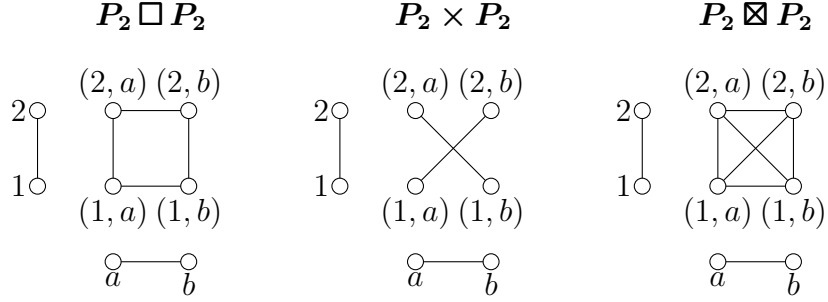


Figure 2.2: $P_2 \square P_2$, $P_2 \times P_2$, and $P_2 \boxtimes P_2$

The Cartesian, direct, and strong products are both commutative and associative. It is clear from the definition that the the semi-strong product is not commutative. In fact, the semi-strong product is not even associative. This was shown in Garman et al. [31] by considering $(P_2 \underline{\times} P_2) \underline{\times} P_2$ and $P_2 \underline{\times} (P_2 \underline{\times} P_2)$. As seen in Figure 2.1, $P_2 \underline{\times} P_2 \cong C_4$. Figure 2.3 shows $C_4 \underline{\times} P_2$ on the left and $P_2 \underline{\times} C_4$ on the right. Since $C_4 \underline{\times} P_2$ is 3-regular, while $P_2 \underline{\times} C_4$ has vertices of degree 4, the two graphs cannot be isomorphic.

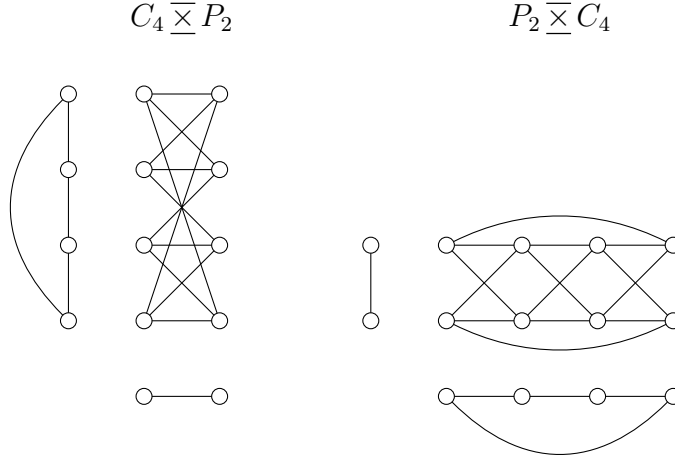


Figure 2.3: $C_4 \underline{\times} P_2$ and $P_2 \underline{\times} C_4$

In a graph product, a set of vertices $V(G) \times v$, for $v \in V(H)$ or $V(H) \times u$, for $u \in V(G)$ is called a **fiber**. In this dissertation, we will follow the convention of making the columns of our product graph copies of $V(G)$ and the rows copies of $V(H)$. For this reason, we will refer to fibers of the form $V(G) \times v$ as vertical fibers and those of the form $V(H) \times u$ as horizontal fibers.

2.1 Distance

A **walk** in a graph is a sequence of vertices where successive vertices are adjacent. A **path** is a walk in which no vertex is repeated. The **distance** between vertices $x, y \in V(G)$, denoted $d(x, y)$, is the longest path between x and y in G . The **eccentricity** of a vertex v , denoted $e(v)$, is the greatest distance between v and a vertex of G . The **radius** of G , denoted $rad(G)$, is the smallest eccentricity among all vertices of G . The **diameter** of G , denoted $diam(G)$, is the largest eccentricity among all vertices of G . The **center** of G , denoted $cen(G)$, is the subgraph induced by all vertices of G whose eccentricity is equal to the radius of G . Such vertices are known as **central vertices**. The periphery of G , denoted $per(G)$, is the subgraph induced by all vertices of G whose eccentricity is equal to the diameter of G . Such vertices are known as **peripheral vertices**. In this subsection, we investigate distance, eccentricity, radius, diameter, center, and periphery for the semi-strong product.

Suppose (g, h) is a vertex of $G \overline{\times} H$. Recall that the semi-strong product contains the horizontal edges from $G \square H$ and all the edges from $G \times H$. This means that we can move from (g, h) to an adjacent vertex using either a horizontal edge or a diagonal edge, but never using a vertical edge. In total, there are 6 possible directions. If we start at the vertex $(0,0)$ in Figure 2.4, we can move diagonally up to $(1,1)$ or right to $(1,0)$. If we begin at $(2,2)$, then we can move in all of the six possible directions.

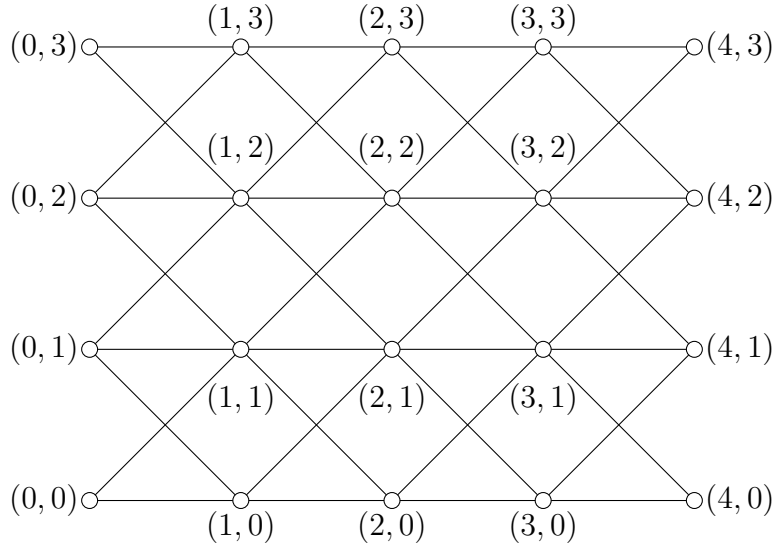


Figure 2.4: Labels of vertices in $P_4 \overline{\times} P_5$

Theorem 2.1.1. *Let (g_1, h_1) and (g_2, h_2) be vertices of $G \overline{\times} H$. The distance between the*

two vertices is determined as follows:

If $|d(g_1, g_2) - d(h_1, h_2)|$ is even, then $d((g_1, h_1), (g_2, h_2)) = \max(d(g_1, g_2), d(h_1, h_2))$

If $|d(g_1, g_2) - d(h_1, h_2)|$ is odd, then $d((g_1, h_1), (g_2, h_2)) = \max(d(g_1, g_2) + 1, d(h_1, h_2))$

Proof. For each case, we will first describe a sequence of moves which forms a shortest path from (g_1, h_1) to (x, y) and then show that such a path is indeed the shortest possible. Since we have adopted the convention of letting the columns of $G \overline{\times} H$ be copies of $V(G)$, while letting the rows consist of copies of $V(H)$, we will refer to $d(g_1, g_2)$ as the vertical distance from (g_1, h_1) to (g_2, h_2) and $d(h_1, h_2)$ as the horizontal distance between these two vertices.

We split this proof into three cases:

Case 1: $d(h_1, h_2) \geq d(g_1, g_2)$:

To construct a path from (g_1, h_1) to (g_2, h_2) , we first take $d(h_1, h_2) - d(g_1, g_2)$ horizontal steps. Each of these steps decreases the horizontal distance by one. We stop at a vertex, (x, y) , whose horizontal and vertical distances to (g_2, h_2) are equal. From the vertex (x, y) , we take $d(g_1, g_2)$ diagonal steps. Altogether we take $d(h_1, h_2) - d(g_1, g_2) + d(g_1, g_2) = d(h_1, h_2)$ steps, forming a path of length $d(h_1, h_2)$. If such a path is not minimum, then we can find a new path of length less than $d(h_1, h_2)$ from (g_1, h_1) to (g_2, h_2) . But, since each movement to form this new path must be either horizontal or diagonal, each movement corresponds to an edge in H and we can use the horizontal coordinates of the vertices in this new path to construct a path from h_1 to h_2 shorter than $d(h_1, h_2)$. But this is a contradiction, so our original path is in fact a shortest path.

An example of this scenario is moving from $(0, 0)$ to $(4, 2)$ in Figure 2.4. We first make three movements to the right, ending at $(2, 0)$. Next, we make two movements diagonally up, ending at $(4, 2)$. This forms a path of length 4, which is $d(0, 4)$.

Case 2: $d(h_1, h_2) < d(g_1, g_2)$ and $d(g_1, g_2) - d(h_1, h_2)$ is even:

We begin by making $d(g_1, g_2) - d(h_1, h_2)$ diagonal movements until the remaining horizontal and vertical distances are equal. Further, we make the movements by “bouncing back and forth” so that while constantly decreasing the vertical distance we follow every movement that carries us closer horizontally by another movement that carries us away horizontally. Recall that diagonal movements change both the horizontal and vertical distances. Since $d(g_1, g_2) - d(h_1, h_2)$ is even, we will make an even number of moves and the horizontal distance will remain unchanged at the end. Lastly, we make $d(h_1, h_2)$ moves where we decrease

both the horizontal and vertical distance by one with each move. In total, these moves form a path of length $d(g_1, g_2) - d(h_1, h_2) + d(h_1, h_2) = d(g_1, g_2)$.

Such a path must be a shortest path. A horizontal move cannot decrease the vertical distance. Thus, the only way to decrease the vertical distance is by making diagonal moves. If we could make fewer than $d(g_1, g_2)$ such moves, we could use the vertical coordinates in this path to form a path from g_1 to g_2 shorter than $d(g_1, g_2)$. (This is still possible even if these diagonal moves have horizontal moves in between them because horizontal moves do not change the vertical fiber). Contradiction.

An example of this is moving from (0,1) to (0,3) in Figure 2.4. We first move diagonally up and to the right to (1,2), then up and left to (0,3).

Case 3: $d(h_1, h_2) < d(g_1, g_2)$ and $d(g_1, g_2) - d(h_1, h_2)$ is odd:

We begin by moving horizontally to a vertex, (x, y) , adjacent to (g_1, h_1) such that $d(x, g_2) - d(y, h_2)$ is even. We then follow the same strategy as in Case 2. First we make as many moves as necessary to equalize the horizontal and vertical distances, bouncing back and forth so that the horizontal distance is unchanged after this sequence of moves. Then, we make diagonal moves to decrease both horizontal and vertical distance simultaneously. In total we make $1 + d(g_1, g_2) - d(h_1, h_2) + d(h_1, h_2) = d(g_1, g_2) + 1$ moves.

As in case 2, if we could reach our target vertex with fewer than $d(g_1, g_2)$ diagonal moves, this would lead to a contradiction. Note that since we need to decrease the vertical distance as well as the horizontal, all steps in such a path must be diagonal. Thus, to show our path is minimum, we must show that we cannot find a path of length $d(g_1, g_2)$ between (g_1, h_1) and (g_2, h_2) . If a path with exactly $d(g_1, g_2)$ diagonal moves existed, then because $d(g_1, g_2) - d(h_1, h_2)$ is odd, it would be impossible for the changes in the horizontal distance to cancel out. Thus, after making $d(g_1, g_2)$ moves, we could not have reached (g_2, h_2) . We need at least one more move. Therefore, $d(g_1, g_2) + 1$ is the minimum number of moves.

An example of this situation in Figure 2.4 occurs when moving from (1,0) to (1,3). The horizontal distance is 0 and the vertical distance is 3. So, we first go back and forth to (0,3) and then right to (1,3), using 4 moves in all.

□

Theorem 2.1.2. *If (g, h) is a vertex of $G \overline{\times} H$, then $e((g, h)) = \max(e(g) + 1, e(h))$.*

Proof. Let (g, h) be a vertex of $G \overline{\times} H$. We claim that if H contains no isolated vertices, then we can always find a vertex whose distance from (g, h) is $e(g) + 1$. If u is a vertex

in G furthest from g , then, by the previous result, $d((g, h), (u, h))$ is $e(g) + 1$ if $e(g)$ is odd and $e(g)$ if $e(g)$ is even. Since H contains no isolated vertices, h is adjacent to some vertex $v \in H$. So, if $e(g)$ is even, then $d((g, u)) - d((h, v)) = e(g) - 1$, which is odd. Hence, $d((g, h), (u, v)) = e(g) + 1$.

Therefore, from the above claim and the previous result, the two possibilities for $e((g, h))$ are $e(g) + 1$ or $e(h)$. Since the eccentricity is the largest possible distance, $e((g, h))$ must be the maximum of $e(g) + 1$ and $e(h)$. Figure 2.5 displays the eccentricities for the vertices $P_4 \overline{\times} P_5$. \square

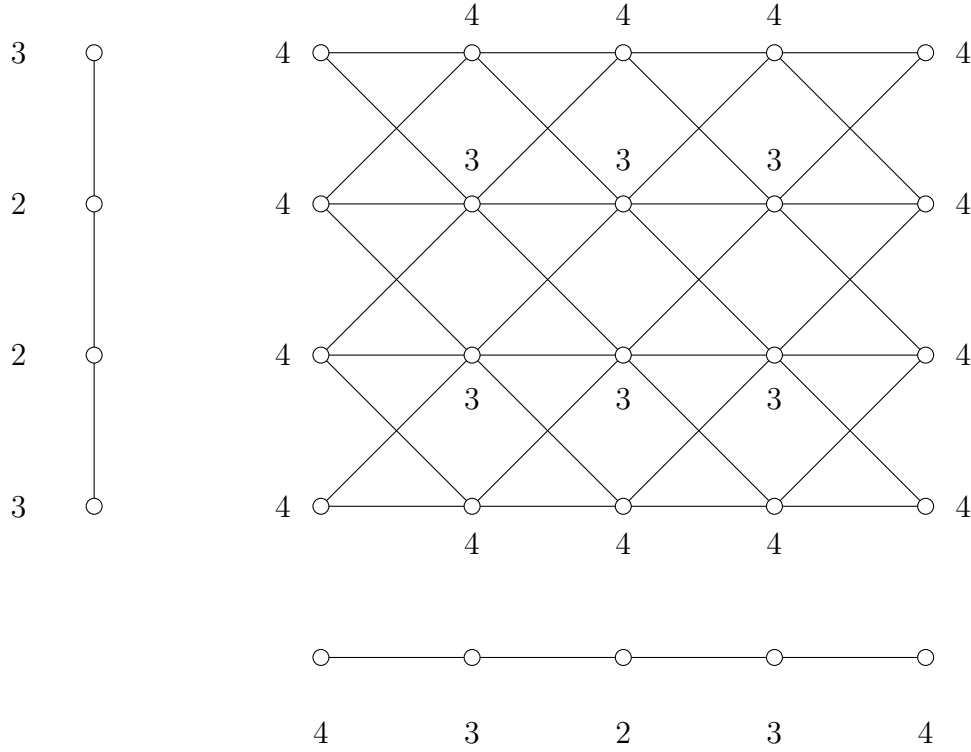


Figure 2.5: Eccentricities of vertices in $P_4 \overline{\times} P_5$

Theorem 2.1.3. $rad(G \overline{\times} H) = \max(rad(G) + 1, rad(H))$.

Proof. In general, if $a \leq x$ and $b \leq y$, then $\max(a, b) \leq \max(x, y)$.

Therefore, $\max(rad(G) + 1, rad(H)) \leq \max(e(g) + 1, e(h))$ for all values of $e(g)$ and $e(h)$. Since there exist vertices (g, h) of $G \overline{\times} H$ where $e(g) = rad(G)$ and $e(h) = rad(H)$, this minimum is attained. \square

Theorem 2.1.4. $\text{diam}(G \underline{\times} H) = \max(\text{diam}(G) + 1, \text{diam}(H))$.

