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SYMMETRY ANALYSIS OF THE CANONICAL CONNECTION ON LIE GROUPS:CO-DIMENSION TWO ABELIAN NILRADICAL WITH ABELIAN AND NON ABELIAN COMPLEMENT

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SYMMETRY ANALYSIS OF THE CANONICAL CONNECTION ON LIE GROUPS:CO-DIMENSION TWO ABELIAN NILRADICAL WITH ABELIAN AND NON ABELIAN COMPLEMENT

A Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Virginia Commonwealth University.

by

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Abstract

SYMMETRY ANALYSIS OF THE CANONICAL CONNECTION ON LIE GROUPS:CO-DIMENSION TWO ABELIAN NILRADICAL WITH ABELIAN AND NON ABELIAN COMPLEMENT

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We consider the symmetry algebra of the geodesic equations of the canonical connection on a Lie groups. We mainly consider the solvable indecomposable four, five and six-dimensional Lie algebras with co-dimension two abelian nilradical, that have an abelian and not abelian complement. In this particular case, we have only one algebra in dimension four namely; $A_{4,12}$, and three algebras in dimension five namely; $A_{5,33}$, $A_{5,34}$, and $A_{5,35}$ In dimension six, based on the list of Lie algebras in Turkowski's list, there are nineteen such algebras namely; $A_{6,1}$- $A_{6,19}$ that have an abelian complement, and there are eight algebras that have a non-abelian complement namely; $A_{6,20}$- $A_{6,27}$. For each algebra, we give the geodesic equations, a basis for the symmetry Lie algebra in terms of vector fields. Finally we examine each case and identify the symmetry Lie algebra.

CHAPTER 1

INTRODUCTION

1.1 Symmetry Analysis of Ordinary Differential Equations

In the latter part of the 19th century, Sophus Lie introduced the concept of continuous groups, today recognized as Lie groups, aiming to unify and expand the variety of specialized methods available for solving ordinary differential equations (ODEs). Inspired by Sylow's lectures in Christiania (now Oslo) on Galois theory and the related works of Abel, Lie embarked on a journey that would profoundly influence the mathematical landscape. The collaboration between Sylow and Lie in 1881, which focused on the meticulous editing of Abel's complete works, set a foundational stone for Lie's groundbreaking contributions.

Lie's innovative approach demonstrated that the order of an ODE could be constructively reduced by one if the equation remained invariant under a one-parameter Lie group of point transformations. This principle systematically connected disparate areas within the study of ODEs, encompassing integrating factors, separable equations, homogeneous equations, order reduction techniques, the methods of undetermined coefficients, variation of parameters for linear equations, solutions to Euler equations, and the application of Laplace transforms.

In 1881, Lie also highlighted how, for linear partial differential equations (PDEs), invariance under a Lie group directly facilitates the superposition of solutions through transforms. A symmetry in this context is defined as a transformation that map any given solution of a differential equation system into another valid solution. Lie's framework conceptualized these transformations as groups reliant on continuous parameters, comprising either point transformations (point symmetries), which act on the variables of the system, or contact transformations (contact symmetries), which additionally influence the first derivatives of dependent variables. Elementary instances of Lie groups include translations, rotations, and scalings. Notably, an autonomous system of first-order ODEs inherently delineates a one-parameter Lie group of point transformations.

Lie elucidated that for any differential equation, whether linear or nonlinear, the corresponding continuous group of point transformations, affecting its space of independent and dependent variables, can be explicitly identified through a computational process known as Lie's algorithm. This profound insight not only bridged various techniques in solving ODEs but also laid the groundwork for the modern application of symmetry analysis in differential equations, revealing the intrinsic connections between algebraic structures and analytical methods [28]. We refer the reader to [3, 4, 5, 6]. Another very accessible reference is [7].

1.2 Symmetry of Ordinary Differential Equations

A symmetry is a transformation that leaves an object unchanged or "invariant" [2]. For example, the rotation of a disk through an angle ϵ. Consider the points (x, y) and $(\bar{x}, \bar{y})$, on the circumference of a circle of radius r. We can write these in terms of the radius r and the angles θ (a reference angle), that is,

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{1.2.1}$$

Also we can write these in terms of the radius r and the angles $\theta + \epsilon$ (after rotation), that is, These then become

$$\bar{x} = r\cos(\theta + \epsilon), \quad \bar{y} = r\sin(\theta + \epsilon). \tag{1.2.2}$$

Eliminating θ

$$\bar{x} = x\cos(\epsilon) - y\sin(\epsilon), \quad \bar{y} = y\cos(\epsilon) + x\sin(\epsilon). \tag{1.2.3}$$

To show invariance of the circle under 1.2.3 is to show that

$$\bar{x}^2 + \bar{y}^2 = r^2. \tag{1.2.4}$$

if

$$x^2 + y^2 = r^2. \tag{1.2.5}$$

Therefore,

$$
\begin{aligned}
\bar{x}^2 + \bar{y}^2 &= (x\cos(\epsilon) - y\sin(\epsilon))^2 + (y\cos(\epsilon) + x\sin(\epsilon))^2 \\
&= x^2\cos^2(\epsilon) - 2xy\sin(\epsilon)\cos(\epsilon) + y^2\cos^2(\epsilon) \\
&+ x^2\sin^2(\epsilon) + 2xy\sin(\epsilon)\cos(\epsilon) + y^2\cos^2(\epsilon) \\
&= x^2 + y^2 \\
&= r^2.
\end{aligned}
$$

A symmetry of a differential equation is a transformation that maps all of the differential equation's solutions to the same equation's solution. The transformation is the standard change of variables in differential equations that does not alter the order. In the 1880s Sophus Lie [8] was able to show that a majority of the techniques used to integrate ODEs could be explained by a theory known as Lie group analysis, where the symmetries of a differential equation could be found and exploited. He demonstrated that a differential equation's symmetries form a group. We can compute the symmetries of DEs that are continuously dependent on a one-parameter group using the Lie approach, that's us heading to review the concept of one-parameter Lie groups [2].

1.3 Lie Groups

This section aims to present a brief overview of Lie groups.

Definition 1.3.1. Consider the set of transformations

$$G = \{\bar{x}_i = f_i(x_j, \epsilon) | i, j = 1, 2, .., n\}. \tag{1.3.1}$$

First G is a group under the composition operation if satisfied the following:

1. G is closure, for all $a, b \in G$, then $\phi(a, b) \in G$.
2. G is Associative, for any $a, b, c \in G$, then $\phi(a, \phi(b, c)) = \phi(\phi(a, b), c)$.
3. Identity, for $a \in G$, there exists an $e \in G$, $s.t$ $\phi(a, e) = \phi(e, a) = a$.
4. Inverse. If $a \in G$, then there exists a unique element $a^{-1} \in G$, $s.t$ $\phi(a, a^{-1}) = \phi(a^{-1}, a) = e$.

Second, they further satisfy the following :

1. f_i is smooth function of the variables x_j.
2. f_i is analytic function in the parameter ϵ, that is, a function with a convergent Taylor series in ϵ.
3. $\epsilon = 0$ can always be chosen to correspond with the identity element ϵ.
4. the law of composition can be taken as $\phi(a, b) = a + b$.

Then G is said to be a one-parameter Lie group.

We starting show invariance of the circle under 1.2.3, now we illustrate that differential equations can be invariant under Lie groups.

Example 1.3.2. Consider

$$\frac{dy}{dx} = y^2 - \frac{y}{x} - \frac{1}{x^2}. \tag{1.3.2}$$

Is invariant under Lie groups

$$\bar{x} = e^{\epsilon} x, \qquad \bar{y} = e^{-\epsilon} y. \tag{1.3.3}$$

Solution:

$$L.H.S = \frac{d\bar{y}}{d\bar{x}} = \frac{\frac{d\bar{y}}{dx}}{\frac{d\bar{x}}{dx}} \implies \frac{d\bar{y}}{dx} = e^{-\epsilon}\frac{dy}{dx}, \quad \frac{d\bar{x}}{dx} = e^{\epsilon} \implies \frac{d\bar{y}}{d\bar{x}} = e^{-2\epsilon}\frac{dy}{dx}. \tag{1.3.4}$$

$$R.H.S = e^{-2\epsilon}y^2 - \frac{e^{-\epsilon}y}{e^{\epsilon}x} - \frac{1}{e^{2\epsilon}x^2} = e^{-2\epsilon}y^2 - \frac{e^{-2\epsilon}y}{x} - \frac{-e^{2\epsilon}}{x^2}. \tag{1.3.5}$$

Now

$$L.H.S = R.H.S$$

$$\frac{d\bar{y}}{d\bar{x}} = e^{-2\epsilon}\frac{dy}{dx} = e^{-2\epsilon}y^2 - \frac{e^{-2\epsilon}y}{x} - \frac{-e^{2\epsilon}}{x^2} \implies \frac{dy}{dx} = y^2 - \frac{y}{x} - \frac{1}{x^2}. \tag{1.3.6}$$

Hence 1.3.2 is invariant under 1.3.3.

How about the complicated ODEs? We need to use change of variables to get a new ODEs involved new variables and separable then easy to solve. And any ODE can be reduced to a new ODE with a new variables is invariant under some Lie groups.

1.4 Infinitesimals Transformations and Lie's Invariance Condition

In previous section we saw how these transformation are important, and how they are work. Consider the general first order ODE

$$\frac{dy}{dx} = F(x, y). \tag{1.4.1}$$

Invariant under Lie groups

$$\bar{x} = f(x, y, \epsilon), \quad \bar{y} = g(x, y, \epsilon). \tag{1.4.2}$$

We end up with very difficult partial differential equation, and unknown functions

f, and g

$$F(f(x,y,\epsilon), g(x,y,\epsilon)) = \frac{g_x + g_y F(x,y)}{f_x + f_y F(x,y)}. \tag{1.4.3}$$

In this case we linearized the Lie groups ϵ =0

$$\bar{x} = f(x,y,\epsilon), \quad \bar{y} = g(x,y,\epsilon). \tag{1.4.4}$$

We obtain

$$\begin{aligned} \bar{x} &= f(x,y,0) + \frac{\partial f}{\partial \epsilon}|_{\epsilon=0} \;\; \epsilon, \quad \& \quad \frac{\partial f}{\partial \epsilon}|_{\epsilon=0} = X(x,y). \\ \bar{y} &= g(x,y,0) + \frac{\partial g}{\partial \epsilon}|_{\epsilon=0} \;\; \epsilon, \quad \& \quad \frac{\partial g}{\partial \epsilon}|_{\epsilon=0} = Y(x,y). \end{aligned} \tag{1.4.5}$$

Hence,

$$\bar{x} = x + X\epsilon, \;\; f(x,y,0) = x \qquad \textit{and} \qquad \bar{y} = y + Y\epsilon, \;\; g(x,y,0) = y. \tag{1.4.6}$$

We called X and Y infinitesimals transformations or symmetries, from X and Y we could construct a transformations that would be lead to separable independent equation. The differential equation is invariant under symmetries X and Y iff satisfied Lie's invariance condition (L.I.C)

$$Y_x + (Y_y - X_x)F - X_y F^2 = XF_x + YF_y. \tag{1.4.7}$$

Example 1.4.1. Consider the ODE 1.3.2 under Lie groups 1.3.3, show that X and Y are satisfied L.I.C.

Solution: We show that 1.3.2 is invariant under 1.3.3. Also we can find the symmetries X and Y from Lie groups 1.3.3.

$$X = \frac{\partial \bar{x}}{\partial \epsilon}|_{\epsilon=0} = x, \quad Y = \frac{\partial \bar{y}}{\partial \epsilon}|_{\epsilon=0} = -y. \tag{1.4.8}$$

Substitute 1.4.8 into 1.4.7 gives

$$(-2)(y^2 - \frac{y}{x} - \frac{1}{x^2}) = x(\frac{y}{x^2} + \frac{2}{x^3}) - y(2y - \frac{1}{x}). \tag{1.4.9}$$

$$-2y^2 + \frac{2y}{x} + \frac{2}{x^2} = -2y^2 + \frac{2y}{x} + \frac{2}{x^2}. \tag{1.4.10}$$

That implies that X and Y are solution of L.I.C.

In case X and Y are unknown it hard to solve L.I.C. we can consider a special solutions such as

$$X = A(x), \qquad Y = B(x)y. \tag{1.4.11}$$

Actually this doesn't work, because it gives a particular solution we need a most general solution.

1.5 Infinitesimal Operator and Higher Order

In order to find all possibilities of X and Y that satisfied L.I.C we need to define infinitesimal operator

$$\Gamma = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y}. \tag{1.5.1}$$

Note: From 1.5.1 we can rewrite R.H.S of L.I.C 1.4.7 as

$$(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y})F = \Gamma F. \tag{1.5.2}$$

We can extend the operator to nth order

$$\Gamma^{(1)}, \Gamma^{(2)}, ..., \Gamma^{(n)}.$$

First order operator

$$\Gamma^{(1)} = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Y_{[x]}\frac{\partial}{\partial y'} \tag{1.5.3}$$

And the extend of symmetry Y define as

$$Y_{[x]} = Y_x + (Y_y - X_x)y' - X_y {y'}^2. \tag{1.5.4}$$

Second order operator

$$\Gamma^{(2)} = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Y_{[x]}\frac{\partial}{\partial y'} + Y_{[xx]}\frac{\partial}{\partial y''}. \tag{1.5.5}$$

1.6 Total Differential Operator

In order to define second order symmetry $Y_{[xx]}$ we need to define total differential operator

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}. \tag{1.6.1}$$

then $Y_{[x]}$ by using 1.6.1

$$Y_{[x]} = Y_x + Y_y y' - (X_x - X_y y')y'.$$

$$= \frac{\partial}{\partial x}Y + y'\frac{\partial}{\partial y}Y - (\frac{\partial}{\partial x}X + y'\frac{\partial}{\partial y}X)y'.$$

$$= D_x(Y) - D_x(X)y'.$$

and

$$Y_{[xx]} = Y_{xx} + (2Y_{xy} - X_{xx})y' + (Y_{yy} - 2X_{xy})y'^2 - X_{yy}y'^3 + (Y_y - 2X_x)y'' - 3X_y y' y''. \quad (1.6.2)$$

If we define Δ such that

$$\Delta = y' - F(x, y) = 0. \quad (1.6.3)$$

We get Lie's invariance condition

$$\Gamma^{(1)}(\Delta)|_{\Delta=0} = 0. \quad (1.6.4)$$

Second order ODE

$$\Delta = y'' - F(x, y, y') = 0, \quad (1.6.5)$$

then L.I.C

$$\Gamma^{(2)}(\Delta)|_{\Delta=0} = 0. \quad (1.6.6)$$

In general, if we define the nth order extension

$$\Delta(x, y, y', ..., y^{(n)}) = 0, \quad (1.6.7)$$

then L.I.C

$$\Gamma^{(n)}(\Delta)|_{\Delta=0} = 0. \quad (1.6.8)$$

1.7 Lie Invariance Condition in Dimension Six

In this section, we explain the algorithm for finding the Lie symmetries of the geodesic equations. In local coordinates and in dimension n, the geodesic equations are given by

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \tag{1.7.1}$$

where Γ^i_{jk} are the connection components or Christoffel symbols, where $i, j, k = 1..n$. In dimension six, let's take our coordinates to be t, p, q, x, y, z, w, where t is the independent variable and p, q, x, y, z, w are the dependant variables, so are functions of t. Define Γ to be

$$\Gamma = T\frac{\partial}{\partial t} + P\frac{\partial}{\partial p} + Q\frac{\partial}{\partial q} + X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z} + W\frac{\partial}{\partial w}, \tag{1.7.2}$$

where T, P, Q, X, Y, Z and W are unknown functions of (t, p, q, x, y, z, w). The first prolongation Γ^1 and second prolongation Γ^2 of Γ are give by

$$\Gamma^1 = \Gamma + P_t\frac{\partial}{\partial \dot{p}} + Q_t\frac{\partial}{\partial \dot{q}} + X_t\frac{\partial}{\partial \dot{x}} + Y_t\frac{\partial}{\partial \dot{y}} + Z_t\frac{\partial}{\partial \dot{z}} + W_t\frac{\partial}{\partial \dot{w}}, \tag{1.7.3}$$

$$\Gamma^2 = \Gamma^1 + P_{tt}\frac{\partial}{\partial \ddot{p}} + Q_{tt}\frac{\partial}{\partial \ddot{q}} + X_{tt}\frac{\partial}{\partial \ddot{x}} + Y_{tt}\frac{\partial}{\partial \ddot{y}} + Z_{tt}\frac{\partial}{\partial \ddot{z}} + W_{tt}\frac{\partial}{\partial \ddot{w}}, \tag{1.7.4}$$

where

$$\begin{aligned}
P_t &= D_t(P) - \dot{p}D_t(T), & P_{tt} &= D_t(P_t) - \ddot{p}D_t(T),\\
Q_t &= D_t(Q) - \dot{q}D_t(T), & Q_{tt} &= D_t(Q_t) - \ddot{q}D_t(T),\\
X_t &= D_t(X) - \dot{x}D_t(T), & X_{tt} &= D_t(X_t) - \ddot{x}D_t(T),\\
Y_t &= D_t(Y) - \dot{y}D_t(T), & Y_{tt} &= D_t(Y_t) - \ddot{y}D_t(T),\\
Z_t &= D_t(Z) - \dot{z}D_t(T), & Z_{tt} &= D_t(Z_t) - \ddot{z}D_t(T),\\
W_t &= D_t(W) - \dot{w}D_t(T), & W_{tt} &= D_t(W_t) - \ddot{w}D_t(T),
\end{aligned} \tag{1.7.5}$$

where D_t is given by

$$D_t = \frac{\partial}{\partial t} + \dot{p}\frac{\partial}{\partial p} + \dot{q}\frac{\partial}{\partial q} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z} + \dot{w}\frac{\partial}{\partial w} + \ddot{p}\frac{\partial}{\partial \dot{p}} + \ddot{q}\frac{\partial}{\partial \dot{q}} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \ddot{y}\frac{\partial}{\partial \dot{y}} + \ddot{z}\frac{\partial}{\partial \dot{z}} + \ddot{w}\frac{\partial}{\partial \dot{w}}. \tag{1.7.6}$$

Finally, Γ is said to be a Lie symmetry of the system of the geodesic equations if

$$\Gamma^2(\Delta_i^{(2)})|_{\Delta_i^{(2)}=0} = 0, \tag{1.7.7}$$

where

$$\Delta_i^{(2)} = \frac{d^2x^i}{dt^2} - f^i(t, x^i), \qquad i = 1, 2, \cdots, 6. \tag{1.7.8}$$

Equation (1.7.7) is called the Lie invariance condition. We equate the coefficients of the linearly independent derivation terms to zero and this yields to an overdetermined system of PDEs. For a good reference on symmetries of differential equations, we refer the reader to [7].

1.8 Lie's Method of Symmetry Analysis

For any ODE with unknown infinitesimals X and Y, we can go through Lie's method of symmetry analysis to figure out all possibilities of X and Y.

Example 1.8.1. Consider the following ODE

$$y'' + yy' + xy^4 = 0. \tag{1.8.1}$$

$$y'' = -yy' - xy^4. \tag{1.8.2}$$

Let

$$\Delta = y'' + yy' + xy^4 = 0. \tag{1.8.3}$$

L.I.C

$$\Gamma^{(2)}\Delta|_{\Delta=0} = 0. \tag{1.8.4}$$

by using 1.5.4, 1.5.5, and 1.6.2 L.I.C 2nd order 1.6.6 gives

$$\Gamma^{(2)}(y'' + yy' + xy^4) = Xy^4 + Y(y' + 4xy^3) + y(Y_x + (Y_y - X_x)y' - X_y y'^2),$$

$$+Y_{xx} + y'(2Y_{xy} - X_{xx}) + y'^2(Y_{yy} - 2X_{xy}) - X_{yy}y'^3 + y''(Y_y - 2X_x) - 3X_y y' y'' = 0. \tag{1.8.5}$$

Substitute by 1.8.3 we obtained

$$\Gamma^{(2)}(y'' + yy' + xy^4)|_{y''=-yy'-xy^4} = Xy^4 + 4xYy^3 + Yy' + Y_x y + y'(Y_y y - X_x y),$$

$$-X_y yy'^2 + Y_{xx} + y'(2Y_{xy} - X_{xx}) + y'^2(Y_{yy} - 2X_{xy}) + X_{yy}y'^3 - Y_y xy^4,$$

$$-Y_y yy' + 2X_x yy' + 2X_x xy^4 + 3X_y yy'^2 + 3X_y xy^4 y' = 0. \tag{1.8.6}$$

Setting the coefficients of y', y'^2, and y'^3 gives

$$Y_{xx} + 4xy^3 Y + yY_x + Xy^4 - xy^4(Y_y - 2X_x) = 0. \tag{1.8.7}$$

$$Y + yX_x + 2Y_{xy} - X_{xx} + 3xy^4 X_y = 0. \tag{1.8.8}$$

$$Y_{yy} - 2X_{xy} + 2yX_y = 0. \tag{1.8.9}$$

$$X_{yy} = 0. \tag{1.8.10}$$

Solve the system of PDEs to find all X, and Y starting by 1.8.10 integrating twice with respect to y, we obtained

$$X(x, y) = a(x)y + b(x). \tag{1.8.11}$$

Where $a(x)$, and $b(x)$ are arbitrary functions. Substitute1.8.11 into 1.8.9 and integrate gives

$$Y(x,y) = a'(x)y^2 - \frac{1}{3}a(x)y^3 + p(x)y + q(x). \tag{1.8.12}$$

Now substitute 1.8.11, and 1.8.12 into 1.8.8 gives

$$a(x) = 0. \tag{1.8.13}$$

$$3a''(x) + b'(x) + p(x) = 0. \tag{1.8.14}$$

$$2p'(x) - b''(x) + q(x) = 0. \tag{1.8.15}$$

From 1.8.13 we see that $a(x) = 0$ which leads to

$$X = b(x), \qquad Y = p(x)y + q(x). \tag{1.8.16}$$

Substituting 1.8.16 into 1.8.7 and recognizing, then set the coefficients of the various power of y to zero, we obtain

$$q(x) = 0, \qquad p'(x) = 0, \qquad 2xb'(x) + b(x) + 3xp(x) = 0. \tag{1.8.17}$$

The solution of the system

$$b(x) = cx, \qquad p(x) = -c, \qquad q(x) = 0. \tag{1.8.18}$$

Where c is a constant. This leads to the symmetry X and Y as

$$X = cx, \qquad Y = -cy. \tag{1.8.19}$$

CHAPTER 2

LIE ALGEBRAS AND CANONICAL CONNECTION

2.1 Introduction

A Lie algebra over the field of real number or complex number is a vector space equipped with an additional operation, it is called the bracket operation. The main example of to keep in mind is $gl(n, \mathbb{R})$, the space of all $n \times n$ real matrices. In this case, the bracket of any two matrices is the commutator. In fact, every finite dimensional Lie algebra is isomorphic to a subalgebra of the general linear group $gl(n, \mathbb{R})$ for some $n \in N$. Lie algebras have been classified into different types such as the simple, semi-simple, nilpotent and solvable Lie algebras. The simple Lie algebras have been classified according to the root system associated to their Cartan subalgebras, and the root systems, in turn are classified by their Dynkin Diagram. Therefore, we obtain the following:

- $A_n : sl(n+1)$, the special linear Lie algebra.
- $B_n : so(2n+1)$, the odd-dimensional special orthogonal Lie algebra.
- $C_n : sp(2n)$, the symplectic Lie algebra.
- $D_n : so(2n)$, the even-dimensional special orthogonal Lie algebra.

A classification of the exceptional Lie algebras are given and they are: $\mathfrak{g}_2$, f_4, e_6, e_7 and e_8. The dimensions of these algebras are: 14, 52, 78, 133 and 248, respectively. A classification of the low dimensional Lie algebras have been given. In 1891, K.A.

Umlauf classified the nilpotent Lie algebras over the complex numbers in dimensions less than or equal to six. In 1894, E. Cartan [9]classified the Semi-simple Lie algebras. A classification of the six-dimensional nilpotent Lie algebras over the field of real numbers was given by Morozov [10] in 1958. Mubarakzyanov [11] classified the solvable Lie algebras in dimension less than or equal to five. There are fort families of indecomposable Lie algebras in dimension five. He also classified the solvable indecomposable six-dimensional Lie algebras with co-dimension one nilradical. There are ninety nine families of Lie algeb ras in this case. In 1988, Turkowski [13] classified the six-dimensional solvable Lie algebras with co-dimension two nilradical and there are forty families of Lie algebras in this case. Finally, Craig Seeley [12] classified the seven-dimensional nilpotent Lie algebras over the field of complex numbers in 1993. In 1998, Ming-Peng Gong [14] classified the seven-dimensional nilpotent Le algebras over the field of real number and there are one hundred fifty five families of Lie algebras.

In this Chapter, we give the necessary definitions and examples about Lie algebras in Section 2.2. We also define and give the main properties of the canonical connection on Lie groups in Section 2.3.

2.2 Definitions and Examples

Definition 2.2.1. A vector space $\mathfrak{g}$ over the field $\mathbb{R}$, with a binary operation $[\cdot,\cdot]$: $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted by $(x, y) \to [x, y]$, is called a Lie algebra over $\mathbb{R}$ with Lie bracket operation.

If it satisfies the following, [1], [15]:

1- Bilinearity:

$$[ax + by, z] = a[x, z] + b[y, z], \quad x, y, z \in \mathfrak{g}, \quad \& \quad a, b \in \mathbb{R}. \tag{2.2.1}$$

2- Skew-symmetry:

$$[y, x] = -[x, y], \quad x, y \in \mathfrak{g}. \tag{2.2.2}$$

3- Jacobi Identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad x, y, z \in \mathfrak{g}. \tag{2.2.3}$$

$[x, y]$ is called the bracket of x with y.

Example 2.2.2. The vector space of all $n \times n$ matrices with real entries, called $gl(n, \mathbb{R})$ with the bracket of two matrices to be defined as the commutator is a Lie algebra.

Example 2.2.3. $sl(n, \mathbb{R})$, the space of all $n \times n$ trace-free matrices is a Lie algebra.

Example 2.2.4. $so(n)$, the space of all $n \times n$ skew-symmetric matrices is a Lie algebra.

Example 2.2.5. Consider the following three vector fields:

$$E_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad E_2 = \frac{\partial}{\partial y}, \quad E_3 = -\frac{\partial}{\partial z}. \tag{2.2.4}$$

Now, we calculate the brackets in the following way:

$$\begin{aligned}
[E_1, E_2] &= (\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial z})(\tfrac{\partial}{\partial y}) - [(\tfrac{\partial}{\partial y})(\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial z})]. \\
&= \tfrac{\partial^2}{\partial y \partial x} + y\tfrac{\partial^2}{\partial y \partial z} - [\tfrac{\partial^2}{\partial x \partial y} + (y\tfrac{\partial^2}{\partial z \partial y} + (1)\tfrac{\partial}{\partial z})]. \\
&= \tfrac{\partial^2}{\partial y \partial x} + y\tfrac{\partial^2}{\partial y \partial z} - \tfrac{\partial^2}{\partial x \partial y} - y\tfrac{\partial^2}{\partial z \partial y} - \tfrac{\partial}{\partial z}. \\
&= -\tfrac{\partial}{\partial z}. \\
&= E_3.
\end{aligned}$$

Similarly, we calculate the brackets of $[E_1, E_3]$ and $[E_2, E3]$ to obtain:

$$[E_1, E_3] = 0, \quad [E_2, E_3] = 0. \tag{2.2.5}$$

We can easily verify that the vector space spanned by E_1, E_2, E_3 satisfies all the conditions of being a Lie algebra.

Definition 2.2.6. The derived algebra of Lie algebra $\mathfrak{g}$, denoted $D_{\mathfrak{g}}$ is the set

$$\{X \in \mathfrak{g} | \text{ for some } Y, Z \in \mathfrak{g},\ X = [Y, Z]\}. \tag{2.2.6}$$

An alternative way to describe the derived algebra is

$$D_{\mathfrak{g}} = [\mathfrak{g}, \mathfrak{g}]. \tag{2.2.7}$$

Where $[\mathfrak{g}, \mathfrak{g}]$ denote all possible Lie brackets between vectors in $\mathfrak{g}$.