Proof. The diameter of $G \underline{\times} H$ is the largest eccentricity over all vertices in the graph. By Theorem 2.1.2, the two possibilities for the eccentricity of a vertex (g, h) are $e(g) + 1$ and $e(h)$. The largest values for these two eccentricities are $\text{diam}(G) + 1$ and $\text{diam}(H)$, respectively. Thus, whichever of the two is greater must be the diameter for the entire graph. \square

Theorem 2.1.5. *If $\text{rad}(G) + 1 = \text{rad}(H)$, then $\text{Cen}(G \underline{\times} H) = \text{Cen}(G) \times \text{Cen}(H)$.*

If $\text{rad}(G) + 1 > \text{rad}(H)$, then $\text{Cen}(G \underline{\times} H) = \text{Cen}(G) \times \{v \in H : e(v) \leq \text{rad}(G) + 1\}$

If $\text{rad}(H) > \text{rad}(G) + 1$, then $\text{Cen}(G \underline{\times} H) = \text{Cen}(H) \times \{u \in G : e(u) + 1 \leq \text{rad}(H)\}$

Proof. Let (g, h) be a vertex of $G \underline{\times} H$. The center of $G \underline{\times} H$, i.e., the set of vertices for which $e((g, h)) = \text{rad}(G \underline{\times} H)$ is precisely the set of vertices for which the following inequality holds:

$$\min(\text{rad}(G) + 1, \text{rad}(H)) \leq \min(e(g) + 1, e(h)) \leq \max(e(g) + 1, e(h)) = \max(\text{rad}(G) + 1, \text{rad}(H))$$

Clearly, neither $e(g) + 1$ nor $e(h)$ can be less than the minimum of $\text{rad}(G) + 1, \text{rad}(H)$ and if either of $e(g) + 1$ or $e(h)$ was larger than the maximum of $\text{rad}(G) + 1, \text{rad}(H)$, the eccentricity of (g, h) would exceed the radius.

When $\text{rad}(G) + 1 = \text{rad}(H)$, the only admissible vertices are those for which $e(g) = \text{rad}(G)$ and $e(h) = \text{rad}(H)$. And those vertices form the set $\text{Cen}(G) \times \text{Cen}(H)$.

If $\text{rad}(G) + 1 > \text{rad}(H)$, then there are some vertices (g, h) for which $\text{rad}(H) < e(h) \leq \text{rad}(G) + 1$. The union of these vertices and the set of vertices for which $e(h) = \text{rad}(H)$ forms the set $\{v \in H : e(v) \leq \text{rad}(G) + 1\}$. Hence, the center is $\text{Cen}(G) \times \{v \in H : e(v) \leq \text{rad}(G) + 1\}$.

If $\text{rad}(H) > \text{rad}(G) + 1$, then by similar reasoning as above, we need to include vertices for which $\text{rad}(G) < e(g) \leq \text{rad}(H)$ in the center. And so, the center is $\text{Cen}(H) \times \{u \in G : e(u) + 1 \leq \text{rad}(H)\}$.

As shown in Figure 2.5, the center of $P_4 \underline{\times} P_5$ is $\text{Cen}(P_4) \times \text{Cen}(P_5)$.

\square

Theorem 2.1.6. *If $\text{diam}(G) + 1 = \text{diam}(H)$, then $\text{Per}(G \underline{\times} H) =$*

$$\text{Per}(G) \times V(H) \cup \text{Per}(H) \times V(G).$$

If $\text{diam}(G) + 1 > \text{diam}(H)$, then $\text{Per}(G \underline{\times} H) = \text{Per}(G) \times V(H)$.

If $\text{diam}(H) > \text{diam}(G) + 1$, then $\text{Per}(G \underline{\times} H) = \text{Per}(H) \times V(G)$

Proof. For every vertex, (g, h) , in $\text{Per}(G \underline{\times} H)$, the following inequality holds:

$$\min(e(g)+1, e(h)) \leq \min(\text{diam}(G)+1, \text{diam}(H)) \leq \max(e(g)+1, e(h)) = \max(\text{diam}(G)+1, \text{diam}(H))$$

In other words, if $\max(e(g) + 1, e(h)) = \text{diam}(G \underline{\times} H)$, then no matter what the value of $\min(e(g) + 1, e(h))$ the eccentricity of (g, h) is still the diameter of the product graph. Therefore, for a vertex to be in the periphery of $G \underline{\times} H$, at least one of $e(g) + 1, e(h)$ must be equal to $\max(\text{diam}(G) + 1, \text{diam}(H))$.

If $\text{diam}(G) + 1 > \text{diam}(H)$, then the vertices for which $e((g, h)) = \text{diam}(G) + 1$ are exactly those when $e(g) = \text{diam}(G)$, regardless of the value of $e(h)$. Thus, the periphery of the product is $\text{Per}(G) \times V(H)$.

Likewise, if $\text{diam}(H) > \text{diam}(G) + 1$, the only allowable vertices are those for which $e(H) = \text{diam}(H)$, so $\text{Per}(G \underline{\times} H) = \text{Per}(H) \times V(G)$.

If $\text{diam}(G) + 1 = \text{diam}(H)$, then unlike before, we have two possibilities. $e((g, h))$ can be equal to $\text{diam}(G \underline{\times} H)$ both when $e(g) + 1 = \text{diam}(G)$ and when $e(h) = \text{diam}(H)$. Hence, the periphery of the product is $\text{Per}(G) \times V(H) \cup \text{Per}(H) \times V(G)$. This occurs in Figure 2.5, where $\text{Per}(P_4 \underline{\times} P_5) = \text{Per}(P_4) \times V(P_5) \cup \text{Per}(P_5) \times V(P_4)$.

□

2.2 Wiener Index

The **Wiener index** of a graph, G , is the sum of the all distances between pairs of vertices of G , which we will denote $W(G)$. This graph invariant was introduced by Wiener in 1947 in the context of investigating the boiling points of paraffins [88]. The Wiener index has been determined for the Cartesian product, by Graovac and Pisanski [34]. If G and H are connected graphs, then $W(G \square H) = |H|^2 W(G) + |G|^2 W(H)$. Yeh and Gutman independently found the formula for the Wiener index of the Cartesian product and also determined the Wiener index of the lexicographic product [89].

The Wiener index of the direct and strong products have also been studied, for instance in [67, 68, 69, 70]. Peterin and Pleteršek give a (fairly complicated) general formula for the Wiener index of the strong product which they state cannot be simplified in general [72].

In this section, we find the Wiener index for the semistrong product of a complete graph with a connected graph, H and the Wiener index for the semistrong product of a complete bipartite graph with a connected graph.

In each calculation, we will split the sum of pairs of distances into three parts and then add these three sums to form the total. The first is pairs of vertices within the same horizontal fiber, the second is pairs of vertices within the same vertical fiber. Lastly, we will calculate the sum of distances between pairs of vertices which are in different horizontal fibers and different vertical fibers.

Theorem 2.2.1. *If H is a connected graph, then $W(K_n \overline{\times} H) = \binom{n+1}{2}W(H) + n(n-1)|H|$*

Proof. Since each horizontal fiber is a copy of H , the sum of distances within a single horizontal fiber is $W(H)$. There are n such fibers, so the total of this sum is $nW(H)$. Vertices in the same vertical fiber have horizontal distance 0 and vertical distance 1, so by Theorem 2.1.1, their distance in the product is 2. There are $\binom{n}{2}$ pairs of vertices in each vertical fiber and $|H|$ vertical fibers, so in total, this sum is $2\binom{n}{2}|H| = n(n-1)|H|$.

Suppose $(g_1, h_1), (g_2, h_2)$ are a pair of vertices in both different horizontal and vertical fibers. The vertical distance between such a pair of vertices is 1. Whatever the horizontal distance is, it must be at least 1, so by Theorem 2.1.1, $d((g_1, h_1), (g_2, h_2)) = d(h_1, h_2)$. Hence, the sum of distances between any pair of horizontal fibers is $W(H)$. There are $\binom{n}{2}$ pairs of fibers and so the total sum is $\binom{n}{2}W(H)$.

Combining these three sums, we find that $W(K_n \overline{\times} H) = \binom{n}{2}W(H) + nW(H) + n(n-1)|H|$. This simplifies to $\binom{n+1}{2}W(H) + n(n-1)|H|$. \square

Theorem 2.2.2. *If H is a connected graph, then*

$$W(K_{n,m} \overline{\times} H) = W(H) \left(2nm + \binom{m}{2} + \binom{n}{2} \right) + |H| \left((n+m)^2 - (n+m) \right) + 2|E(H)| \left(\binom{m}{2} + \binom{n}{2} \right)$$

Proof. Each horizontal fiber is a copy of H , and there are nm horizontal fibers, so our first sum is $nmW(H)$. If (g_1, h) and (g_2, h) is a pair of vertices in the same vertical fiber, then we have two cases. If the vertices are in the same partite set, then the vertical distance is 2. If the vertices are in different partite sets, then the vertical distance is 1. In either case,

since the horizontal distance is 0, the distance in the product is 2. Within each vertical fiber, there are $\binom{n}{2} + \binom{m}{2} + nm$ pairs of vertices and $|H|$ horizontal fibers altogether, so our second sum is $2|H|(\binom{n}{2} + \binom{m}{2} + nm) = |H|((n+m)^2 - (n+m))$.

Suppose that $(g_1, h_1), (g_2, h_2)$ are vertices in both different vertical and horizontal fibers. Further, suppose that each vertex is contained in a fiber associated with the same partite set of $K_{n,m}$. Then, the vertical distance is 2. If h_1 is adjacent to h_2 , then the horizontal distance is 1. If not, it is at least 2. Summing up all pairs of distances for this product yields $W(H)$ plus an additional amount equal to the degree of each vertex. Since there are $\binom{n}{2} + \binom{m}{2}$ pairs of fibers in the same partite set, this sum in total is $(2|E(H)| + W(H))(\binom{n}{2} + \binom{m}{2})$.

Now, suppose that $(g_1, h_1), (g_2, h_2)$ are contained in fibers associated with different partite sets. If $(g_1, h_1) \sim (g_2, h_2)$, both the horizontal and vertical distances are 1. Alternatively, if $(g_1, h_1) \not\sim (g_2, h_2)$, then the horizontal distance is at least 2 and the vertical distance is 1. Either $|d(g_1, g_2) - d(h_1, h_2)|$ is even or $d(h_1, h_2) \geq d(g_1, g_2)$, so by Theorem 2.1.1 the total sum over such a pair of fibers is $W(H)$. There are nm such pairs so in total our sum is $nmW(H)$.

Combining these three expressions yields

$$W(H)\left(2nm + \binom{n}{2} + \binom{m}{2}\right) + |H|\left((n+m)^2 - (n+m)\right) + 2|E(H)|\left(\binom{n}{2} + \binom{m}{2}\right)$$

□

2.3 Coloring and Clique Number

A **proper coloring** of a graph, G , is an assignment of colors to the vertices of G such that no adjacent vertices receive the same color. The **chromatic number** of a graph, denoted $\chi(G)$, is the minimum number of colors used among all proper colorings of G . The chromatic number is one of the most well known graph invariants. It originated with attempts to prove the famous 4-color problem, that if G is a planar graph, then $\chi(G) \leq 4$.

In [31], Garman, Ringelsen, and White stated the following without proof. The result is straightforward, but we present a proof here for the sake of completeness:

In the proof of this result, we use the following elementary lemma concerning graph coloring

Lemma 2.3.1. *If P is a subgraph of G , then $\chi(G) \geq \chi(P)$.*

Theorem 2.3.2. $\chi(G \overline{\times} H) = \chi(H)$.

Proof. Since each horizontal fiber of the product is a copy of H , by Lemma 2.3.1 we need at least $\chi(H)$ colors to form a proper coloring of the product. Hence, $\chi(G \overline{\times} H) \geq \chi(H)$.

Let C be a proper coloring of H . Color a single fiber of $G \overline{\times} H$ using C and then repeat this coloring exactly in every other horizontal fiber. This produces a proper coloring of $G \overline{\times} H$. The reason is because if two vertices are colored the same and are in different fibers, then they could only be joined by a direct product edge. But, direct product edges also correspond to edges in H . Thus, if two adjacent vertices (u, x) and (v, y) are in different fibers and assigned the same color, (u, x) would also be adjacent to a vertex (v, x) in its horizontal fiber. But, we assumed C was a proper coloring, so this is impossible. Therefore, $\chi(G \overline{\times} H) \leq \chi(H)$. \square

The **clique number** of a graph G , denoted $\omega(G)$ is the size of the largest complete subgraph of G . Since a clique is a subgraph of G , $\chi(G) \geq \omega(G)$. We show that, analogous to the chromatic number of the semi-strong product, the clique number of the semi-strong product is determined by the clique number of H .

Theorem 2.3.3. $\omega(G \overline{\times} H) = \omega(H)$.

Proof. Let C be a largest clique of H . Since every horizontal fiber of the product is a copy of H , we can find a clique of size $|C| = \omega(H)$ in any fiber of the product. Thus, $\omega(G \overline{\times} H) \geq \omega(H)$.

In order to demonstrate an upper bound, we must show that no clique larger than $\omega(H)$ can be found in the product. Let K be any complete subgraph of $G \overline{\times} H$. No two vertices of K can share the same vertical fiber because no vertices in the same vertical fiber can be adjacent. Since every vertex of K is in a different vertical fiber, we can project every vertex of K onto H to form a subgraph of H . Since we assumed K was a clique, every vertex in K is adjacent to every other vertex in K . If two vertices $(u, x), (v, y)$ are joined by horizontal edges, then $x, y \in H$ are also adjacent.

Likewise, if $(u, x), (v, y)$ are joined by direct product edges, x, y are also joined in H . Therefore, our projection is also a clique, so $|K| \leq \omega(H)$. Since K was an arbitrary clique of the product, $\omega(G \overline{\times} H) \leq \omega(H)$. \square

2.4 Eulerian circuits and Hamiltonicity

A **trail** is a walk in a graph in which no edge is repeated. A **closed walk** is a walk with the same beginning and ending vertex. An **open walk** is a walk with different starting and ending vertices. An **Eulerian circuit** is a closed trail in a graph which traverses each edge in the graph exactly once. An **Eulerian trail** is an open trail in a graph which traverses each edge exactly once. A **Hamiltonian path** is a path which contains every vertex in a graph. A **Hamiltonian cycle** is a cycle which contains every vertex in a graph. If a graph contains a Hamiltonian cycle, then we say the graph is **Hamiltonian**.

Garman et. al. stated the following proposition in [31]:

Proposition 2.4.1. *If (g, h) is a vertex in $G \overline{\times} H$, then $d((g, h)) = (d(g) + 1)d(h)$*

Proof. First we consider the direct product edges. Let $v \in G$ be some vertex adjacent to g . Then, in every horizontal fiber $\{v\} \times V(H)$, (g, h) is adjacent to all vertices of the form (v, w) , where $w \in H$ is adjacent to h . There are $d(g)$ such horizontal fibers. So, (g, h) is adjacent to $d(g)d(h)$ vertices via direct product edges. In addition, (g, h) is adjacent to $d(h)$ vertices within its own horizontal fiber. Altogether, (g, h) is adjacent $d(g)d(h) + d(h) = (d(g) + 1)d(h)$ vertices. \square

With this proposition and the following famous theorems due to Euler and Hierholzer [28, 42], we can characterize those semi-strong products which have Eulerian circuits and Eulerian trails:

Theorem 2.4.2. *A nontrivial connected graph, G , is Eulerian if and only if every vertex of G has even degree.*

Theorem 2.4.3. *A connected graph G contains an Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these vertices and ends at another.*

Theorem 2.4.4. *$G \overline{\times} H$ is Eulerian if and only if G has all odd degrees or H is Eulerian.*

Proof. The product of k natural numbers is even when at least one of the factors is even. Hence, if G has all odd degrees or H is Eulerian, at least one of $d(g) + 1, d(h)$ will be even for all $g \in G, h \in H$. Therefore, $(d(g) + 1)d(h)$ will be even for all product vertices (g, h) . Now suppose that G has at least one even degree and H is not Eulerian. Then, for some

vertex (u, v) , both $d(u) + 1$ and $d(v)$ are odd and thus their product is also odd. So, $G \overline{\times} H$ has at least one odd degree and hence is not Eulerian. \square

Theorem 2.4.5. *$G \overline{\times} H$ has an Eulerian trail if and only if G has exactly one even vertex and H has an Eulerian trail.*

Proof. A graph has an Eulerian trail if exactly two of its vertices have odd degree. $G \overline{\times} H$ has two odd vertices if and only if two of the products $(d(g) + 1)(d(h))$ are odd. There are two ways for this to happen: G has two even vertices and H has one odd vertex or G has one even vertex and H has two odd vertices. The first situation is impossible because no graph can have an odd number of odd degree vertices. Therefore, if G has one even vertex and H contains an Eulerian trail, then $G \overline{\times} H$ contains an Eulerian trail and the condition is also necessary. \square

We now present a sufficient condition for a Hamiltonian cycle in the product.