Definition 2.2.7. The derived series, given by a series of $D^i_{\mathfrak{g}}$, for all $i \in N$, where each $D^i_{\mathfrak{g}}$ is defined by

$$D^i_{\mathfrak{g}} = [D^{i-1}_{\mathfrak{g}}, D^{i-1}_{\mathfrak{g}}] \quad \text{with} \quad D^1_{\mathfrak{g}} = D_{\mathfrak{g}} = [\mathfrak{g}, \mathfrak{g}]. \tag{2.2.8}$$

Another useful series is the lower central series, given by a series of $D_{i\mathfrak{g}}$, for all $i \in N$, where each $D_{i\mathfrak{g}}$ is defined by

$$D_{i\mathfrak{g}} = D_{i\mathfrak{g}} = [\mathfrak{g}, D_{i\mathfrak{g}}], \quad \text{with} \quad D_{1\mathfrak{g}} = D_{\mathfrak{g}} = [\mathfrak{g}, \mathfrak{g}]. \tag{2.2.9}$$

Note that for any $i \in N$, $D^i_{\mathfrak{g}} \subseteq D_{i\mathfrak{g}}$.

Definition 2.2.8. Lie algebra is solvable if the derived series of $\mathfrak{g}$ eventually arrive to zero sub-algebra, for some

$$i \in N, \ D^i_{\mathfrak{g}} = \vec{0}. \tag{2.2.10}$$

Definition 2.2.9. Lie algebra is nilpotent if the lower central series of $\mathfrak{g}$ eventually arrive to zero sub-algebra, for some

$$i \in N, \ D_{i\mathfrak{g}} = \vec{0}. \tag{2.2.11}$$

Then, it is obvious that all nilpotent Lie algebra are solvable, because $D^i_{\mathfrak{g}} \subseteq D_{i\mathfrak{g}}$.

Definition 2.2.10. A linear subspace K of $\mathfrak{g}$ is called an ideal if

$$[K, \mathfrak{g}] \subset K. \tag{2.2.12}$$

Thus an ideal is stronger than a sub-algebra.

Definition 2.2.11. The radical of Lie algebra $\mathfrak{g}$, is its maximal solvable ideal denote by $R(\mathfrak{g})$. And the nilradical of lie algebra has same idea of radical, which is defined to be the maximal nilpotent ideal denote by $NR(\mathfrak{g})$. Then the co-dimension of the nilradical of Lie algebra $\mathfrak{g}$ is the difference between the dimension of Lie algebra and the dimension of the nilradical that is

$$dim(\mathfrak{g}) - dimNR(\mathfrak{g}). \tag{2.2.13}$$

Definition 2.2.12. The Levi decomposition Lie algebra is a semi-direct product of its radical (solvable ideal) and a semisimple Lie algebra. Simesimple Lie algebra if its radical is zero.

Theorem 2.2.1. Ado's theorem states that every finite-dimensional Lie algebra $\mathfrak{g}$ over a field $\mathbb{R}$ of characteristic zero can be viewed as a Lie algebra of square matrices under the commutator bracket.

2.3 The Canonical Lie Group Connection

On left invariant vector fields X and Y the canonical symmetric connection ∇ , [9] on a Lie group G is defined by

$$\nabla_X Y = \frac{1}{2}\,[X, Y] \tag{2.3.1}$$

and then extended to arbitrary vector fields using linearity and the Leibnitz rule. Clearly ∇ is left-invariant. One could just as well use right-invariant vector fields to define ∇, but one must check that ∇ is well defined, a fact that we will prove next.

Proposition 2.3.1. In the definition of ∇ we can equally assume that X and Y are right-invariant vector fields and hence ∇ is also left-invariant and hence bi-invariant. Moreover ∇ is symmetric, that is, its torsion is zero.

Proof. The fact that ∇ is symmetric is obvious from eq.(2.3.1). Now we choose a fixed basis in the tangent space at the identity T_IG. We shall denote its left and right invariant extensions by $\{X_1, X_2, ..., X_n\}$ and $\{Y_1, Y_2, ..., Y_n\}$, respectively. Then there must exist a non-singular matrix A of functions on G such that $Y_i = a_i^j X_j$. We shall suppose that

$$[X_i, X_j] = C_{ij}^k X_k. \tag{2.3.2}$$

Changing from the left-invariant basis to the right gives

$$C_{ij}^k a_k^p = a_i^k a_j^m C_{km}^p. \tag{2.3.3}$$

Next, we use the fact that left and right vector fields commute to deduce that

$$a_j^k C_{ik}^m + X_i a_j^m = 0, \tag{2.3.4}$$

where the second term in 2.3.4 denotes directional derivative. We note that necessarily

$$[Y_i, Y_j] = -C_{ij}^k Y_k. \tag{2.3.5}$$

Now we compute

$$\nabla_{Y_i} Y_j + \frac{1}{2} C_{ij}^k Y_k = \frac{1}{2} a_i^k a_j^m C_{km}^p + a_i^k (X_k a_i^p) + \frac{1}{2} C_{ij}^k a_k^p. \tag{2.3.6}$$

Next we use 2.3.4 to replace the second term on the right hand side of 2.3.6 so as to obtain

$$\nabla_{Y_i} Y_j + \frac{1}{2} C_{ij}^k Y_k = \frac{1}{2} a_i^k a_j^m C_{km}^p - a_i^k a_j^m C_{km}^p + \frac{1}{2} C_{ij}^k a_k^p. \tag{2.3.7}$$

However, the right hand side of 2.3.7 is seen to be zero by virtue of 2.3.3. Thus

$$\nabla_X Y = \frac{1}{2}[X, Y] \tag{2.3.8}$$

whenever X and Y are right invariant vector fields. □

An alternative proof of Proposition (2.3.1) uses the inversion map ψ defined by, for $S \in G$,

$$\psi(S) = S^{-1}. \tag{2.3.9}$$

As such, one checks that ψ_{*_I} maps a left-invariant vector field evaluated at I to minus its right-invariant counterpart evaluated at I. Then ψ_{*_I} is an isomorphism and there is no change of sign in the structure constants, as compared with eq.(2.3.5). Since there are two minus signs in eq.(2.3.1) the same condition eq.(2.3.1) applies also to right-invariant vector fields.

Proposition 2.3.2. (i) An element in the center of $\mathfrak{g}$ engenders a bi-invariant vector field.

(ii) A vector field in the center of $\mathfrak{g}$ is parallel.

(iii) A bi-invariant differential k-form θ is closed and so defines an element of the cohomology group $H^k(M, \mathbb{R})$.

Proof. (i) Suppose that $Z \in T_I G$ is in the center of $\mathfrak{g}$ and let $\exp(tZ)$ be the associated one-parameter subgroup of G so that Z corresponds to the equivalence class of curves $[\exp(tZ)]$ based at I. Let $S \in G$; then $L_{S*}Z$ corresponds to the equivalence class of curves $[S\exp(tZ)]$ based at S. Since Z is in the center of $\mathfrak{g}$ then $\exp(tZ)$ will be in the center of G and hence $[S\exp(tZ)] = [\exp(tZ)S]$. It follows that any element in the center of $\mathfrak{g}$ engenders a bi-invariant vector field.

(ii) Obvious from eq.(2.3.1).

(iii) A proof can be found in [33]. Spivak shows that $\psi^*(\theta) = (-1)^k\theta$, whereas $d\theta$, which is also bi-invariant, changes by $\psi^*(d\theta) = (-1)^{k+1}d\theta$. It follows that $d\theta = 0$.

□

Proposition 2.3.3. (i) The curvature tensor, which is also bi-invariant, on vector fields X, Y, Z is given by

$$R(X,Y)Z = \frac{1}{4}\,[[X,Y],Z]. \tag{2.3.10}$$

(ii) The connection ∇ is flat if and only if the Lie algebra $\mathfrak{g}$ of G is two-step nilpotent.
(iii) The tensor R is parallel in the sense that $\nabla_W R(X,Y)Z = 0$, where W is a fourth right invariant vector field, so that G is in a sense a symmetric space.
(iv) The Ricci tensor R_{ij} of ∇ is given by

$$R_{ij} = \frac{1}{4}\,C^l_{jm}C^m_{il} \tag{2.3.11}$$

and is symmetric and bi-invariant and is obtained by translating to the left or right one quarter of the Killing form. It engenders a bi-invariant pseudo-Riemannian metric if and only if the Lie algebra $\mathfrak{g}$ is semi-simple.

Proof. (i) Is obvious and applies to arbitrary vector fields since it is a tensorial object.
(ii) Is obvious.

(iii) This fact follows from a series of implications:

$$
\begin{aligned}
&4\nabla_W R(X,Y)Z + 4R(\nabla_W X,Y)Z + 4R(X,\nabla_W Y)Z + 4R(X,Y)\nabla_W Z = \nabla_W[[X,Y],Z], \\
&4\nabla_W R(X,Y)Z + 2R([W,X],Y)Z + 2R(X,[W,Y])Z + 2R(X,Y)[W,Z] - \tfrac{1}{2}[W,[[X,Y],Z] = 0, \\
&4\nabla_W R(X,Y)Z + \tfrac{1}{2}[[W,X],Y],Z] + \tfrac{1}{2}[X,[W,Y]],Z] + \tfrac{1}{2}[[X,Y],[W,Z]] - \tfrac{1}{2}[W,[[X,Y],Z] = 0, \\
&4\nabla_W R(X,Y)Z + \tfrac{1}{2}[[W,X],Y],Z] + \tfrac{1}{2}[X,[W,Y]],Z] - \tfrac{1}{2}[Z,[[X,Y],W] = 0, \\
&\nabla_W R(X,Y)Z = 0.
\end{aligned}
\tag{2.3.12}
$$

(iv) The formula eq.(2.3.11) is obvious from eqs. (2.3.1) and (2.3.10). The last remark follows from Cartan's criterion. □

Proposition 2.3.4. (i) Any left or right-invariant vector field is geodesic.
(ii) Any geodesic curve emanating from the identity is a one-parameter subgroup..
(iii) An arbitrary geodesic curve is a translation, to the left or right, of a one-parameter subgroup.

Proof. (i) Is obvious because of the skew-symmetry in eq.(2.3.1).
(ii) By definition the curve $t \mapsto [S\exp(tX)]$ integrates a geodesic field X.
(iii) If the geodesic curve at $t = 0$ starts at S, translate the curve to I by multiplying on the left or right by S^{-1} and apply (ii). □

Proposition 2.3.5. (i) A left or right-invariant vector field is a symmetry, a.k.a. affine collineation, of ∇.
(ii) Any left or right-invariant one-form engenders a first integral of the geodesic system of ∇.

Proof. (i) The following condition for vector fields X and Y says that vector field W is a symmetry or, affine collineation, of a symmetric linear connection:

$$\nabla_X \nabla_Y W - \nabla_{\nabla_X Y} W - R(W, X)Y = 0. \tag{2.3.13}$$

In the case at hand of the canonical connection, this condition just reduces to the Jacobi identity when W, X and Y are all left or right-invariant.
(ii) A one-form α is a *Killing one-form*, if the following condition holds:

$$\langle \nabla_X \alpha, Y \rangle + \langle X, \nabla_Y \alpha \rangle = 0. \tag{2.3.14}$$

In the case of the canonical connection, if X and Y are right-invariant and α is right-invariant then eq.(2.3.1) gives

$$\langle X, \nabla_Y \alpha \rangle = \frac{1}{2} \langle [X, Y], \alpha \rangle. \tag{2.3.15}$$

Clearly, (2.3.15) implies (2.3.14) so that every left or right-invariant one-form engenders a first integral of the geodesics: if the one-form is given in a coordinate system as $\alpha_i dx^i$ on G, the first integral is $\alpha_i u^i$ viewed as a function on the tangent bundle TG that is linear in the fibers. □

Proposition 2.3.6. Any left or right-invariant one-form α is closed if and only if $\langle [\mathfrak{g}, \mathfrak{g}], \alpha \rangle = 0$, that is, α annihilates the derived algebra of $\mathfrak{g}$.

Proof. Consider the identity

$$d\alpha(X, Y) = X \langle Y, \alpha \rangle - Y \langle X, \alpha \rangle - \langle [X, Y], \alpha \rangle. \tag{2.3.16}$$

If α is left-invariant and we take X and Y left-invariant, then the first and second terms in eq.(2.3.16) are zero. Now the conclusion of the Proposition is obvious. The proof for right-invariant one-forms is similar. □

Proposition 2.3.7. Consider the following conditions for a one-form α on G:

(i) α is bi-invariant.

(ii) α is right-invariant and closed.

(iii) α is left-invariant and closed.

(iv) α is parallel.

Then we have the following implications: (i), (ii) and (iii) are equivalent and any one of them implies (iv).

Proof. The fact that (i) implies (ii) and (iii) follows from Proposition (2.3.2) part (iii). Now suppose that (iii) holds and let X and Y be right and left-invariant vector fields, respectively. Then consider again the identity

$$d\alpha(X,Y) = X\langle Y,\alpha\rangle - Y\langle X,\alpha\rangle - \langle [X,Y],\alpha\rangle. \tag{2.3.17}$$

Assuming that α is closed, then either because $[X,Y] = 0$ or by using Proposition 2.3.6, we find that eq.(2.3.17) reduces to

$$X\langle Y,\alpha\rangle = Y\langle X,\alpha\rangle. \tag{2.3.18}$$

Now the left hand side of eq.(2.3.18) is zero, since Y and α are left-invariant. Hence $\langle X,\alpha\rangle$ is constant, which implies that α is right-invariant and hence bi-invariant. Thus (iii) implies (i). The proof that (ii) implies (i) is similar. Finally, supposing that (ii) or (iii) holds we show that (iv) holds. Then as with any symmetric connection, the closure condition may be written, for arbitrary vector fields X and Y, as

$$\langle \nabla_X \alpha, Y\rangle - \langle X, \nabla_Y \alpha\rangle = 0. \tag{2.3.19}$$

Clearly eq.(2.3.14) and eq.(2.3.19)imply that α is parallel. So a closed, invariant one-form is parallel. □

Of course, it may well be the case that there are no bi-invariant one-forms on G, for example if G is semi-simple so that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. However, there must be at least one such one-form if G is solvable and at least two if G is nilpotent.

If we choose a basis of dimension dim $\mathfrak{g}$-dim $[\mathfrak{g}, \mathfrak{g}]$ for the bi-invariant one-forms on G, it may be used to obtain a partial coordinate system on G, since each such form is closed. Such a partial coordinate system is significant in terms of the geodesic system, in that it gives rise to second order differential equations that resemble the system in Euclidean space.

Proposition 2.3.8. Each of the bi-invariant one-forms on G projects to a one-form on the quotient space $G/[G,G]$, assuming that the commutator subgroup $[G,G]$ is closed topologically in G. Furthermore the canonical connection ∇ on G projects to a flat connection on $G/[G,G]$ and the induced system of one-forms on $G/[G,G]$ comprises a "flat" coordinate system.

Proof. The fact that a bi-invariant one-form on G projects to a one-form on $G/[G,G]$ follows because each such form annihilates the vertical distribution of the principal right $[G,G]$-bundle $G \to G/[G,G]$ and furthermore the equivariance, or Lie-derivative condition along the fibers, is trivially satisfied since the one-form is closed. The fact that ∇ projects to $G/[G,G]$ follows because $[G,G] \triangleleft G$, as was noted in [25]. □

CHAPTER 3

FOUR AND FIVE DIMENSIONAL LIE ALGEBRAS

3.1 Introduction

This chapter focuses on the the symmetry algebras of geodesic equations related to solvable Lie algebras with co-dimension two abelian nilradical that have an abelian complement in dimensions four and five. In dimension four, there is only one Lie algebra to consider, namely $A_{4,12}$ in Winternitz list [27]. In dimension five, there are three families of Lie algebras to consider, namely, $A_{5,33}$, $A_{5,34}$, and $A_{5,35}$. For each of these Lie algebras, we calculated the geodesic equations, a basis for the symmetry Lie algebra in terms of vector fields and identify the Lie algebras. We work out the four-dimensional case in details: we construct the Lie invariance condition, and we integrate the symmetries to obtain corresponding Lie groups. We finally summarize our results in Table 1.

3.2 The Four Dimensional Lie Algebra $A_{4,12}$

The non-zero brackets for $A_{4,12}$ are given by

$$[e_2, e_3] = e_2, \quad [e_1, e_3] = e_1, \quad [e_1, e_4] = -e_2, \quad [e_2, e_4] = e_1. \tag{3.2.1}$$

From the non-zero Lie brackets, we calculate the nilradical for algebra $A_{4,12}$ by $\{e_1, e_2\}$, and the co-dimension two abelian nilradical by $\{e_3, e_4\}$.

3.2.1 Geodesic Equations for the Lie Algebra $A_{4,12}$

In this section we consider special systems of second order ordinary differential equations, known as geodesic equations. The geodesic equation is a second order differential equation, where the independent variable is t, which represents time, and the equation itself represents the motion of a particle moving in a curved space. In Riemannian geometry, a geodesic curve gives the shortest path between two points in that space. In general relativity, geodesic equations describe the motion in spacetime. In the case of a differentiable manifold with a connection, geodesic curves provide a generalization of straight lines in Euclidean space. On a smooth manifold, where (x^i) are a system of local coordinates, and Γ^i_{jk} are the connection components or Christoffel symbols. Given an n-dimensional Lie algebra $\mathfrak{g}$, the system of geodesic equations is given by

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0, \quad (i, j, k = 1, 2, ..., n). \tag{3.2.2}$$

Now, we compute the geodesic equations for the Lie algebra $A_{4,12}$. In order to calculate the connection components Γ^i_{jk}, we need to calculate the covariant derivatives of the vector fields given. We will also use the definition of the canonical connection given by

$$\nabla_X Y = \frac{1}{2}\,[X, Y]. \tag{3.2.3}$$

We can find representation of Lie algebra in terms of vector fields.

$$X = \tfrac{\partial}{\partial x} + z\tfrac{\partial}{\partial z} + w\tfrac{\partial}{\partial w}, \quad Y = \tfrac{\partial}{\partial y} + w\tfrac{\partial}{\partial z} - z\tfrac{\partial}{\partial w}, \quad Z = \tfrac{\partial}{\partial z}, \quad W = \tfrac{\partial}{\partial w}. \tag{3.2.4}$$

By using change of basis we obtained:

$$E_1 = \frac{\partial}{\partial z}, \quad E_2 = \frac{\partial}{\partial w}, \quad E_3 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}, \quad E_4 = \frac{\partial}{\partial y} + w\frac{\partial}{\partial z} - z\frac{\partial}{\partial w}. \tag{3.2.5}$$

It is easy to verify that the commutators of the above vector fields give the same relations as equation

$$\begin{aligned}
[E_1, E_3] &= Z(X) - X(Z). \\
&= \frac{\partial^2}{\partial z \partial x} + \frac{\partial z}{\partial z}\frac{\partial}{\partial z} + z\frac{\partial^2}{\partial z^2} + \frac{\partial w}{\partial z}\frac{\partial}{\partial w}, \\
&+ w\frac{\partial^2}{\partial z \partial w} - \frac{\partial^2}{\partial z \partial x} - z\frac{\partial^2}{\partial z^2} - w\frac{\partial^2}{\partial w \partial z}. \\
&= \frac{\partial}{\partial z}. \\
&= E_1.
\end{aligned} \tag{3.2.6}$$

$$\begin{aligned}
[E_2, E_3] &= W(X) - X(W). \\
&= \frac{\partial^2}{\partial w \partial x} + \frac{\partial z}{\partial w}\frac{\partial}{\partial z} + z\frac{\partial^2}{\partial w \partial z} + \frac{\partial w}{\partial w}\frac{\partial}{\partial w}, \\
&+ w\frac{\partial^2}{\partial w^2} - \frac{\partial^2}{\partial w \partial x} - z\frac{\partial^2}{\partial z \partial w} - w\frac{\partial^2}{\partial w^2}. \\
&= \frac{\partial}{\partial w}. \\
&= E_2.
\end{aligned} \tag{3.2.7}$$

$$
\begin{aligned}
[E_1, E_4] &= Z(Y) - Y(Z). \\
&= \tfrac{\partial^2}{\partial z \partial y} + \tfrac{\partial w}{\partial z}\tfrac{\partial}{\partial z} + w\tfrac{\partial^2}{\partial z^2} - \tfrac{\partial z}{\partial z}\tfrac{\partial}{\partial w}, \\
&- z\tfrac{\partial^2}{\partial z \partial w} - \tfrac{\partial^2}{\partial z \partial y} - w\tfrac{\partial^2}{\partial z^2} + z\tfrac{\partial^2}{\partial w \partial z}. \\
&= -\tfrac{\partial}{\partial w}. \\
&= -E_2.
\end{aligned} \tag{3.2.8}
$$

$$
\begin{aligned}
[E_2, E_4] &= W(Y) - Y(W). \\
&= \tfrac{\partial^2}{\partial w \partial y} + \tfrac{\partial w}{\partial w}\tfrac{\partial}{\partial z} + w\tfrac{\partial^2}{\partial w \partial z} - \tfrac{\partial z}{\partial w}\tfrac{\partial}{\partial w}, \\
&- z\tfrac{\partial^2}{\partial w^2} - \tfrac{\partial^2}{\partial w \partial y} - w\tfrac{\partial^2}{\partial z \partial w} + z\tfrac{\partial^2}{\partial w^2}. \\
&= \tfrac{\partial}{\partial z}. \\
&= E_1.
\end{aligned} \tag{3.2.9}
$$

We have the following covariant derivations:

$$
\begin{aligned}
&\nabla_{E_1} E_1, \quad \nabla_{E_2} E_1, \quad \nabla_{E_1} E_3, \quad \nabla_{E_1} E_4, \quad \nabla_{E_2} E_2, \quad \nabla_{E_2} E_3, \quad \nabla_{E_2} E_4, \quad \nabla_{E_3} E_3, \\
&\nabla_{E_4} E_3, \quad \nabla_{E_4} E_4.
\end{aligned} \tag{3.2.10}
$$

Simply let

$$
E_1 = \partial z, \quad E_2 = \partial w, \quad E_3 = \partial x + z\partial z + w\partial w, \quad E_4 = \partial y + w\partial z - z\partial w. \tag{3.2.11}
$$

For any two vectors E_i, E_j by symmetric

$$\nabla_{E_i} E_j = -\nabla_{E_j} E_i, \quad (i, j = 1, 2, 3, 4 \;\; \& \;\; i \neq j). \tag{3.2.12}$$

This will result in a system of equations with Γ^i_{jk} to be the unknowns. To illustrate this, we will show how to obtain the equations in the system.

1- $\nabla_{E_1} E_1$

$$\nabla_{E_1} E_1 = \frac{1}{2}\, [E_1, E_1]. \tag{3.2.13}$$

The left hand side of 3.2.13 gives the following:

$$\begin{aligned} \nabla_{E_1} E_1 &= \nabla_{\partial z}(\partial z). \\ &= \left(\Gamma^k_{33} \partial x^k\right). \\ &= \Gamma^1_{33} \partial x + \Gamma^2_{33} \partial y + \Gamma^3_{33} \partial z + \Gamma^4_{33} \partial w. \end{aligned} \tag{3.2.14}$$

The right hand side of 3.2.13 gives

$$\frac{1}{2}\, [E_1, E_1] = 0. \tag{3.2.15}$$

Then, we obtain the following system of equations:

$$\Gamma^1_{33} = 0, \quad \Gamma^2_{33} = 0, \quad \Gamma^3_{33} = 0, \quad \Gamma^4_{33} = 0. \tag{3.2.16}$$

2 - $\nabla_{E_2} E_1$

$$\nabla_{E_2} E_1 = \frac{1}{2}\, [E_2, E_1]. \tag{3.2.17}$$

The left hand side of 3.2.17 gives the following:

$$\begin{aligned}\nabla_{E_2} E_1 &= \nabla_{\partial w} (\partial z) . \\ &= \left(\Gamma^k_{34} \partial x^k\right) . \\ &= \Gamma^1_{34} \partial x + \Gamma^2_{34} \partial y + \Gamma^3_{34} \partial z + \Gamma^4_{34} \partial w.\end{aligned} \tag{3.2.18}$$

The right hand side of 3.2.17 gives

$$\frac{1}{2} \, [E_2, E_1] = 0. \tag{3.2.19}$$

Then, we obtain the following system of equations:

$$\Gamma^1_{34} = 0, \quad \Gamma^2_{34} = 0, \quad \Gamma^3_{34} = 0, \quad \Gamma^4_{34} = 0. \tag{3.2.20}$$

3 - $\nabla_{E_1} E_3$

$$\nabla_{E_1} E_3 = \frac{1}{2} \, [E_1, E_3]. \tag{3.2.21}$$

The left hand side of 3.2.21 gives the following:

$$\begin{aligned}
\nabla_{E_1} E_3 &= \nabla_{\partial z} \left(\partial x + z \partial z + w \partial w \right). \\
&= \Gamma^k_{13} \partial x^k + \nabla_{\partial z}(z \partial z) + \nabla_{\partial z}(w \partial w). \\
&= \Gamma^1_{13} \partial x + \Gamma^2_{13} \partial y + \Gamma^3_{13} \partial z + \Gamma^4_{13} \partial w + \tfrac{\partial}{\partial z}(z) \partial z, \\
&+ z \nabla_{\partial z}(\partial z) + \tfrac{\partial}{\partial z}(w) \partial w + w \nabla_{\partial z}(\partial w). \\
&= \Gamma^1_{13} \partial x + \Gamma^2_{13} \partial y + \Gamma^3_{13} \partial z + \Gamma^4_{13} \partial w + \partial z, \qquad (3.2.22) \\
&+ z \left(\Gamma^1_{33} \partial x + \Gamma^2_{33} \partial y + \Gamma^3_{33} \partial z + \Gamma^4_{33} \partial w \right), \\
&+ w \left(\Gamma^1_{34} \partial x + \Gamma^2_{34} \partial y + \Gamma^3_{34} \partial z + \Gamma^4_{34} \partial w \right). \\
&= \left(\Gamma^1_{13} + z \Gamma^1_{33} + w \Gamma^1_{34} \right) \partial x + \left(\Gamma^2_{13} + z \Gamma^2_{33} + w \Gamma^2_{34} \right) \partial y, \\
&+ \left(\Gamma^3_{13} + 1 + z \Gamma^3_{33} + w \Gamma^3_{34} \right) \partial z + \left(\Gamma^4_{13} + z \Gamma^4_{33} + w \Gamma^4_{34} \right) \partial w.
\end{aligned}$$

The right hand side of the 3.2.21 gives

$$\frac{1}{2} \left[E_1, E_3 \right] = \frac{1}{2} \, \partial z. \tag{3.2.23}$$

and so equating the sides give the following equation:

$$\begin{aligned}
&\left(\Gamma^1_{13} + z\Gamma^1_{33} + w\Gamma^1_{34}\right)\partial x + \left(\Gamma^2_{13} + z\Gamma^2_{33} + w\Gamma^2_{34}\right)\partial y,\\
&+ \left(\Gamma^3_{13} + \tfrac{1}{2} + z\Gamma^3_{33} + w\Gamma^3_{34}\right)\partial z + \left(\Gamma^4_{13} + z\Gamma^4_{33} + w\Gamma^4_{34}\right)\partial w.\\
&= 0.
\end{aligned} \tag{3.2.24}$$

Then, we obtain the following system of equations:

$$\begin{aligned}
&\Gamma^1_{13} + z\Gamma^1_{33} + w\Gamma^1_{34} = 0, \qquad \Gamma^2_{13} + z\Gamma^2_{33} + w\Gamma^2_{34} = 0,\\
&\Gamma^3_{13} + \tfrac{1}{2} + z\Gamma^3_{33} + w\Gamma^3_{34} = 0, \quad \Gamma^4_{13} + z\Gamma^4_{33} + w\Gamma^4_{34} = 0.
\end{aligned} \tag{3.2.25}$$

4 - $\nabla_{E_1} E_4$

$$\nabla_{E_1} E_4 = \frac{1}{2}\,[E_1, E_4]. \tag{3.2.26}$$

The left hand side of 3.2.26 gives the following:

$$
\begin{aligned}
\nabla_{E_1} E_4 &= \nabla_{\partial z} \left(\partial y + w\partial z - z\partial w \right). \\
&= \Gamma^k_{23} \partial x^k + \nabla_{\partial z}(w\partial z) - \nabla_{\partial z}(z\partial w). \\
&= \Gamma^1_{23}\partial x + \Gamma^2_{23}\partial y + \Gamma^3_{23}\partial z + \Gamma^4_{23}\partial w + \tfrac{\partial}{\partial z}(w)\partial z, \\
&+ w\nabla_{\partial z}(\partial z) - \tfrac{\partial}{\partial z}(z)\partial w - z\nabla_{\partial z}(\partial w). \\
&= \Gamma^1_{23}\partial x + \Gamma^2_{23}\partial y + \Gamma^3_{23}\partial z + \Gamma^4_{23}\partial w - \partial w, \\
&+ w\left(\Gamma^1_{33}\partial x + \Gamma^2_{33}\partial y + \Gamma^3_{33}\partial z + \Gamma^4_{33}\partial w \right), \\
&- z\left(\Gamma^1_{34}\partial x + \Gamma^2_{34}\partial y + \Gamma^3_{34}\partial z + \Gamma^4_{34}\partial w \right). \\
&= \left(\Gamma^1_{23} + w\Gamma^1_{33} - z\Gamma^1_{34} \right)\partial x + \left(\Gamma^2_{23} + w\Gamma^2_{33} - z\Gamma^2_{34} \right)\partial y \\
&+ \left(\Gamma^3_{23} + w\Gamma^3_{33} - z\Gamma^3_{34} \right)\partial z + \left(\Gamma^4_{23} + w\Gamma^4_{33} - 1 - z\Gamma^4_{34} \right)\partial w.
\end{aligned}
\tag{3.2.27}
$$