Theorem 2.4.6. *If G has a Hamiltonian path and H contains a Hamiltonian cycle, then $G \overline{\times} H$ contains a Hamiltonian cycle.*

Proof. Suppose $|G| = n$ and $|H| = m$. Since each horizontal fiber corresponds to a vertex of G , label all the horizontal fibers in sequence corresponding to the vertices of a Hamiltonian path of G as follows: H_1, H_2, \dots, H_n . Then, within each fiber, label the vertices corresponding to their position in a copy of a Hamiltonian cycle, C_i , of H_i . Vertices within H_1 would be labeled as follows: $v_{1,1}, v_{1,2}, \dots, v_{1,m}$. The vertical coordinate is first and the horizontal coordinate is second.

We have three cases:

Case 1: $n = m$

In this instance, we find a Hamiltonian cycle in $G \overline{\times} H$ as follows: Begin with $v_{1,1}$ and follow C_1 up to $v_{1,m-1}$. Then, move to $v_{2,m}$ and move along C_2 , ending at $v_{2,m-2}$, the third to last vertex. In general, within the i^{th} horizontal fiber, we move horizontally along C_i (wrapping around when necessary) and ending at $v_{i,m-i}$. We then move up to $v_{i+1,m-i+1}$ in the next horizontal fiber. During this process, within each fiber we pass through all vertices except $v_{i,m-i+1}$. Since $n = m$, the first vertex we reach in H_n is $v_{n,2}$. We then follow C_n to $v_{n,1}$ and then follow the left to right diagonal through all the vertices we avoided until we reach $v_{1,m}$. Then we wrap around from $v_{1,m}$ to $v_{1,1}$.

In other words, within each horizontal fiber we include each vertex except the vertices of the diagonal formed by the top left to bottom right vertices. We then follow this diagonal down to the bottom right corner and return to the bottom left corner. Figure 2.6 gives an example of how such a cycle is constructed. The labels on the vertices are the order of the Hamiltonian cycle, with 9 returning back to 1.

Case 2: $n < m$

This case is similar to the previous case, except that now the vertices we avoid are on the diagonal from the bottom right corner up to $v_{n,m-n+1}$. Then, as before, we follow that diagonal down to $v_{1,m}$ and then move from $v_{1,m}$ to $v_{1,1}$.

Case 3: $n > m$

In this case, we must do something different. Starting at the bottom right corner, we form a path of vertices diagonally up and left, until we reach $v_{m,1}$. Then, we move diagonally up and right until we reach $v_{2m,m}$ and so on, switching from left to right each time. These are the vertices that we avoid when moving horizontally through each fiber. Then, once we have reached the highest row, we follow this path down to $v_{1,m}$ and move from $v_{1,m}$ to $v_{1,1}$.

That these three procedures yield Hamiltonian cycles follows from the fact that they are formed only of allowable moves within $P_n \overline{\times} C_m$. If G and H satisfy the hypotheses of this theorem, then G contains a subgraph isomorphic to P_n and H contains a subgraph isomorphic to C_m , so $G \overline{\times} H$ contains a subgraph isomorphic to $P_n \overline{\times} C_m$. Within such a graph, we can move horizontally following the vertices of C_m and we can move diagonally between horizontal fibers corresponding to adjacent vertices of P_n .

□

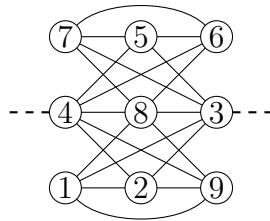


Figure 2.6: $P_3 \overline{\times} C_3$

2.5 Independence and Matching

An **independent set** of vertices in a graph G is a set of vertices, $I \subseteq V(G)$ such that no vertex in I is adjacent to any other vertex of I . If I' is an independent set of G such that no independent set of G contains more vertices than I' (though some may have the same amount), then we say that I' is a **maximum independent set** of G . The cardinality of a maximum independent set of a graph G , denoted $\alpha(G)$, is the **independence number** of G .

In this section, we determine the independence number of the semi-strong product and the matching number for some graph classes of the semi-strong product. We also include results on the vertex covering number and edge covering number of the semi-strong strong product.

Geller and Stahl found a formula for the independence number of the lexicographic product of two graphs: $\alpha(G \circ H) = \alpha(G)\alpha(H)$ [32]. Determining the independence number of the other main graph products is challenging in general. For instance, in [56], it is remarked that $\alpha(K_k \square G) = \alpha_k(G)$, where $\alpha_k(G)$ is the maximum cardinality of a set of vertices of G , no two of which are distance k or less apart. So, $\alpha_1(G)$ would be the ordinary independence number. Determining $\alpha_k(G)$ is NP-complete [58]. Vizing [87] showed the following bounds

$$\alpha(G)\alpha(H) + \min\{|G| - \alpha(G), |H| - \alpha(H)\} \leq \alpha(G \square H) \leq \min\{\alpha(G)|H|, \alpha(H)|G|\}$$

Other papers investigating $\alpha(G \square H)$ are [1, 19, 37, 53]. The independence number of the direct product of graphs has been investigated in [53, 54, 55, 71]. The strong product of graphs is connected to the Shannon Capacity [80], which is $\lim_{k \rightarrow \infty} \sqrt[k]{\chi_{i=1}^k G}$. Some papers investigating the independence number of the strong product of graphs are [38, 78, 86]. It is perhaps somewhat surprising that, as with the lexicographic product, there is a formula for the independence number of the semi-strong product of graphs:

Theorem 2.5.1. $\alpha(G \overline{\times} H) = |G|\alpha(H)$.

Proof. First we will describe a lower bound. Choose one horizontal fiber of $G \overline{\times} H$ and then select a maximum independent set of H from this fiber. If we then copy this independent set in every other horizontal fiber, the resulting set, I , is an independent set of $G \overline{\times} H$. Suppose v, w are two vertices of I . If v and w are in the same horizontal fiber, then, by assumption, v and w are not adjacent. If v and w are in different horizontal fibers and different vertical fibers, then the only edge that could exist between the two is a direct product edge. But, a

direct product edge only exists if the corresponding vertices of *both* G and H are adjacent. But the corresponding vertices from H are not adjacent because they correspond to vertices of an independent set. Recall that there are no edges between vertices of the same vertical fiber in the semi-strong product. I consists of $|G|$ maximum independent sets of H , so $|I| = |G|\alpha(H)$. The red vertices in Figure 2.7 give an example of this construction. $|P_3| = 3$ and $\alpha(K_3) = 1$, so we form a maximum independent set of $P_3 \overline{\times} K_3$ with 3 vertices.

For the upper bound, note that in any independent set of the semi-strong product, the maximum number of vertices in any horizontal fiber is $\alpha(H)$. If we had more, then the vertices would no longer form a maximum independent set within that fiber. And so, no set of vertices of $G \overline{\times} H$ with more than $|G|\alpha(H)$ vertices can be independent because, by the pigeonhole principle, at least one horizontal fiber must have more than $\alpha(H)$ vertices. Since the upper and lower bounds match, $\alpha(G \overline{\times} H) = |G|\alpha(H)$.

□

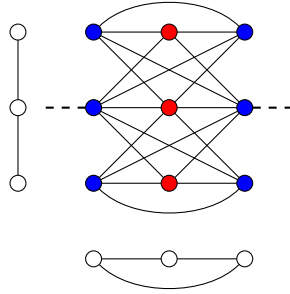


Figure 2.7: $P_3 \overline{\times} K_3$

A **vertex cover** of a graph G is a set of vertices C such that every edge of G is incident to at least one vertex of C . The smallest number of vertices in a vertex cover of G is the **vertex covering number** of G , denoted $\beta(G)$. In a linear programming sense, this is the dual problem to finding the independence number of G .

Theorem 2.5.2. $\beta(G \overline{\times} H) = |G|\beta(H)$

Proof. First we describe a lower bound. Since Cartesian edges can only be covered within a horizontal fiber, we need at least $\beta(H)$ vertices in every horizontal fiber to form a vertex cover of $G \overline{\times} H$. So, $\beta(G \overline{\times} H) \geq |G|\beta(H)$. For the upper bound, consider the following construction: copy the same minimum vertex cover of H in every horizontal fiber of the product, using a total of $|G|\beta(H)$ vertices. Figure 2.7 shows an example of this construction

using the blue vertices. $\beta(K_3) = 2$ and $|P_3| = 3$, so $\beta(P_3 \overline{\times} K_3) = 6$. If we can prove that this construction is a vertex cover of the product, then this will provide an upper bound matching the lower bound.

Let C be a minimum vertex cover of H and let C' be union of all the copies of C over all horizontal fibers of the product. Since each horizontal fiber contains a vertex cover, all Cartesian edges are covered. Let w be some vertex of G . Consider the horizontal fiber $\{w\} \times V(H)$. For any direct product edge, e , incident with a vertex of $\{w\} \times V(H)$, there are two possibilities: e is incident with a vertex of the form (w, x) , $x \in C$ or e is incident with a vertex (w, y) , $y \notin C$.

If e is incident with a vertex of the first kind, then e is covered by C' because $x \in C$, so $(w, x) \in C'$. If e is incident with a vertex of the second kind, then e is not covered by a vertex of $\{w\} \times V(H)$. However, consider the other endpoint of e , (y, z) , where $y \in V(G)$ and $z \in H$. yz is a copy of an edge of H and since yz is not covered by y , yz must be covered by z . Hence, $z \in C$ and $(y, z) \in C'$. Therefore, e is covered by (y, z) .

Since $\{w\} \times V(H)$ was an arbitrary horizontal fiber of H , we can conclude that the diagonal edges associated with any horizontal fiber of $G \overline{\times} H$ are covered. Hence, C' is indeed a vertex cover of $G \overline{\times} H$. Our upper and lower bounds match, so we conclude that $\beta(G \overline{\times} H) = |G|\beta(H)$. \square

An **independent set of edges** or a **matching** of G is a set of edges of G such that no two share an endpoint. The maximum possible number of edges in such a set is the **edge independence number** of G , denoted $\nu(G)$ or the **matching number** of G . The matching number can also be denoted $\alpha'(G)$, but we use $\nu(G)$ to avoid confusion with the independence number of G . A **perfect matching** or a **1-factor** is a matching of G in which every vertex of G is incident with an edge of the matching. An **edge covering** of G is a set of edges of G such that every vertex of G is adjacent to an edge of the edge covering. The **edge covering number** of G , denoted $\beta'(G)$ is the minimum number of edges in an edge cover of G .

Matchings in graph products have not been studied as extensively as independent sets. Klavzar and Jha and Paulraja and Varadarajan have studied the matching number of the direct product in [55] and [71], respectively. Perfect matchings in the Cartesian product have been studied in [59]. Perfect matchings in the direct product of graphs are studied in [7]. Sufficient conditions on perfect matchings in the lexicographic product, direct product, and even the semi-strong product are investigated in [74]. In a series of papers, Sitthiwirattam

along with others has investigated matchings in the direct and strong products [79, 81, 82, 83]. Lindeberg and Hellmuth construct matchings in the Cartesian, direct, strong, and lexicographic products in [63].

It is more difficult to determine the matching number of the semi-strong product than the independence number. We do not have a formula for all graphs, but we can find the matching number of the semi-strong product for some graph classes.

We use a lemma along with the König-Egerváry and König-Rado Theorems to determine these parameters for the semi-strong product under certain circumstances. We will first prove the following proposition:

Proposition 2.5.3. *If I is an independent set of H , then we can stack $|G|$ copies of I in $G \underline{\times} H$ to form an independent set, I' , in the product.*

Proof. Suppose $v, w \in I'$. If v, w are in the same horizontal fiber of $G \underline{\times} H$, then they are not adjacent because they correspond to an independent set of H . If v, w are in the same vertical fiber but different horizontal fibers, then since this is a semi-strong product v and w are not adjacent. If v and w are in different horizontal and vertical fibers, then the only way v and w could be adjacent is by a direct product edge. But, since the projections of v and w onto H are also not adjacent, v and w are not adjacent in the product. \square

In [31], it is stated that If H is bipartite, then $G \underline{\times} H$ is bipartite. In fact, $G \underline{\times} H$ is bipartite if and only if H is bipartite. For the sake of completeness, we include a proof of this fact.

Lemma 2.5.4. *$G \underline{\times} H$ is bipartite if and only if H is also bipartite.*

Proof. We will first prove sufficiency. Assume H is bipartite. Then, we can split $V(H)$ into two sets X and Y such that every edge in H is incident with a vertex in X and a vertex in Y . Another way to say this is that X and Y are such that $X \cup Y = V(H)$, $X \cap Y = \emptyset$ and both X and Y are independent sets of H .

$V(G \underline{\times} H) = V(G) \times V(H)$. Therefore, split up $V(G \underline{\times} H)$ as follows: Let X' consist of the union of all copies of X over each horizontal fiber of the product. Let Y' be defined likewise. Then, by construction, $X' \cup Y' = V(G \underline{\times} H)$ and $X' \cap Y' = \emptyset$. Further, by Proposition 2.5.3, both X' and Y' are independent. Hence, $G \underline{\times} H$ is bipartite.

Now, we will show that if $G \underline{\times} H$ is bipartite, then H is bipartite using the contrapositive. Assume that H is not bipartite. Then, H contains an odd cycle. Since every horizontal fiber is a copy of H , $G \underline{\times} H$ also contains an odd cycle. Hence, $G \underline{\times} H$ is not bipartite.

□

Theorem 2.5.5 (König-Egerváry Theorem). *If G is a bipartite graph, then $\beta(G) = \nu(G)$.*

Theorem 2.5.6. *If H is bipartite, then $\nu(G \overline{\times} H) = |G|\nu(H)$.*

Proof. By Lemma 2.5.4, if H is bipartite, $G \overline{\times} H$ is also bipartite. Since the product is bipartite, we can apply the König-Egerváry Theorem to conclude that $\nu(G \overline{\times} H) = \beta(G \overline{\times} H)$. By Theorem 2.5.2, $\beta(G \overline{\times} H) = |G|\beta(H)$. We can then apply the König-Egerváry Theorem to H and conclude that $|G|\beta(H) = |G|\nu(H)$. □

Consider a graph G and assign a number between 1 and 0 to every vertex of G such that the sum of the endpoints over each edge of G is no more than 1. Such an assignment is called a **fractional independent set** of G and the largest possible sum of vertex assignments for a given G is known as the **fractional independence number** and denoted $\alpha_f(G)$. A graph is *KE* or König-Egerváry if $\alpha(G) + \nu(G) = |G|$. The significance of KE graphs is that such graphs is the class of graphs for which the König-Egerváry Theorem is true. In general, the following holds:

Theorem 2.5.7 (Gallai's Theorem). *For any graph, G , $\alpha(G) + \beta(G) = |G|$ and $\nu(G) + \beta'(G) = |G|$.*

In addition, $\alpha(G) + \nu(G) \leq |G|$ for all G . Let M be a maximum matching of G and I be a maximum independent set of G . We can include at most 1 vertex from each edge of the matching in I . We can also possibly include all unmatched vertices in I . Thus, $\alpha(G) \leq \nu(G) + |G| - 2\nu(G)$, which implies that $\alpha(G) + \nu(G) \leq |G|$. By Gallai's Theorem $\beta(G) = |G| - \alpha(G)$, so in general $\nu(G) \leq |G| - \alpha(G) = \beta(G)$. Hence, the graphs for which $\nu(G) = \beta(G)$ are also the graphs for which $\alpha(G) + \nu(G) = |G|$. We know that all bipartite graphs are KE, but the KE graphs also contain non-bipartite graphs. For instance, the graphs consisting of an odd cycle with a single leaf are non-bipartite but are KE.

In 2012, Larson proved the following theorem characterizing KE graphs in terms of their fractional independence numbers:

Theorem 2.5.8 (KE Characterization Theorem [60]). *A graph G is KE if and only if $\alpha(G) = \alpha_f(G)$.*

Using the following lemma and the KE characterization theorem, we can determine the fractional independence number for a particular class of semi-strong product graphs:

Lemma 2.5.9. *If H is bipartite, then $G \underline{\times} H$ is KE.*

Proof. By Theorem 2.5.1, $\alpha(G \underline{\times} H) = |G|\alpha(H)$. If H is bipartite, then $\nu(G \underline{\times} H) = |G|\beta(H)$. But then, $\alpha(G \underline{\times} H) + \nu(G \underline{\times} H) = |G|\alpha(H) + |G|\beta(H) = |G|(\alpha(H) + \beta(H))$. By Gallai's Theorem, $\alpha(H) + \beta(H) = |H|$. Thus, $|G|(\alpha(H) + \beta(H)) = |G||H| = |G \underline{\times} H|$. Therefore, $G \underline{\times} H$ is KE. \square

Corollary 2.5.10. *If H is bipartite, $\alpha_f(G \underline{\times} H) = |G|\alpha(H)$.*

Proof. By Lemma 2.5.9, $G \underline{\times} H$ is KE, so by the KE characterization theorem, $\alpha_f(G \underline{\times} H) = \alpha(G \underline{\times} H) = |G|\alpha(H)$. \square

We now determine $\beta'(G \underline{\times} H)$ when H is bipartite. First, we will prove the following proposition:

Proposition 2.5.11. *$G \underline{\times} H$ has no isolated vertices if and only if H has no isolated vertices.*

Proof. First, we will show that if the product has no isolated vertices, H also has no isolated vertices. The proof will be by contrapositive. Assume H has an isolated vertex, x . Then, any vertex in the product of the form (a, x) must also be isolated. Because x is isolated, (a, x) is not joined to any other vertices by Cartesian edges. Further, since direct product edges require a vertex to be adjacent to another in both components, (a, x) cannot be joined to another vertex by direct product edges either.

Now, we will show the other direction, that if H has no isolated vertices, $G \underline{\times} H$ has no isolated vertices. If H has no isolated vertices, then every vertex of H has at least one neighbor. Therefore, every vertex in the product has at least one neighbor in its own horizontal fiber. \square

Lemma 2.5.12. *If H is bipartite and has no isolated vertices, then $G \underline{\times} H$ is also bipartite with no isolated vertices.*

Proof. Assume H is bipartite with no isolated vertices. By Lemma 2.5.4, $G \underline{\times} H$ is also bipartite. By Proposition 2.5.11, $G \underline{\times} H$ also has no isolated vertices. \square

Theorem 2.5.13 (König-Rado Theorem). *If G is a bipartite graph without isolated vertices, then $\alpha(G) = \beta'(G)$.*

Theorem 2.5.14. *If H is bipartite without isolated vertices, then $\beta'(G \underline{\times} H) = |G|\beta'(H)$.*

Proof. Suppose H is bipartite with no isolated vertices. Then, by Lemma 2.5.12 and Theorem 2.5.13, $\beta'(G \underline{\times} H) = \alpha(G \underline{\times} H)$. By Theorem 2.5.1, $\alpha(G \underline{\times} H) = |G|\alpha(H)$. We then apply the König-Rado Theorem to H to conclude $|G|\alpha(H) = |G|\beta'(H)$. \square

Earlier, we showed that $\nu(G \underline{\times} H) = |G|\nu(H)$ when H is bipartite. In fact, $|G|\nu(H)$ is a general lower bound for $\nu(G \underline{\times} H)$. We simply stack $|G|$ copies of a maximum matching of H , using only the Cartesian edges of H . This ensures that each matching is indeed a matching within each horizontal fiber. Furthermore, since Cartesian edges are only contained within a particular horizontal fiber of the product, the edges are independent between different horizontal fibers. Hence, these $|G|$ copies of a maximum matching of H form a matching of G . We can encapsulate this observation in the following proposition:

Proposition 2.5.15. *Suppose M is a matching of H . Let M' be the set of edges of $G \underline{\times} H$ consisting of $|G|$ copies of M , in which each copy of M consists of the edges of a single horizontal fiber of the product. Then, M' is a matching of G , containing $|G|\nu(H)$ edges.*

As shown in Theorem 2.5.6, this lower bound is attained when H is bipartite. This lower bound is also attained for the wider class of KE graphs:

Lemma 2.5.16. *If H is KE, then $G \underline{\times} H$ is KE*

Proof. In general, $\alpha(G \underline{\times} H) + \nu(G \underline{\times} H) \leq |G||H|$. Furthermore, $\nu(G \underline{\times} H) \geq |G|\nu(H)$, so we have $|G|\alpha(H) + |G|\nu(H) \leq \alpha(G \underline{\times} H) + \nu(G \underline{\times} H)$. We can rewrite the left hand side of the inequality as $|G|(\alpha(H) + \nu(H))$. But since H is KE, $\alpha(H) + \nu(H) = |H|$, which yields

$$|G||H| \leq \alpha(G \underline{\times} H) + \nu(G \underline{\times} H) \leq |G||H|$$

Hence, $G \underline{\times} H$ is KE. \square

Theorem 2.5.17. *If H is KE, then $\nu(G \underline{\times} H) = |G|\nu(H)$*

Proof. By Lemma 2.5.16, $G \underline{\times} H$ is KE, so $\alpha(G \underline{\times} H) + \nu(G \underline{\times} H) = |G||H|$. This implies that $|G|\alpha(H) + \nu(G \underline{\times} H) = |G||H|$, which yields $\nu(G \underline{\times} H) = |G|(|H| - \alpha(H))$. But we assumed that H is KE, so $|H| - \alpha(H) = \nu(H)$ and therefore, $\nu(G \underline{\times} H) = |G|\nu(H)$. \square

This condition is not necessary, i.e., there exist non-KE graphs, H for which $\nu(G \underline{\times} H) = |G|\nu(H)$. For instance, K_4 is not KE: $\nu(K_4) + \alpha(K_4) = 2 + 1 = 3 < 4$. However, since

K_4 contains a perfect matching, we can simply copy this perfect matching in the Cartesian edges of $G \overline{\times} K_4$ to make a perfect matching, M , in $G \overline{\times} K_4$. Since M is perfect, M must be maximum and hence $\nu(G \overline{\times} K_4) = |G|\nu(K_4)$.

We can extend this example to the following theorem:

Theorem 2.5.18. *If H has a perfect matching, then $G \overline{\times} H$ also contains a perfect matching.*

Proof. By Proposition 2.5.15, we can stack a perfect matchings of H in each horizontal fiber to form a matching, M , in the product. This matching contains $|G|\nu(H) = \frac{|G||H|}{2}$ edges. Hence, M is a perfect matching of $G \overline{\times} H$. \square

2.6 Domination

A **dominating set**, D , of G is a subset of $V(G)$ where every vertex of G is either contained in D or is adjacent to a vertex in D . The **domination number** of G , denoted $\gamma(G)$ is the cardinality of a minimum dominating set of G . A **total dominating set**, T , of G is a subset of $V(G)$ where ever vertex of G is adjacent to a vertex in T . The **total domination number** of G , denoted $\gamma_t(G)$ is the cardinality of a minimum total dominating set of G . In this section, we investigate the domination number of some semi-strong product graphs. We determine the domination number of the semi-strong product of a complete graph with a connected graph, a complete bipartite graph with a connected graph, and a cycle or a path with a connected graph. We also prove a general upper bound for the domination number of the semi-strong product.

Both domination and total domination in graph products have been extensively studied. Vizing conjectured in [87] that for all simple graphs G and H , $\gamma(G)\gamma(H) \leq \gamma(G \square H)$. Vizing's conjecture has been the subject of much research. An overview is provided in [40]. Research has also been undertaken on the domination number of specific product graph classes. Chang studied the domination number of the Cartesian product of paths in his dissertation in 1992 [23]. This is a challenging problem and was not completely solved until 2011 [33]. Some other classes of graphs that have been studied are products of cycles and trees. The domination number of Cartesian products of cycles, paths, and trees were studied by Jacobson and Kinch in [47, 48].

Domination in the direct product of paths was studied by Gravier et. al. in 1995 [25]. The domination number of the strong product of paths was found by Nowakowski and Rall [65]. For the strong product, the opposite of Vizing's conjecture holds, i.e., $\gamma(G \boxtimes H) \leq \gamma(G)\gamma(H)$

[29]. A **fractional dominating set** is an assignment of a rational number from 0 to 1 to each vertex of a graph such that the total sum over each closed neighborhood is at least 1. The minimum possible sum over all vertices of a fractional dominating set is called the **fractional domination number** and denoted $\gamma_f(G)$. In [30], it is shown that $\gamma_f(G \boxtimes H) = \gamma_f(G)\gamma_f(H)$ and that Vizing's conjecture holds for the fractional domination number of the Cartesian product, i.e., $\gamma_f(G \square H) \geq \gamma_f(G)\gamma_f(H)$.

We start our research on domination by finding the domination number of the semi-strong product of two complete graphs:

Proposition 2.6.1. $\gamma(K_n \overline{\times} K_m) = 2$ unless $n = 1$ or $m = 1$. If $n = 1$, $K_n \overline{\times} K_m = K_m$, hence $\gamma(K_n \overline{\times} K_m) = \gamma(K_m) = 1$. If $m = 1$, the graph is a collection of n isolated vertices and $\gamma = n$.

Proof. The cases where $n = 1$ or $m = 1$ were dealt with in the statement of the theorem. Suppose $n \geq 2$, $m \geq 2$. In the semistrong product of two complete graphs, every vertex is adjacent to every other vertex in the graph except those in the same vertical fiber. So we may select any vertex, v , to be in the dominating set, D . Then, v dominates every vertex in the graph except those vertices in the same vertical fiber. Since $n \geq 2$, there is at least one vertex in the same vertical fiber as v . Thus, $\gamma(K_n \overline{\times} K_m) \geq 2$. Next, select any vertex, u in the same horizontal fiber as v . Since $m \geq 2$, there is at least one additional vertex in this fiber. u dominates all vertices in the graph except those in contained in the vertical fiber of u . Which means that, in particular, u dominates those vertices in the vertical fiber containing v . Since we have shown a lower bound and a matching upper bound, $\gamma(K_n \overline{\times} K_m) = 2$ when $n \geq 2$, $m \geq 2$. \square

Figure 2.8 depicts a dominating set of $K_3 \overline{\times} K_3$. Envision the dotted edges as wrapping around behind the figure and connecting to the dotted edges in the same row or diagonal. We see that the leftmost black vertex in the product does not dominate the two red vertices above and below but does dominate every other vertex in the graph. Once we add one more vertex into the dominating set, every vertex in the product is dominated by these two vertices.

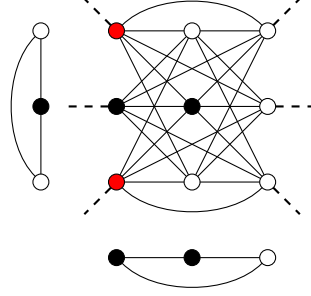


Figure 2.8: A dominating set of $K_3 \overline{\times} K_3$

Observe that the dominating sets constructed in the proof of Theorem 2.6.1 are in the same horizontal fiber. Since each horizontal fiber of $K_n \overline{\times} K_m$ is a copy of a complete graph, the two vertices in these dominating sets are adjacent, so they also form a total dominating set. A total dominating set needs at least two vertices, so these vertices must also form a minimum total dominating set of $K_n \overline{\times} K_m$ and we can conclude that $\gamma_t(K_n \overline{\times} K_m) = 2$ when $n, m \geq 2$. When $m = 1$, $K_n \overline{\times} K_m$ is a collection of isolated vertices, for which it is impossible to find a total dominating set. When $n = 1$, $K_n \overline{\times} K_m = K_m$, so in that case $\gamma_t(K_n \overline{\times} K_m) = \gamma_t(K_m) = 2$.

In general, if we can find a minimum dominating set of a graph that is also a total dominating set, then the domination number of that graph equals the total domination number:

Proposition 2.6.2. *If G is a graph with a minimum dominating set, T , such that T is also a total dominating set of G , then $\gamma_t(G) = \gamma(G)$.*

Proof. A minimum dominating set of a graph, G , is the minimum over *all* dominating sets of G . Since a total dominating set of G is also a dominating set of G , $\gamma(G) \leq \gamma_t(G)$. Suppose we can find a total dominating set, T , where $|T| = \gamma(G)$. Since T is a total dominating set, we know that $\gamma_t(G) \leq |T|$. Hence, $\gamma_t(G) \leq \gamma(G)$ and therefore, $\gamma_t(G) = |T| = \gamma(G)$. \square

Observe that the dominating set of $K_3 \overline{\times} K_3$ in Figure 2.8 is the Cartesian product of a dominating set of K_3 and a total dominating set of K_3 . In general, the Cartesian product of a dominating set of G and a total dominating set of H produces a dominating set for $G \overline{\times} H$. This provides an upper bound on the domination number of the semi-strong product of graphs:

Theorem 2.6.3. *If $|H| \geq 2$ and H contains no isolated vertices, then $\gamma(G \overline{\times} H) \leq \gamma(G)\gamma_t(H)$.*

Proof. Let A be a minimum dominating set of G and B be a minimum total dominating set of H . (Note that we need H to have no isolated vertices because a graph with isolated vertices cannot contain a total dominating set). Consider $D = A \times B \subseteq V(G \overline{\times} H)$. Suppose $(g, h) \in V(G \overline{\times} H)$, g_1 dominates g in G , and h_1 dominates h in H . If $g_1 \neq g$ and $h_1 \neq h$, $(g_1, h_1) \sim (g, h)$ by a direct product edge, so (g_1, h_1) dominates (g, h) . If $g_1 = g, h_1 \neq h$, then (g_1, h_1) is adjacent to $(g_1, h) = (g, h)$ by a Cartesian product edge and so (g_1, h_1) dominates (g, h) . If $g_1 \neq g$ and $h = h_1$, then (g_1, h_1) does not dominate (g, h) because (g_1, h_1) is in the vertical fiber above (g, h) . However, since B is a total dominating set, there is some vertex h_2 such that h_2 dominates h_1 in H . Then, (g_1, h_2) dominates $(g, h_1) = (g, h)$ via a direct product edge. These are all the possible cases, so we can see that D is indeed a dominating set of $G \overline{\times} H$. Therefore, $\gamma(G \overline{\times} H) \leq |D| = |A||B| = \gamma(G)\gamma_t(H)$. \square

Not only does the Cartesian product of a dominating set of G and a total dominating set of H form a dominating set of $G \overline{\times} H$, it also forms a total dominating set of $G \overline{\times} H$. This is because each horizontal fiber contains a total dominating set of H , so each vertex of the dominating set is dominated within its own horizontal fiber. By Proposition 2.6.2, if $\gamma(G \overline{\times} H) = \gamma(G)\gamma_t(H)$, then $\gamma_t(G) = \gamma(G)\gamma_t(H)$:

Corollary 2.6.4. *If $|H| \geq 2$, H contains no isolated vertices, and $\gamma(G \overline{\times} H) = \gamma(G)\gamma_t(H)$, then $\gamma_t(G \overline{\times} H) = \gamma(G \overline{\times} H) = \gamma(G)\gamma_t(H)$.*

We now determine the domination number of the semi-strong product of a complete graph and an arbitrary connected graph:

Theorem 2.6.5. *If H is connected, $\gamma(K_n \overline{\times} H) = \gamma_t(K_n \overline{\times} H) = \gamma_t(H)$, unless $n = 1$ or $H = K_1$. We have already dealt with the case where $H = K_1$ above. If $n = 1$, then $K_n \overline{\times} H = H$ and the domination number of the product is $\gamma(H)$.*

Proof. By Theorem 2.6.3, $\gamma(K_n \overline{\times} H) \leq \gamma(K_n)\gamma_t(H) = \gamma_t(H)$. Suppose D is a minimum dominating set of $K_n \overline{\times} H$. Since there is at least one horizontal fiber in $K_n \overline{\times} H$, any dominating set must have at least $\gamma(H)$ vertices (because every horizontal fiber is a copy of H). Since $n \geq 2$, for any $v \in D$, there exists a vertex, x , in the same vertical fiber as v , which v does not dominate. Hence, there must be some vertex $u \in D$ such that u dominates x . But then, each vertex in D has a vertex in its strong neighborhood which is dominated by at least one other vertex. Thus, there are no isolated strong neighborhoods of vertices in D , and $|D| \geq \gamma_t(H)$.