The right hand side of 3.2.26 gives

$$
\frac{1}{2}\ [E_1, E_4] = -\frac{1}{2}\ E_2 = -\frac{1}{2}\ \partial w. \tag{3.2.28}
$$

and so equating the sides give the following equation:

$$
\begin{aligned}
&\left(\Gamma^1_{23} + w\Gamma^1_{33} - z\Gamma^1_{34}\right)\partial x + \left(\Gamma^2_{23} + w\Gamma^2_{33} - z\Gamma^2_{34}\right)\partial y, \\
&+ \left(\Gamma^3_{23} + w\Gamma^3_{33} - z\Gamma^3_{34}\right)\partial z + \left(\Gamma^4_{23} + w\Gamma^4_{33} - z\Gamma^4_{34} - \tfrac{1}{2}\right)\partial w. \\
&= 0.
\end{aligned} \tag{3.2.29}
$$

Then, we obtain the following system of equations:

$$
\begin{aligned}
&\Gamma^1_{23} + w\Gamma^1_{33} - z\Gamma^1_{34} = 0, \quad \Gamma^2_{23} + w\Gamma^2_{33} - z\Gamma^2_{34} = 0, \\
&\Gamma^3_{23} + w\Gamma^3_{33} - z\Gamma^3_{34} = 0, \quad \Gamma^4_{23} + w\Gamma^4_{33} - z\Gamma^4_{34} - \tfrac{1}{2} = 0.
\end{aligned} \tag{3.2.30}
$$

5 -$\nabla_{E_2}E_2$

$$\nabla_{E_2}E_2 = \frac{1}{2}\,[E_2, E_2]. \tag{3.2.31}$$

The left hand side of 3.2.31 gives the following:

$$
\begin{aligned}
\nabla_{E_2}E_2 &= \nabla_{\partial w}\left(\partial w\right). \\
&= \left(\Gamma^k_{44}\partial x^k\right). \\
&= \Gamma^1_{44}\partial x + \Gamma^2_{44}\partial y + \Gamma^3_{44}\partial z + \Gamma^4_{44}\partial w.
\end{aligned} \tag{3.2.32}
$$

The right hand side of 3.2.31 gives

$$\frac{1}{2}\,[E_2, E_2] = 0. \tag{3.2.33}$$

Then, we obtain the following system of equations:

$$\Gamma^1_{44} = 0, \quad \Gamma^2_{44} = 0, \quad \Gamma^3_{44} = 0, \quad \Gamma^4_{44} = 0. \tag{3.2.34}$$

6 - $\nabla_{E_2} E_3$

$$\nabla_{E_2} E_3 = \frac{1}{2}\ [E_2, E_3]. \tag{3.2.35}$$

The left hand side of 3.2.35 gives the following:

$$\begin{aligned}
\nabla_{E_2} E_3 \ &= \nabla_{\partial w}\left(\partial x + z\partial z + w\partial w\right). \\
&= \Gamma^k_{14}\partial x^k + \nabla_{\partial w}(z\partial z) + \nabla_{\partial w}(w\partial w). \\
&= \Gamma^1_{14}\partial x + \Gamma^2_{14}\partial y + \Gamma^3_{14}\partial z + \Gamma^4_{14}\partial w + \tfrac{\partial}{\partial w}(z)\partial z, \\
&+ z\nabla_{\partial w}(\partial z) + \tfrac{\partial}{\partial w}(w)\partial w + w\nabla_{\partial w}(\partial w). \\
&= \Gamma^1_{14}\partial x + \Gamma^2_{14}\partial y + \Gamma^3_{14}\partial z + \Gamma^4_{14}\partial w, \\
&+ z\left(\Gamma^1_{34}\partial x + \Gamma^2_{34}\partial y + \Gamma^3_{34}\partial z + \Gamma^4_{34}\partial w\right) + \partial w, \\
&+ w\left(\Gamma^1_{44}\partial x + \Gamma^2_{44}\partial y + \Gamma^3_{44}\partial z + \Gamma^4_{44}\partial w\right). \\
&= \left(\Gamma^1_{14} + z\Gamma^1_{34} + w\Gamma^1_{44}\right)\partial x + \left(\Gamma^2_{14} + z\Gamma^2_{34} + w\Gamma^2_{44}\right)\partial y, \\
&+ \left(\Gamma^3_{14} + z\Gamma^3_{34} + w\Gamma^3_{44}\right)\partial z + \left(\Gamma^4_{14} + z\Gamma^4_{34} + w\Gamma^4_{34} + 1\right)\partial w.
\end{aligned} \tag{3.2.36}$$

The right hand side of the 3.2.35 gives

$$\frac{1}{2}\ [E_2, E_3] = \frac{1}{2}\ E_2 = \frac{1}{2}\ \partial w. \tag{3.2.37}$$

and so equating the sides give the following equation:

$$
\begin{aligned}
&\left(\Gamma^1_{14} + z\Gamma^1_{34} + w\Gamma^1_{44}\right)\partial x + \left(\Gamma^2_{14} + z\Gamma^2_{34} + w\Gamma^2_{44}\right)\partial y,\\
&+ \left(\Gamma^3_{14} + z\Gamma^3_{34} + w\Gamma^3_{44}\right)\partial z + \left(\Gamma^4_{14} + z\Gamma^4_{34} + w\Gamma^4_{44} + \tfrac{1}{2}\right)\partial w.\\
&= 0.
\end{aligned}
\tag{3.2.38}
$$

Then, we obtain the following system of equations:

$$
\begin{aligned}
&\Gamma^1_{14} + z\Gamma^1_{34} + w\Gamma^1_{44} = 0, \quad \Gamma^2_{14} + z\Gamma^2_{34} + w\Gamma^2_{44} = 0,\\
&\Gamma^3_{14} + z\Gamma^3_{34} + w\Gamma^3_{44} = 0, \quad \Gamma^4_{14} + z\Gamma^4_{34} + w\Gamma^4_{44} + \tfrac{1}{2} = 0.
\end{aligned}
\tag{3.2.39}
$$

7 - $\nabla_{E_2} E_4$

$$
\nabla_{E_2} E_4 = \frac{1}{2}\,[E_2, E_4]. \tag{3.2.40}
$$

The left hand side of 3.2.40 gives the following:

$$\begin{aligned}
\nabla_{E_2} E_4 \quad &= \nabla_{\partial w} \left(\partial y + w \partial z - z \partial w \right). \\
&= \Gamma^k_{24} \partial x^k + \nabla_{\partial w}(w \partial z) - \nabla_{\partial w}(z \partial w). \\
&= \Gamma^1_{24} \partial x + \Gamma^2_{24} \partial y + \Gamma^3_{24} \partial z + \Gamma^4_{24} \partial w + \tfrac{\partial}{\partial w}(w) \partial z, \\
&+ w \nabla_{\partial w}(\partial z) - \tfrac{\partial}{\partial w}(z) \partial w - z \nabla_{\partial w}(\partial w). \\
&= \Gamma^1_{24} \partial x + \Gamma^2_{24} \partial y + \Gamma^3_{24} \partial z + \Gamma^4_{24} \partial w + \partial z, \qquad (3.2.41) \\
&+ w \left(\Gamma^1_{34} \partial x + \Gamma^2_{34} \partial y + \Gamma^3_{34} \partial z + \Gamma^4_{34} \partial w \right), \\
&- z \left(\Gamma^1_{44} \partial x + \Gamma^2_{44} \partial y + \Gamma^3_{44} \partial z + \Gamma^4_{44} \partial w \right). \\
&= \left(\Gamma^1_{24} + w \Gamma^1_{34} - z \Gamma^1_{44} \right) \partial x + \left(\Gamma^2_{24} + w \Gamma^2_{34} - z \Gamma^2_{44} \right) \partial y, \\
&+ \left(\Gamma^3_{24} + w \Gamma^3_{34} - z \Gamma^3_{44} + 1 \right) \partial z + \left(\Gamma^4_{24} + w \Gamma^4_{34} - z \Gamma^4_{44} \right) \partial w.
\end{aligned}$$

The right hand side of 3.2.40 gives

$$\frac{1}{2}\ [E_2, E_4] = \frac{1}{2}\ E_1 = \frac{1}{2}\ \partial z. \qquad (3.2.42)$$

and so equating the sides give the following equation:

$$\begin{aligned}
&\left(\Gamma^1_{24} + w\Gamma^1_{34} - z\Gamma^1_{44}\right)\partial x + \left(\Gamma^2_{24} + w\Gamma^2_{34} - z\Gamma^2_{44}\right)\partial y,\\
&+ \left(\Gamma^3_{24} + w\Gamma^3_{34} - z\Gamma^3_{44} + \tfrac{1}{2}\right)\partial z + \left(\Gamma^4_{24} + w\Gamma^4_{34} - z\Gamma^4_{44}\right)\partial w.\\
&= 0.
\end{aligned} \tag{3.2.43}$$

Then, we obtain the following system of equations:

$$\begin{aligned}
&\Gamma^1_{24} + w\Gamma^1_{34} - z\Gamma^1_{44} = 0, \qquad &&\Gamma^2_{24} + w\Gamma^2_{34} - z\Gamma^2_{44} = 0,\\
&\Gamma^3_{24} + w\Gamma^3_{34} - z\Gamma^3_{44} + \tfrac{1}{2} = 0, &&\Gamma^4_{24} + w\Gamma^4_{34} - z\Gamma^4_{44} = 0.
\end{aligned} \tag{3.2.44}$$

8 - $\nabla_{E_3} E_3$

$$\nabla_{E_3} E_3 = \frac{1}{2}\,[E_3, E_3]. \tag{3.2.45}$$

The left hand side of 3.2.45 gives the following:

$$\begin{aligned}
\nabla_{E_3} E_3 &= \nabla_{(\partial x + z\partial z + w\partial w)}\left(\partial x + z\partial z + w\partial w\right).\\
&= \nabla_{(\partial x + z\partial z + w\partial w)}\left(\partial x\right) + \nabla_{(\partial x + z\partial z + w\partial w)}\left(z\partial z\right) + \nabla_{(\partial x + z\partial z + w\partial w)}\left(w\partial w\right).
\end{aligned} \tag{3.2.46}$$

$$
\begin{aligned}
\nabla_{(\partial x+z\partial z+w\partial w)}\left(\partial x\right) &= \nabla_{\partial x}\left(\partial x\right)+z\nabla_{\partial z}\left(\partial x\right)+w\nabla_{\partial w}\left(\partial x\right). \\
&= \Gamma^{k}_{11}\partial x^{k}+z\Gamma^{k}_{13}\partial x^{k}+w\Gamma^{k}_{14}\partial x^{k}. \\
&= \Gamma^{1}_{11}\partial x+\Gamma^{2}_{11}\partial y+\Gamma^{3}_{11}\partial z+\Gamma^{4}_{11}\partial w, \\
&+z\left(\Gamma^{1}_{13}\partial x+\Gamma^{2}_{13}\partial y+\Gamma^{3}_{13}\partial z+\Gamma^{4}_{13}\partial w\right), \\
&+w\left(\Gamma^{1}_{14}\partial x+\Gamma^{2}_{14}\partial y+\Gamma^{3}_{14}\partial z+\Gamma^{4}_{14}\partial w\right). \\
&= \left(\Gamma^{1}_{11}+z\Gamma^{1}_{13}+w\Gamma^{1}_{14}\right)\partial x+\left(\Gamma^{2}_{11}+z\Gamma^{2}_{13}+w\Gamma^{2}_{14}\right)\partial y, \\
&+\left(\Gamma^{3}_{11}+z\Gamma^{3}_{13}+w\Gamma^{3}_{14}\right)\partial z+\left(\Gamma^{4}_{11}+z\Gamma^{4}_{13}+w\Gamma^{4}_{14}\right)\partial w.
\end{aligned}
\tag{3.2.47}
$$

$$
\begin{aligned}
\nabla_{(\partial x+z\partial z+w\partial w)}\left(z\partial z\right) &= \nabla_{\partial x}\left(z\partial z\right)+z\nabla_{\partial z}\left(z\partial z\right)+w\nabla_{\partial w}\left(z\partial z\right).\\
&= \left(\tfrac{\partial}{\partial x}(z)\partial z+z\nabla_{\partial x}\left(\partial z\right)\right)+z\left(\tfrac{\partial}{\partial z}(z)\partial z+z\nabla_{\partial z}(\partial z)\right),\\
&+w\left(\tfrac{\partial}{\partial w}(z)\partial z+z\nabla_{\partial w}(\partial z)\right).\\
&= z\Gamma^{k}_{31}\partial x^{k}+z\partial z+z^{2}\Gamma^{k}_{33}\partial x^{k}+zw\Gamma^{k}_{34}\partial x^{k}.\\
&= z\left(\Gamma^{1}_{31}\partial x+\Gamma^{2}_{31}\partial y+\Gamma^{3}_{31}\partial z+\Gamma^{4}_{31}\partial w\right)+z\partial z,\\
&+z^{2}\left(\Gamma^{1}_{33}\partial x+\Gamma^{2}_{33}\partial y+\Gamma^{3}_{33}\partial z+\Gamma^{4}_{33}\partial w\right),\\
&+zw\left(\Gamma^{1}_{34}\partial x+\Gamma^{2}_{34}\partial y+\Gamma^{3}_{34}\partial z+\Gamma^{4}_{34}\partial w\right).\\
&= \left(z\Gamma^{1}_{31}+z^{2}\Gamma^{1}_{33}+zw\Gamma^{1}_{34}\right)\partial x+\left(z\Gamma^{2}_{31}+z^{2}\Gamma^{2}_{33}+zw\Gamma^{2}_{34}\right)\partial y,\\
&+\left(z\Gamma^{3}_{31}+z^{2}\Gamma^{3}_{33}+zw\Gamma^{3}_{34}+z\right)\partial z+\left(z\Gamma^{4}_{31}+z^{2}\Gamma^{4}_{33}+zw\Gamma^{4}_{34}\right)\partial w.
\end{aligned}
\tag{3.2.48}
$$

$$
\begin{aligned}
\nabla_{(\partial x+z\partial z+w\partial w)}(w\partial w) &= \nabla_{\partial x}(w\partial w) + z\nabla_{\partial z}(w\partial w) + w\nabla_{\partial w}(w\partial w)\,. \\
&= \left(\tfrac{\partial}{\partial x}(w)\partial w + w\nabla_{\partial x}(\partial w)\right) + z\left(\tfrac{\partial}{\partial z}(w)\partial w + w\nabla_{\partial z}(\partial w)\right), \\
&+w\left(\tfrac{\partial}{\partial w}(w)\partial w + w\nabla_{\partial w}(\partial w)\right). \\
&= w\Gamma^k_{41}\partial x^k + zw\Gamma^k_{43}\partial x^k + w\partial w + w^2\Gamma^k_{44}\partial x^k. \\
&= w\left(\Gamma^1_{41}\partial x + \Gamma^2_{41}\partial y + \Gamma^3_{41}\partial z + \Gamma^4_{41}\partial w\right), \\
&+zw\left(\Gamma^1_{43}\partial x + \Gamma^2_{43}\partial y + \Gamma^3_{43}\partial z + \Gamma^4_{43}\partial w\right), \\
&+w\partial w + w^2\left(\Gamma^1_{44}\partial x + \Gamma^2_{44}\partial y + \Gamma^3_{44}\partial z + \Gamma^4_{44}\partial w\right). \\
&= \left(w\Gamma^1_{41} + zw\Gamma^1_{43} + w^2\Gamma^1_{44}\right)\partial x + \left(w\Gamma^2_{41} + zw\Gamma^2_{43} + w^2\Gamma^2_{44}\right)\partial y, \\
&+\left(w\Gamma^3_{41} + zw\Gamma^3_{43} + w^2\Gamma^3_{44}\right)\partial z + \left(w\Gamma^4_{41} + zw\Gamma^4_{43} + w^2\Gamma^4_{44} + w\right)\partial w.
\end{aligned}
\tag{3.2.49}
$$

The right hand side of the 3.2.45 gives

$$
\frac{1}{2}\left[E_3, E_3\right] = 0. \tag{3.2.50}
$$

by adding 3.2.47, 3.2.48, and 3.2.49 and so equating the sides give the following equation:

$$
\begin{aligned}
&(\Gamma^1_{11} + 2z\Gamma^1_{13} + 2w\Gamma^1_{14} + z^2\Gamma^1_{33} + 2zw\Gamma^1_{43} + w^2\Gamma^1_{44})\,\partial x,\\
&+ (\Gamma^2_{11} + 2z\Gamma^2_{13} + 2w\Gamma^2_{14} + z^2\Gamma^2_{33} + 2zw\Gamma^2_{43} + w^2\Gamma^2_{44})\,\partial y,\\
&+ (\Gamma^3_{11} + 2z\Gamma^3_{13} + 2w\Gamma^3_{14} + z^2\Gamma^3_{33} + 2zw\Gamma^3_{43} + w^2\Gamma^3_{44} + z)\,\partial z,\\
&+ (\Gamma^4_{11} + 2z\Gamma^4_{13} + 2w\Gamma^4_{14} + z^2\Gamma^4_{33} + 2zw\Gamma^4_{43} + w^2\Gamma^4_{44} + w)\,\partial w.\\
&= 0.
\end{aligned}
\tag{3.2.51}
$$

Then, we obtain the following system of equations:

$$
\begin{aligned}
\Gamma^1_{11} + 2z\Gamma^1_{31} + 2w\Gamma^1_{14} + z^2\Gamma^1_{33} + 2zw\Gamma^1_{43} + w^2\Gamma^1_{44} &= 0.\\
\Gamma^2_{11} + 2z\Gamma^2_{31} + 2w\Gamma^2_{14} + z^2\Gamma^2_{33} + 2zw\Gamma^2_{43} + w^2\Gamma^2_{44} &= 0.\\
\Gamma^3_{11} + 2z\Gamma^3_{31} + 2w\Gamma^3_{14} + z^2\Gamma^3_{33} + 2zw\Gamma^3_{43} + w^2\Gamma^3_{44} + z &= 0.\\
\Gamma^4_{11} + 2z\Gamma^4_{31} + 2w\Gamma^4_{14} + z^2\Gamma^4_{33} + 2zw\Gamma^4_{43} + w^2\Gamma^4_{44} + w &= 0.
\end{aligned}
\tag{3.2.52}
$$

9 - $\nabla_{E_4} E_3$

$$\nabla_{E_4} E_3 = \frac{1}{2}\,[E_4, E_3]. \tag{3.2.53}$$

The left hand side of 3.2.53 gives the following:

$$\begin{aligned}\nabla_{E_4} E_3 &= \nabla_{(\partial y + w\partial z - z\partial w)}\left(\partial x + z\partial z + w\partial w\right). \\ &= \nabla_{(\partial y + w\partial z - z\partial w)}\left(\partial x\right) + \nabla_{(\partial y + w\partial z - z\partial w)}\left(z\partial z\right) + \nabla_{(\partial y + w\partial z - z\partial w)}\left(w\partial w\right).\end{aligned} \tag{3.2.54}$$

$$\begin{aligned}\nabla_{(\partial y + w\partial z - z\partial w)}\left(\partial x\right) &= \nabla_{\partial y}\left(\partial x\right) + w\nabla_{\partial z}\left(\partial x\right) - z\nabla_{\partial w}\left(\partial x\right). \\ &= \Gamma^k_{12}\partial x^k + w\Gamma^k_{13}\partial x^k - z\Gamma^k_{14}\partial x^k. \\ &= \Gamma^1_{12}\partial x + \Gamma^2_{12}\partial y + \Gamma^3_{12}\partial z + \Gamma^4_{12}\partial w, \\ &+ w\left(\Gamma^1_{13}\partial x + \Gamma^2_{13}\partial y + \Gamma^3_{13}\partial z + \Gamma^4_{13}\partial w\right), \\ &- z\left(\Gamma^1_{14}\partial x + \Gamma^2_{14}\partial y + \Gamma^3_{14}\partial z + \Gamma^4_{14}\partial w\right). \\ &= \left(\Gamma^1_{12} + w\Gamma^1_{13} - z\Gamma^1_{14}\right)\partial x + \left(\Gamma^2_{12} + w\Gamma^2_{13} - z\Gamma^2_{14}\right)\partial y, \\ &+ \left(\Gamma^3_{12} + w\Gamma^3_{13} - z\Gamma^3_{14}\right)\partial z + \left(\Gamma^4_{12} + w\Gamma^4_{13} - z\Gamma^4_{14}\right)\partial w.\end{aligned} \tag{3.2.55}$$

$$
\begin{aligned}
\nabla_{(\partial y+w\partial z-z\partial w)}\left(z\partial z\right) &= \nabla_{\partial y}\left(z\partial z\right)+w\nabla_{\partial z}\left(z\partial z\right)-z\nabla_{\partial w}\left(z\partial z\right).\\
&= \left(\tfrac{\partial}{\partial y}(z)\partial z+z\nabla_{\partial y}(\partial z)\right)+w\left(\tfrac{\partial}{\partial z}(z)\partial z+z\nabla_{\partial z}(\partial z)\right),\\
&-z\left(\tfrac{\partial}{\partial w}(z)\partial z+z\nabla_{\partial w}(\partial z)\right).\\
&= z\Gamma^{k}_{32}\partial x^{k}+w\partial z+zw\Gamma^{k}_{33}\partial x^{k}-z^{2}\Gamma^{k}_{34}\partial x^{k}.\\
&= z\left(\Gamma^{1}_{32}\partial x+\Gamma^{2}_{32}\partial y+\Gamma^{3}_{32}\partial z+\Gamma^{4}_{32}\partial w\right)+w\partial z,\\
&+wz\left(\Gamma^{1}_{33}\partial x+\Gamma^{2}_{33}\partial y+\Gamma^{3}_{33}\partial z+\Gamma^{4}_{33}\partial w\right),\\
&-z^{2}\left(\Gamma^{1}_{34}\partial x+\Gamma^{2}_{34}\partial y+\Gamma^{3}_{34}\partial z+\Gamma^{4}_{34}\partial w\right).\\
&= \left(z\Gamma^{1}_{32}+zw\Gamma^{1}_{33}-z^{2}\Gamma^{1}_{34}\right)\partial x+\left(z\Gamma^{2}_{32}+zw\Gamma^{2}_{33}-z^{2}\Gamma^{2}_{34}\right)\partial y,\\
&+\left(z\Gamma^{3}_{32}+zw\Gamma^{3}_{33}-z^{2}\Gamma^{3}_{34}+w\right)\partial z+\left(z\Gamma^{4}_{32}+zw\Gamma^{4}_{33}-z^{2}\Gamma^{4}_{34}\right)\partial w.
\end{aligned} \tag{3.2.56}
$$

$$
\begin{aligned}
\nabla_{(\partial y + w\partial z - z\partial w)}(w\partial w) &= \nabla_{\partial y}(w\partial w) + w\nabla_{\partial z}(w\partial w) - z\nabla_{\partial w}(w\partial w). \\
&= \left(\tfrac{\partial}{\partial y}(w)\partial w + w\nabla_{\partial y}(\partial w)\right) + w\left(\tfrac{\partial}{\partial z}(w)\partial w + w\nabla_{\partial z}(\partial w)\right), \\
&-z\left(\tfrac{\partial}{\partial w}(w)\partial w + w\nabla_{\partial w}(\partial w)\right). \\
&= w\Gamma^{k}_{42}\partial x^{k} + w^{2}\Gamma^{k}_{43}\partial x^{k} - z\partial w - zw\Gamma^{k}_{44}\partial x^{k}. \\
&= w\left(\Gamma^{1}_{42}\partial x + \Gamma^{2}_{42}\partial y + \Gamma^{3}_{42}\partial z + \Gamma^{4}_{42}\partial w\right) - z\partial w, \\
&+w^{2}\left(\Gamma^{1}_{43}\partial x + \Gamma^{2}_{43}\partial y + \Gamma^{3}_{43}\partial z + \Gamma^{4}_{43}\partial w\right), \\
&-wz\left(\Gamma^{1}_{44}\partial x + \Gamma^{2}_{44}\partial y + \Gamma^{3}_{44}\partial z + \Gamma^{4}_{44}\partial w\right). \\
&= \left(w\Gamma^{1}_{42} + w^{2}\Gamma^{1}_{43} - zw\Gamma^{1}_{44}\right)\partial x + \left(w\Gamma^{2}_{42} + w^{2}\Gamma^{2}_{43} - zw\Gamma^{2}_{44}\right)\partial y, \\
&+ \left(w\Gamma^{3}_{42} + w^{2}\Gamma^{3}_{43} - zw\Gamma^{3}_{44}\right)\partial z + \left(w\Gamma^{4}_{42} + w^{2}\Gamma^{4}_{43} - zw\Gamma^{4}_{44} - z\right)\partial w.
\end{aligned} \tag{3.2.57}
$$

The right hand side of the 3.2.53 gives

$$
\frac{1}{2}\left[E_4, E_3\right] = 0. \tag{3.2.58}
$$

by adding 3.2.55, 3.2.56, and 3.2.57 and so equating the sides give the following equation:

$$
\begin{aligned}
&(\Gamma^1_{12} + w\Gamma^1_{13} - z\Gamma^1_{14} + z\Gamma^1_{32} + zw\Gamma^1_{33} - z^2\Gamma^1_{34} + w\Gamma^1_{42} + w^2\Gamma^1_{43} - zw\Gamma^1_{44})\,\partial x,\\
&+(\Gamma^2_{12} + w\Gamma^2_{13} - z\Gamma^2_{14} + z\Gamma^2_{32} + zw\Gamma^2_{33} - z^2\Gamma^2_{34} + w\Gamma^2_{42} + w^2\Gamma^2_{43} - zw\Gamma^2_{44})\,\partial y,\\
&+(\Gamma^3_{12} + w\Gamma^3_{13} - z\Gamma^3_{14} + z\Gamma^3_{32} + zw\Gamma^3_{33} - z^2\Gamma^3_{34} + w\Gamma^3_{42} + w^2\Gamma^3_{43} - zw\Gamma^3_{44} + w)\,\partial z,\\
&+(\Gamma^4_{12} + w\Gamma^4_{13} - z\Gamma^4_{14} + z\Gamma^4_{32} + zw\Gamma^4_{33} - z^2\Gamma^4_{34} + w\Gamma^4_{42} + w^2\Gamma^4_{43} - zw\Gamma^4_{44} - z)\,\partial w = 0.
\end{aligned}
\tag{3.2.59}
$$

Then, we obtain the following system of equations:

$$
\begin{aligned}
\Gamma^1_{12} + w\Gamma^1_{13} - z\Gamma^1_{14} + z\Gamma^1_{32} + zw\Gamma^1_{33} - z^2\Gamma^1_{34} + w\Gamma^1_{42} + w^2\Gamma^1_{43} - zw\Gamma^1_{44} &= 0.\\
\Gamma^2_{12} + w\Gamma^2_{13} - z\Gamma^2_{14} + z\Gamma^2_{32} + zw\Gamma^2_{33} - z^2\Gamma^2_{34} + w\Gamma^2_{42} + w^2\Gamma^2_{43} - zw\Gamma^2_{44} &= 0.\\
\Gamma^3_{12} + w\Gamma^3_{13} - z\Gamma^3_{14} + z\Gamma^3_{32} + zw\Gamma^3_{33} - z^2\Gamma^3_{34} + w\Gamma^3_{42} + w^2\Gamma^3_{43} - zw\Gamma^3_{44} + w &= 0.\\
\Gamma^4_{12} + w\Gamma^4_{13} - z\Gamma^4_{14} + z\Gamma^4_{32} + zw\Gamma^4_{33} - z^2\Gamma^4_{34} + w\Gamma^4_{42} + w^2\Gamma^4_{43} - zw\Gamma^4_{44} - z &= 0.
\end{aligned}
\tag{3.2.60}
$$

10 - $\nabla_{E_4} E_4$

$$\nabla_{E_4} E_4 = \frac{1}{2}\,[E_4, E_4]. \tag{3.2.61}$$

The left hand side of 3.2.61 gives the following:

$$\begin{aligned}\nabla_{E_4} E_4 &= \nabla_{(\partial y + w\partial z - z\partial w)} \left(\partial y + w\partial z - z\partial w\right). \\ &= \nabla_{(\partial y + w\partial z - z\partial w)} \left(\partial y\right) + \nabla_{(\partial y + w\partial z - z\partial w)} \left(w\partial z\right) + \nabla_{(\partial y + w\partial z - z\partial w)} \left(-z\partial w\right).\end{aligned} \tag{3.2.62}$$

$$\begin{aligned}\nabla_{(\partial y + w\partial z - z\partial w)} \left(\partial y\right) &= \nabla_{\partial y} \left(\partial y\right) + w\nabla_{\partial z} \left(\partial y\right) - z\nabla_{\partial w} \left(\partial y\right). \\ &= \Gamma^k_{22}\partial x^k + w\Gamma^k_{23}\partial x^k - z\Gamma^k_{24}\partial x^k. \\ &= \Gamma^1_{22}\partial x + \Gamma^2_{22}\partial y + \Gamma^3_{22}\partial z + \Gamma^4_{22}\partial w, \\ &+ w\left(\Gamma^1_{23}\partial x + \Gamma^2_{23}\partial y + \Gamma^3_{23}\partial z + \Gamma^4_{23}\partial w\right), \\ &- z\left(\Gamma^1_{24}\partial x + \Gamma^2_{24}\partial y + \Gamma^3_{24}\partial z + \Gamma^4_{24}\partial w\right). \\ &= \left(\Gamma^1_{22} + w\Gamma^1_{23} - z\Gamma^1_{24}\right)\partial x + \left(\Gamma^2_{22} + w\Gamma^2_{23} - z\Gamma^2_{24}\right)\partial y, \\ &+ \left(\Gamma^3_{22} + w\Gamma^3_{23} - z\Gamma^3_{24}\right)\partial z + \left(\Gamma^4_{22} + w\Gamma^4_{23} - z\Gamma^4_{24}\right)\partial w.\end{aligned} \tag{3.2.63}$$

$$
\begin{aligned}
\nabla_{(\partial y + w\partial z - z\partial w)}\left(w\partial z\right) &= \nabla_{\partial y}\left(w\partial z\right) + w\nabla_{\partial z}\left(w\partial z\right) - z\nabla_{\partial w}\left(w\partial z\right). \\
&= \left(\frac{\partial}{\partial y}(w)\partial z + w\nabla_{\partial y}(\partial z)\right) + w\left(\frac{\partial}{\partial z}(w)\partial z + w\nabla_{\partial z}(\partial z)\right), \\
&-z\left(\frac{\partial}{\partial w}(w)\partial z + w\nabla_{\partial w}(\partial z)\right). \\
&= w\Gamma^k_{32}\partial x^k + w^2\Gamma^k_{33}\partial x^k - z\partial z - zw\Gamma^k_{34}\partial x^k. \\
&= w\left(\Gamma^1_{32}\partial x + \Gamma^2_{32}\partial y + \Gamma^3_{32}\partial z + \Gamma^4_{32}\partial w\right) - z\partial z, \\
&+w^2\left(\Gamma^1_{33}\partial x + \Gamma^2_{33}\partial y + \Gamma^3_{33}\partial z + \Gamma^4_{33}\partial w\right), \\
&-zw\left(\Gamma^1_{34}\partial x + \Gamma^2_{34}\partial y + \Gamma^3_{34}\partial z + \Gamma^4_{34}\partial w\right). \\
&= \left(w\Gamma^1_{32} + w^2\Gamma^1_{33} - zw\Gamma^1_{34}\right)\partial x + \left(w\Gamma^2_{32} + w^2\Gamma^2_{33} - zw\Gamma^2_{34}\right)\partial y, \\
&+\left(w\Gamma^3_{32} + w^2\Gamma^3_{33} - zw\Gamma^3_{34} - z\right)\partial z + \left(w\Gamma^4_{32} + w^2\Gamma^4_{33} - zw\Gamma^4_{34}\right)\partial w.
\end{aligned}
\tag{3.2.64}
$$

$$
\begin{aligned}
\nabla_{(\partial y+w\partial z-z\partial w)}(-z\partial w) \;&= \nabla_{\partial y}(-z\partial w) + w\nabla_{\partial z}(-z\partial w) - z\nabla_{\partial w}(-z\partial w)\,.\\
&= \left(\tfrac{\partial}{\partial y}(-z)\partial w - z\nabla_{\partial y}(\partial w)\right) + w\left(\tfrac{\partial}{\partial z}(-z)\partial w - z\nabla_{\partial z}(\partial w)\right),\\
&-z\left(\tfrac{\partial}{\partial w}(-z)\partial w - z\nabla_{\partial w}(\partial w)\right).\\
&= -z\Gamma^k_{42}\partial x^k - w\partial w - zw\Gamma^k_{43}\partial x^k + z^2\Gamma^k_{44}\partial x^k.\\
&= -z\left(\Gamma^1_{42}\partial x + \Gamma^2_{42}\partial y + \Gamma^3_{42}\partial z + \Gamma^4_{42}\partial w\right) - w\partial w,\\
&-zw\left(\Gamma^1_{43}\partial x + \Gamma^2_{43}\partial y + \Gamma^3_{43}\partial z + \Gamma^4_{43}\partial w\right),\\
&+z^2\left(\Gamma^1_{44}\partial x + \Gamma^2_{44}\partial y + \Gamma^3_{44}\partial z + \Gamma^4_{44}\partial w\right).\\
&= \left(-z\Gamma^1_{42} - zw\Gamma^1_{43} + z^2\Gamma^1_{44}\right)\partial x,\\
&+\left(-z\Gamma^2_{42} - zw\Gamma^2_{43} + z^2\Gamma^2_{44}\right)\partial y,\\
&+\left(-z\Gamma^3_{42} - zw\Gamma^3_{43} + z^2\Gamma^3_{44}\right)\partial z,\\
&+\left(-z\Gamma^4_{42} - zw\Gamma^4_{43} + z^2\Gamma^4_{44} - w\right)\partial w.
\end{aligned}
\tag{3.2.65}
$$

The right hand side of the 3.2.61 gives

$$
\frac{1}{2}\,[E_4, E_4] = 0. \tag{3.2.66}
$$

by adding 3.2.63, 3.2.64, and 3.2.65 and so equating the sides give the following equation:

$$
\begin{aligned}
&(\Gamma^1_{22} + 2w\Gamma^1_{23} - 2z\Gamma^1_{24} + w^2\Gamma^1_{33} - 2zw\Gamma^1_{43} + z^2\Gamma^1_{44})\,\partial x,\\
&+(\Gamma^2_{22} + 2w\Gamma^2_{23} - 2z\Gamma^2_{24} + w^2\Gamma^2_{33} - 2zw\Gamma^2_{43} + z^2\Gamma^2_{44})\,\partial y,\\
&+(\Gamma^3_{22} + 2w\Gamma^3_{23} - 2z\Gamma^3_{24} + w^2\Gamma^3_{33} - 2zw\Gamma^3_{43} + z^2\Gamma^3_{44} - z)\,\partial z,\\
&+(\Gamma^4_{22} + 2w\Gamma^4_{23} - 2z\Gamma^4_{24} + w^2\Gamma^4_{33} - 2zw\Gamma^4_{43} + z^2\Gamma^4_{44} - w)\,\partial w = 0.
\end{aligned}
\tag{3.2.67}
$$

Then, we obtain the following system of equations:

$$
\begin{aligned}
\Gamma^1_{22} + 2w\Gamma^1_{23} - 2z\Gamma^1_{24} + w^2\Gamma^1_{33} - 2zw\Gamma^1_{43} + z^2\Gamma^1_{44} &= 0.\\
\Gamma^2_{22} + 2w\Gamma^2_{23} - 2z\Gamma^2_{24} + w^2\Gamma^2_{33} - 2zw\Gamma^2_{43} + z^2\Gamma^2_{44} &= 0.\\
\Gamma^3_{22} + 2w\Gamma^3_{23} - 2z\Gamma^3_{24} + w^2\Gamma^3_{33} - 2zw\Gamma^3_{43} + z^2\Gamma^3_{44} - z &= 0.\\
\Gamma^4_{22} + 2w\Gamma^4_{23} - 2z\Gamma^4_{24} + w^2\Gamma^4_{33} - 2zw\Gamma^4_{43} + z^2\Gamma^4_{44} - w &= 0.
\end{aligned}
\tag{3.2.68}
$$

From 3.2.44 we obtained:

$$
\Gamma^1_{24} = 0, \quad \Gamma^2_{24} = 0, \quad \Gamma^3_{24} = -\tfrac{1}{2}, \quad \Gamma^4_{24} = 0. \tag{3.2.69}
$$

Similarly by symmetry we have:

$$\begin{aligned} \Gamma^K_{42} &= 0, \quad (k = 1, 2, 4). \\ \Gamma^3_{42} &= -\tfrac{1}{2}. \end{aligned} \tag{3.2.70}$$

From 3.2.30 we obtained:

$$\Gamma^1_{23} = 0, \quad \Gamma^2_{23} = 0, \quad \Gamma^3_{23} = 0, \quad \Gamma^4_{23} = \tfrac{1}{2}. \tag{3.2.71}$$

Similarly by symmetry we have:

$$\begin{aligned} \Gamma^K_{32} &= 0, \quad (k = 1, 2, 3). \\ \Gamma^4_{32} &= \tfrac{1}{2}. \end{aligned} \tag{3.2.72}$$

From 3.2.68 we obtained:

$$\Gamma^1_{22} = 0, \quad \Gamma^2_{22} = 0, \quad \Gamma^3_{22} = 0, \quad \Gamma^4_{22} = 0. \tag{3.2.73}$$

From 3.2.39 we obtained:

$$\Gamma^1_{14} = 0, \quad \Gamma^2_{14} = 0, \quad \Gamma^3_{14} = 0, \quad \Gamma^4_{14} = -\tfrac{1}{2}. \tag{3.2.74}$$

Similarly by symmetry we have:

$$\Gamma^{K}_{41} = 0, \quad (k = 1, 2, 3).$$
$$\Gamma^{4}_{41} = -\tfrac{1}{2}. \tag{3.2.75}$$

From 3.2.25 we obtained:

$$\Gamma^{1}_{13} = 0, \quad \Gamma^{2}_{13} = 0, \quad \Gamma^{3}_{13} = -\tfrac{1}{2}, \quad \Gamma^{4}_{13} = 0. \tag{3.2.76}$$

Similarly by symmetry we have:

$$\Gamma^{K}_{31} = 0, \quad (k = 1, 2, 4).$$
$$\Gamma^{3}_{31} = -\tfrac{1}{2}. \tag{3.2.77}$$

From 3.2.60 we obtained:

$$\Gamma^{1}_{12} = 0, \quad \Gamma^{2}_{12} = 0, \quad \Gamma^{3}_{12} = 0, \quad \Gamma^{4}_{12} = 0. \tag{3.2.78}$$

Similarly by symmetry we have:

$$\Gamma^{K}_{21} \quad = 0, \quad (k = 1, 2, 3, 4). \tag{3.2.79}$$

From 3.2.52 we obtained:

$$\Gamma^{1}_{11} = 0, \quad \Gamma^{2}_{11} = 0, \quad \Gamma^{3}_{11} = 0, \quad \Gamma^{4}_{11} = 0. \tag{3.2.80}$$

After we applied all covariant derivatives, and use the definition of the canonical connection. We obtain the following components of the connection given by:

$$\Gamma^1_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \Gamma^3_{ij} = \begin{bmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{bmatrix},$$

$$\Gamma^2_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \Gamma^4_{ij} = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{bmatrix}. \tag{3.2.81}$$

Substitute the values of Γ^k_{ij} into the geodesic equations 3.2.2 for each $i, j, k = 1, 2, 3, 4$. Therefore we obtain the following system of geodesic equations:

$$\ddot{x} = 0, \quad \ddot{y} = 0, \quad \ddot{z} = \dot{x}\dot{z} + \dot{y}\dot{w}, \quad \ddot{w} = \dot{x}\dot{w} - \dot{y}\dot{z}. \tag{3.2.82}$$

3.2.2 The Lie Symmetry Conditions

In this section we compute the Lie Invariance Condition (L.I.C) to the system of ordinary differential equations given by 3.2.82. If Γ is an infinitesimal symmetry given by

$$\Gamma = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z} + W\frac{\partial}{\partial w}. \tag{3.2.83}$$

where T, X, Y, Z, W are functions of t, x, y, z, w. The the Lie invariance condition gives the following system of PFE's:

$$-2T_{x,z} - T_z = 0, \qquad 2W_{w,y} + Z_w - X_y + W_z - 2T_{t,y} = 0,$$

$$2W_{t,w} - X_t - T_{t,t} = 0, \qquad 2Z_{t,z} - X_t - T_{t,t} = 0,$$

$$W_{w,w} - X_w - 2T_{t,w} = 0, \qquad 2X_{w,x} + X_w - 2T_{t,w} = 0,$$

$$2W_{w,x} - X_x - 2T_{t,x} = 0, \qquad 2Z_{x,z} - X_x - 2T_{t,x} = 0,$$

$$Z_{z,z} - X_z - 2T_{t,z} = 0, \qquad 2X_{x,z} + X_z - 2T_{t,z} = 0,$$

$$2Y_{y,z} + Y_w - 2T_{t,z} = 0, \qquad 2Y_{w,y} + Y_z - 2T_{t,w} = 0,$$

$$2W_{x,y} + Z_x - W_y = 0, \qquad 2Z_{x,y} - Z_y - W_x = 0,$$

$$2W_{w,z} + Y_w - X_z - 2T_{t,z} = 0, \qquad 2Z_{w,y} + Z_z - W_w - Y_y = 0,$$

$$2Z_{w,z} - Y_z - X_w - 2T_{t,w} = 0, \qquad 2W_{y,z} + Z_z + Y_y - W_w = 0,$$

$$2Z_{y,z} - X_y - W_z - Z_w - 2T_{t,y} = 0,$$

$$
\begin{aligned}
&2X_{t,x} - T_{t,t} = 0, \quad X_{x,x} - 2T_{t,x} = 0, \quad 2Y_{t,y} - T_{t,t} = 0,\\
&2Y_{x,y} - 2T_{t,x} = 0, \quad Y_{y,y} - 2T_{t,y} = 0, \quad 2X_{x,y} - 2T_{t,y} = 0,\\
&-2T_{w,y} - T_z = 0, \quad -2T_{y,z} + T_w = 0, \quad 2W_{t,x} - W_t = 0,\\
&2Z_{t,y} - W_t = 0, \quad W_{x,x} - W_x = 0, \quad Z_{y,y} - W_y = 0,\\
&2X_{y,z} - X_w = 0, \quad 2X_{w,y} + X_z = 0, \quad 2Z_{t,w} - Y_t = 0,\\
&2W_{t,z} + Y_t = 0, \quad Z_{w,w} - Y_w = 0, \quad 2Y_{w,x} + Y_w = 0,\\
&2Z_{w,x} - Y_x = 0, \quad 2W_{x,z} + Y_x = 0, \quad W_{z,z} + Y_z = 0,\\
&2Y_{x,z} + Y_z = 0, \quad 2Z_{t,x} - Z_t = 0, \quad 2W_{t,y} + Z_t = 0,\\
&Z_{x,x} - Z_x = 0, \quad W_{y,y} + Z_y = 0, \quad -2T_{w,x} - T_w = 0,\\
&W_{t,t} = 0, \quad X_{t,t} = 0, \quad X_{t,w} = 0,\\
&X_{t,y} = 0, \quad X_{t,z} = 0, \quad X_{w,w} = 0,\\
&X_{w,z} = 0, \quad X_{y,y} = 0, \quad X_{z,z} = 0,\\
&Y_{t,t} = 0, \quad Y_{t,w} = 0, \quad Y_{t,x} = 0,\\
&Y_{t,z} = 0, \quad Y_{w,w} = 0, \quad Y_{w,z} = 0,\\
&Y_{x,x} = 0, \quad Y_{z,z} = 0, \quad Z_{t,t} = 0,\\
&T_{w,w} = 0, \quad T_{w,z} = 0, \quad T_{x,x} = 0,\\
&T_{x,y} = 0, \quad T_{y,y} = 0, \quad T_{z,z} = 0.
\end{aligned}
\tag{3.2.84}
$$

The most general solution to the above system of PDE's is giveb by

$$X = c_1,$$

$$Y = c_2,$$

$$Z = c_7 z - c_8\, w + (c_{11}\, \sin y - c_{10}\, \cos y)e^x + c_{12},$$

$$W = c_7\, w + c_8\, z + c_9 + c_{10}(\,\sin y + c_{11}\, \cos y)e^x,$$

$$T = c_3\, t + c_4\, x + c_5\, y + c_6.$$

3.2.3 The Infinitesimal Lie Symmetries

As we have seen in section (3.2.2), the solution to the PDE system has twelve parameters and so we obtain a basis for the solution given by

$$\begin{aligned}
e_1 &= D_x, & e_2 &= D_z,\\
e_3 &= D_y, & e_4 &= tD_t,\\
e_5 &= D_t, & e_6 &= D_w,\\
e_7 &= xD_t, & e_8 &= yD_t,\\
e_9 &= zD_z + wD_w, & e_{10} &= zD_w - wD_z,\\
e_{11} &= e^x \sin(y)D_z + e^x \cos(y)D_w, & e_{12} &= e^x \sin(y)D_w - e^x \cos(y)D_z.
\end{aligned} \tag{3.2.85}$$

Then we consider the change of basis for the nilradical $\{e_2, e_5, e_6, e_7, e_8, e_{11}, e_{12}\}$, and the complement $\{e_1, e_3, e_4, e_9, e_{10}\}$ as following

$$\begin{aligned}
&\overline{e}_1 = e_2, \quad \overline{e}_2 = e_5, \quad \overline{e}_3 = e_6, \quad \overline{e}_4 = e_7, \quad \overline{e}_5 = e_8, \quad \overline{e}_6 = e_{11}, \\
&\overline{e}_7 = e_{12}, \quad \overline{e}_8 = e_1, \quad \overline{e}_9 = e_3, \quad \overline{e}_{10} = e_4, \quad \overline{e}_{11} = e_9, \quad \overline{e}_{12} = e_{10}.
\end{aligned} \tag{3.2.86}$$

The non-zero Lie brackets of the symmetry algebra are given by:

$$\begin{aligned}
&[e_1, e_{11}] = e_1, \quad [e_1, e_{12}] = e_3, \quad [e_2, e_{10}] = e_2, \quad [e_3, e_{11}] = e_3, \quad [e_3, e_{12}] = -e_1, \\
&[e_4, e_8] = -e_2, \quad [e_4, e_{10}] = e_4, \quad [e_5, e_9] = -e_2, \quad [e_5, e_{10}] = e_5, \quad [e_6, e_8] = -e_6, \\
&[e_6, e_9] = e_7, \quad [e_6, e_{11}] = e_6, \quad [e_6, e_{12}] = e_7, \quad [e_7, e_8] = -e_7, \quad [e_7, e_9] = -e_6, \\
&[e_7, e_{11}] = e_7, \quad [e_7, e_{12}] = -e_6.
\end{aligned} \tag{3.2.87}$$

Proposition 3.2.1. We investigate the above symmetry algebra, and conclude that it's solvable with seven dimensional nilradical spanned by $e_1 - e_7$, and an abelian five dimensional complement spanned by $e_8 - e_{12}$. Therefore, the symmetry algebra can be identified as $(\mathbb{R}^7 \rtimes \mathbb{R}^5)$.

3.2.4 Lie Groups Corresponding to Symmetries

For four dimensional Lie algebra $A_{4,12}$, we can find Lie groups $\bar{X}$, $\bar{Y}$, $\bar{Z}$, and $\bar{W}$ by solving the system of PDE's

$$\begin{aligned} &\tfrac{\partial \overline{X}}{\partial \epsilon} = X(\overline{x}, \overline{y}, \overline{z}, \overline{w}), \quad \overline{X}|_{\epsilon=0} = x, \quad \tfrac{\partial \overline{Y}}{\partial \epsilon} = Y(\overline{x}, \overline{y}, \overline{z}, \overline{w}), \quad \overline{Y}|_{\epsilon=0} = y, \\ &\tfrac{\partial \overline{Z}}{\partial \epsilon} = Z(\overline{x}, \overline{y}, \overline{z}, \overline{w}), \quad \overline{Z}|_{\epsilon=0} = z, \quad \tfrac{\partial \overline{W}}{\partial \epsilon} = W(\overline{x}, \overline{y}, \overline{z}, \overline{w}), \quad \overline{W}|_{\epsilon=0} = w. \end{aligned} \tag{3.2.88}$$

using general form of symmetry:

$$X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z} + W\frac{\partial}{\partial w}. \tag{3.2.89}$$

Example 3.2.2. Consider

$$e_{10} = z\frac{\partial}{\partial w} - w\frac{\partial}{\partial z} \tag{3.2.90}$$

by comparison with 3.2.89 we obtained

$$X = 0, \quad Y = 0, \quad Z = -w, \quad W = z. \tag{3.2.91}$$

By using 3.2.88, we obtian

$$\begin{aligned} &\tfrac{\partial \overline{X}}{\partial \epsilon} = X(\overline{x}, \overline{y}, \overline{z}, \overline{w}) = 0, \quad \Longrightarrow \overline{X} = f(x, y, z, w), \quad \overline{X}|_{\epsilon=0} = x, \\ &\Longrightarrow f(x, y, z, w) = x, \quad \Longrightarrow \overline{X} = x. \end{aligned} \tag{3.2.92}$$

Similarly

$$\frac{\partial \overline{Y}}{\partial \epsilon} = Y(\overline{x}, \overline{y}, \overline{z}, \overline{w}) = 0, \quad \Longrightarrow \overline{Y} = f(x,y,z,w), \quad \overline{Y}|_{\epsilon=0} = y,$$
$$\Longrightarrow f(x,y,z,w) = y, \quad \Longrightarrow \overline{Y} = y. \tag{3.2.93}$$

$$\frac{\partial \overline{Z}}{\partial \epsilon} = Z(\overline{x}, \overline{y}, \overline{z}, \overline{w}) = -\overline{w}, \quad \Longrightarrow \overline{Z} = -\overline{w}\epsilon + f(x,y,z,w), \quad \overline{Z}|_{\epsilon=0} = z,$$
$$\Longrightarrow \overline{Z} = -\overline{w}\epsilon + z. \tag{3.2.94}$$

$$\frac{\partial \overline{W}}{\partial \epsilon} = W(\overline{x}, \overline{y}, \overline{z}, \overline{w}) = \bar{z}, \quad \Longrightarrow \overline{W} = \overline{z}\epsilon + f(x,y,z,w), \quad \overline{W}|_{\epsilon=0} = w,$$
$$\Longrightarrow \overline{W} = \overline{z}\epsilon + w. \tag{3.2.95}$$

Solve 3.2.94 and 3.2.95 to find $\overline{Z}$:

$$\overline{Z} = -(\overline{Z}\epsilon + w)\epsilon + z, \quad \Longrightarrow \overline{Z} = -\overline{Z}\epsilon^2 - w\epsilon + z, \quad \Longrightarrow \overline{Z} + \overline{Z}\epsilon^2 = z - w\epsilon,$$
$$\Longrightarrow \overline{Z} = \frac{z - w\epsilon}{1+\epsilon^2}. \tag{3.2.96}$$

Now find $\overline{W}$ by solving 3.2.95 and 3.2.96

$$\overline{W} = (\frac{z - w\epsilon}{1+\epsilon^2})\epsilon + w. \tag{3.2.97}$$

For

$$e_{11} = e^x \sin(y)\frac{\partial}{\partial z} + e^x \cos(y)\frac{\partial}{\partial w} \tag{3.2.98}$$

by comparison with 3.2.89, we obtained

$$X = 0, \quad Y = 0, \quad Z = e^{x}\sin(y), \quad W = e^{x}\cos(y). \tag{3.2.99}$$

By using 3.2.88, we obtian

$$\begin{aligned} &\tfrac{\partial \overline{X}}{\partial \epsilon} = X(\overline{x}, \overline{y}, \overline{z}, \overline{w}) = 0, \quad \Longrightarrow \overline{X} = f(x, y, z, w), \quad \overline{X}|_{\epsilon=0} = x, \\ &\Longrightarrow f(x, y, z, w) = x, \quad \Longrightarrow \overline{X} = x. \end{aligned} \tag{3.2.100}$$

Similarly

$$\begin{aligned} &\tfrac{\partial \overline{Y}}{\partial \epsilon} = Y(\overline{x}, \overline{y}, \overline{z}, \overline{w}) = 0, \quad \Longrightarrow \overline{Y} = f(x, y, z, w), \quad \overline{Y}|_{\epsilon=0} = y, \\ &\Longrightarrow f(x, y, z, w) = y, \quad \Longrightarrow \overline{Y} = y. \end{aligned} \tag{3.2.101}$$

$$\begin{aligned} &\tfrac{\partial \overline{Z}}{\partial \epsilon} = Z(\overline{x}, \overline{y}, \overline{z}, \overline{w}) = e^{\overline{x}}\sin(\overline{y}), \quad \Longrightarrow \overline{Z} = e^{\overline{x}}\sin(\overline{y})\epsilon + f(x, y, z, w), \quad \overline{Z}|_{\epsilon=0} = z, \\ &\Longrightarrow \overline{Z} = e^{\overline{x}}\sin(\overline{y})\epsilon + z. \end{aligned} \tag{3.2.102}$$

$$\begin{aligned} &\tfrac{\partial \overline{W}}{\partial \epsilon} = W(\overline{x}, \overline{y}, \overline{z}, \overline{w}) = e^{\overline{x}}\cos(\overline{y}), \quad \Longrightarrow \overline{W} = e^{\overline{x}}\cos(\overline{y})\epsilon + f(x, y, z, w), \quad \overline{W}|_{\epsilon=0} = w, \\ &\Longrightarrow \bar{W} = e^{\overline{x}}\cos(\overline{y})\epsilon + w. \end{aligned} \tag{3.2.103}$$

Now by substitute 3.2.100 and 3.2.101 into 3.2.102 and 3.2.103, we obtained

$$\overline{Z} = f(x, y, z, w), \quad \overline{W} = f(x, y, z, w). \tag{3.2.104}$$

$$\overline{Z} = e^x \sin(y)\epsilon + z, \quad \overline{W} = e^x \cos(y)\epsilon + w. \tag{3.2.105}$$

When $\epsilon = 0$ all lie groups $\overline{X}$, $\overline{Y}$, $\overline{Z}$, and $\overline{W}$ end up with the identity

$$\overline{X} = x, \quad \overline{Y} = y, \quad \overline{Z} = z, \quad \overline{W} = w. \tag{3.2.106}$$

3.3 Five Dimensional Lie Algebra

3.4 Introduction

In this section we consider the five dimensional Lie algebras with co-dimension two abelian nilradical. There are only three algebras; $A^{ab}_{5,33}$, $A^{a}_{5,34}$ and $A^{ab}_{5,35}$.

3.4.1 Algebra $A^{ab}_{5,33}, (a^2 + b^2 \neq 0)$

The non-zero brackets for the algebra $A^{ab}_{5,33}$ are given by

$$[e_1, e_4] = e_1, \quad [e_3, e_4] = be_3, \quad [e_2, e_5] = e_2, \quad [e_3, e_5] = ae_3. \tag{3.4.1}$$

The geodesic equations are given by

$$\ddot{q} = \dot{q}\dot{z}, \quad \ddot{x} = \dot{x}\dot{w}, \quad \ddot{y} = a\dot{y}\dot{w} + b\dot{y}\dot{z}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{3.4.2}$$

For the general case $A_{5,33}^{a\neq 0, b\neq 0}$, the symmetry Lie algebra is spanned by:

$$\begin{aligned}
&e_1 = D_w, \quad e_2 = D_z, \quad e_3 = tD_t, \quad e_4 = D_t, \quad e_5 = D_y,\\
&e_6 = D_q, \quad e_7 = D_x, \quad e_8 = yD_y, \quad e_9 = wD_t, \quad e_{10} = zD_t,\\
&e_{11} = qD_q, \quad e_{12} = xD_x, \quad e_{13} = e^z D_q, \quad e_{14} = e^w D_x, \quad e_{15} = e^{aw} e^{bz} D_y.
\end{aligned} \tag{3.4.3}$$

We make the following change of basis:

$$\begin{aligned}
&\overline{e}_1 = e_4, \quad \overline{e}_2 = e_5, \quad \overline{e}_3 = e_6, \quad \overline{e}_4 = e_7, \quad \overline{e}_5 = e_9, \quad \overline{e}_6 = e_{10}, \quad \overline{e}_7 = e_{13},\\
&\overline{e}_8 = e_{14}, \quad \overline{e}_9 = e_{15}, \quad \overline{e}_{10} = e_1, \quad \overline{e}_{11} = e_2, \quad \overline{e}_{12} = e_3, \quad \overline{e}_{13} = e_8, \quad \overline{e}_{14} = e_{11},\\
&\overline{e}_{15} = e_{12}.
\end{aligned} \tag{3.4.4}$$

The non-zero brackets of the symmetry algebra are given by:

$$\begin{aligned}
&[e_1, e_{12}] = e_1, \quad [e_2, e_{13}] = e_2, \quad [e_3, e_{14}] = e_3, \quad [e_4, e_{15}] = e_4, \quad [e_5, e_{10}] = -e_1,\\
&[e_5, e_{12}] = e_5, \quad [e_6, e_{11}] = -e_1, \quad [e_6, e_{12}] = e_6, \quad [e_7, e_{11}] = -e_7, \quad [e_7, e_{14}] = e_7,\\
&[e_8, e_{10}] = -e_8, \quad [e_8, e_{15}] = e_8, \quad [e_9, e_{10}] = -ae_9, \quad [e_9, e_{11}] = -be_9, \quad [e_9, e_{13}] = e_9.
\end{aligned} \tag{3.4.5}$$

We describe the symmetry algebra by the following proposition:

Proposition 3.4.1. The symmetry Lie algebra is a fifteen dimensional solvable Lie algebra, with nine-dimensional nilradical spanned by $e_1 - e_9$, and an abelian six-dimensional complement spanned by $e_{10} - e_{15}$. Therefore, the symmetry algebra can be identified as $(\mathbb{R}^9 \rtimes \mathbb{R}^6)$.