In addition, there is a total dominating set of $K_n \overline{\times} H$ with $\gamma(K_n \overline{\times} H)$ vertices, namely the set formed by placing a total dominating set of H in a single horizontal fiber. Hence, $\gamma_t(K_n \overline{\times} H) = \gamma(K_n \overline{\times} H) = \gamma_t(H)$. \square

$\gamma(K_n \overline{\times} H)$ is equal to the bound of Theorem 2.6.3, but this is not true for $G \overline{\times} K_m$. In general, it is not true that $\gamma(G \overline{\times} K_m) = \gamma(G)\gamma_t(K_m) = 2\gamma(G)$. Figure 2.9 provides a counterexample. In this figure the dotted edges represent cycles that join the leftmost vertex in each row to the rightmost vertex in each row. The domination number of P_4 is 2, so $2\gamma(P_4) = 4$, but we can dominate $P_4 \overline{\times} K_3$ with 3 vertices.

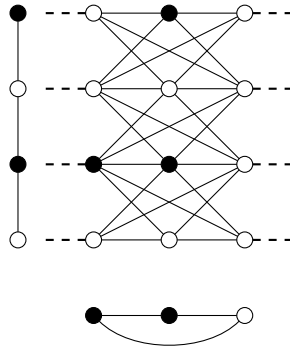


Figure 2.9: A dominating set of $P_4 \overline{\times} K_3$

In fact, this type of construction works in general. Suppose G is a graph with a minimum dominating set D . Further suppose that there is a vertex $v \in D$ such that $D \setminus \{v\}$ dominates $V(G) \setminus \{v\}$. Then, if H is a graph without isolated vertices, we can dominate $G \overline{\times} H$ with less than $\gamma(G)\gamma_t(H)$ vertices as follows: Let T be a minimum total dominating set of H . Then, by Theorem 2.6.3 we use $(D \setminus \{v\}) \times T$ to dominate all vertices of the product except for the horizontal fiber $\{v\} \times V(H)$. Since we have only a single horizontal fiber remaining, we can dominate it with $\gamma(H)$ vertices. Hence, $\gamma(G \overline{\times} H) < \gamma(G)\gamma_t(H)$. If there are multiple vertices of a minimum dominating set with this property, then we can follow the same construction and dominate $G \overline{\times} H$ with even fewer vertices.

There is also another way that we can dominate the semi-strong product with fewer than $\gamma(G)\gamma_t(H)$ vertices. Consider the graph Q in Figure 2.10. The domination number of Q is 3 and the filled vertices provide a minimum dominating set of Q . Consider $Q \overline{\times} P_3$. Since $\gamma_t(P_3) = 2$, $\gamma(Q)\gamma_t(P_3)$ yields 6 as an upper bound. However, we can dominate this product with only 4 vertices. Depicting the entire product would be messy, so I have instead decided to show only some subgraphs of the product. There are $3|Q| = 24$ vertices in all, and each

vertex occurs in exactly one of $N[x]$, $N[y]$ or $N[z]$. Observe that the black vertex $N[x] \overline{\times} P_3$ dominates all of this subgraph except for the two blue vertices and, likewise, the black vertex in $N[y] \overline{\times} P_3$ dominates all of this subgraph except the two blue vertices. Similarly, we can dominate all of the vertices in $N[z] \overline{\times} P_3$ except the blue vertex with the black vertex in the upper path. The black vertex in the lower path of $N[z] \overline{\times} P_3$ dominates all of the blue vertices. It dominates the blue vertex in its same horizontal fiber with Cartesian product edges and dominates the other four with direct product edges.

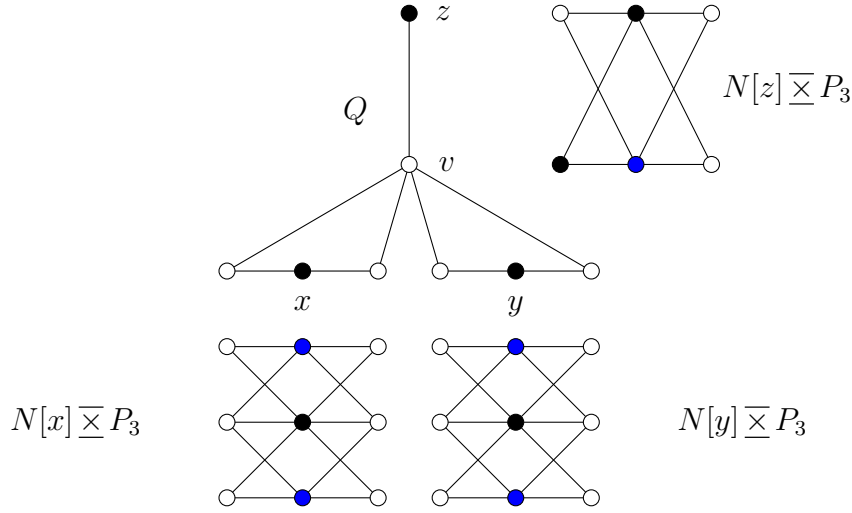


Figure 2.10: Dominating $Q \overline{\times} P_3$ with fewer than $\gamma(Q)\gamma_t(P_3)$ vertices

We can use this method in general to dominate $Q \overline{\times} H$, where H is any connected graph. In this case, we use $3\gamma(H)$ vertices for the horizontal fibers corresponding to x, y and z . Then, we use $\gamma_t(H) - \gamma(H)$ vertices in the horizontal fiber corresponding to v to dominate the remaining vertices. Altogether, we use $3\gamma(H) + \gamma_t(H) - \gamma(H) = 2\gamma(H) + \gamma_t(H)$, which is always less than or equal to $3\gamma_t(H) = \gamma(Q)\gamma_t(H)$.

After considering an upper bound for the domination number of the semi-strong product, we now consider the problem of finding a lower bound. It would seem reasonable to conjecture a Vizing-like bound for the semi-strong product, i.e., $\gamma(G \overline{\times} H) \geq \gamma(G)\gamma(H)$. When $G = nK_1$, $G \overline{\times} H$ is n copies of H , so $\gamma(G \overline{\times} H) = |G|\gamma(H) = \gamma(G)\gamma(H)$. This situation is shown in Figure 2.11.

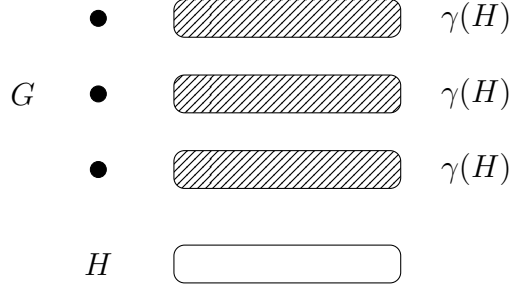


Figure 2.11: $G \overline{\times} H$ when G is nK_1

In addition, we have the following proposition:

Proposition 2.6.6. *If S is the set of horizontal fibers which contain dominating vertices in some dominating set, D , of $G \overline{\times} H$, then S must contain at least $\gamma(G)$ fibers.*

Proof. Let S be as described in the statement of the proposition. Let C be the vertices of G corresponding to horizontal fibers of S .

Claim: C must be a dominating set of G . A horizontal fiber $\{v\} \times H$ of $G \overline{\times} H$ can only be dominated by a horizontal fiber $\{w\} \times H$ where either $w = v$ or w is a vertex adjacent to v in G . Hence, if C was not a dominating set of G , then some horizontal fiber of the product would be undominated by any horizontal fiber of S . But then D would not be a dominating set of $G \overline{\times} H$, contradiction. But if C is a dominating set of G , then $|C| \geq \gamma(G)$, hence S contains at least $\gamma(G)$ horizontal fibers. \square

In 1995, Gravier and Khelladi conjectured that a Vizing bound holds for the direct product of graphs, i.e., $\gamma(G \times H) \geq \gamma(G)\gamma(H)$ in [35]. This conjecture was shown to be false by Nowakowski and Rall in [65]. Nowakowski and Rall showed that the octahedral graph, which is also $K_{2,2,2}$, provides a counterexample as $\gamma(K_{2,2,2}) = 2$, but $\gamma(K_{2,2,2} \times K_{2,2,2}) = 3$. The octahedral graph is also K_6 with a perfect matching removed and in [76], Rall states that for $n \geq 6$, the direct product of K_n with a perfect matching removed with itself falls below the Vizing bound. In [57], Klavžar and, Zmazek provide a family of graphs, G_k , such that for any $k \geq 0$, $\gamma(G_k \times G_k) \gamma(G_k) \gamma(G_k) - k$.

These results for the direct product are significant for the semi-strong product as well because $E(G \times H) \subseteq E(G \overline{\times} H)$. Dominating the semi-strong product can only be easier than dominating the direct product, so $\gamma(G \overline{\times} H) \leq \gamma(G \times H)$. Which means that the Vizing-like bound also fails for the semi-strong product and in fact all of these results hold for the semi-strong product as well. For instance, $\gamma(K_{2,2,2} \overline{\times} K_{2,2,2})$ is also equal to 3.

Now, we investigate the domination number of the semi-strong product of complete bipartite graphs. We begin with stars.

Theorem 2.6.7. *If G is a star other than K_1 and H is a connected graph with no isolated vertices, then $\gamma(G \underline{\times} H) = \gamma_t(H)$.*

Proof. $\gamma(K_{1,t}) = 1$. By Theorem 2.6.3, $\gamma(G \underline{\times} H) \leq \gamma_t(H)$. The lower bound follows the same argument as in the determination of $\gamma(K_n \underline{\times} H)$. Hence the domination number of the product is $\gamma_t(H)$. Furthermore, by Corollary 2.6.4, $\gamma_t(G \underline{\times} H) = \gamma(G \underline{\times} H) = \gamma_t(H)$. \square

Observe that the upper bound of Theorem 2.6.3 and the lower bound argument given in Theorem 2.6.7 apply to any semi-strong product $G \underline{\times} H$, where G is a graph with $|G| \geq 2$ and a vertex v such that v dominates all of G and H contains no isolated vertices. Furthermore, Corollary 2.6.4 also applies. So, we have the following theorem:

Theorem 2.6.8. *Suppose G is a graph such that $|G| \geq 2$ and G contains a vertex v such that v dominates all of G . Then, if H is a connected graph with no isolated vertices, $\gamma_t(G \underline{\times} H) = \gamma(G \underline{\times} H) = \gamma(G)\gamma_t(H) = \gamma_t(H)$.*

Now, we consider complete bipartite graphs other than stars, i.e., complete bipartite graphs of the form $K_{s,t}$ where $s, t \geq 2$. For these graphs, because drawing the full product is very messy, it is convenient to think about the graphs in a different way. The only way for one horizontal fiber to dominate another in the semi-strong product is with the direct product edges. These are determined by the adjacencies of both G (the vertical graph) and H (the horizontal graph whose edges are included as the Cartesian edges), but in particular, if there are no edges between vertices of G , then there will be no edges between the corresponding horizontal fibers of the product. So, one way to investigate products with more edges is to draw G and associate each vertex of G with a horizontal fiber of H . Then, next to each vertex of G , we can write how many dominating vertices are in the associated horizontal fiber in the product. An example is shown in Figure 2.12.

In the semi-strong product of a complete bipartite graph, $K_{s,t}$, we will denote the partite sets by A and B , with A being the set with fewer vertices. Likewise, we will use these letters to denote the corresponding two sets of horizontal fibers in the product. One set contains s copies of H and the other contains t copies of H . Technically this is an abuse of notation, but it will be clear from the context because we are only concerned with dominating the product.

The following proposition will be useful in determining the domination number of the semi-strong product of a complete bipartite graph with a connected graph:

Proposition 2.6.9. *For any graph H , $\gamma_t(H) \leq 2\gamma(H)$ and this bound is sharp.*

Proof. Suppose D is a dominating set of H . To form a total dominating set D' from D , at most, we need to dominate each vertex of D with one other vertex. Thus, D' requires no more than twice as many vertices as D . Equality is achieved for complete graphs K_n , $n \geq 2$, where $\gamma(K_n) = 1$ and $\gamma_t(K_n) = 2$. \square

Before stating any theorems, we will first discuss some general considerations about dominating the semi-strong product of the complete bipartite graph. Consider the product graph $K_{s,t} \overline{\times} H$ (where $s, t \geq 2$ and let A and B be the partite sets of horizontal fibers, as described above. There are two ways to dominate any horizontal fiber: either from within its partite set or from outside its partite set. If we dominate from within the partite set, we must dominate each horizontal fiber separately. While if we dominate from outside the partite set, we can dominate multiple horizontal fibers at once. To dominate any horizontal fiber from within requires at least $\gamma(H)$ vertices, while to dominate a horizontal fiber entirely from outside requires at least $\gamma_t(H)$ vertices.

If either A or B contains at least two horizontal fibers, then the most efficient way to dominate these horizontal fibers is from outside because $\gamma_t(H) \leq 2\gamma(H)$. It is also possible that some horizontal fibers are dominated from both within and without, but since the above inequality holds for any subset of $V(H)$, the most efficient method is still to dominate from outside.

Once we have dominated one partite set of horizontal fibers, it now remains for us to dominate the remaining partite set. Without loss of generality, let $|A| \leq |B|$. It is always more efficient to place the $\gamma_t(H)$ vertices in the smaller of the two partite sets because these vertices then dominate more horizontal fibers. Figure 2.12 provides an example using the graph $K_{2,3} \overline{\times} H$. The rectangles represent horizontal fibers. If we dominate B from A , only one more fiber remains to be dominated, while if we dominate A from B , two fibers remain to be dominated.

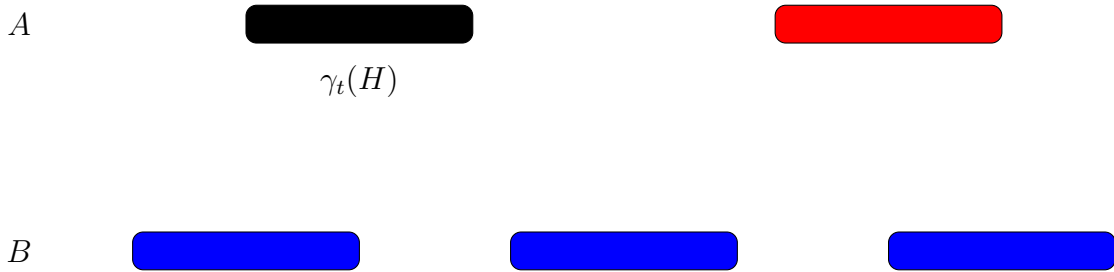


Figure 2.12: Dominating $K_{2,3} \overline{\times} H$

It is important to note that because $\gamma(H) \leq \gamma_t(H) \leq 2\gamma(H)$, the $\gamma_t(H)$ vertices in A are sufficient to dominate at least one fiber and possibly two. In Figure 2.12, the black fiber dominates the three blue fibers in B , while the red fiber in A remains undominated. Since we want a lower bound, assume that the $\gamma_t(H)$ vertices in A are sufficient to dominate two horizontal fibers. If we dominate the remaining horizontal fibers in A from within A , then we need at least $(|A| - 2)\gamma(H)$ dominating vertices. To dominate these remaining fibers from B requires $\gamma_t(H)$ vertices, so we can only *guarantee* that it is more efficient to dominate from B if $(|A| - 2)\gamma(H) \geq 2\gamma(H)$, which implies that $|A| \geq 4$. Since we assumed that $|A| \leq |B|$, the above reasoning gives the following:

Theorem 2.6.10. $\gamma(K_{s,t} \overline{\times} H) = 2\gamma_t(H)$ if $s, t \geq 4$

Proof. We have just shown that $2\gamma_t(H)$ is both a lower and upper bound for the domination number of $K_{s,t} \overline{\times} H$, so $2\gamma_t(H)$ is the domination number of the product. Furthermore, if we dominate one horizontal fiber in A and one horizontal fiber in B with a total dominating set, then our minimum dominating set is also a total dominating set, so by Proposition 2.6.2, $\gamma_t(K_{s,t} \overline{\times} H) = 2\gamma_t(H)$. \square

Theorem 2.6.11. Suppose G is a complete bipartite graph of the form $K_{2,t}$, $t \geq 2$. Then, $2\gamma(H) \leq \gamma(G \overline{\times} H) \leq \gamma_t(H) + \gamma(H)$. Furthermore, both the upper and lower bounds are sharp.