3.4.2 Algebra $A^a_{5,34}$

The non-zero brackets for the algebra $A^a_{5,34}$ are given by:

$$[e_1, e_4] = ae_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_3, \quad [e_1, e_5] = e_1, \quad [e_3, e_5] = e_2. \tag{3.4.6}$$

The geodesics equations are given by:

$$\ddot{q} = a\dot{q}\dot{z} + \dot{q}\dot{w}, \quad \ddot{x} = \dot{x}\dot{z} + \dot{y}\dot{w}, \quad \ddot{y} = \dot{y}\dot{z}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{3.4.7}$$

For the general case $A^{a\neq 0}_{5,34}$, the symmetry Lie algebra is spanned by:

$$\begin{array}{lllll} e_1 = D_t, & e_2 = D_x, & e_3 = D_q, & e_4 = D_y, & e_5 = D_w, \\ e_6 = D_z, & e_7 = tD_t, & e_8 = qD_q, & e_9 = wD_t, & e_{10} = zD_t, \\ e_{11} = yD_x, & e_{12} = xD_x + yD_y, & e_{13} = e^z D_x, & e_{14} = we^z D_x + e^z D_y, & e_{15} = e^w e^{az} D_q. \end{array} \tag{3.4.8}$$

We make the following change of basis:

$$\begin{array}{lllllll} \overline{e}_1 = e_1, & \overline{e}_2 = e_2, & \overline{e}_3 = e_3, & \overline{e}_4 = e_4, & \overline{e}_5 = e_9, & \overline{e}_6 = e_{10}, & \overline{e}_7 = e_{11}, \\ \overline{e}_8 = e_{13}, & \overline{e}_9 = e_{14}, & \overline{e}_{10} = e_{15}, & \overline{e}_{11} = e_5, & \overline{e}_{12} = e_6, & \overline{e}_{13} = e_7, & \overline{e}_{14} = e_8, \\ \overline{e}_{15} = e_{12}. & & & & & & \end{array} \tag{3.4.9}$$

The non-zero brackets of the symmetry algebra are given by:

$$
\begin{aligned}
&[e_1, e_{13}] = e_1, \quad [e_2, e_{15}] = e_2, \quad [e_3, e_{14}] = e_3, \quad [e_4, e_7] = e_2, \quad [e_4, e_{15}] = e_4, \\
&[e_5, e_{11}] = -e_1, \quad [e_5, e_{13}] = e_5, \quad [e_6, e_{12}] = -e_1, \quad [e_6, e_{13}] = e_6, \quad [e_7, e_9] = -e_8, \\
&[e_8, e_{12}] = -e_8, \quad [e_8, e_{15}] = e_8, \quad [e_9, e_{11}] = -e_8, \quad [e_9, e_{12}] = -e_9, \quad [e_9, e_{15}] = e_9, \\
&[e_{10}, e_{11}] = -e_{10}, \quad [e_{10}, e_{12}] = -ae_{10}, \quad [e_{10}, e_{14}] = e_{10}.
\end{aligned}
\tag{3.4.10}
$$

We describe the symmetry algebra by the following proposition:

Proposition 3.4.2. The symmetry Lie algebra is a fifteen dimensional solvable Lie algebra, with ten dimensional nilradical spanned by $e_1 - e_{10}$, and an abelian five dimensional complement spanned by $e_{11} - e_{15}$. In fact, the nilradical is a direct sum of $A_{5,1}$ in Winternitz [27] and $\mathbb{R}^5$. Therefore, the symmetry algebra can be identified as $(A_{5,1} \oplus \mathbb{R}^5) \rtimes \mathbb{R}^5$

3.4.3 Algebra $A_{5,35}^{ab}$

The non-zero brackets for the algebra $A_{5,35}^{ab}, (a^2 + b^2 \neq 0)$ are given by:

$$
\begin{aligned}
&[e_1, e_4] = be_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_3, \quad [e_1, e_5] = ae_1, \quad [e_2, e_5] = -e_3, \\
&[e_3, e_5] = e_2.
\end{aligned}
\tag{3.4.11}
$$

The geodesics are given by:

$$
\ddot{q} = b\dot{q}\dot{z} + a\dot{q}\dot{w}, \quad \ddot{x} = \dot{x}\dot{z} + \dot{y}\dot{w}, \quad \ddot{y} = -\dot{x}\dot{w} + \dot{y}\dot{z}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{3.4.12}
$$

For the general case $A_{5,35}^{a\neq 0,b\neq 0}$, the symmetry Lie algebra is spanned by:

$$\begin{aligned}
&e_1 = D_y, && e_2 = D_x, && e_3 = D_q,\\
&e_4 = D_w, && e_5 = D_z, && e_6 = tD_t,\\
&e_7 = D_t, && e_8 = qD_q, && e_9 = wD_t,\\
&e_{10} = zD_t, && e_{11} = xD_x + yD_y, && e_{12} = xD_y - yD_x,\\
&e_{13} = e^{aw}e^{bz}D_q, && e_{14} = \sin(w)e^z D_x + \cos(w)e^z D_y, && e_{15} = \sin(w)e^z D_y - \cos(w)e^z D_x.
\end{aligned} \tag{3.4.13}$$

We make the following change of basis:

$$\begin{aligned}
&\overline{e}_1 = e_1, \quad \overline{e}_2 = e_2, \quad \overline{e}_3 = e_3, \quad \overline{e}_4 = e_7, \quad \overline{e}_5 = e_9, \quad \overline{e}_6 = e_{10}, \quad \overline{e}_7 = e_{13},\\
&\overline{e}_8 = e_{14}, \quad \overline{e}_9 = e_{15}, \quad \overline{e}_{10} = e_4, \quad \overline{e}_{11} = e_5, \quad \overline{e}_{12} = e_6, \quad \overline{e}_{13} = e_8, \quad \overline{e}_{14} = e_{11},\\
&\overline{e}_{15} = e_{12}.
\end{aligned} \tag{3.4.14}$$

The non-zero brackets of the symmetry algebra are given by:

$$
\begin{array}{lllll}
[e_1, e_{14}] = e_1, & [e_1, e_{15}] = -e_2, & [e_2, e_{14}] = e_2, & [e_2, e_{15}] = e_1, & [e_3, e_{13}] = e_3, \\
[e_4, e_{12}] = e_4, & [e_5, e_{10}] = -e_4, & [e_5, e_{12}] = e_5, & [e_6, e_{11}] = -e_4, & [e_6, e_{12}] = e_6, \\
[e_7, e_{10}] = -ae_7, & [e_7, e_{11}] = -be_7, & [e_7, e_{13}] = e_7, & [e_8, e_{10}] = e_9, & [e_8, e_{11}] = -e_8, \\
[e_8, e_{14}] = e_8, & [e_8, e_{15}] = e_9, & [e_9, e_{10}] = -e_8, & [e_9, e_{11}] = -e_9, & [e_9, e_{14}] = e_9, \\
[e_9, e_{15}] = -e_8. & & & &
\end{array}
\tag{3.4.15}
$$

We describe the symmetry algebra by the following proposition:

Proposition 3.4.3. The symmetry Lie algebra is a fifteen dimensional solvable Lie algebra, with nine-dimensional nilradical spanned by $e_1 - e_9$, and an abelian six-dimensional complement spanned by $e_{10} - e_{15}$. Therefore, the symmetry algebra can be identified as $(\mathbb{R}^9 \rtimes \mathbb{R}^6)$.

3.5 Conclusion

We summarize the result of five dimension in the following table:

Five-dimensional Lie Algebras	Dimension	Identification
$A_{5,33}^{a\neq 0, b\neq 0}$	15	$(\mathbb{R}^9 \rtimes \mathbb{R}^6)$
$A_{5,34}^{a\neq 0}$	15	$(A_{5,1} \oplus \mathbb{R}^5) \rtimes \mathbb{R}^5$
$A_{5,35}^{ab}$	15	$(\mathbb{R}^9 \rtimes \mathbb{R}^6)$

Table 1. Five dimensional Lie Algebras and Identification of the Symmetry Algebra .

CHAPTER 4

CLASSIFICATION OF THE SYMMETRY LIE ALGEBRAS FOR SIX-DIMENSIONAL CO-DIMENSION TWO ABELIAN NILRADICAL WITH ABELIAN COMPLEMENT LIE ALGEBRAS

4.1 Introduction

In this chapter, we consider the six-dimensional solvable Lie algebras that have a four-dimensional abelian nilradical and a two-dimensional abelian complement. In this case, there are nineteen such algebras, namely, $A_{6,19} - A_{6,19}$ in Turkowski's list [13]. For each Lie algebra, we give the geodesic equations, a basis for the symmetry Lie algebra in terms of vector fields, and finally we identify the symmetry Lie algebra.

4.2 Algebra $A_{6,1}^{abcd}(abcd : ab \neq 0,\ c^2 + d^2 \neq 0)$

The non-zero brackets for the algebra $A_{6,1}^{abcd}$ are given by

$$\begin{gathered}
[e_1, e_3] = ae_3, \quad [e_1, e_4] = ce_4, \quad [e_1, e_6] = e_6, \\
[e_2, e_3] = be_3, \quad [e_2, e_4] = de_4, \quad [e_2, e_5] = e_5.
\end{gathered} \tag{4.2.1}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}\dot{w}, \quad \ddot{q} = \dot{q}\dot{z}, \quad \ddot{x} = \dot{x}(d\dot{z} + c\dot{w}), \quad \ddot{y} = \dot{y}(b\dot{z} + a\dot{w}), \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.2.2}$$

For the general case $A_{6,1}^{a\neq 0,b\neq 0,c\neq 0,d\neq 0}$, the symmetry Lie algebra is spanned by:

$$\begin{aligned}
&e_1 = D_q, && e_2 = D_t, && e_3 = D_x, && e_4 = D_y,\\
&e_5 = D_p, && e_6 = wD_t, && e_7 = zD_t, && e_8 = e^z D_q,\\
&e_9 = e^w D_p, && e_{10} = e^{cw+dz} D_x, && e_{11} = e^{aw+bz} D_y, && e_{12} = D_w,\\
&e_{13} = D_z, && e_{14} = tD_t, && e_{15} = xD_x, && e_{16} = qD_q,\\
&e_{17} = yD_y, && e_{18} = pD_p.
\end{aligned} \tag{4.2.3}$$

The non-zero brackets of the symmetry algebra are given by:

$$\begin{aligned}
&[e_1, e_{16}] = e_1, && [e_2, e_{14}] = e_2, && [e_3, e_{15}] = e_3, && [e_4, e_{17}] = e_4,\\
&[e_5, e_{18}] = e_5, && [e_6, e_{12}] = -e_2, && [e_6, e_{14}] = e_6, && [e_7, e_{13}] = -e_2,\\
&[e_7, e_{14}] = e_7, && [e_8, e_{13}] = -e_8, && [e_8, e_{16}] = e_8, && [e_9, e_{12}] = -e_9,\\
&[e_9, e_{18}] = e_9, && [e_{10}, e_{12}] = -ce_{10}, && [e_{10}, e_{13}] = de_{10}, && [e_{10}, e_{15}] = e_{10},\\
&[e_{11}, e_{12}] = -ae_{11}, && [e_{11}, e_{13}] = -be_{11}, && [e_{11}, e_{17}] = e_{11}.
\end{aligned} \tag{4.2.4}$$

In this case, based on the Lie invariance condition, we have to consider eight subcases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are: $A_{6,1}^{a=1,b\neq 0}$, $A_{6,1}^{a=c,a\neq 0}$, $A_{6,1}^{b=1,a\neq 0}$, $A_{6,1}^{b=d,b\neq 0}$, $A_{6,1}^{c=0,d\neq 0}$, $A_{6,1}^{c=1}$, $A_{6,1}^{d=0,c\neq 0}$ and $A_{6,1}^{d=1}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition:

Proposition 4.2.1. The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semi-direct product of eleven-dimensional abelian nilradical spanned by e_1–e_{11} and a seven-dimensional abelian complement spanned by e_{12}–e_{18}. Hence, it can be described as $\mathbb{R}^{11} \rtimes \mathbb{R}^7$.

4.3 Algebra $A_{6,2}^{abc}, (a^2 + b^2 \neq 0)$

The non-zero brackets for the algebra $A_{6,2}^{abc}$ are given by

$$\begin{gathered}[e_1, e_4] = e_4, \quad [e_1, e_5] = e_6, \quad [e_2, e_5] = e_5, \quad [e_2, e_6] = e_6, \quad [e_2, e_3] = be_3, \\ [e_2, e_4] = ce_4, \quad [e_1, e_3] = ae_3. \end{gathered} \tag{4.3.1}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}\dot{z}, \quad \ddot{q} = \dot{p}\dot{w} + \dot{q}\dot{z}, \quad \ddot{x} = \dot{x}(c\dot{z} + \dot{w}), \quad \ddot{y} = \dot{y}(b\dot{z} + a\dot{w}), \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.3.2}$$

For the general case $A_{6,2}^{a\neq 0, b\neq 0, c\neq 0}$, the symmetry Lie algebra is spanned by

$$\begin{array}{llll} e_1 = e^z D_q, & e_2 = D_q, & e_3 = e^z D_p + w e^z D_q, & e_4 = D_p, \\ e_5 = p D_q, & e_6 = D_t, & e_7 = D_x, & e_8 = D_y, \\ e_9 = x D_x, & e_{10} = x D_t, & e_{11} = z D_t, & e_{12} = w^{w+cz} D_x, \\ e_{13} = e^{aw+bz} D_y, & e_{14} = D_w, & e_{15} = D_z, & e_{16} = t D_t, \\ e_{17} = y D_y, & e_{18} = p D_p + q D_q. & & \end{array} \tag{4.3.3}$$

The non-zero brackets of the symmetry algebra are given by:

$$\begin{array}{llll} [e_1, e_{15}] = e_1, & [e_2, e_{18}] = e_2, & [e_3, e_{16}] = e_3, & [e_4, e_{17}] = e_4, \\ [e_5, e_8] = e_2, & [e_5, e_{18}] = e_5, & [e_6, e_{13}] = -e_1, & [e_6, e_{15}] = e_6, \\ [e_7, e_{14}] = -e_1, & [e_7, e_{15}] = e_7, & [e_8, e_{10}] = -e_9, & [e_9, e_{14}] = -e_9, \\ [e_9, e_{18}] = e_9, & [e_{10}, e_{13}] = -e_9, & [e_{10}, e_{14}] = -e_{10}, & [e_{10}, e_{18}] = e_{10}, \\ [e_{11}, e_{13}] = -e_{11}, & [e_{11}, e_{14}] = -c e_{11}, & [e_{11}, e_{16}] = e_{11}, & [e_{12}, e_{13}] = -a e_{12}, \\ [e_{12}, e_{14}] = -b e_{12}, & [e_{12}, e_{17}] = e_{12}. & & \end{array} \tag{4.3.4}$$

In this case, based on the Lie invariance condition, we have to consider seven sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are: $A_{6,2}^{a=0, b\neq 0}$,

$A_{6,2}^{a=1}$, $A_{6,2}^{b=0,a\neq0}$, $A_{6,2}^{b=1}$, $A_{6,2}^{a\neq0,b=c}$, $A_{6,2}^{c=0}$, and $A_{6,2}^{c=1}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 4.3.1. The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of twelve-dimensional decomposable nilradical spanned by e_1–e_{12} and a six-dimensional abelian complement spanned by e_{12}–e_{18}. In fact, the nilradical is a direct sum of $A_{5,1}$ in Winternitz list [27] and $\mathbb{R}^7$. Hence, symmetry algebra can be can be described as $(A_{5,1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$ where the non-zero brackets of $A_{5,1}$ are given by

$$[e_3, e_5] = e_1, [e_4, e_5] = e_2. \tag{4.3.5}$$

4.3.1 Algebra $A_{6,3}^a$

The non-zero brackets for the algebra $A_{6,3}^a$ are given by

$$\begin{aligned} &[e_1, e_3] = e_3, \quad [e_1, e_4] = e_4, \quad [e_1, e_5] = e_6, \quad [e_2, e_5] = e_5, \\ &[e_2, e_4] = ae_4, \quad [e_2, e_3] = ae_3 + e_4, \quad [e_2, e_6] = e_6. \end{aligned} \tag{4.3.6}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}\dot{z} + \dot{q}\dot{w}, \quad \ddot{q} = \dot{q}\dot{z}, \quad \ddot{x} = \dot{x}(a\dot{z} + \dot{w}) + \dot{y}\dot{z}, \quad \ddot{y} = \dot{y}(a\dot{z} + \dot{w}), \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.3.7}$$

For the general case $A_{6,3}^{a\neq 0}$, the symmetry Lie algebra is spanned by

$$\begin{array}{lll} e_1 = D_t, & e_2 = D_x, & e_3 = D_p, \\ e_4 = D_y, & e_5 = D_q, & e_6 = wD_t, \\ e_7 = zD_t, & e_8 = yD_x, & e_9 = qD_p, \\ e_{10} = e^z D_p, & e_{11} = we^z D_p + e^z D_q, & e_{12} = \frac{e^w e^{az} D_x}{a}, \\ e_{13} = \frac{(az-1)e^{az+w} D_x}{a} + e^w e^{az} D_y, & e_{14} = D_w, & e_{15} = D_z, \\ e_{16} = tD_t, & e_{17} = xD_x + yD_y, & e_{18} = pD_p + qD_q. \end{array} \tag{4.3.8}$$

The non-zero brackets of the symmetry algebra are given by:

$$\begin{array}{lll} [e_1, e_{16}] = e_1, & [e_2, e_{17}] = e_2, & [e_3, e_{18}] = e_3, \\ [e_4, e_8] = e_2, & [e_4, e_{17}] = e_4, & [e_5, e_9] = e_3, \\ [e_5, e_{18}] = e_5, & [e_6, e_{14}] = -e_1, & [e_6, e_{16}] = e_6, \\ [e_7, e_{15}] = -e_1, & [e_7, e_{16}] = e_7, & [e_8, e_{13}] = -ae_{12}, \\ [e_9, e_{11}] = -e_{10}, & [e_{10}, e_{15}] = -e_{10}, & [e_{10}, e_{18}] = e_{10}, \\ [e_{11}, e_{14}] = -e_{10}, & [e_{11}, e_{15}] = -e_{11}, & [e_{11}, e_{18}] = e_{11}, \\ [e_{12}, e_{14}] = -e_{12}, & [e_{12}, e_{15}] = -ae_{12}, & [e_{12}, e_{17}] = e_{12}, \\ [e_{13}, e_{14}] = -e_{13}, & [e_{13}, e_{15}] = -ae_{12} - ae_{13}, & [e_{13}, e_{17}] = e_{13}. \end{array} \tag{4.3.9}$$

In this case, based on the Lie invariance condition, we have to consider two sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A_{6,3}^{a=0}$ and $A_{6,3}^{a=1}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition:

Proposition 4.3.2. The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra with thirteen-dimensional nilradical spanned by e_1–e_{13} and five-dimensional abelian complement spanned by e_{14}–e_{18}. In fact, the nilradical is a direct sum of two copies of $A_{5,1}$ and $\mathbb{R}^3$. Hence, the symmetry algebra is $(A_{5,1} \oplus A_{5,1} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$, where $A_{5,1}$ is given by Eq (4.3.5).

4.3.2 Algebra $A_{6,4}^{ab}, (a \neq 0)$

The non-zero brackets for the algebra $A_{6,4}^{ab}$ are given by

$$\begin{aligned} &[e_1, e_3] = e_3, \quad [e_1, e_4] = e_4, \quad [e_1, e_5] = e_6, \quad [e_2, e_3] = e_4, \\ &[e_2, e_4] = -e_3, \quad [e_2, e_5] = ae_5 + be_6, \quad [e_2, e_6] = ae_6. \end{aligned} \tag{4.3.10}$$

The geodesic equations are given by

$$\ddot{p} = \dot{z}(a\dot{p} + b\dot{q}) + \dot{q}\dot{w}, \quad \ddot{q} = a\dot{q}\dot{z}, \quad \ddot{x} = \dot{x}\dot{w} - \dot{y}\dot{z}, \quad \ddot{y} = \dot{x}\dot{z} + \dot{y}\dot{w}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.3.11}$$

For the general case $A_{6,4}^{a\neq0,b\neq0}$, the symmetry Lie algebra is spanned by

$$
\begin{aligned}
&e_1 = D_x, && e_2 = D_y,\\
&e_3 = D_p, && e_4 = D_q,\\
&e_5 = D_t, && e_6 = qD_p,\\
&e_7 = wD_t, && e_8 = zD_t,\\
&e_9 = \tfrac{e^{az}D_p}{a}, && e_{10} = e^w\cos(z)D_x + e^w\sin(z)D_y,\\
&e_{11} = e^w\sin(z)D_x - e^w\cos(z)D_y, && e_{12} = \tfrac{((bz+w)a-b)e^{az}D_p}{a} + e^{az}D_q,\\
&e_{13} = D_w, && e_{14} = D_z,\\
&e_{15} = tD_t, && e_{16} = pD_p + qD_q,\\
&e_{17} = xD_x + yD_y, && e_{18} = yD_x - xD_y.
\end{aligned}
\tag{4.3.12}
$$

The non-zero brackets of the symmetry algebra are given by:

$$\begin{array}{llll}
[e_1, e_{17}] = e_1, & [e_1, e_{18}] = -e_2, & [e_2, e_{17}] = e_2, & [e_2, e_{18}] = e_1, \\
[e_3, e_{16}] = e_3, & [e_4, e_6] = e_3, & [e_4, e_{16}] = e_4, & [e_5, e_{15}] = e_5, \\
[e_6, e_{12}] = -ae_9, & [e_7, e_{13}] = -e_5, & [e_7, e_{15}] = e_7, & [e_8, e_{14}] = -e_5, \\
[e_8, e_{15}] = e_8, & [e_9, e_{14}] = -ae_9, & [e_9, e_{16}] = e_9, & [e_{10}, e_{13}] = -e_{10}, \\
[e_{10}, e_{14}] = e_{11}, & [e_{10}, e_{17}] = e_{10}, & [e_{10}, e_{18}] = e_{11}, & [e_{11}, e_{13}] = -e_{11}, \\
[e_{11}, e_{14}] = -e_{10}, & [e_{11}, e_{17}] = e_{11}, & [e_{11}, e_{18}] = -e_{10}, & [e_{12}, e_{13}] = -ae_9, \\
[e_{12}, e_{14}] = -abe_9 - ae_{12}, & [e_{12}, e_{16}] = e_{12}. & &
\end{array} \tag{4.3.13}$$

In this case, based on the Lie invariance condition, we have to consider one sub-case based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The only case we consider is $A_{6,4}^{b=0}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition:

Proposition 4.3.3. The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of twelve-dimensional decomposable nilradical spanned by e_1–e_{12} and a six-dimensional abelian complement spanned by e_{12}–e_{18}. In fact, the nilradical is a direct sum of $A_{5,1}$ in Winternitz list [27] and $\mathbb{R}^7$. Hence, symmetry algebra can be can be described as $(A_{5,1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$, where $A_{5,1}$ is given by Eq (4.3.5).

4.3.3 Algebra $A_{6,5}^{ab}, (ab \neq 0)$

The non-zero brackets for the algebra $A_{6,5}^{ab}$ are given by

$$\begin{aligned}
&[e_1, e_6] = e_6, \quad [e_2, e_3] = be_3, \quad [e_2, e_4] = e_4, \quad [e_1, e_5] = e_5 + e_6, \\
&[e_1, e_3] = ae_3.
\end{aligned} \tag{4.3.14}$$

The geodesic equations are given by

$$\ddot{p} = \dot{w}(\dot{p} + \dot{q}), \quad \ddot{q} = \dot{q}\dot{w}, \quad \ddot{x} = \dot{x}(b\dot{z} + a\dot{w}), \quad \ddot{y} = \dot{y}\dot{z}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.3.15}$$

For the general case $A_{6,5}^{a\neq 0, b\neq 0}$, the symmetry Lie algebra is spanned by

$$\begin{aligned}
&e_1 = D_t, && e_2 = D_p, && e_3 = D_x, \\
&e_4 = D_y, && e_5 = D_q, && e_6 = wD_t, \\
&e_7 = zD_t, && e_8 = qD_p, && e_9 = e^w D_p, \\
&e_{10} = e^z D_y, && e_{11} = (w-1)e^w D_p + e^w D_q, && e_{12} = e^{aw} e^{bz} D_x, \\
&e_{13} = D_z, && e_{14} = D_w, && e_{15} = tD_t, \\
&e_{16} = xD_x, && e_{17} = yD_y, && e_{18} = pD_p + qD_q.
\end{aligned} \tag{4.3.16}$$

The non-zero brackets of the symmetry algebra are given by:

$$\begin{array}{llll}
[e_1, e_{15}] = e_1, & [e_2, e_{18}] = e_2, & [e_3, e_{16}] = e_3, & [e_4, e_{17}] = e_4, \\
[e_5, e_8] = e_2, & [e_5, e_{18}] = e_5, & [e_6, e_{14}] = -e_1, & [e_6, e_{15}] = e_6, \\
[e_7, e_{13}] = -e_1, & [e_7, e_{15}] = e_7, & [e_8, e_{11}] = -e_9, & [e_9, e_{14}] = -e_9, \\
[e_9, e_{18}] = e_9, & [e_{10}, e_{13}] = -e_{10}, & [e_{10}, e_{17}] = e_{10}, & [e_{11}, e_{14}] = -e_{11} - e_9, \\
[e_{11}, e_{18}] = e_{11}, & [e_{12}, e_{13}] = -be_{12}, & [e_{12}, e_{14}] = -ae_{12}, & [e_{12}, e_{16}] = e_{12}.
\end{array} \tag{4.3.17}$$

In this case, based on the Lie invariance condition, we have to consider two sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A_{6,5}^{a=1,b\neq 0}$ and $A_{6,5}^{a\neq 0,b=1}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition:

Proposition 4.3.4. The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of twelve-dimensional decomposable nil-radical spanned by e_1–e_{12} and a six-dimensional abelian complement spanned by e_{12}–e_{18}. In fact, the nilradical is a direct sum of $A_{5,1}$ in Winternitz list [27] and $\mathbb{R}^7$. Hence, symmetry algebra can be can be described as $(A_{5,1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$, where $A_{5,1}$ is given by Eq (4.3.5).

4.3.4 Algebra $A_{6,6}^{ab}, (a^2 + b^2 \neq 0)$

The non-zero brackets for the algebra $A_{6,6}^{ab}$ are given by

$$\begin{aligned} &[e_1, e_3] = ae_3, \quad [e_1, e_4] = ae_4, \quad [e_2, e_4] = e_4, \quad [e_1, e_6] = e_6, \\ &[e_2, e_3] = e_3 + e_4, \quad [e_1, e_5] = e_5 + e_6, \quad [e_2, e_5] = be_6. \end{aligned} \tag{4.3.18}$$

The geodesic equations are given by

$$\ddot{p} = \dot{w}(\dot{p} + \dot{q}) + b\dot{q}\dot{z}, \quad \ddot{q} = \dot{q}\dot{w}, \quad \ddot{x} = \dot{x}(\dot{z} + a\dot{w}) + \dot{y}\dot{z}, \quad \ddot{y} = \dot{y}(\dot{z} + a\dot{w}), \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.3.19}$$

For the general case $A_{6,6}^{a\neq 0,b\neq 0}$, the symmetry Lie algebra is spanned by

$$\begin{aligned}
&e_1 = D_t, && e_2 = D_x, \\
&e_3 = D_p, && e_4 = D_y, \\
&e_5 = D_q, && e_6 = wD_t, \\
&e_7 = zD_t, && e_8 = yD_x, \\
&e_9 = qD_p, && e_{10} = e^w D_p, \\
&e_{11} = e^z e^{aw} D_x, && e_{12} = (bz + w - 1)e^w D_p + e^w D_q, \\
&e_{13} = (z-1)e^{aw+z} D_x + e^z e^{aw} D_y, && e_{14} = D_z, \\
&e_{15} = D_w, && e_{16} = tD_t, \\
&e_{17} = xD_x + yD_y, && e_{18} = pD_p + qD_q.
\end{aligned} \tag{4.3.20}$$

The non-zero brackets of the symmetry algebra are given by

$$\begin{array}{llll}
[e_1, e_{16}] = e_1, & [e_2, e_{17}] = e_2, & [e_3, e_{18}] = e_3, & [e_4, e_8] = e_2, \\
[e_4, e_{17}] = e_4, & [e_5, e_9] = e_3, & [e_5, e_{18}] = e_5, & [e_6, e_{15}] = -e_1, \\
[e_6, e_{16}] = e_6, & [e_7, e_{14}] = -e_1, & [e_7, e_{16}] = e_7, & [e_8, e_{13}] = -e_{11}, \\
[e_{10}, e_{15}] = -e_{10}, & [e_{10}, e_{18}] = e_{10}, & [e_{11}, e_{14}] = -e_{11}, & [e_{11}, e_{15}] = -ae_{11}, \\
[e_{11}, e_{17}] = e_{11}, & [e_{12}, e_{14}] = -be_{10}, & [e_{12}, e_{15}] = -e_{10} - e_{12}, & [e_{12}, e_{18}] = e_{12}, \\
[e_{13}, e_{14}] = -e_{11} - e_{13}, & [e_{13}, e_{15}] = -ae_{13}, & [e_{13}, e_{17}] = e_{13}, & [e_{19}, e_{12}] = -e_{10}.
\end{array} \tag{4.3.21}$$

In this case, based on the Lie invariance condition, we have to consider three sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A_{6,6}^{a=0,b\neq 0}$, $A_{6,6}^{a=1}$ and $A_{6,6}^{a\neq 0,b=0}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 4.3.5. The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra with thirteen-dimensional nilradical spanned by e_1–e_{13} and five-dimensional abelian complement spanned by e_{14}–e_{18}. In fact, the nilradical is a direct sum of two copies of $A_{5,1}$ and $\mathbb{R}^3$. Hence, the symmetry algebra is $(A_{5,1} \oplus A_{5,1} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$, where $A_{5,1}$ is given by Eq (4.3.5).

4.3.5 Algebra $A_{6,7}^{abc}, (a^2 + b^2 \neq 0)$

The non-zero brackets for the algebra $A_{6,7}^{abc}$ are given by

$$\begin{aligned} &[e_1, e_3] = ae_3, \quad [e_1, e_4] = ae_4, \quad [e_1, e_5] = e_5 + e_6, \quad [e_1, e_6] = e_6, \\ &[e_2, e_3] = ce_3 + e_4, \quad [e_2, e_4] = -e_3 + ce_4, \quad [e_2, e_5] = be_6. \end{aligned} \tag{4.3.22}$$

The geodesic equations are given by

$$\begin{aligned} &\ddot{p} = \dot{p}(c\dot{z} + a\dot{w}) + \dot{q}\dot{z}, \quad \ddot{q} = \dot{z}(-\dot{p} + c\dot{q}) + a\dot{q}\dot{w}, \quad \ddot{x} = \dot{x}\dot{w}, \quad \ddot{y} = \dot{x}(b\dot{z} + \dot{w}) + \dot{y}\dot{w}, \\ &\ddot{z} = 0, \quad \ddot{w} = 0. \end{aligned} \tag{4.3.23}$$

For the general case $A_{6,7}^{a\neq 0,b\neq 0,c\neq 0}$, the symmetry Lie algebra is spanned by

$$
\begin{aligned}
&e_1 = D_q, && e_2 = D_p,\\
&e_3 = D_y, && e_4 = D_x,\\
&e_5 = D_t, && e_6 = xD_y,\\
&e_7 = wD_t, && e_8 = zD_t,\\
&e_9 = e^w D_y, && e_{10} = e^w D_x + (bz + w - 1)e^w D_y,\\
&e_{11} = \sin(z)e^{aw+cz}D_p + e^{cz}\cos(z)e^{aw}D_q, && e_{12} = -\cos(z)e^{aw+cz}D_p + e^{cz}\sin(z)e^{aw}D_q,\\
&e_{13} = D_z, && e_{14} = D_w,\\
&e_{15} = tD_t, && e_{16} = xD_x + yD_y,\\
&e_{17} = pD_p + qD_q, && e_{18} = -qD_p + pD_q.
\end{aligned}
\tag{4.3.24}
$$

The non-zero brackets of the symmetry algebra are given by:

$$
\begin{array}{llll}
[e_1, e_{17}] = e_1, & [e_1, e_{18}] = -e_2, & [e_2, e_{17}] = e_2, & [e_2, e_{18}] = e_1, \\
[e_3, e_{16}] = e_3, & [e_4, e_6] = e_3, & [e_4, e_{16}] = e_4, & [e_5, e_{15}] = e_5, \\
[e_6, e_{10}] = -e_9, & [e_7, e_{14}] = -e_5, & [e_7, e_{15}] = e_7, & [e_8, e_{13}] = -e_5, \\
[e_8, e_{15}] = e_8, & [e_9, e_{14}] = -e_9, & [e_9, e_{16}] = e_9, & [e_{10}, e_{13}] = -be_9, \\
[e_{10}, e_{14}] = -e_{10} - e_9, & [e_{10}, e_{16}] = e_{10}, & [e_{11}, e_{13}] = -ce_{11} + e_{12}, & [e_{11}, e_{14}] = -ae_{11}, \\
[e_{11}, e_{17}] = e_{11}, & [e_{11}, e_{18}] = e_{12}, & [e_{12}, e_{13}] = -ce_{12} - e_{11}, & [e_{12}, e_{14}] = -ae_{12}, \\
[e_{12}, e_{17}] = e_{12}, & [e_{12}, e_{18}] = -e_{11}. & &
\end{array}
\tag{4.3.25}
$$

In this case, based on the Lie invariance condition, we have to consider four sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A_{6,7}^{a=0,b\neq 0}$, $A_{6,7}^{a=1}$, $A_{6,7}^{a\neq 0,b=0}$ and $A_{6,7}^{c=0}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition:

Proposition 4.3.6. The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of twelve-dimensional decomposable nilradical spanned by e_1–e_{12} and a six-dimensional abelian complement spanned by e_{12}–e_{18}. In fact, the nilradical is a direct sum of $A_{5,1}$ in Winternitz list [27] and $\mathbb{R}^7$. Hence, symmetry algebra can be can be described as $(A_{5,1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$, where $A_{5,1}$ is given by Eq (4.3.5).

4.3.6 Algebra $A_{6,8}$

The non-zero brackets for the algebra $A_{6,8}$ are given by

$$[e_1, e_3] = e_3, \quad [e_1, e_4] = e_6, \quad [e_2, e_5] = e_5 + e_6, \quad [e_2, e_6] = e_6, \quad [e_2, e_4] = e_4. \tag{4.3.26}$$

The geodesic equations are given by

$$\ddot{p} = \dot{z}, (\dot{p} + \dot{y}) + \dot{q}\dot{w}, \quad \ddot{q} = \dot{q}\dot{z}, \quad \ddot{x} = \dot{x}\dot{w}, \quad \ddot{y} = \dot{y}\dot{z}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.3.27}$$

The symmetry Lie algebra is spanned by

$$\begin{aligned}
&e_1 = D_t, && e_2 = D_p, && e_3 = D_x,\\
&e_4 = D_q, && e_5 = D_y, && e_6 = wD_t,\\
&e_7 = zD_t, && e_8 = qD_p, && e_9 = yD_p,\\
&e_{10} = e^z D_p, && e_{11} = e^w D_x, && e_{12} = we^z D_p + e^z D_q,\\
&e_{13} = (z-1)e^z D_p + e^z D_y, && e_{14} = D_z, && e_{15} = tD_t,\\
&e_{16} = D_w, && e_{17} = xD_x, && e_{18} = pD_p + qD_q + yD_y.
\end{aligned} \tag{4.3.28}$$

The non-zero brackets of the symmetry algebra are given by:

$$
\begin{aligned}
&[e_1, e_{15}] = e_1, && [e_2, e_{18}] = e_2, && [e_3, e_{17}] = e_3, && [e_4, e_8] = e_2,\\
&[e_4, e_{18}] = e_4, && [e_5, e_9] = e_2, && [e_5, e_{18}] = e_5, && [e_6, e_{15}] = e_6,\\
&[e_6, e_{16}] = -e_1, && [e_7, e_{14}] = -e_1, && [e_7, e_{15}] = e_7, && [e_8, e_{12}] = -e_{10},\\
&[e_9, e_{13}] = -e_{10}, && [e_{10}, e_{14}] = -e_{10}, && [e_{10}, e_{18}] = e_{10}, && [e_{11}, e_{16}] = -e_{11},\\
&[e_{11}, e_{17}] = e_{11}, && [e_{12}, e_{14}] = -e_{12}, && [e_{12}, e_{16}] = -e_{10}, && [e_{12}, e_{18}] = e_{12},\\
&[e_{13}, e_{14}] = -e_{10} - e_{13}, && [e_{13}, e_{18}] = e_{13}.
\end{aligned}
\tag{4.3.29}
$$

Proposition 4.3.7. The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra with thirteen-dimensional nilradical spanned by e_1–e_{13}, which is a direct sum of an eight-dimensional indecomposable solvable Lie algebra $B_{8(a=0)}$ and a five-dimensional abelian Lie algebra. The complement of the nilradical is a another five-dimensional abelian Lie algebra spanned by e_{14}–e_{18}. Therefore, the symmetry algebra can be identified as: $(B_{8(a=0)} \oplus \mathbb{R}^5) \rtimes \mathbb{R}^5$, where the non-zero brackets of B_{8a} are given by Eq (4.3.34).

4.3.7 Algebra $A^a_{6,9}$

The non-zero brackets for the algebra $A^a_{6,9}$ are given by

$$
[e_1, e_3] = e_3, \quad [e_1, e_4] = e_6, \quad [e_2, e_6] = e_6, \quad [e_2, e_4] = e_4 + e_5, \quad [e_2, e_5] = e_5 + ae_6.
\tag{4.3.30}
$$

The geodesic equations are given by

$$\ddot{p} = \dot{z}(\dot{p} + a\dot{y}) + \dot{q}\dot{w}, \quad \ddot{q} = \dot{q}\dot{z}, \quad \ddot{x} = \dot{x}\dot{w}, \quad \ddot{y} = \dot{z}(\dot{q} + \dot{y}), \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.3.31}$$

For the general case $A_{6,9}^{a\neq 0}$, the symmetry Lie algebra is spanned by

$$\begin{aligned}
&e_1 = D_t, && e_2 = D_y,\\
&e_3 = D_p, && e_4 = D_x,\\
&e_5 = D_q, && e_6 = wD_t,\\
&e_7 = zD_t, && e_8 = qD_p,\\
&e_9 = e^z D_p, && e_{10} = e^w D_x,\\
&e_{11} = ayD_p + qD_y, && e_{12} = (z-1)ae^z D_p + e^z D_y,\\
&e_{13} = [\tfrac{(z^2-2z+2)a}{2} + w]e^z D_p + e^z D_q + (z-1)e^z D_y, && e_{14} = D_w,\\
&e_{15} = D_z, && e_{16} = tD_t,\\
&e_{17} = xD_x, && e_{18} = pD_p + qD_q + yD_y.
\end{aligned} \tag{4.3.32}$$

The non-zero brackets of the symmetry algebra are given by:

$$
\begin{array}{llll}
[e_1, e_{16}] = e_1, & [e_2, e_{11}] = ae_3, & [e_2, e_{18}] = e_2, & [e_3, e_{18}] = e_3, \\
[e_4, e_{17}] = e_4, & [e_5, e_8] = e_3, & [e_5, e_{11}] = e_2, & [e_5, e_{18}] = e_5, \\
[e_6, e_{14}] = -e_1, & [e_6, e_{16}] = e_6, & [e_7, e_{15}] = -e_1, & [e_7, e_{16}] = e_7, \\
[e_8, e_{13}] = -e_9, & [e_9, e_{15}] = -e_9, & [e_9, e_{18}] = e_9, & [e_{10}, e_{14}] = -e_{10}, \\
[e_{10}, e_{17}] = e_{10}, & [e_{11}, e_{12}] = -ae_9, & [e_{11}, e_{13}] = -e_{12}, & [e_{12}, e_{15}] = -ae_9 - e_{12}, \\
[e_{12}, e_{18}] = e_{12}, & [e_{13}, e_{14}] = -e_9, & [e_{13}, e_{15}] = -e_{12} - e_{13}, & [e_{13}, e_{18}] = e_{13}.
\end{array}
\tag{4.3.33}
$$

In this case, based on the Lie invariance condition, we have to consider one sub-case based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The only case we consider is $A_{6,9}^{a=0}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 4.3.8. The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra with thirteen-dimensional nilradical spanned by e_1–e_{13}, which is a direct sum of an eight-dimensional indecomposable solvable Lie algebra B_8 and a five-dimensional abelian Lie algebra. The complement of the nilradical is a another five-dimensional abelian Lie algebra spanned by e_{14}–e_{18}. Therefore, the symmetry algebra can be identified as $(B_8 \oplus \mathbb{R}^5) \rtimes \mathbb{R}^5$, where B_8 is the following solvable indecomposable eight-dimensional Lie algebra given by the non-zero brackets

$$[e3, e4] = e2, [e3, e6] = e1, [e4, e8] = -e5, [e6, e7] = -ae5, [e6, e8] = -e7. \tag{4.3.34}$$

4.3.8 Algebra $A^{ab}_{6,10}$

The non-zero brackets for the algebra $A^{ab}_{6,10}$ are given by

$$\begin{aligned} &[e_1, e_6] = e_6, \quad [e_2, e_3] = e_3, \qquad [e_2, e_4] = e_5, \quad [e_2, e_5] = e_6, \\ &[e_1, e_3] = ae_3, \quad [e_1, e_4] = e_4 + be_6, \quad [e_1, e_5] = e_5. \end{aligned} \tag{4.3.35}$$

The geodesic equations are given by

$$\ddot{p} = \dot{w}(\dot{p} + b\dot{x}) + \dot{q}\dot{z}, \quad \ddot{q} = \dot{q}\dot{w} + \dot{x}\dot{z}, \quad \ddot{x} = \dot{x}\dot{w}, \quad \ddot{y} = \dot{y}(\dot{z} + a\dot{w}), \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.3.36}$$

For the general case $A_{6,10}^{a\neq 0,b\neq 0}$, the symmetry Lie algebra is spanned by

$$\begin{aligned}
&e_1 = D_t, && e_2 = D_q,\\
&e_3 = D_p, && e_4 = D_y,\\
&e_5 = D_x, && e_6 = wD_t,\\
&e_7 = zD_t, && e_8 = xD_p,\\
&e_9 = qD_p + xD_q, && e_{10} = e^w D_p, \qquad (4.3.37)\\
&e_{11} = ze^w D_p + e^w D_q, && e_{12} = e^{aw} e^z D_y,\\
&e_{13} = [\tfrac{(2w-2)b}{2} + \tfrac{z^2}{2}]e^w D_p + ze^w D_q + e^w D_x, e_{14} = D_z, && e_{15} = D_w,\\
&e_{16} = tD_t, && e_{17} = yD_y,\\
&e_{18} = pD_p + qD_q + xD_x.
\end{aligned}$$

The non-zero brackets of the symmetry algebra are given by:

$$
\begin{array}{llll}
[e_1, e_{16}] = e_1, & [e_2, e_9] = e_3, & [e_2, e_{18}] = e_2, & [e_3, e_{18}] = e_3, \\
[e_4, e_{17}] = e_4, & [e_5, e_8] = e_3, & [e_5, e_9] = e_2, & [e_5, e_{18}] = e_5, \\
[e_6, e_{15}] = -e_1, & [e_6, e_{16}] = e_6, & [e_7, e_{14}] = -e_1, & [e_7, e_{16}] = e_7, \\
[e_8, e_{13}] = -e_{10}, & [e_9, e_{11}] = -e_{10}, & [e_9, e_{13}] = -e_{11}, & [e_{10}, e_{15}] = -e_{10}, \\
[e_{10}, e_{18}] = e_{10}, & [e_{11}, e_{14}] = -e_{10}, & [e_{11}, e_{15}] = -e_{11}, & [e_{11}, e_{18}] = e_{11}, \\
[e_{12}, e_{14}] = -e_{12}, & [e_{12}, e_{15}] = -ae_{12}, & [e_{12}, e_{17}] = e_{12}, & [e_{13}, e_{14}] = -e_{11}, \\
[e_{13}, e_{15}] = -be_{10} - e_{13}, & [e_{13}, e_{18}] = e_{13}. & &
\end{array}
\tag{4.3.38}
$$

In this case, based on the Lie invariance condition, we have to consider three sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A_{6,10}^{a=0}$, $A_{6,10}^{a=1}$ and $A_{6,10}^{b=0}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 4.3.9. The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra with thirteen-dimensional nilradical spanned by e_1–e_{13}, which is a direct sum of an eight-dimensional indecomposable solvable Lie algebra $B_{8(a=1)}$ and a five-dimensional abelian Lie algebra. The complement of the nilradical is a another five-dimensional abelian Lie algebra spanned by e_{14}–e_{18}. Therefore, the symmetry algebra can be identified as $(B_{8(a=1)} \oplus \mathbb{R}^5) \rtimes \mathbb{R}^5$, where the non-zero brackets of B_{8a} are given by Eq (4.3.34).

4.3.9 Algebra $A_{6,11}^{a}$

The non-zero brackets for the algebra $A_{6,11}^{a}$ are given by

$$[e_1, e_3] = e_4, \quad [e_2, e_4] = e_4, \quad [e_1, e_6] = e_6, \quad [e_2, e_3] = e_3, \quad [e_2, e_5] = ae_5,$$
$$[e_1, e_5] = e_5 + e_6, \quad [e_2, e_6] = ae_6. \tag{4.3.39}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}(a\dot{z} + \dot{w}) + \dot{q}\dot{w}, \quad \ddot{q} = \dot{q}(a\dot{z} + \dot{w}), \quad \ddot{x} = \dot{x}\dot{z}, \quad \ddot{y} = \dot{x}\dot{w} + \dot{z}\dot{y}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.3.40}$$

For the general case $A_{6,11}^{a\neq 0}$, the symmetry Lie algebra is spanned by

$$\begin{aligned}
&e_1 = D_t, && e_2 = D_p, && e_3 = D_y,\\
&e_4 = D_q, && e_5 = D_x, && e_6 = wD_t,\\
&e_7 = zD_t, && e_8 = qD_p, && e_9 = xD_y,\\
&e_{10} = e^z D_y, && e_{11} = e^z D_x + we^z D_y, && e_{12} = e^w e^{az} D_p,\\
&e_{13} = we^w e^{az} D_p + e^w e^{az} D_q, && e_{14} = D_w, && e_{15} = D_z,\\
&e_{16} = tD_t, && e_{17} = pD_p + qD_q, && e_{18} = xD_x + yD_y.
\end{aligned} \tag{4.3.41}$$

The non-zero brackets of the symmetry algebra are given by

$$\begin{array}{llll}
[e_1, e_{16}] = e_1, & [e_2, e_{17}] = e_2, & [e_3, e_{18}] = e_3, & [e_4, e_8] = e_2, \\
[e_4, e_{17}] = e_4, & [e_5, e_9] = e_3, & [e_5, e_{18}] = e_5, & [e_6, e_{14}] = -e_1, \\
[e_6, e_{16}] = e_6, & [e_7, e_{15}] = -e_1, & [e_7, e_{16}] = e_7, & [e_8, e_{13}] = -e_{12}, \\
[e_9, e_{11}] = -e_{10}, & [e_{10}, e_{15}] = -e_{10}, & [e_{10}, e_{18}] = e_{10}, & [e_{11}, e_{14}] = -e_{10}, \\
[e_{11}, e_{15}] = -e_{11}, & [e_{11}, e_{18}] = e_{11}, & [e_{12}, e_{14}] = -e_{12}, & [e_{12}, e_{15}] = -ae_{12}, \\
[e_{12}, e_{17}] = e_{12}, & [e_{13}, e_{14}] = -e_{12} - e_{13}, & [e_{13}, e_{15}] = -ae_{13}, & [e_{13}, e_{17}] = e_{13}.
\end{array} \tag{4.3.42}$$

In this case, based on the Lie invariance condition, we have to consider two sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A_{6,11}^{a=0}$ and $A_{6,11}^{a=1}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 4.3.10. The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra with thirteen-dimensional nilradical spanned by e_1–e_{13} and five-dimensional abelian complement spanned by e_{14}–e_{18}. In fact, the nilradical is a direct sum of two copies of $A_{5,1}$ and $\mathbb{R}^3$. Hence, the symmetry algebra is $(A_{5,1} \oplus A_{5,1} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$, where $A_{5,1}$ is given by Eq (4.3.5).

4.3.10 Algebra $A^{ab}_{6,12}$

The non-zero brackets for the algebra $A^{ab}_{6,12}$ are given by

$$\begin{aligned} &[e_1, e_4] = e_4, && [e_1, e_5] = e_5 + e_6, && [e_1, e_6] = e_6, \\ &[e_2, e_3] = ae_4 + e_5 - be_6, && [e_1, e_3] = e_3 + e_4, && [e_2, e_4] = e_6, \\ &[e_2, e_5] = -e_3 + be_4 + ae_6, && [e_2, e_6] = -e_4. && \end{aligned} \tag{4.3.43}$$

The geodesic equations are given by

$$\begin{aligned} &\ddot{p} = \dot{z}(\dot{p} + \dot{x}) + \dot{w}(a\dot{x} + b\dot{y} - \dot{q}), && \ddot{q} = \dot{z}(\dot{q} + \dot{y}) + \dot{w}(a\dot{y} - b\dot{x} + \dot{p}), && \ddot{x} = \dot{x}\dot{z} - \dot{y}\dot{w}, \\ &\ddot{y} = \dot{y}\dot{z} + \dot{x}\dot{w}, && \ddot{z} = 0, && \ddot{w} = 0. \end{aligned} \tag{4.3.44}$$

For the general case $A^{a\neq 0, b\neq 0}_{6,12}$, the symmetry Lie algebra is spanned by

$$\begin{aligned} &e_1 = D_y, && e_2 = D_p, \\ &e_3 = D_q, && e_4 = D_t, \\ &e_5 = D_x, && e_6 = wD_t, \\ &e_7 = zD_t, && e_8 = xD_p + yD_q, \\ &e_9 = yD_p - xD_q, && e_{10} = \cos(w)e^z D_p + \sin(w)e^z D_q, \\ &e_{11} = \sin(w)e^z D_p - \cos(w)e^z D_q, && \end{aligned}$$

$$e_{12} = ((aw+b+z-1)\ \cos(w) + bw\ \sin(w))\ e^z D_p + ((aw+b+z-1)\sin(w),$$

$$-w\ \cos(w)b)\ e^z D_q + \cos(w)\ e^z D_x + \sin(w)\ e^z D_y,$$

$$e_{13} = ((-bw+a)\ \cos(w), + \sin(w)(aw+z-1))\ e^z D_p + (-\cos(w)(aw+z-1),$$

$$-(bw-a)\sin(w))\ e^z D_q + \sin(w)\ e^z D_x - \cos(w)\ e^z D_y,$$

$$e_{14} = tD_t, \qquad e_{15} = D_z, \qquad e_{16} = D_w,$$

$$e_{17} = pD_p + qD_q + xD_x + yD_y, \quad e_{18} = qD_p - pD_q + yD_x - xD_y. \tag{4.3.45}$$

The non-zero brackets of the symmetry algebra are given by

$$
\begin{array}{lll}
[e_1, e_8] = e_3, & [e_1, e_9] = e_2, & [e_1, e_{17}] = e_1, \\
[e_1, e_{18}] = e_5, & [e_2, e_{17}] = e_2, & [e_2, e_{18}] = -e_3, \\
[e_3, e_{17}] = e_3, & [e_3, e_{18}] = e_2, & [e_4, e_{14}] = e_4, \\
[e_5, e_8] = e_2, & [e_5, e_9] = -e_3, & [e_5, e_{17}] = e_5, \\
[e_5, e_{18}] = -e_1, & [e_6, e_{14}] = e_6, & [e_6, e_{16}] = -e_4, \\
[e_7, e_{14}] = e_7, & [e_7, e_{15}] = -e_4, & [e_8, e_{12}] = -e_{10}, \\
[e_8, e_{13}] = -e_{11}, & [e_9, e_{12}] = -e_{11}, & [e_9, e_{13}] = e_{10}, \\
[e_{10}, e_{15}] = -e_{10}, & [e_{10}, e_{16}] = e_{11}, & [e_{10}, e_{17}] = e_{10}, \\
[e_{10}, e_{18}] = e_{11}, & [e_{11}, e_{15}] = -e_{11}, & [e_{11}, e_{16}] = -e_{10}, \\
[e_{11}, e_{17}] = e_{11}, & [e_{11}, e_{18}] = -e_{10}, & [e_{12}, e_{15}] = -e_{10} - e_{12}, \\
[e_{12}, e_{16}] = -2ae_{10} + e_{13}, & [e_{12}, e_{17}] = e_{12}, & [e_{12}, e_{18}] = -ae_{10} + be_{11} + e_{13}, \\
[e_{13}, e_{15}] = -e_{11} - e_{13}, & [e_{13}, e_{16}] = 2be_{10} - e_{12}, & [e_{13}, e_{17}] = e_{13}, \\
[e_{13}, e_{18}] = ae_{11} + be_{10} - e_{12}. & &
\end{array}
\tag{4.3.46}
$$

In this case, based on the Lie invariance condition, we have to consider two sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A_{6,12}^{a=0}$ and $A_{6,12}^{b=0}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 4.3.11. The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra with thirteen-dimensional nilradical spanned by e_1–e_{13} and five-dimensional abelian complement spanned by e_{14}–e_{18}. In fact, the nilradical is a direct sum of ten-dimensional solvable Lie algebra C_{10} and $\mathbb{R}^3$. Hence, the symmetry algebra is $(C_{10} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$, where C_{10} is an indecoposable ten-dimensional solvable Lie algebra given by the non-zero brackets

$$[e_1, e_5] = e_3, \quad [e_1, e_6] = e_2, \quad [e_4, e_5] = e_2, \quad [e_4, e_6] = -e_3,$$
$$[e_5, e_9] = -e_7, \quad [e_5, e_{10}] = -e_8, \quad [e_6, e_9] = -e_8, \quad [e_6, e_{10}] = e_7. \tag{4.3.47}$$

4.3.11 Algebra $A_{6,13}^{abcd}(abcd : a^2 + b^2 \neq 0,\ c^2 + d^2 \neq 0)$

The non-zero brackets for the algebra $A_{6,13}^{abcd}$ are given by

$$\begin{aligned} &[e_1, e_3] = ae_3, \quad [e_1, e_4] = ce_4, \quad [e_1, e_5] = e_6, \quad [e_1, e_6] = -e_5, \\ &[e_2, e_3] = be_3, \quad [e_2, e_4] = de_4, \quad [e_2, e_6] = e_6, \quad [e_2, e_5] = e_5. \end{aligned} \tag{4.3.48}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}\dot{z} - \dot{q}\dot{w}, \quad \ddot{q} = \dot{p}\dot{w} + \dot{q}\dot{z}, \quad \ddot{x} = \dot{x}(d\dot{z} + c\dot{w}), \quad \ddot{y} = \dot{y}(b\dot{z} + a\dot{w}), \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.3.49}$$

For the general case $A_{6,13}^{a\neq 0,b\neq 0,c\neq 0,d\neq 0}$, the symmetry Lie algebra is spanned by

$$\begin{array}{lll}
e_1 = D_t, & e_2 = D_p, & e_3 = D_q, \\
e_4 = D_x, & e_5 = D_y, & e_6 = wD_t, \\
e_7 = zD_t, & e_8 = e^{cw}e^{dz}D_x, & e_9 = e^{aw}e^{bz}D_y, \\
e_{10} = \cos(w)e^zD_p + \sin(w)e^zD_q, & e_{11} = \sin(w)e^zD_p - \cos(w)e^zD_q, & e_{12} = D_w, \\
e_{13} = D_z, & e_{14} = tD_t, & e_{15} = xD_x, \\
e_{16} = yD_y, & e_{17} = pD_p + qD_q & e_{18} = qD_p - pD_q.
\end{array} \tag{4.3.50}$$

The non-zero brackets of the symmetry algebra are given by

$$\begin{array}{llll}
[e_1, e_{14}] = e_1, & [e_2, e_{17}] = e_2, & [e_2, e_{18}] = -e_3, & [e_3, e_{17}] = e_3, \\
[e_3, e_{18}] = e_2, & [e_4, e_{15}] = e_4, & [e_5, e_{16}] = e_5, & [e_6, e_{12}] = -e_1, \\
[e_6, e_{14}] = e_6, & [e_7, e_{13}] = -e_1, & [e_7, e_{14}] = e_7, & [e_8, e_{12}] = -ce_8, \\
[e_8, e_{13}] = -de_8, & [e_8, e_{15}] = e_8, & [e_9, e_{12}] = -ae_9, & [e_9, e_{13}] = -be_9, \\
[e_9, e_{16}] = e_9, & [e_{10}, e_{12}] = e_{11}, & [e_{10}, e_{13}] = -e_{10}, & [e_{10}, e_{17}] = e_{10}, \\
[e_{10}, e_{18}] = e_{11}, & [e_{11}, e_{12}] = -e_{10}, & [e_{11}, e_{13}] = -e_{11}, & [e_{11}, e_{17}] = e_{11}, \\
[e_{11}, e_{18}] = -e_{10}. & & &
\end{array} \tag{4.3.51}$$

In this case, based on the Lie invariance condition, we have to consider eight sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are: $A_{6,13}^{a=0,b\neq 0}$, $A_{6,13}^{a=c,a\neq 0}$, $A_{6,13}^{b=0,a\neq 0}$, $A_{6,13}^{b=1}$, $A_{6,13}^{b=d,b\neq 0}$, $A_{6,13}^{c=0,d\neq 0}$, $A_{6,13}^{d=1,}$ and $A_{6,13}^{d=0,c\neq 0}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition:

Proposition 4.3.12. The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of eleven-dimensional abelian nilradical spanned by e_1–e_{11} and a seven-dimensional abelian complement spanned by e_{12}–e_{18}. Hence, it can be described as $\mathbb{R}^{11} \rtimes \mathbb{R}^7$.