Proof. We make the assumption that $t \geq 2$ because if not, then G would be a star. In graphs of the form $K_{2,t}$, no matter which vertices we choose in a dominating set, at least two vertices must be dominated separately. In other words, unlike stars, no single vertex can dominate the entire graph. Hence, we must dominate at least two horizontal fibers separately in the product; we must place dominating vertices in at least two horizontal fibers. Since a horizontal fiber cannot be dominated with less than $\gamma(H)$ vertices, we need at least $2\gamma(H)$ vertices to dominate the product.

This lower bound is attained when we have graphs where either a minimum dominating set is also a minimum total dominating set or H is such that the union of two minimum dominating sets forms a total dominating set. An example of this scenario is $K_{2,t} \overline{\times} K_3$.

The upper bound is attained as follows. We place all dominating vertices in A . Next, we dominate one copy of H with a minimum total dominating set. This then dominates that copy of H as well as all copies of H in B . Lastly, we dominate the other copy of H in A with $\gamma(H)$ vertices. Figure 2.13 is a diagram of both situations (the lower and upper

bound), with $K_{2,2} \overline{\times} H$ where the labels to the left and right of the filled vertices show how many vertices we are using to dominate each horizontal fiber.

It is possible to dominate a product using a number of vertices between $2\gamma(H)$ and $\gamma_t(H) + \gamma(H)$. This is the case when H is such that the minimum dominating set of a *subset* of $V(H)$ is either a minimum total dominating set or is the union of two minimum total dominating sets. An example is $K_{2,2} \overline{\times} (K_3 + P_5)$. In this case $2\gamma(K_3 + P_5) = 6$ and $\gamma_t(K_3 + P_5) + \gamma(K_3 + P_5) = 8$. However, if we dominate each of the K_3 's in A with different vertices, then the union of these minimum dominating sets is a total dominating set of K_3 . Further, if we dominate one P_5 with a total dominating set (needed for the copies of P_5 in B) and the other with a minimum dominating set, then we use $1 + 3 + 1 + 2 = 7$ vertices.

If H is such that no total dominating set of H is a union of minimum total dominating sets, then we cannot spread a total dominating set among two graphs: it must be contained within a single graph. In that case, the upper bound is also a lower bound and in this case, the domination number of the product must be $\gamma_t(H) + \gamma(H)$.

□

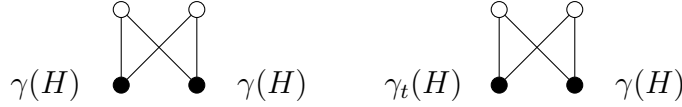


Figure 2.13: Dominating $K_{2,2} \overline{\times} H$

Theorem 2.6.12. *Suppose G is a graph of the form $K_{3,t}$, $t \geq 3$. Then, $3\gamma(H) \leq \gamma(G \overline{\times} H) \leq 2\gamma_t(H)$ and these bounds are sharp.*

Proof. By the reasoning proceeding Theorem 2.6.10, since $|B| = t \geq 3$, a minimum dominating set of the product dominates B entirely from outside. (Another way to say this is that using a minimum total dominating set contained in A to dominate every horizontal fiber in B is the most efficient “first move” in dominating the product). Since $\gamma_t(H) \leq 2\gamma(H)$, we know that at most 2 horizontal fibers are dominated in A , so we need at least $\gamma(H)$ more vertices to dominate every horizontal fiber in A . $K_{3,t} \overline{\times} K_3$ attains this bound. We choose a different vertex to dominate each copy of K_3 in A . Then, the union of these three minimum dominating sets forms a total dominating set of K_3 , so B is also dominated. In general, a sufficient condition for the lower bound occurs when a minimum dominating set of H is also a total dominating set of H or there are either two or three minimum dominating sets of H whose union is a total dominating set of H .

The upper bound occurs because the combination of a minimum total dominating set in one horizontal fiber of A and another in one horizontal fiber of B dominates all of the product. As before, it is also possible that we can dominate the product with a number of vertices between these bounds. Consider $K_{3,3} \overline{\times} (2K_3 + P_5)$. $3\gamma(2K_3 + P_5) = 3(2 + 2) = 12$. $2\gamma_t(2K_3 + P_5) = 2(4 + 3) = 14$. However, it is only necessary to dominate one of the P_5 's with a total dominating set and we can also dominate two of the K_3 's with different vertices (spreading the total dominating set across two graphs). Then, we use a total of 13 vertices. A sufficient condition for this is when some subset, S , of $V(H)$ is such that a total dominating set of S can be formed from either one or the union of two, or three minimum dominating sets of H . \square

Next, we consider the domination number of the semi-strong product of paths and cycles, i.e., $P_n \overline{\times} H$ and $C_n \overline{\times} H$. There are three cases, based on whether n is congruent to 0, 1, or 2 (mod 3).

Before beginning these theorems, consider dominating a graph G , not necessarily a product. An upper bound for the domination number of G is the order of G , $|G|$, since every vertex can dominate itself. One way to think of a minimum dominating set is as a dominating set D of G where $|G| - |D|$ is as large as possible, in other words, where we reduce the upper bound by as much as possible while still dominating G . We will use this idea in proving lower bounds for these products.

Theorem 2.6.13. *If $n \equiv 0 \pmod{3}$, then $\gamma(P_n \overline{\times} H) = \gamma(C_n \overline{\times} H) = \frac{n}{3}\gamma_t(H)$*

Proof. For the upper bound, partition the vertex set of P_n and C_n into sets of three consecutive vertices. Then, select the middle vertex of each of these sets and dominate the corresponding horizontal fibers with a minimum total dominating set. A diagram is shown in Figure 2.14. An example of the full product graph is shown in Figures 2.15 and 2.17.

Another possible upper bound is to dominate every horizontal fiber with a separate minimum dominating set, which will require $|G|\gamma(H)$ vertices. A minimum dominating set of $G \overline{\times} H$ is a dominating set of $G \overline{\times} H$, $|D|$, in which the difference between $|G|\gamma(H)$ and $|D|$ is as large as possible.

Consider all horizontal fibers of the product associated with vertices of maximum degree in G , vertices of degree $\Delta(G)$. If we can dominate any such horizontal fiber with a minimum total dominating set, then this will reduce $|G|\gamma(H)$ by the greatest amount possible. This is because we reduce $|G|\gamma(H)$ by $(\Delta(G) + 1) - \gamma_t(H)$. We could also dominate any of these horizontal fibers internally or dominate them partially inside or partially outside, but

neither option will reduce the upper bound by more than the method of using a minimum total dominating set to dominate the entire fiber neighborhood.

Furthermore, if it is possible to dominate every horizontal fiber of $G \overline{\times} H$ in this way and in such a way that each horizontal fiber is dominated by one and only one minimum total dominating set (no overlap among horizontal fiber neighborhoods), then we can be certain that this is a minimum dominating set of $G \overline{\times} H$. This is because each step of domination was the best possible and there was no overlap between steps, so at the end of this process, we have reduced $|G|\gamma(H)$ by as much as possible. But the method of domination described in the upper bound is exactly this same process.

Since there are $n/3$ middle horizontal fibers, the domination number is $\frac{n}{3}\gamma_t(H)$. Also observe that this method produces a total dominating set of $G \overline{\times} H$, so by Proposition 2.6.2 $\gamma_t(G \overline{\times} H) = \gamma(G \overline{\times} H) = \frac{n}{3}\gamma_t(H)$.

□

Theorem 2.6.14. *If $n \equiv 2 \pmod{3}$, then $\gamma(P_n \overline{\times} H) = \gamma(C_n \overline{\times} H) = \frac{n+1}{3}\gamma_t(H)$.*

Proof. The upper bound is similar to Theorem 2.6.13. In this case, we partition the vertices of P_n and C_n into as many sets of three consecutive vertices as possible, either from bottom to top or top to bottom. However, in this case we will have a set of two consecutive vertices left over. As before, we dominate the horizontal fibers associated with the sets of three consecutive vertices by dominating the middle fiber with a minimum total dominating set. Then, we dominate either of the remaining horizontal fibers with a minimum total dominating set. This will dominate both horizontal fibers and so the entire graph is now dominated. A diagram is shown in Figure 2.14, while an example of the full product is shown in Figures 2.15 and 2.17.

For the lower bound, we first dominate a sequence of horizontal fibers associated with vertices of $\Delta(G)$ with minimum total dominating sets such that no horizontal fiber is dominated more than once by a minimum total dominating set. Since each of these dominations reduces $|G|\gamma(H)$ by the largest amount possible and there is no overlap, if we can dominate whatever remains after a sequence of such dominations in the most efficient way possible, then we will have found a minimum dominating set.

In this case, what remains is $K_2 \overline{\times} H$. The most efficient way to dominate this graph (by Theorem 2.6.5) is by dominating one of the horizontal fibers with a minimum total dominating set. But now, we have used $\frac{n+1}{3}$ minimum total dominating sets, so $\gamma(G \overline{\times} H) =$

$\frac{n+1}{3}\gamma_t(H)$. As described in the upper bound, this is also a total dominating set of the product, so $\gamma_t(G \overline{\times} H) = \gamma(G \overline{\times} H)$. \square

Theorem 2.6.15. *If $n \equiv 1 \pmod{3}$, then $\gamma(P_n \overline{\times} H) = \gamma(C_n \overline{\times} H) = \frac{n-1}{3}\gamma_t(H) + \gamma(H)$*

Proof. The upper bound is found in the same way as the previous two cases. We partition the vertices of G into as many sets of three consecutive vertices as possible. In this case, we will have a single vertex left over. We then dominate the horizontal fibers corresponding to these sets of three vertices in the same way (using a minimum total dominating set). Next, we dominate the last horizontal fiber with a minimum dominating set of H , which uses $\gamma(H)$ vertices. A diagram is shown in Figure 2.14, while an example of a dominating set using the full product is shown in Figures 2.16 and 2.18.

The lower bound is found in the same way. Dominating the sets of three consecutive horizontal fibers in this way is the most efficient way to reduce $|G|\gamma(H)$ and after doing this, we are left with a single horizontal fiber. To dominate a single horizontal fiber, we cannot use less than $\gamma(H)$ vertices. Hence, we have chosen the most efficient option at each step, so the final result must also be the most efficient. We used $\frac{n-1}{3}$ minimum total dominating sets H and one minimum dominating set, so in total, we have used $\frac{n-1}{3}\gamma_t(H) + \gamma(H)$ vertices. In this case, the dominating set is not a total dominating set because the single horizontal fiber is not totally dominated, so the total domination number in this case is yet to be found. \square

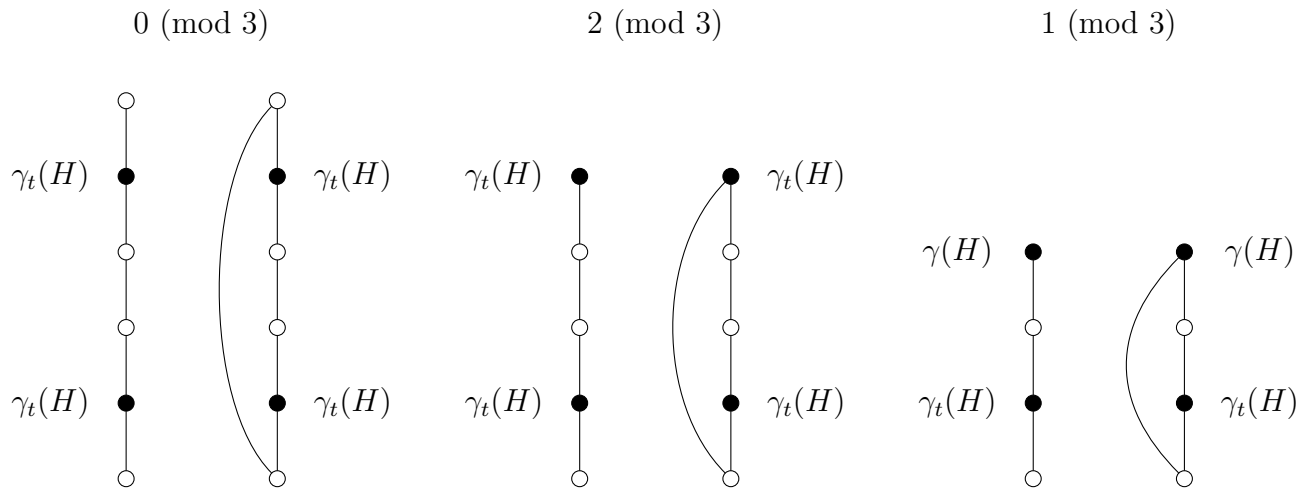


Figure 2.14: Dominating the semi-strong product of paths and cycles

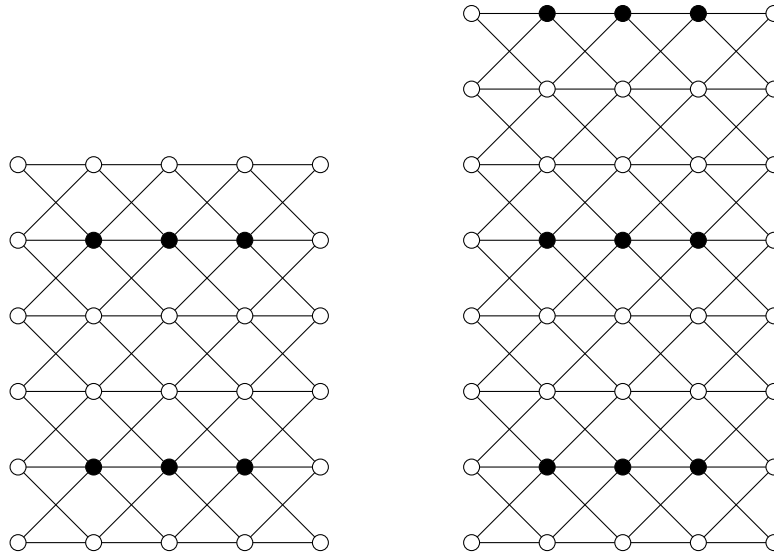


Figure 2.15: Dominating sets of $P_6 \overline{\times} P_5$ and $P_8 \overline{\times} P_5$, respectively

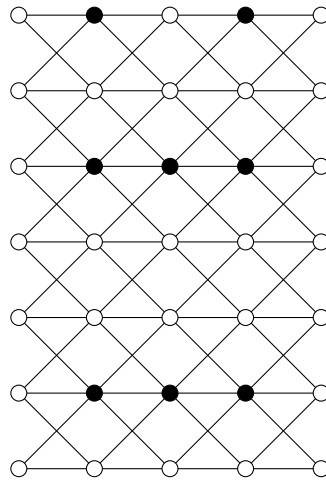


Figure 2.16: A dominating set of $P_7 \overline{\times} P_5$

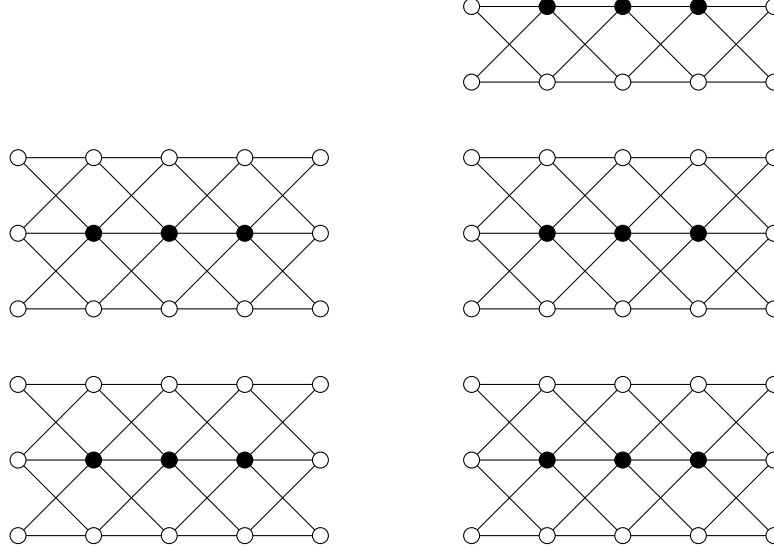


Figure 2.17: $P_6 \overline{\times} P_5$ and $P_8 \overline{\times} P_5$ with the edges connecting the subgraphs of the partition removed