4.3.12 Algebra $A_{6,14}^{abc}, (ab \neq 0)$

The non-zero brackets for the algebra $A_{6,14}^{abc}$ are given by

$$[e_1, e_5] = ce_5 + e_6, \quad [e_1, e_6] = e_5 + ce_6, \quad [_1, e_3] = ae_3, \quad [e_2, e_3] = be_3, \\ [e_2, e_4] = e_4. \tag{4.3.52}$$

The geodesic equations are given by

$$\ddot{p} = \dot{w}(c\dot{p} + \dot{q}), \quad \ddot{q} = \dot{w}(-\dot{p} + c\dot{q}), \quad \ddot{x} = \dot{x}\dot{z}, \quad \ddot{y} = \dot{y}(b\dot{z} + a\dot{w}), \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.3.53}$$

For the general case $A_{6,14}^{a\neq 0,b\neq 0,c\neq 0}$, the symmetry Lie algebra is spanned by

$$
\begin{aligned}
e_1 &= D_t, & e_2 &= D_q, \\
e_3 &= D_p, & e_4 &= D_y, \\
e_5 &= D_x, & e_6 &= wD_t, \\
e_7 &= zD_t, & e_8 &= e^z D_x, \\
e_9 &= e^{aw}e^{bz}D_y, & e_{10} &= e^{cw}\sin(w)D_p + e^{cw}\cos(w)D_q, \\
e_{11} &= -e^{cw}\cos(w)D_p + e^{cw}\sin(w)D_q, & e_{12} &= D_z, \\
e_{13} &= D_w, & e_{14} &= tD_t, \\
e_{15} &= yD_y, & e_{16} &= xD_x, \\
e_{17} &= pD_p + qD_q, & e_{18} &= -qD_p + pD_q.
\end{aligned}
\tag{4.3.54}
$$

The non-zero brackets of the symmetry algebra are given by

$$
\begin{aligned}
&[e_1, e_{14}] = e_1, && [e_2, e_{17}] = e_2, && [e_2, e_{18}] = -e_3, && [e_3, e_{17}] = e_3,\\
&[e_3, e_{18}] = e_2, && [e_4, e_{15}] = e_4, && [e_5, e_{16}] = e_5, && [e_6, e_{13}] = -e_1,\\
&[e_6, e_{14}] = e_6, && [e_7, e_{14}] = e_7, && [e_7, e_{12}] = -e_1, && [e_8, e_{12}] = -e_8,\\
&[e_8, e_{16}] = e_8, && [e_9, e_{12}] = -be_9, && [e_9, e_{13}] = -ae_9, && [e_9, e_{15}] = e_9,\\
&[e_{10}, e_{13}] = -ce_{10} + e_{11}, && [e_{10}, e_{17}] = e_{10}, && [e_{10}, e_{18}] = e_{11}, && [e_{11}, e_{13}] = -ce_{11} - e_{10},\\
&[e_{11}, e_{17}] = e_{11}, && [e_{11}, e_{18}] = -e_{10}.
\end{aligned}
\tag{4.3.55}
$$

In this case, based on the Lie invariance condition, we have to consider two sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are: $A_{6,14}^{b=1,a\neq 0}$ and $A_{6,14}^{a=c,a\neq 0}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 4.3.13. The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of eleven-dimensional abelian nilradical spanned by e_1–e_{11} and a seven-dimensional abelian complement spanned by e_{12}–e_{18}. Hence, it can be described as $\mathbb{R}^{11} \rtimes \mathbb{R}^7$.

4.3.13 Algebra $A_{6,15}^{abcd}(abcd : b \neq 0)$

The non-zero brackets for the algebra $A_{6,15}^{abcd}$ are given by

$$
\begin{aligned}
&[e_1, e_3] = e_3, && [e_1, e_4] = e_4, && [e_1, e_5] = ae_5 + be_6, && [e_1, e_6] = -be_5 + ae_6, \\
&[e_2, e_3] = ce_3 + e_4, && [e_2, e_4] = -e_3 + ce_4, && [e_2, e_6] = de_6, && [e_2, e_5] = de_5.
\end{aligned}
\tag{4.3.56}
$$

The geodesic equations are given by

$$
\begin{aligned}
&\ddot{p} = \dot{p}(c\dot{z} + \dot{w}) - \dot{q}\dot{z}, && \ddot{q} = \dot{z}(\dot{p} + c\dot{q}) + \dot{q}\dot{w}, && \ddot{x} = \dot{x}(d\dot{z} + a\dot{w}) - b\dot{y}\dot{w}, \\
&\ddot{y} = b\dot{x}\dot{w} + \dot{y}(d\dot{z} + q\dot{w}), && \ddot{z} = 0, && \ddot{w} = 0.
\end{aligned}
\tag{4.3.57}
$$

For the general case $A_{6,15}^{a\neq 0, b\neq 0, c\neq 0, d\neq 0}$, the symmetry Lie algebra is spanned by

$$
\begin{aligned}
&e_1 = D_p, \quad e_2 = D_t, \quad e_3 = D_w, \quad e_4 = D_z, \quad e_5 = D_x, \\
&e_6 = D_y, \quad e_7 = wD_t, \quad e_8 = zD_t, \quad e_9 = tD_t, \quad e_{10} = xD_x + yD_y.
\end{aligned}
\tag{4.3.58}
$$

The non-zero brackets of the symmetry algebra are given by

$$
\begin{aligned}
&[e_2, e_9] = e_2, \quad [e_3, e_7] = e_2, \quad [e_4, e_8] = e_2, \quad [e_5, e_{10}] = e_5, \quad [e_6, e_{10}] = e_6, \\
&[e_7, e_9] = e_7, \quad [e_8, e_9] = e_8.
\end{aligned}
\tag{4.3.59}
$$

In this case, based on the Lie invariance condition, we have to consider five sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A_{6,15}^{a=0}$, $A_{6,15}^{a=1}$, $A_{6,15}^{c=0}$, $A_{6,15}^{d=0}$

and $A_{6,15}^{c=d}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 4.3.14. The symmetry Lie algebra is a ten-dimensional solvable Lie algebra with eight -dimensional not abelian nilradical spanned by e_1–e_8 and two-dimensional abelian complement spanned by e_9 and e_{10}. In fact, The nilradical is a direct sum of $A_{5,4}$ and $\mathbb{R}^3$. Hence, the symmetry Lie algebra can be identified as $(A_{5,4} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^2$, where the non-zero brackets of $A_{5,4}$ are given by

$$[e_2, e_4] = e_1, \ [e_3, e_5] = e_1. \tag{4.3.60}$$

4.3.14 Algebra $A_{6,16}^{ab}$

The non-zero brackets for the algebra $A_{6,16}^{ab}$ are given by

$$\begin{aligned} &[e_1, e_3] = e_4, && [e_2, e_4] = e_4, \quad [e_1, e_5] = ae_5 + e_6, \quad [e_2, e_3] = e_3, \\ &[e_1, e_6] = -e_5 + ae_6, && [e_2, e_5] = be_5, \quad [e_2, e_6] = be_6. \end{aligned} \tag{4.3.61}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}(b\dot{z} + a\dot{w}) - \dot{q}\dot{w}, \quad \ddot{q} = \dot{p}\dot{w} + \dot{q}(b\dot{z} + a\dot{w}), \quad \ddot{x} = \dot{x}\dot{z}, \quad \ddot{y} = \dot{y}\dot{z}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.3.62}$$

For the general case $A_{6,16}^{a\neq 0, b\neq 0}$, the symmetry Lie algebra is spanned by

$$
\begin{aligned}
e_1 &= D_t, & e_2 &= D_p,\\
e_3 &= D_q, & e_4 &= D_x,\\
e_5 &= D_y, & e_6 &= wD_t,\\
e_7 &= zD_t, & e_8 &= e^z D_x,\\
e_9 &= e^z D_y, & e_{10} &= e^{bz}\cos(w)e^{aw}D_p + \sin(w)e^{aw+bz}D_q,\\
e_{11} &= e^{bz}\sin(w)e^{aw}D_p - \cos(w)e^{aw+bz}D_q, & e_{12} &= D_z,\\
e_{13} &= tD_t, & e_{14} &= D_w,\\
e_{15} &= xD_x, & e_{16} &= yD_x,\\
e_{17} &= xD_y, & e_{18} &= yD_y,\\
e_{19} &= pD_p + qD_q, & e_{20} &= qD_p - pD_q.
\end{aligned}
\tag{4.3.63}
$$

The non-zero brackets of the symmetry algebra are given by

$$
\begin{array}{llll}
[e_1, e_{13}] = e_1, & [e_2, e_{19}] = e_2, & [e_2, e_{20}] = -e_3, & [e_3, e_{19}] = e_3, \\
[e_3, e_{20}] = e_2, & [e_4, e_{15}] = e_4, & [e_4, e_{17}] = e_5, & [e_5, e_{16}] = e_4, \\
[e_5, e_{18}] = e_5, & [e_6, e_{13}] = e_6, & [e_6, e_{14}] = -e_1, & [e_7, e_{12}] = -e_1, \\
[e_7, e_{13}] = e_7, & [e_8, e_{12}] = -e_8, & [e_8, e_{15}] = e_8, & [e_8, e_{17}] = e_9, \\
[e_9, e_{12}] = -e_9, & [e_9, e_{16}] = e_8, & [e_9, e_{18}] = e_9, & [e_{10}, e_{12}] = -be_{10}, \\
[e_{10}, e_{14}] = -ae_{10} + e_{11}, & [e_{10}, e_{19}] = e_{10}, & [e_{10}, e_{20}] = e_{11}, & [e_{11}, e_{12}] = -be_{11}, \\
[e_{11}, e_{14}] = -ae_{11} - e_{10}, & [e_{11}, e_{19}] = e_{11}, & [e_{11}, e_{20}] = -e_{10}, & [e_{15}, e_{16}] = -e_{16}, \\
[e_{15}, e_{17}] = e_{17}, & [e_{16}, e_{17}] = -e_{15} + e_{18}, & [e_{16}, e_{18}] = -e_{16}, & [e_{17}, e_{18}] = e_{17}.
\end{array}
\tag{4.3.64}
$$

In this case, based on the Lie invariance condition, we have to consider three sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A_{6,16}^{a=0}$, $A_{6,16}^{b=0}$ and $A_{6,16}^{b=1}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition:

Proposition 4.3.15. The symmetry Lie algebra is a twenty-dimensional semi direct product of seventeen solvable Lie algebra and three -dimensional semi-simple $sl(2, \mathbb{R})$. Furthermore, the symmetry Lie algebra has eleven dimensional abelian nilradical and nine-dimensional complement. Therefore, the symmetry Lie algebra can be identified as $(\mathbb{R}^{11} \rtimes \mathbb{R}^6) \rtimes sl(2, \mathbb{R})$.

4.3.15 Algebra $A_{6,17}^{a}$

The non-zero brackets for the algebra $A_{6,17}^{a}$ are given by

$$[e_1, e_3] = ae_3 + e_4, \quad [e_1, e_4] = ae_4, \quad [e_1, e_5] = e_6, \quad [e_1, e_6] = -e_5, \quad [e_2, e_5] = e_5,$$
$$[e_2, e_6] = e_6. \tag{4.3.65}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}\dot{z} - \dot{q}\dot{w}, \quad \ddot{q} = \dot{p}\dot{w} + \dot{q}\dot{z}, \quad \ddot{x} = \dot{w}(a\dot{x} + \dot{y}), \quad \ddot{y} = a\dot{y}\dot{w}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{4.3.66}$$

Based on the system of PDE's obtained from the Lie Invarince Condition we consider the following subcases of certain values of the parameters.

4.3.15.1 Case 1: $A_{6,17}^{a\neq 0}$

The symmetry Lie algebra is spanned by

$$
\begin{aligned}
&e_1 = D_x, && e_2 = D_p,\\
&e_3 = D_q, && e_4 = D_y,\\
&e_5 = D_t, && e_6 = yD_x,\\
&e_7 = wD_t, && e_8 = zD_t,\\
&e_9 = \tfrac{e^{aw}D_x}{a}, && e_{10} = \cos(w)e^z D_p + \sin(w)e^z D_q,\\
&e_{11} = \sin(w)e^z D_p - \cos(w)e^z D_q, && e_{12} = \tfrac{(aw-1)e^{aw}D_x}{a} + e^{aw}D_y,\\
&e_{13} = D_z, && e_{14} = D_w,\\
&e_{15} = tD_t, && e_{16} = xD_x + yD_y,\\
&e_{17} = pD_p + qD_q, && e_{18} = qD_p - pD_q.
\end{aligned}
\tag{4.3.67}
$$

The non-zero brackets of the symmetry algebra are given by

$$\begin{aligned}
&[e_1, e_{16}] = e_1, && [e_2, e_{17}] = e_2, && [e_3, e_{18}] = -e_3, && [e_3, e_{17}] = e_3,\\
&[e_3, e_{18}] = e_2, && [e_4, e_6] = e_1, && [e_4, e_{16}] = e_4, && [e_5, e_{15}] = e_5,\\
&[e_6, e_{12}] = -ae_9, && [e_7, e_{14}] = -e_5, && [e_7, e_{15}] = e_7, && [e_8, e_{13}] = -e_5,\\
&[e_8, e_{15}] = e_8, && [e_9, e_{14}] = -ae_9, && [e_9, e_{16}] = e_9, && [e_{10}, e_{13}] = -e_{10},\\
&[e_{10}, e_{14}] = e_{11}, && [e_{10}, e_{17}] = e_{10}, && [e_{10}, e_{18}] = e_{11}, && [e_{11}, e_{13}] = -e_{11},\\
&[e_{11}, e_{14}] = -e_{10}, && [e_{11}, e_{17}] = e_{11}, && [e_{11}, e_{18}] = -e_{10}, && [e_{12}, e_{14}] = -ae_{12} - ae_9,\\
&[e_{12}, e_{16}] = e_{12}.
\end{aligned} \tag{4.3.68}$$

Proposition 4.3.16. The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of twelve-dimensional decomposable nilradical spanned by e_1–e_{12} and a six-dimensional abelian complement spanned by e_{12}–e_{18}. In fact, the nilradical is a direct sum of $A_{5,1}$ in Winternitz list [27] and $\mathbb{R}^7$. Hence, symmetry algebra can be can be described as $(A_{5,1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$, where $A_{5,1}$ is given by Eq (4.3.5).

4.3.15.2 Case 2: $A_{6,17}^{a=0}$

The symmetry Lie algebra is spanned by

$$
\begin{aligned}
&e_1 = D_t, && e_2 = D_x,\\
&e_3 = D_p, && e_4 = D_q,\\
&e_5 = D_y, && e_6 = wD_t,\\
&e_7 = zD_t, && e_8 = zD_x,\\
&e_9 = wD_x, && e_{10} = \tfrac{1}{2}\ \ w^2D_x + wD_y,\\
&e_{11} = \tfrac{1}{2}\ \ wzD_x + zD_y, && e_{12} = \cos(w)e^zD_p + \sin(w)e^zD_q,\\
&e_{13} = \sin(w)e^zD_p - \cos(w)e^zD_q, && e_{14} = tD_x,\\
&e_{15} = D_z, && e_{16} = D_w,\\
&e_{17} = tD_t, && e_{18} = yD_t,\\
&e_{19} = yD_x, && e_{20} = pD_p + qD_q,\\
&e_{21} = qD_p - pD_q, && e_{22} = \tfrac{1}{2}\ \ twD_x + tD_y,\\
&e_{23} = (wy - 2x)D_t, && e_{24} = \tfrac{1}{2}\ \ wyD_x + yD_y,\\
&e_{25} = (x - \tfrac{wy}{2})D_x, && e_{26} = (\tfrac{1}{2}\ yw^2 - wx)D_x + (wy - 2x)D_y.
\end{aligned}
\tag{4.3.69}
$$

The non-zero brackets of the symmetry algebra are given by

$$
\begin{array}{lll}
[e_1, e_{14}] = e_2, & [e_1, e_{17}] = e_1, & [e_1, e_{22}] = e_5 + \frac{e_9}{2}, \\
[e_2, e_{23}] = -2e_1, & [e_2, e_{25}] = e_2, & [e_2, e_{26}] = -2e_5 - e_9, \\
[e_3, e_{20}] = e_3, & [e_3, e_{21}] = -e_4, & [e_4, e_{20}] = e_4, \\
[e_4, e_{21}] = e_3, & [e_5, e_{18}] = e_1, & [e_5, e_{19}] = e_2, \\
[e_5, e_{23}] = e_6, & [e_5, e_{24}] = e_5 + \frac{e_9}{2}, & [e_5, e_{25}] = \frac{e_9}{2}, \\
[e_5, e_{26}] = e_{10}, & [e_6, e_{14}] = e_9, & [e_6, e_{16}] = -e_1, \\
[e_6, e_{17}] = e_6, & [e_6, e_{22}] = e_{10}, & [e_7, e_{14}] = e_8, \\
[e_7, e_{15}] = -e_1, & [e_7, e_{17}] = e_7, & [e_7, e_{22}] = e_{11}, \\
[e_8, e_{15}] = -e_2, & [e_8, e_{23}] = -2e_7, & [e_8, e_{25}] = e_8, \\
[e_8, e_{26}] = -2e_{11}, & [e_9, e_{16}] = -e_2, & [e_9, e_{23}] = -2e_6, \\
[e_9, e_{25}] = e_9, & [e_9, e_{26}] = -2e_{10}, & [e_{10}, e_{16}] = -e_5 - e_9, \\
[e_{10}, e_{18}] = e_6, & [e_{10}, e_{19}] = e_9, & [e_{10}, e_{24}] = e_{10}, \\
[e_{11}, e_{15}] = -e_5 - \frac{e_9}{2}, & [e_{11}, e_{16}] = -\frac{e_8}{2}, & [e_{11}, e_{18}] = e_7, \\
[e_{11}, e_{19}] = e_8, & [e_{11}, e_{24}] = e_{11}, & [e_{12}, e_{15}] = -e_{12}, \\
[e_{12}, e_{16}] = e_{13}, & [e_{12}, e_{20}] = e_{12}, & [e_{12}, e_{21}] = e_{13},
\end{array}
$$

$$
\begin{array}{lll}
[e_{13}, e_{15}] = -e_{13}, & [e_{13}, e_{16}] = -e_{12}, & [e_{13}, e_{20}] = e_{13}, \\
[e_{13}, e_{21}] = -e_{12}, & [e_{14}, e_{17}] = -e_{14}, & [e_{14}, e_{18}] = -e_{19}, \\
[e_{14}, e_{23}] = -2e_{17} + 2e_{25}, & [e_{14}, e_{25}] = e_{14}, & [e_{14}, e_{26}] = -2e_{22}, \\
[e_{16}, e_{22}] = \frac{e_{14}}{2}, & [e_{16}, e_{23}] = e_{18}, & [e_{16}, e_{24}] = \frac{e_{19}}{2}, \\
[e_{16}, e_{25}] = -\frac{e_{19}}{2}, & [e_{16}, e_{26}] = e_{24} - e_{25}, & [e_{17}, e_{18}] = -e_{18}, \\
[e_{17}, e_{22}] = e_{22}, & [e_{17}, e_{23}] = -e_{23}, & [e_{18}, e_{22}] = -e_{17} + e_{24}, \\
[e_{18}, e_{24}] = -e_{18}, & [e_{18}, e_{26}] = -e_{23}, & [e_{19}, e_{22}] = -e_{14}, \\
[e_{19}, e_{23}] = -2e_{18}, & [e_{19}, e_{24}] = -e_{19}, & [e_{19}, e_{25}] = e_{19}, \\
[e_{19}, e_{26}] = -2e_{24} + 2e_{25}, & [e_{22}, e_{23}] = -e_{26}, & [e_{22}, e_{24}] = e_{22}, \\
[e_{23}, e_{25}] = -e_{23}, & [e_{24}, e_{26}] = -e_{26}, & [e_{25}, e_{26}] = e_{26}.
\end{array}
\tag{4.3.70}
$$

Proposition 4.3.17. The symmetry Lie algebra is a twenty six-dimensional semi direct product of eighteen solvable Lie algebra and eight-dimensional semi-simple $sl(3, \mathbb{R})$. Furthermore, the symmetry Lie algebra has thirteen-dimensional complement. Therefore, the symmetry algebra can be identified as $(\mathbb{R}^{13} \rtimes \mathbb{R}^5) \rtimes sl(3, \mathbb{R})$.

4.3.16 Algebra $A^{abc}_{6,18}, (b \neq 0)$

The non-zero brackets for the algebra $A^{abc}_{6,18}$ are given by

$$\begin{aligned} &[e_1, e_3] = e_4, && [e_1, e_4] = -e_3, && [e_1, e_5] = ae_5 + be_6, && [e_2, e_3] = e_3, \\ &[e_1, e_6] = -be_5 + ae_6, && [e_2, e_4] = e_4, && [e_2, e_5] = ce_5, && [e_2, e_6] = ce_6. \end{aligned} \tag{4.3.71}$$

The geodesic equations are given by

$$\begin{aligned} &\ddot{p} = \dot{p}\dot{z} + \dot{q}\dot{w}, \quad \ddot{q} = -\dot{p}\dot{w} + \dot{q}\dot{z}, \quad \ddot{x} = \dot{x}(c\dot{z} - a\dot{w}) + b\dot{y}\dot{w}, \quad \ddot{y} = -b\dot{x}\dot{w} + \dot{y}(c\dot{z} - a\dot{w}), \\ &\ddot{z} = 0, \qquad \ddot{w} = 0. \end{aligned} \tag{4.3.72}$$

For the general case $A_{6,18}^{a\neq 0,b\neq 0,c\neq 0}$, the symmetry Lie algebra is spanned by

$$
\begin{aligned}
&e_1 = D_y, && e_2 = D_q,\\
&e_3 = D_p, && e_4 = D_t,\\
&e_5 = D_x, && e_6 = wD_t,\\
&e_7 = zD_t, && e_8 = \sin(w)e^z D_p + \cos(w)e^z D_q,\\
&e_9 = -\cos(w)e^z D_p + \sin(w)e^z D_q, && e_{10} = e^{cz}\sin(bw)e^{-aw}D_x,\\
&+\cos(bw)e^{-aw+cz}D_y, && e_{11} = e^{cz}\cos(bw)e^{-aw}D_x,\\
&-\sin(bw)e^{-aw+cz}D_y, && e_{12} = D_z,\\
&e_{13} = tD_t, && e_{14} = D_w,\\
&e_{15} = pD_p + qD_q, && e_{16} = xD_x + yD_y,\\
&e_{17} = -qD_p + pD_q, && e_{18} = yD_x - xD_y.
\end{aligned}
\tag{4.3.73}
$$

The non-zero brackets of the symmetry algebra are given by

$$\begin{array}{llll}
[e_1, e_{16}] = e_1, & [e_1, e_{18}] = e_5, & [e_2, e_{15}] = e_2, & [e_2, e_{17}] = -e_3, \\
[e_3, e_{15}] = e_3, & [e_3, e_{17}] = e_2, & [e_4, e_{13}] = e_4, & [e_5, e_{16}] = e_5, \\
[e_5, e_{18}] = -e_1, & [e_6, e_{13}] = e_6, & [e_6, e_{14}] = -e_4, & [e_7, e_{12}] = -e_4, \\
[e_7, e_{13}] = e_7, & [e_8, e_{12}] = -e_8, & [e_8, e_{14}] = e_9, & [e_8, e_{15}] = e_8, \\
[e_8, e_{17}] = e_9, & [e_9, e_{12}] = -e_9, & [e_9, e_{14}] = -e_8, & [e_9, e_{15}] = e_9, \\
[e_9, e_{17}] = -e_8, & [e_{10}, e_{12}] = -ce_{10}, & [e_{10}, e_{14}] = ae_{10} - be_{11}, & [e_{10}, e_{16}] = e_{10}, \\
[e_{10}, e_{18}] = e_{11}. & [e_{11}, e_{12}] = -ce_{11}, & [e_{11}, e_{14}] = ae_{11} + be_{10}, & [e_{11}, e_{16}] = e_{11}, \\
[e_{11}, e_{18}] = -e_{10}. & & &
\end{array} \tag{4.3.74}$$

In this case, based on the Lie invariance condition, we have to consider three sub-cases based on the values of the parameters to see if taking certain values will generate new solutions to the system of PDE. The cases we consider are $A^{a=0}_{6,18}$, $A^{c=0}_{6,18}$ and $A^{c=1}_{6,18}$. In the generic and sub-cases, we find that the structure of the symmetry Lie algebras are the same. We summarize the results in the following proposition.

Proposition 4.3.18. The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of eleven-dimensional abelian nilradical spanned by e_1–e_{11} and a seven-dimensional abelian complement spanned by e_{12}–e_{18}. Hence, it can be described as $\mathbb{R}^{11} \rtimes \mathbb{R}^7$.