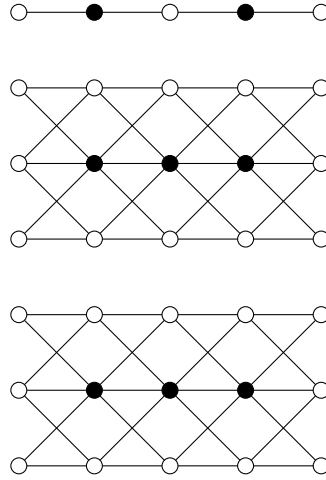


Figure 2.18: $P_7 \overline{\times} P_5$ with the edges connecting the subgraphs of the partition removed

2.6.1 Chessboard Problems and Product Domination

The above theorems on the domination number of the semi-strong product of a path with a path is related to a well-known chessboard problem. Consider an $n \times m$ chessboard. We want to determine the minimum number of kings so that every square of the chessboard either contains a king or is attacked by a king. Since a king can attack any square one space away, either vertically, horizontally, or diagonally, this is equivalent to the domination

number of the strong product of paths, i.e., $\gamma(P_n \boxtimes P_m)$. In [65], Nowakowski and Rall proved that if T is a tree, then $\gamma(T \boxtimes H) = \gamma(T)\gamma(H)$, which implies the following:

Theorem 2.6.16. $\gamma(P_n \boxtimes P_m) = \lceil n/3 \rceil \lceil m/3 \rceil$

Proof. For the upper bound, we partition the horizontal fibers (from top to bottom or bottom to top) into as many copies of $P_3 \boxtimes P_m$ as possible. Then, dominate the middle horizontal fiber of each path by selecting the center vertex of every three consecutive vertices. If $m \equiv 0 \pmod{3}$, nothing will be left over, but if $n \equiv 1$ or $2 \pmod{3}$, then select any of the remaining two or one vertices as a dominating vertex. An example of this, dominating $P_7 \boxtimes P_5$ and an example of the partitions is shown in Figure 2.19.

If $n \equiv 0 \pmod{3}$, then the entire product can be partitioned into copies of $P_3 \boxtimes P_m$. If not, then what remains is either $P_2 \boxtimes P_m$ or P_m . If $P_2 \boxtimes P_m$, then dominate either the upper or lower path in the same way as the middle paths of the $P_3 \boxtimes P_m$. If P_m , then dominate with a minimum dominating set of P_m .

For the lower bound, note that to dominate the product, we must cover a distance of n vertically and m horizontally. Since each dominating vertex in the product can cover at most distance 3 either horizontally or vertically, we need at least $\lceil n/3 \rceil$ vertices to cover the vertical distance ($n/3$ for every set of 3 consecutive vertices and the ceiling is for any left over if n is not divisible by 3). Likewise, we need at least $\lceil m/3 \rceil$ to cover the horizontal distance. Since we need to cover both at the same time, we need at least the product of these two numbers. But this is how many we used in the upper bound, hence the domination number is in fact, $\lceil n/3 \rceil \lceil m/3 \rceil$.

□

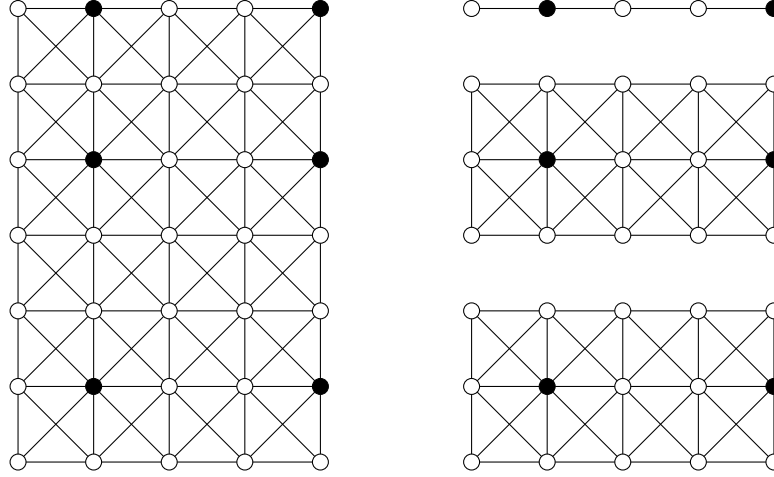


Figure 2.19: A minimum dominating set of $P_7 \boxtimes P_5$

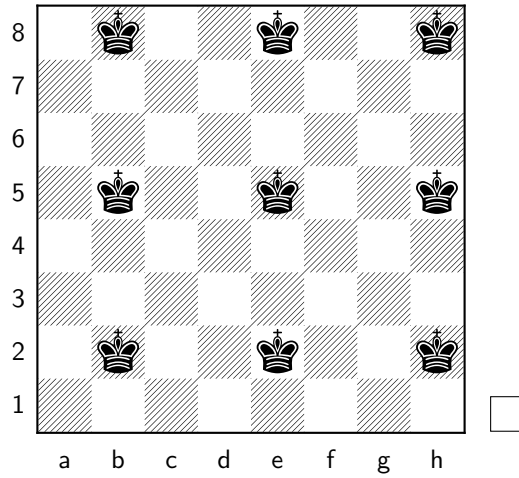


Figure 2.20: A minimum king dominating set of an 8×8 chessboard

Figure 2.20 shows an example of a minimum number of kings placed on a regular 8×8 chessboard. Comparing Figures 2.19 and 2.20, we can see that the strong product of paths does indeed correspond to the King's graph, the graph corresponding to the $n \times m$ chessboard where two vertices u, v are adjacent if a king on u can attack v or vice versa.

Indeed, the above reasoning about the strong product also applies to the k -fold strong product of paths, which would correspond to a k -dimensional chessboard where a king can attack any square which is a unit distance away in any of the k -dimensions.

Theorem 2.6.17. $\gamma(P_{n_1} \boxtimes P_{n_2} \boxtimes \dots \boxtimes P_{n_k}) = \lceil n_1/3 \rceil \lceil n_2/3 \rceil \dots \lceil n_k/3 \rceil$

Proof. The upper bound is the same. For the 3 dimensional case, imagine n_3 copies of $P_{n_1} \boxtimes P_{n_2}$ connected according to the strong product edges. Then, dominate the middle copy of every set of consecutive $P_{n_1} \boxtimes P_{n_2}$'s as determined in Theorem 2.6.16 and this will dominate the two on either side. If we continue this process, we can see by induction, that this method will dominate the k -fold product.

For the lower bound, we need to cover n_d distance in every dimension d . The most a vertex can cover in any dimension is $n_d/3$ and we need to cover them all at the same time, so we need at least $\lceil n_1/3 \rceil \lceil n_2/3 \rceil \dots \lceil n_k/3 \rceil$ vertices. The upper and lower bounds match, so this is the domination number. \square

The connection between chess and the semi-strong product is because we can envision the semi-strong product of paths as a chessboard problem. Consider a piece that can attack every square a king can attack except for the square directly above and below. We might call such a piece a “prince.” As with the king, we can ask, what is the minimum number of princes needed so that every square of an $n \times m$ chessboard either contains or is attacked by a prince. This problem is equivalent to determining the domination number of the semi-strong product of paths. The corresponding problem for two cycles is equivalent to determining the number of princes needed to dominate an $n \times m$ chessboard imbedded on a torus.

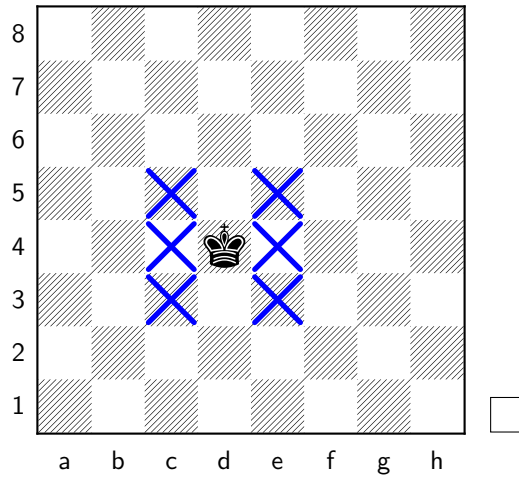


Figure 2.21: The squares dominated by a “prince” on a regular chessboard

Using the above theorems about the semi-strong product of paths with any graph H , we can find formulas in terms of the domination and total domination numbers of P_n and P_m :

Theorem 2.6.18. *If $n \equiv 0, 2 \pmod{3}$, then $\gamma(P_n \boxtimes P_m) = \gamma(P_n)\gamma_t(P_m)$.*

Theorem 2.6.19. *If $n \equiv 1 \pmod{3}$, then $\gamma(P_n \overline{\times} P_m) = \gamma(P_{n-1})\gamma_t(P_m) + \gamma(P_m)$.*

Chapter 3

Bootstrap Percolation

A natural research problem in bootstrap percolation is finding the cardinality of a minimum r -percolating set for a particular class of graphs. The cardinality of a minimum percolating set for a graph G is denoted $m(G, r)$. In this chapter, we find a sufficient condition for when $m(G, r) = r$ for a specific class of Kneser graphs and bipartite Kneser graphs. We begin by giving some background on earlier research into minimum percolating sets.

One class of graphs which has been actively studied is the d -dimensional square lattice, denoted $[n]^d$. This graph is a product graph: it is the d^{th} Cartesian power of P_n , i.e., $\square_{i=1}^d P_{n,i}$. Balogh and Pete sketch a proof that $m([n]^d, d) = n^{d-1}$ [14]. Przykucki and Shelton provide a more detailed proof of this result [75]. Balogh, Bollobás, and Morris proved that $m([n]^d, 2) = \left\lceil \frac{d(n-1)}{2} \right\rceil$ [12]. Morrison and Noel investigated the minimum size of a percolating set in the hypercube, demonstrating that $m([2]^d, 3) = \left\lceil \frac{d(d+3)}{6} \right\rceil + 1$ [64]. They also proved that if r is fixed and $d \rightarrow \infty$, then $m([2]^d, r) = \frac{1+o(1)}{r} \binom{d}{r-1}$. Hambardzumyan, Hatami, and Qian gave a simplified proof of this result [39].

Coja-Oghlan, Feige, and Krivelevich investigated $m(G, r)$ for d -regular graphs in terms of girth and the second-largest eigenvalue [27]. Riedl established bounds for percolating sets in trees while Beeler, Keaton, and Norwood extended this line of investigation [16, 77].

Coelho et. al. investigated $m(G, 2)$ for the lexicographic, Cartesian, and strong products [26]. Coelho et. al. used different terminology: they framed this problem in terms of of the P_3 hull number. When $S \subseteq V(G)$, the P_3 -interval $I[S]$ is S along with every vertex of G which contains at least 2 neighbors in S . We can recursively define $I^p[S]$ to be $I^{p-1}[S]$ together with every vertex outside $I^{p-1}[S]$ with at least two neighbors in $I^{p-1}[S]$. The P_3 -convex hull of S , denoted $H(S)$, is the set formed by $I^p[S]$ after $I^p[S]$ stops growing, i.e., when $I^p[S] = I^{p+1}[S]$. If $H(S) = V(G)$, then S is a P_3 -hull set of G . The number of vertices

in a minimum S for which $H(S) = V(G)$ is the P_3 -hull number of G , denoted $h_{P_3}(G)$.

From these definitions, we can see that the process of selecting S and finding $H[S]$ is the same as 2-bootstrap percolation on G . Hence, $h_{P_3}(G) = m(G, 2)$. Grippo et. al. study the P_3 -hull number of Kneser graphs and Bresar and Valencia-Pabon study the P_3 -hull number of Hamming graphs (i.e., the Cartesian products of complete graphs) in and respectively [36, 18]. Hedzetz and Henning and Hedzetz and Bresar have also studied bootstrap percolation in graph products [41, 17].

Interestingly enough, the semi-strong product even makes an appearance in bootstrap percolation. In [46], it is shown that when the cardinality of an initial percolating set is less than $2r$, the number of rounds to percolation is at least $\lceil d/2 \rceil + 1$. A graph which attains this bound is $K_r \times P_n$. These graphs also have the minimum number of edges among graphs with percolating sets of size r .

The vertex set of the Kneser graph, denoted $K(n, k)$, is all cardinality k subsets of the set with n elements, denoted $[n]$. We will use the notation $\binom{[m]}{k}$ to refer to the set formed by all cardinality k subsets of a set of m elements. Vertices are adjacent if their respective subsets are disjoint. We are primarily concerned with Kneser graphs when $n \geq 2k + 1$. When $n = 2k$, the Kneser graph is k copies of K_2 . When $n < 2k$, $K(n, k)$ is a collection of isolated vertices. Two special classes of Kneser graphs are those when $k = 1$ and when $n = 2k + 1$. Then Kneser graph $K(n, 1)$ is just the complete graph K_n . The graph $K(2k + 1, k)$ is also called the odd graph and denoted O_k .

The author of this dissertation found that $m(K(n, k), 2) = 2$ exactly when $n = 2k + 3$ or for $K(6, 2)$ and $K(3, 1)$. Otherwise, a minimum 2-percolating set is larger than 2 [21]. Independently, Grippo et. al. determined that $m(K(2k + 3), 2) = 2$ and $m(K(2k + 2), 2) = 3$ when $k \geq 3$ [36]. For $K(6, 2)$ a minimum percolating set contains 2 vertices. They also found an upper bound for $m(K(2k + 1, k))$. The Kneser graph is disconnected for $n \leq 2k$ and since percolation cannot occur between components, the above graphs are the cases of interest.

In this dissertation, we continue this investigation. We determine a sufficient condition for when $m(K(n, k), r) = r$. This is also a sufficient condition for when the bipartite Kneser graph $H(n, k)$ contains an r -percolating set of cardinality r . In the bipartite Kneser graph, each set of the bipartition is a copy of the vertex set of $K(n, k)$. Vertices (subsets) are adjacent to disjoint vertices (subsets) in the opposite partite set. In fact, $H(n, k)$ is a product graph, namely $K(n, k) \times K_2$. Figure 3.1 shows the Kneser graph $K(5, 2)$, which is also the Petersen graph.

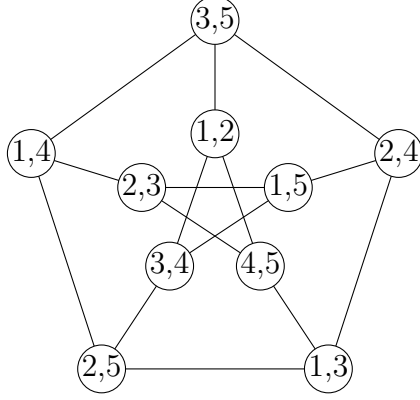


Figure 3.1: The Petersen graph, $K(5, 2)$

Let μ be the minimum number of elements in the union of r subsets of $[n]$ of cardinality k . μ is the smallest natural number such that $\binom{\mu}{k} \geq r$. For example, if $r = 2, k \geq 2$, then $\mu = k + 1$. If $k = 3$ and $r = 4$ then $\mu = 4$ because $\binom{4}{3} = 4$. But if we increase r to 5, then we need at least one additional element in our union and $\mu = 5$.

Theorem 3.0.1. *Let μ be as described. Then, $m(K(2\mu + 1, k), r) = r$.*

Proof. Let A_0 be an initial set of infected vertices of cardinality r . Select A_0 such that a total of μ elements are used among all r vertices, i.e., $|\bigcup_{i=1}^r (v_i)| = \mu$ for $v_i \in A_0$. Since μ elements of $[n]$ are used in A_0 , we know that $\binom{\mu-1}{k} < r \leq \binom{\mu}{k}$.

For $v_i \in A_0$, let $S = \bigcup_{i=1}^r (v_i)$, i.e., the set of μ elements used by A_0 . Let $T = [n] \setminus S$. Partition $V(K(2\mu + 1, k))$ as follows: Let S_i be the set of vertices which contain i elements from S and $k - i$ elements from T . Then, $V(K(2\mu + 1, k)) = S_k, S_{k-1}, \dots, S_1, S_0$. Observe that $A_0 \subseteq S_k$.

Since vertices of $K(n, k)$ are adjacent when they are disjoint, the set of vertices adjacent to A_0 is the vertices which use no elements of S , i.e., S_0 . So, A_0 infects S_0 . Since $|S_0| = \binom{\mu+1}{k} > \binom{\mu}{k} \geq r$, S_0 contains at least r vertices, so S_0 then infects the rest of S_k (if A_0 is a proper subset of S_k).

If v be a vertex of S_{k-1} then v is adjacent to those vertices in S_0 which do not use the single element of T contained by v . Hence, v is adjacent to $\binom{\mu+1-1}{k} = \binom{\mu}{k} \geq r$ vertices in S_0 . Hence, S_0 infects S_{k-1} . We now prove the following:

Claim 1: If $v \in S_i$ and $i \leq k/2$, then v is adjacent to at least r vertices in S_{k-i} .

To form a subset disjoint from v in S_{k-i} , we select $k - i$ elements from S , excluding the i elements used by v and i elements from T , excluding the $k - i$ elements of T used by v . Thus,

v is adjacent to $\binom{\mu-i}{k-i}\binom{\mu-k+i+1}{i}$ vertices in S_{k-i} . If we can show that $\binom{\mu}{k} \leq \binom{\mu-i}{k-i}\binom{\mu-k+i+1}{i}$, this would imply that v is adjacent to at least r vertices in S_{k-i} . Since $\binom{\mu-k+i}{i} \leq \binom{\mu-k+i+1}{1}$, it suffices to show that $\binom{\mu}{k} \leq \binom{\mu-i}{k-i}\binom{\mu-k+i}{i}$.

We now consider this product of binomial coefficients in another way: $\binom{\mu-i}{k-i}$ counts the subsets of $\binom{[\mu]}{k}$ which contain some particular i elements of $[\mu]$. We first select i elements, which we can then combine with any $k-i$ of the remaining elements of $[\mu]$. Likewise, $\binom{\mu-k+i}{i} = \binom{\mu-k+i}{k-(k-i)}$ counts the subsets of $\binom{[\mu]}{k}$ which contain some particular $k-i$ elements of $[\mu]$. We begin by choosing our $k-i$ elements, which we can combine with any $k-(k-i) = i$ of the remaining elements of $[\mu]$.

Fix $[i] \subseteq [\mu]$ and $[k-i] \subseteq [\mu]$ such that $[i] \cap [k-i] = \emptyset$ and let P be the set formed by concatenating each element of $\binom{[\mu-i]}{k-i}$ with each element of $\binom{[\mu-k+i]}{i}$. Some sample elements of P when $k = 4, n = 8$ and $[i] = \{1, 2\}, [k-i] = \{3, 4\}$ are shown in Figure 3.2. We show that $\binom{\mu}{k} \leq \binom{\mu-i}{k-i}\binom{\mu-k+i}{i}$ by finding an injection from $\binom{[\mu]}{k}$ to P . The elements of P contain $2k$ elements of $[\mu]$, k fixed elements (namely, $[i] \cup [k-i]$) and k elements which vary. For each $x \in P$, denote the set of elements which are fixed with a and the set of elements which vary with b , so x is formed from both a and b . In some elements of P , b includes repetitions. Consider $Q \subseteq P$ where for each $x \in Q$, b does not include repetitions. Since b has no repetitions, b forms an element of $\binom{[\mu]}{k}$.

Since each b is a distinct element of $\binom{[\mu]}{k}$, we can map each $y \in \binom{[\mu]}{k}$ to a single $p \in P$ such that b of p is y . There may be more than one such p (see, for example, the bottom two entries of Figure 3.2), but no two y 's can be mapped to the same p . Therefore, we have an injective function from $\binom{[\mu]}{k}$ to P and $\binom{\mu}{k} \leq \binom{\mu-i}{k-i}\binom{\mu-k+i}{i}$.

Claim 2: If $v \in S_{k-i-1}$ then v is adjacent to at least r vertices in S_i .

By similar reasoning to Claim 1, if $v \in S_{k-i-1}$, then v is adjacent to $\binom{\mu-k+i+1}{i}\binom{\mu+1-(i+1)}{k-i} = \binom{\mu-k+i+1}{i}\binom{\mu-i}{k-i}$ vertices in S_i . We already showed in Claim 1 that $\binom{\mu}{k} \leq \binom{\mu-i}{k-i}\binom{\mu-k+i}{i} \leq \binom{\mu-i}{k-i}\binom{\mu-k+i+1}{i}$, so Claim 2 holds.

Using Claim 1 and Claim 2, we can infect the rest of the graph. First, by Claim 1, S_{k-1} infects S_1 . Then, we can show by induction that all S_j 's become infected by showing that for any remaining i , we can infect S_{i+1} . By Claim 2, S_i infects S_{k-i-1} . Then, by Claim 1, S_{k-i-1} infects $S_{k-(k-i-1)} = S_{i+1}$. Hence, our initial set of infected vertices is a percolating set. \square

$$\begin{aligned}
& \{(1, 2)(5, 6)\} \{(1, 2)(3, 4)\} \\
& \{(1, 2)(5, 6)\} \{(2, 5)(3, 4)\} \\
& \{(1, 2)(3, 5)\} \{(2, 6)(3, 4)\} \\
& \{(1, 2)(3, 5)\} \{(5, 6)(3, 4)\} \\
& \{(1, 2)(7, 8)\} \{(5, 6)(3, 4)\} \\
& \{(1, 2)(5, 8)\} \{(6, 7)(3, 4)\}
\end{aligned}$$

Figure 3.2: Elements of P when $k = 4, n = 8$ and $[i] = \{1, 2\}, [k - i] = \{3, 4\}$

Figure 3.3 shows a 2-percolating set in $K(7, 2)$. We select vertices $(1, 2)$ and $(2, 3)$ in the initial round. In round 2, these vertices infect all vertices of S_0 , i.e., the vertices containing 4, 5, 6, or 7. In Round 3, $(1, 3)$, the remaining vertex of S_2 is infected by vertices of S_0 . In addition, all the vertices of S_1 are infected by vertices of S_0 . This is shown on the right hand side of Figure 3.3.

Figure 3.4 provides another example of a percolating set infecting a Kneser graph where $n \geq 2\mu + 1$. As in the previous example, our percolation threshold is 2. $K(11, 4)$ has 330 vertices, so rather than depict the entire graph, we use a diagram. Since $k = 4$, we have 5 sets in our vertex partition: S_4, S_3, S_2, S_1, S_0 . Our initial infected vertices are $(1, 2, 3, 4)$ and $(2, 3, 4, 5)$. These infect all of S_0 . In round 2, S_0 infects the remaining vertices of S_4 as well as infecting S_3 . S_3 infects both S_1 and S_2 .

We now discuss minimum percolating sets in the bipartite Kneser graph, $H(n, k)$. In general, the size of a minimum percolating set of the bipartite Kneser graph, $H(n, k)$ is no more than twice the size of a minimum percolating set of $K(n, k)$.

Proposition 3.0.2. $m(H(n, k), r) \leq 2m(K(n, k), r)$

Proof. Suppose A_0 is a minimum r -percolating set of $K(n, k)$. Infect a copy of A_0 in each of the partite sets of $H(n, k)$ and refer to these two sets as A_0^1 and A_0^2 . Recall that A_t is the set of vertices infected after t rounds. A_0^2 together with the vertices newly infected by A_0^1 forms A_1^2 , a copy of A_1 in the second partite set of $H(n, k)$. Likewise, A_0^1 together with the vertices newly infected by A_0^2 forms A_1^1 . At each round t , A_t^1 infects A_{t+1}^2 and A_t^2 infects A_{t+1}^1 . Since A_0 is a percolating set of $K(n, k)$, we eventually infect two copies of $V(K(n, k))$. Hence, $A_0 \times K_2$ is a percolating set of $H(n, k)$. \square

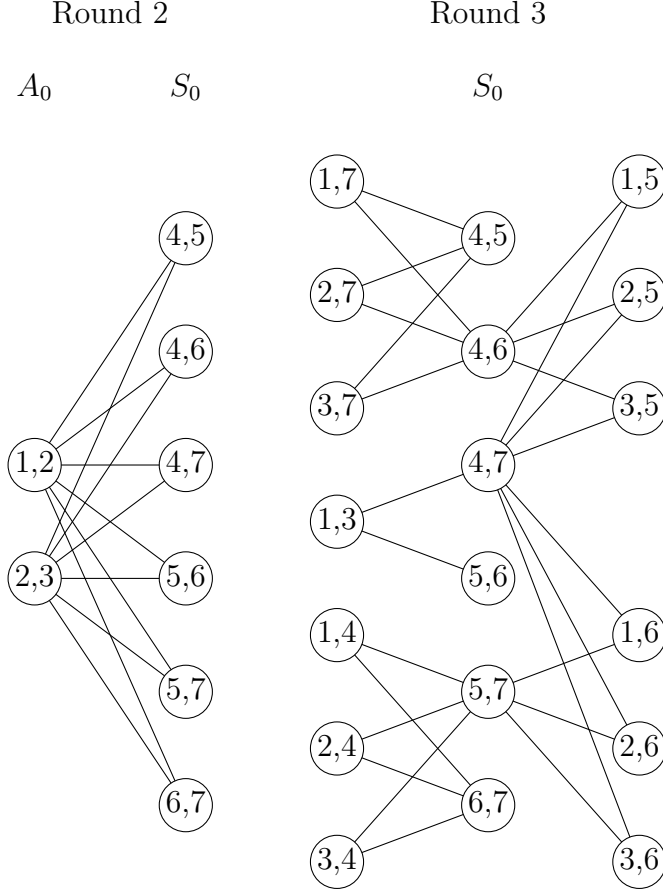


Figure 3.3: Infecting $K(7, 2)$ with a 2-percolating set

In at least one case, this is indeed the size of a minimum percolating set of the bipartite Kneser graph. Grippo et. al. report that for the Petersen graph, $m(K(5, 2), 2) = 3$ and $m(H(5, 2), 2) = 6$ [36]. They conjecture that for the odd graph,

$$\left\lceil \frac{m(H(2k+1, k), 2)}{2} \right\rceil = m(K(2k+1, k), r)$$

However, when $n = 2\mu + 1$, we can do much better. In this instance, $m(H(n, k), r) = r$.

Theorem 3.0.3. $m(H(2\mu + 1, k), r) = r$

Proof. Observe that in the proof of Theorem 3.0.1, the infection across the S_i 's proceeds as follows: $S_k, S_0, S_{k-1}, S_1, S_{k-2}, S_2, S_{k-3}, S_3, \dots$

For each S_i we alternate by infecting S_{k-i} and then S_{k-i-1} . This yields a sequence of S_i 's, where those S_i 's with $i > k/2$ alternate with S_i 's with $i < k/2$. If k is even, then we

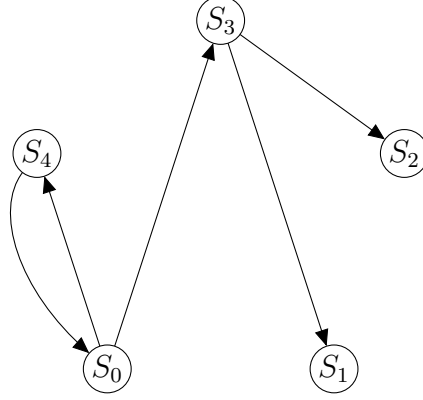


Figure 3.4: Infecting $K(11, 4)$ with a 2-percolating set

have an odd number of S_i 's in total and the sequence ends with $S_{\frac{k+1}{2}}, S_{\frac{k-1}{2}}$. And if k is odd, then the sequence ends with $S_{\frac{k}{2}+1}, S_{\frac{k}{2}-1}, S_{\frac{k}{2}}$. The S_i 's where $i > k/2$ decrease by one, while the S_i 's where $i < k/2$ increase by one.

Using this observation, as well as Claim 1 and Claim 2 of the proof of Theorem 3.0.1, we can show that $m(H(2\mu + 1, k), r) = r$. Suppose $A_0 \subseteq S_k$ is a percolating set containing r vertices of $K(2\mu + 1, k)$, which uses μ elements in total. Let S_i^1 and S_i^2 be as described in the proof of Proposition 3.0.2. Select $A_0^1 \subseteq S_k^1$ to be an initial set of infected vertices of $H(2\mu + 1, k)$. A_0^1 infects S_0^2 , which then infects the rest of S_k^1 as well as S_{k-1}^1 . The infection proceeds as follows: $S_1^2, S_{k-2}^1, S_2^2, \dots$. If k is odd, then we will eventually reach $S_{\frac{k-1}{2}}^2$. But $k - \frac{k-1}{2} - 1 = \frac{k+1}{2} - 1 = \frac{k-1}{2}$, so by Claim 2, $S_{\frac{k-1}{2}}^2$ can infect its counterpart $S_{\frac{k-1}{2}}^1$.

We continue to alternately infect S_{k-i} and S_{k-i-1} as the infection goes backwards through the S_i 's with $S_{\frac{k+1}{2}}^2$ next infected and ending at S_k^2 . If k is even, then the infection is slightly different. The midpoint is $S_{\frac{k}{2}}^1$. But since $k - \frac{k}{2} = \frac{k}{2}$, by Claim 1, $S_{\frac{k}{2}}^1$ infects its counterpart $S_{\frac{k}{2}}^2$ and the infection proceeds through the remaining uninfected S_i^j 's. □

For an example of the process in the proof of Theorem 3.0.3, let $k = 5$ and let $r = 7$. Then, $\mu = 7$. $H(15, 5)$ is infected as follows: $S_5^1, S_0^2, S_4^1, S_1^2, S_3^1, S_2^2, S_2^1, S_3^2, S_1^1, S_4^2, S_0^1, S_5^2$.

We only proved Theorems 3.0.1 and 3.0.3 for $n = 2\mu + 1$, but in fact, these results apply for all $n \geq 2\mu + 1$ because Claim 1 and Claim 2 only use the fact that n is at least $2\mu + 1$. However, we can say something more general:

Lemma 3.0.4. *If $K(n, k)$ contains an r -percolating set of cardinality p , then $K(n + 1, k)$ also contains an r -percolating set of cardinality p .*

Proof. Suppose that $K(n, k)$ contains an r -percolating set of cardinality p and let A_0 be such a percolating set. $K(n + 1, k)$ contains $K(n, k)$ as a subgraph, because every cardinality k subset of $[n]$ is also a subset of $[n + 1]$. Since $A_0 \subseteq V(K(n + 1, k))$, select A_0 to be an initial set of infected vertices for $K(n + 1, k)$.

When A_0 percolates in $K(n, k)$, at every stage A_t infects A_{t+1} . In other words, the vertices in A_{t+1} are adjacent to at least r vertices in A_t , i.e., disjoint from at least r vertices in A_t . If we swap out any element of these vertices for $n + 1$, then they will still be disjoint from r vertices of $K(n, k)$. So, at each stage of the infection process in $K(n + 1, k)$, some vertices containing $n + 1$ are infected along with vertices only containing elements of $[n]$. Eventually all such vertices are infected because all vertices of $K(n, k)$ are infected and all vertices containing $n + 1$ are formed from swapping out $n + 1$ with an element of $[n]$. □

We now have the following corollary:

Corollary 3.0.5. *If $n \geq 2\mu + 1$, then $m(K(n, k), r) = m(H(n, k), r) = r$.*

Although $n \geq 2\mu + 1$ is sufficient to guarantee a percolating set of size r in $K(n, k)$ and $H(n, k)$, the condition is not necessary. For example, when $k = 2$ and $r = 2$, $\mu = 3$, but $(1, 2), (1, 3)$ 2-percolates in $K(6, 2)$. From looking at examples, it appears that n can fall arbitrarily far below $2\mu + 1$ and $K(n, k)$ can still contain a percolating set of size r . In general, even determining when $m(K(n, k), r) = r$ appears challenging.

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