4.3.17 Algebra $A_{6,19}$

The non-zero brackets for the algebra $A_{6,19}$ are given by

$$\begin{aligned} &[e_1, e_3] = e_4 + e_5, \quad [e_1, e_5] = e_6, \quad [e_1, e_6] = -e_5, \quad [e_2, e_3] = e_3, \\ &[e_1, e_4] = -e_3 + e_6, \quad [e_2, e_4] = e_4, \quad [e_2, e_5] = e_5, \quad [e_2, e_6] = e_6. \end{aligned} \tag{4.3.75}$$

The geodesic equations are given by

$$\begin{aligned} &\ddot{p} = \dot{p}\dot{z} + \dot{w}(\dot{x} - \dot{q}), \quad \ddot{q} = \dot{w}(\dot{y} - \dot{p}) + \dot{q}\dot{z}, \quad \ddot{x} = \dot{x}\dot{z} - \dot{y}\dot{w}, \\ &\ddot{y} = \dot{y}\dot{z} + \dot{x}\dot{w}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \end{aligned} \tag{4.3.76}$$

The symmetry Lie algebra is spanned by

$$
\begin{aligned}
&e_1 = D_y, && e_2 = D_p,\\
&e_3 = D_q, && e_4 = D_t,\\
&e_5 = D_x, && e_6 = wD_t,\\
&e_7 = zD_t, && e_8 = e^z e^w D_p - e^{z+w} D_q,\\
&e_9 = e^z e^{-w} D_p + e^{z-w} D_q, && e_{10} = \sin(w) e^z D_p,\\
&+\cos(w) e^z D_x + \sin(w) e^z D_y, && e_{11} = -\cos(w) e^z D_p,\\
&+\sin(w) e^z D_x - \cos(w) e^z D_y, && e_{12} = D_z,\\
&e_{13} = tD_t, && e_{14} = D_w,\\
&e_{15} = pD_p + qD_q + xD_x + yD_y, && e_{16} = qD_p + (-y+p)D_q,\\
&e_{17} = -xD_p + yD_x - xD_y, && e_{18} = (y-p)D_p - qD_q.
\end{aligned}
\tag{4.3.77}
$$

The non-zero brackets of the symmetry algebra are given by

$$
\begin{aligned}
&[e_1, e_{15}] = e_1, && [e_1, e_{16}] = -e_3, && [e_1, e_{17}] = e_5, && [e_1, e_{18}] = e_2,\\
&[e_2, e_{15}] = e_2, && [e_2, e_{16}] = e_3, && [e_2, e_{18}] = -e_2, && [e_3, e_{15}] = e_3,\\
&[e_3, e_{16}] = e_2, && [e_3, e_{18}] = -e_3, && [e_4, e_{13}] = e_4, && [e_5, e_{15}] = e_5,\\
&[e_5, e_{17}] = -e_1 - e_2, && [e_6, e_{13}] = e_6, && [e_6, e_{14}] = -e_4, && [e_7, e_{12}] = -e_4,\\
&[e_7, e_{13}] = e_7, && [e_8, e_{12}] = -e_8, && [e_8, e_{14}] = e_8, && [e_8, e_{15}] = e_8,\\
&[e_8, e_{16}] = e_8, && [e_8, e_{18}] = -e_8, && [e_9, e_{12}] = -e_9, && [e_9, e_{14}] = -e_9,\\
&[e_9, e_{15}] = e_9, && [e_9, e_{16}] = -e_9, && [e_9, e_{18}] = -e_9, && [e_{10}, e_{12}] = -e_{10},\\
&[e_{10}, e_{14}] = e_{11}, && [e_{10}, e_{15}] = e_{10}, && [e_{10}, e_{17}] = e_{11}, && [e_{11}, e_{12}] = -e_{11},\\
&[e_{11}, e_{14}] = -e_{10}, && [e_{11}, e_{15}] = e_{11}, && [e_{11}, e_{17}] = -e_{10}.
\end{aligned}
\tag{4.3.78}
$$

Proposition 4.3.19. The symmetry Lie algebra is an eighteen-dimensional solvable Lie algebra which is the semidirect product of eleven-dimensional abelian nilradical spanned by e_1–e_{11} and a seven-dimensional abelian complement spanned by e_{12}–e_{18}. Hence, it can be described as $\mathbb{R}^{11} \rtimes \mathbb{R}^7$.

4.4 Conclusion

In this work, we have investigated the symmetry Lie algebra of the geodesic equations of the canonical connection on a Lie group. More precisely, we have considered six-dimensional indecomposable solvable Lie algebras with co-dimension two abelian nilradical and abelian complement. In dimension six, there are nineteen such algebras, namely, $A_{6,1} - A_{6,19}$ in [13]. In each case, we list the non-zero brackets of the Lie algebra, the geodesic equations and a basis for the symmetry Lie algebra in terms of vector fields. We also analyze the nilradical of the symmetry Lie algebra. In every case, we identify the symmetry Lie algebra, and a summary of our results is given in Table 2.

Six-dimensional Lie Algebras	Dimension	Identification
$A_{6,1}^{a\neq 0,b\neq 0,c\neq 0,d\neq 0}$	18	$\mathbb{R}^{11} \rtimes \mathbb{R}^7$
$A_{6,2}^{a\neq 0,b\neq 0,c\neq 0}$	18	$(A_{5.1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$
$A_{6,3}^{a\neq 0}$	18	$(A_{5.1} \oplus A_{5.1} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$
$A_{6,4}^{a\neq 0,b\neq 0}$	18	$(A_{5.1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$
$A_{6,5}^{a\neq 0,b\neq 0}$	18	$(A_{5.1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$
$A_{6,6}^{a\neq 0,b\neq 0}$	18	$(A_{5.1} \oplus A_{5.1} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$
$A_{6,7}^{a\neq 0,b\neq 0,c\neq 0}$	18	$(A_{5.1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$
$A_{6,8}$	18	$(B_{8(a=0)} \oplus \mathbb{R}^5) \rtimes \mathbb{R}^5$
$A_{6,9}^{a\neq 0}$	18	$(B_8 \oplus \mathbb{R}^5) \rtimes \mathbb{R}^5$
$A_{6,10}^{a\neq 0,b\neq 0}$	18	$(B_{8(a=1)} \oplus \mathbb{R}^5) \rtimes \mathbb{R}^5$
$A_{6,11}^{a\neq 0}$	18	$(A_{5.1} \oplus A_{5.1} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$
$A_{6,12}^{a\neq 0,b\neq 0}$	18	$(C_{10} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$
$A_{6,13}^{a\neq 0,b\neq 0,c\neq 0,d\neq 0}$	18	$\mathbb{R}^{11} \rtimes \mathbb{R}^7$
$A_{6,14}^{a\neq 0,b\neq 0,c\neq 0}$	18	$\mathbb{R}^{11} \rtimes \mathbb{R}^7$
$A_{6,15}^{a\neq 0,b\neq 0,c\neq 0,d\neq 0}$	10	$(A_{5.4} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^2$
$A_{6,16}^{a\neq 0,b\neq 0}$	20	$(\mathbb{R}^{11} \rtimes \mathbb{R}^6) \rtimes sl(2,\mathbb{R})$
$A_{6,17}^{a\neq 0}$ $A_{6,17}^{a=0}$	18 26	$(A_{5.1} \oplus \mathbb{R}^7) \rtimes \mathbb{R}^6$ $(\mathbb{R}^{13} \rtimes \mathbb{R}^5) \rtimes sl(3,\mathbb{R})$
$A_{6,18}^{a\neq 0,b\neq 0,c\neq 0}$	18	$\mathbb{R}^{11} \rtimes \mathbb{R}^7$
$A_{6,19}$	18	$\mathbb{R}^{11} \rtimes \mathbb{R}^7$

Table 2. Six Dimensional Lie Algebras with Abelian Complement and Identification of the Symmetry Algebra.

CHAPTER 5

SYMMETRY ANALYSIS OF THE CANONICAL CONNECTION ON LIE GROUPS: SIX-DIMENSIONAL CASE WITH A NON-ABELIAN COMPLEMENT

5.1 Introduction

In this chapter, we consider the six-dimensional solvable Lie algebras that have a four-dimensional abelian nilradical and a two-dimensional non-abelian complement. In this case, there are eight such algebras, namely, $A_{6,20} - A_{6,27}$ in Turkowski's list [13]. For each Lie algebra, we give the geodesic equations, a basis for the symmetry Lie algebra in terms of vector fields, and finally we identify the symmetry Lie algebra.

5.2 Algebra $A^{ab}_{6,20}(ab : a^2 + b^2 \neq 0)$

The non-zero brackets for the algebra $A^{ab}_{6,20}$ are given by

$$[e_1, e_4] = ae_4, \quad [e_1, e_6] = e_6, \quad [e_2, e_4] = be_4, \quad [e_1, e_2] = e_3, \quad [e_2, e_5] = e_5. \tag{5.2.1}$$

The geodesic equations are given by:

$$\ddot{p} = \dot{p}(a\dot{z} + b\dot{w}), \quad \ddot{q} = \dot{q}\dot{z}, \quad \ddot{x} = \dot{x}\dot{w}, \quad \ddot{y} = \dot{z}\dot{w}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{5.2.2}$$

For the general case $A^{a\neq0,b\neq0}_{6,20}$, the symmetry Lie algebra is spanned by:

$$
\begin{array}{llll}
e_1 = D_w, & e_2 = D_z, & e_3 = tD_t, & e_4 = D_t, \\
e_5 = tD_y, & e_6 = D_p, & e_7 = D_y, & e_8 = D_q, \\
e_9 = D_x, & e_{10} = pD_p, & e_{11} = wD_t, & e_{12} = zD_t, \\
e_{13} = wD_y, & e_{14} = zD_y, & e_{15} = qD_q, & e_{16} = xD_x, \\
e_{17} = e^z D_q, & e_{18} = e^w D_x, & e_{19} = (wz - 2y)D_t, & e_{20} = (wz - 2y)D_y, \\
e_{21} = e^{bw} e^{az} D_p. & & &
\end{array}
\tag{5.2.3}
$$

We make the following change of basis:

$$
\begin{array}{llll}
\overline{e}_1 = e_4, & \overline{e}_2 = e_6, & \overline{e}_3 = e_7, & \overline{e}_4 = e_8, \\
\overline{e}_5 = e_9, & \overline{e}_6 = e_{11}, & \overline{e}_7 = e_{12}, & \overline{e}_8 = e_{13}, \\
\overline{e}_9 = e_{14}, & \overline{e}_1 = e_{17}, & \overline{e}_1 = e_{18}, & \overline{e}_{12} = e_{21}, \\
\overline{e}_1 = e_1 + \frac{e_{14}}{2}, & \overline{e}_1 = e_2 + \frac{e_{13}}{2}, & \overline{e}_{15} = e_3 - \frac{e_{20}}{2}, & \overline{e}_{16} = e_{10}, \\
\overline{e}_{17} = e_{15}, & \overline{e}_{18} = e_{16}, & \overline{e}_{19} = e_3 + \frac{e_{20}}{2}, & \overline{e}_{20} = e_5, \\
\overline{e}_{21} = e_{19}. & & &
\end{array}
\tag{5.2.4}
$$

The non-zero brackets of the symmetry algebra are given by:

$$
\begin{array}{llll}
[e_1, e_{15}] = e_1, & [e_1, e_{19}] = e_1, & [e_1, e_{20}] = e_3, & [e_2, e_{16}] = e_2, \\
[e_3, e_{15}] = e_3, & [e_3, e_{19}] = -e_3, & [e_3, e_{21}] = -2e_1, & [e_4, e_{17}] = e_4, \\
[e_5, e_{18}] = e_5, & [e_6, e_{13}] = -e_1, & [e_6, e_{15}] = e_6, & [e_6, e_{19}] = e_6, \\
[e_6, e_{20}] = e_8, & [e_7, e_{14}] = -e_1, & [e_7, e_{15}] = e_7, & [e_7, e_{19}] = e_7, \\
[e_7, e_{20}] = e_9, & [e_8, e_{13}] = -e_3, & [e_8, e_{15}] = e_8, & [e_8, e_{19}] = -e_8, \\
[e_8, e_{21}] = -2e_6, & [e_9, e_{14}] = -e_3, & [e_9, e_{15}] = e_9, & [e_9, e_{19}] = -e_9, \\
[e_9, e_{21}] = -2e_7, & [e_{10}, e_{14}] = -e_{10}, & [e_{10}, e_{17}] = e_{10}, & [e_{11}, e_{13}] = -e_{11}, \\
[e_{11}, e_{18}] = e_{11}, & [e_{12}, e_{13}] = -be_{12}, & [e_{12}, e_{14}] = -ae_{12}, & [e_{12}, e_{16}] = e_{12}, \\
[e_{19}, e_{20}] = 2e_{20}, & [e_{19}, e_{21}] = -2e_{21}, & [e_{20}, e_{21}] = -2e_{19}. &
\end{array}
\tag{5.2.5}
$$

We describe the symmetry algebra by the following proposition:

Proposition 5.2.1. The symmetry Lie algebra is a twenty one dimensional Lie algebra. It is a semi direct product of eighteen-dimensional solvable Lie algebra and $sl(2, \mathbb{R})$. The solvable part is $(\mathbb{R}^{12} \rtimes \mathbb{R}^6)$ a semi-direct product of $\mathbb{R}^{12}$ and $\mathbb{R}^6$. Therefore, the symmetry algebra can be identified as $(\mathbb{R}^{12} \rtimes \mathbb{R}^6) \rtimes sl(2, \mathbb{R})$.

5.3 Algebra $A^a_{6,21}$

The non-zero brackets for the algebra $A^a_{6,21}$ are given by

$$
\begin{aligned}
&[e_1, e_4] = e_4, \quad [e_1, e_5] = e_6, \quad [e_2, e_4] = ae_4, \quad [e_2, e_5] = e_5, \quad [e_2, e_6] = e_6, \\
&[e_1, e_2] = e_3.
\end{aligned} \tag{5.3.1}
$$

The geodesic equations are given by:

$$
\ddot{p} = \dot{p}(\dot{z} + a\dot{w}), \quad \ddot{q} = \dot{w}(\dot{q} - x\dot{z}) + \dot{z}\dot{x}, \quad \ddot{x} = \dot{x}\dot{w}, \quad \ddot{y} = \dot{z}\dot{w}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{5.3.2}
$$

For the general case $A_{6,21}^{a\neq 0}$, the symmetry Lie algebra is spanned by:

$$
\begin{aligned}
&e_1 = D_t, && e_2 = tD_y, && e_3 = D_y, && e_4 = D_p, \\
&e_5 = D_q, && e_6 = D_z, && e_7 = D_w, && e_8 = tD_t, \\
&e_9 = pD_p, && e_{10} = wD_t, && e_{11} = zD_t, && e_{12} = wD_y, \\
&e_{13} = zD_y, && e_{14} = xD_q, && e_{15} = zD_q + D_x, && e_{16} = qD_q + xD_x, \\
&e_{17} = e^w D_q, && e_{18} = e^w D_x, && e_{19} = (wz - 2y)D_t, && e_{20} = (wz - 2y)D_y, \\
&e_{21} = e^{aw} e^z D_p.
\end{aligned} \tag{5.3.3}
$$

We make the following change of basis:

$$\begin{array}{llll}
\overline{e}_1 = e_1, & \overline{e}_2 = e_3, & \overline{e}_3 = e_4, & \overline{e}_4 = e_5, \\
\overline{e}_5 = e_{10}, & \overline{e}_6 = e_{11}, & \overline{e}_7 = e_{12}, & \overline{e}_8 = e_{13}, \\
\overline{e}_9 = e_{14}, & \overline{e}_{10} = e_{15}, & \overline{e}_{11} = e_{17}, & \overline{e}_{12} = e_{18}, \\
\overline{e}_{13} = e_{21}, & \overline{e}_{14} = e_6 + \frac{e_{21}}{2}, & \overline{e}_{15} = e_7 + \frac{e_{13}}{2}, & \overline{e}_{16} = e_8 - \frac{e_{20}}{2}, \\
\overline{e}_{17} = e_9, & \overline{e}_{18} = e_{16}, & \overline{e}_{19} = e_2, & \overline{e}_{20} = e_8 + \frac{e_{20}}{2}, \\
\overline{e}_{21} = e_{19}. & & &
\end{array} \tag{5.3.4}$$

The non-zero brackets of the symmetry algebra are given by:

$$
\begin{array}{llll}
[e_1, e_{15}] = -e_1, & [e_1, e_{18}] = e_1, & [e_2, e_{18}] = e_2, & [e_3, e_5] = e_1, \\
[e_3, e_{15}] = -e_3, & [e_3, e_{18}] = e_3, & [e_4, e_5] = e_2, & [e_4, e_{14}] = -e_2, \\
[e_4, e_{18}] = e_4, & [e_6, e_{14}] = -e_9, & [e_6, e_{16}] = e_6, & [e_6, e_{19}] = e_8, \\
[e_6, e_{20}] = e_6, & [e_7, e_{15}] = -e_{10}, & [e_7, e_{16}] = e_7, & [e_7, e_{20}] = -e_7, \\
[e_7, e_{21}] = -2e_{12}, & [e_8, e_{14}] = -e_{10}, & [e_8, e_{16}] = e_8, & [e_8, e_{20}] = -e_8, \\
[e_8, e_{21}] = -2e_6, & [e_9, e_{16}] = e_9, & [e_9, e_{19}] = e_{10}, & [e_9, e_{20}] = e_9, \\
[e_{10}, e_{16}] = e_{10}, & [e_{10}, e_{20}] = -e_{10}, & [e_{10}, e_{21}] = -2e_9, & [e_{11}, e_{17}] = e_{11}, \\
[e_{12}, e_{15}] = -e_9, & [e_{12}, e_{16}] = e_{12}, & [e_{12}, e_{19}] = e_7, & [e_{12}, e_{20}] = e_{12}, \\
[e_{13}, e_{14}] = -e_{13}, & [e_{13}, e_{15}] = -ae_{13}, & [e_{13}, e_{17}] = e_{13}, & [e_{19}, e_{20}] = -2e_{19}, \\
[e_{19}, e_{21}] = -2e_{20}, & [e_{20}, e_{21}] = -2e_{21}. & &
\end{array}
\tag{5.3.5}
$$

We describe the symmetry algebra by the following proposition:

Proposition 5.3.1. The symmetry Lie algebra is a twenty one dimensional Lie algebra. It is a semi direct product of eighteen-dimensional solvable Lie algebra and $sl(2,\mathbb{R})$. The nilradical is thirteen-dimensional decomposable Lie algebra. In fact, the nilradical is a direct sum of $A_{5,1}$ in Winternitz [27] and $\mathbb{R}^8$. The nilradical has a five-dimensional abelian complement. Therefore, the symmetry algebra can be identified as $((A_{5,1} \oplus \mathbb{R}^8) \rtimes \mathbb{R}^5) \rtimes sl(2,\mathbb{R})$, where the non-zero brackets of $A_{5.1}$ are given by:

$$[e_3, e_5] = e_1, [e_4, e_5] = e_2. \tag{5.3.6}$$

5.4 Algebra $A_{6,22}^{a\epsilon} (a\epsilon : a^2 + \epsilon^2 \neq 0, \epsilon = 0, 1)$

The non-zero brackets for the algebra $A_{6,22}^{a\epsilon}$ are given by

$$[e_1, e_3] = e_3, \quad [e_1, e_5] = e_6, \quad [e_2, e_4] = e_4, \quad [e_2, e_3] = ae_3, \quad [e_1, e_2] = \epsilon e_5. \tag{5.4.1}$$

5.4.1 $A_{6,22}^{\epsilon=0}$

The geodesic equations are given by:

$$\ddot{p} = \dot{z}\dot{y}, \quad \ddot{q} = \dot{w}\dot{q}, \quad \ddot{x} = \dot{x}(\dot{z} + a\dot{w}), \quad \ddot{y} = 0, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{5.4.2}$$

For the general case $A_{6,22}^{a\neq0,\epsilon=0}$, the symmetry Lie algebra is spanned by:

$$
\begin{array}{lll}
e_1 = D_y, & e_2 = D_w, & e_3 = D_z, \\
e_4 = tD_t, & e_5 = D_x, & e_6 = D_t, \\
e_7 = tD_p, & e_8 = D_p, & e_9 = D_q, \\
e_{10} = xD_x, & e_{11} = wD_t, & e_{12} = yD_t, \\
e_{13} = zD_t, & e_{14} = wD_p, & e_{15} = yD_p, \\
e_{16} = zD_p, & e_{17} = qD_q, & e_{18} = pD_p + yD_y, \\
e_{19} = e^w D_q, & e_{20} = \frac{z^2}{2}D_p + zD_y, & e_{21} = \frac{zt}{2}D_p + tD_y, \\
e_{22} = (yz-2p)D_t, & e_{23} = (yz-2p)D_p, & e_{24} = \frac{wz}{2}D_p + wD_y,
\end{array}
$$

$$
e_{25} = e^{aw}e^z D_x, \quad , \quad e_{26} = (\frac{yz^2}{2} - pz)D_p + (yz-2p)D_y. \tag{5.4.3}
$$

We consider the following change of basis:

$$\begin{array}{llll}
\overline{e}_1 = e_1, & \overline{e}_2 = e_5, & \overline{e}_3 = e_6, & \overline{e}_4 = e_8, \\
\overline{e}_5 = e_9, & \overline{e}_6 = e_{11}, & \overline{e}_7 = e_{13}, & \overline{e}_8 = e_{14}, \\
\overline{e}_9 = e_{16}, & \overline{e}_{10} = e_{19}, & \overline{e}_{11} = e_{20}, & \overline{e}_{12} = e_{24}, \\
\overline{e}_{13} = e_{25}, & \overline{e}_{14} = e_2, & \overline{e}_{15} = e_3 + \frac{e_{15}}{2}, & \overline{e}_{16} = e_4 + e_{18}, \\
\overline{e}_{17} = e_{10}, & \overline{e}_{18} = e_{17}, & \overline{e}_{19} = e_4 + \frac{e_{23}}{2}, & \overline{e}_{20} = e_7, \\
\overline{e}_{21} = e_{12}, & \overline{e}_{22} = e_{15}, & \overline{e}_{23} = e_{18} + e_{23}, & \overline{e}_{24} = e_{21}, \\
\overline{e}_{25} = e_{22}, & \overline{e}_{26} = e_{26}. & &
\end{array} \tag{5.4.4}$$

The non-zero brackets of the symmetry algebra are given by:

$[e_1, e_{15}] = \frac{e_4}{2}$, $[e_1, e_{16}] = e_1$, $[e_1, e_{19}] = \frac{e_9}{2}$,

$[e_1, e_{21}] = e_3$, $[e_1, e_{22}] = e_4$, $[e_1, e_{23}] = e_1 + e_9$,

$[e_1, e_{25}] = e_7$, $[e_1, e_{26}] = e_{11}$, $[e_2, e_{17}] = e_2$,

$[e_3, e_{16}] = e_3$, $[e_3, e_{19}] = e_3$, $[e_3, e_{20}] = e_4$,

$[e_3, e_{24}] = e_1 + \frac{e_9}{2}$, $[e_4, e_{16}] = e_4$, $[e_4, e_{19}] = -e_4$,

$[e_4, e_{23}] = -e_4$, $[e_4, e_{25}] = -2e_3$, $[e_4, e_{26}] = -2e_1 - e_9$,

$[e_5, e_{18}] = e_5$, $[e_6, e_{14}] = -e_3$, $[e_6, e_{16}] = e_6$,

$[e_6, e_{19}] = e_6$, $[e_6, e_{20}] = e_8$, $[e_6, e_{24}] = e_{12}$,

$[e_7, e_{15}] = -e_3$, $[e_7, e_{16}] = e_7$, $[e_7, e_{19}] = e_7$,

$[e_7, e_{20}] = e_9$, $[e_7, e_{24}] = e_{11}$, $[e_8, e_{14}] = -e_4$,

$[e_8, e_{16}] = e_8$, $[e_8, e_{19}] = -e_8$, $[e_8, e_{23}] = -e_8$,

$[e_8, e_{25}] = -2e_6$, $[e_8, e_{26}] = -2e_{12}$, $[e_9, e_{15}] = -e_4$,

$[e_9, e_{16}] = e_9$, $[e_9, e_{19}] = -e_9$, $[e_9, e_{23}] = -e_9$,

$[e_9, e_{25}] = -2e_7$, $[e_9, e_{26}] = -2e_{11}$, $[e_{10}, e_{14}] = -e_{10}$,

$$
\begin{array}{lll}
[e_{10}, e_{18}] = e_{10}, & [e_{11}, e_{15}] = -e_1 - \frac{9}{2}, & [e_{11}, e_{16}] = e_{11}, \\
[e_{11}, e_{21}] = e_7, & [e_{11}, e_{22}] = e_9, & [e_{11}, e_{23}] = e_{11}, \\
[e_{12}, e_{14}] = -e_1 - \frac{9}{2}, & [e_{12}, e_{16}] = e_{12}, & [e_{12}, e_{21}] = e_6, \\
[e_{12}, e_{22}] = e_8, & [e_{12}, e_{23}] = e_{12}, & [e_{13}, e_{14}] = -ae_{13}, \\
[e_{13}, e_{15}] = -e_{13}, & [e_{13}, e_{17}] = e_{13}, & [e_{19}, e_{20}] = 2e_{20}, \\
[e_{19}, e_{21}] = -e_{21}, & [e_{19}, e_{22}] = e_{22}, & [e_{19}, e_{24}] = e_{24}, \\
[e_{19}, e_{25}] = -2e_{25}, & [e_{19}, e_{26}] = -e_{26}, & [e_{20}, e_{21}] = -e_{22}, \\
[e_{20}, e_{23}] = -e_{20}, & [e_{20}, e_{25}] = -2e_{19}, & [e_{20}, e_{26}] = -2e_{24}, \\
[e_{21}, e_{23}] = -e_{21}, & [e_{21}, e_{24}] = -e_{19} + e_{23}, & [e_{21}, e_{26}] = -e_{25}, \\
[e_{22}, e_{23}] = -2e_{22}, & [e_{22}, e_{24}] = -e_{20}, & [e_{22}, e_{25}] = -2e_{21}, \\
[e_{22}, e_{26}] = -2e_{23}, & [e_{23}, e_{24}] = -e_{24}, & [e_{23}, e_{25}] = -e_{25}, \\
[e_{23}, e_{26}] = -2e_{26}, & [e_{24}, e_{25}] = -e_{26}. &
\end{array}
\tag{5.4.5}
$$

We describe the symmetry algebra by the following proposition:

Proposition 5.4.1. The symmetry Lie algebra is a twenty six dimensional Lie algebra. It is a semi direct product of an eighteen-dimensional solvable Lie algebra and $sl(3,\mathbb{R})$. The solvable part is $(\mathbb{R}^{13} \rtimes \mathbb{R}^5)$ a semi-direct product of $\mathbb{R}^{13}$ and $\mathbb{R}^5$. Therefore, the symmetry algebra can be identified as $(\mathbb{R}^{13} \rtimes \mathbb{R}^5) \rtimes sl(3,\mathbb{R})$.

5.4.2 $A_{6,22}^{\epsilon=1}$

The geodesic equations are given by:

$$\ddot{p} = \dot{p}(a\dot{z} + \dot{w}), \quad \ddot{q} = \dot{q}\dot{z}, \quad \ddot{x} = \dot{y}\dot{w}, \quad \ddot{y} = \dot{z}\dot{w}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{5.4.6}$$

For the general case $A_{6,22}^{a\neq 0,\epsilon=1}$, the symmetry Lie algebra is spanned by:

$$\begin{aligned}
&e_1 = D_t, && e_2 = tD_x, && e_3 = D_p,\\
&e_4 = D_x, && e_5 = D_y, && e_6 = D_q,\\
&e_7 = D_w, && e_8 = D_z, && e_9 = tD_t,\\
&e_{10} = pD_p, && e_{11} = wD_t, && e_{12} = zD_t,\\
&e_{13} = zD_x, && e_{14} = wD_x, && e_{15} = qD_q,\\
&e_{16} = yD_x + zD_y, && e_{17} = e^z D_q, && e_{18} = twD_x + 2tD_y,\\
&e_{19} = \tfrac{w^2}{2}D_x + wD_y, && e_{20} = wzD_x + 2zD_y, && e_{21} = (yz - 2y)D_t,\\
&e_{22} = e^w e^{az} D_p, \quad , && e_{23} = (wy - \tfrac{w^2 z}{2})D_x + (-wz + 2y)D_y. &&
\end{aligned} \tag{5.4.7}$$

We consider the following change of basis:

$$\begin{array}{llll}
\overline{e}_1 = e_1, & \overline{e}_2 = e_2, & \overline{e}_3 = e_3, & \overline{e}_4 = e_4, \\
\overline{e}_5 = e_5, & \overline{e}_6 = e_6, & \overline{e}_7 = e_{11}, & \overline{e}_8 = e_{12}, \\
\overline{e}_9 = e_{13}, & \overline{e}_{10} = e_{14}, & \overline{e}_{11} = e_{16}, & \overline{e}_{12} = e_{17}, \\
\overline{e}_{13} = e_{19}, & \overline{e}_{14} = e_{20}, & \overline{e}_{15} = e_{22}, & \overline{e}_{16} = e_7, \\
\overline{e}_{18} = e_9 + \frac{e_{23}}{2}, & \overline{e}_{19} = e_{10}, & \overline{e}_{20} = e_{15}, & \\
\overline{e}_{21} = e_9 - \frac{e_{23}}{2}, & \overline{e}_{22} = e_{18}, & \overline{e}_{23} = e_{21}. &
\end{array} \tag{5.4.8}$$

The non-zero brackets of the symmetry algebra are given by:

$$
\begin{array}{lll}
[e_1, e_2] = e_4, & [e_1, e_{18}] = e_1, & [e_1, e_{21}] = e_1, \\
[e_1, e_{22}] = e_{10} + 2e_5, & [e_2, e_7] = -e_{10}, & [e_2, e_8] = -e_9, \\
[e_2, e_{18}] = -e_2, & [e_2, e_{21}] = -e_2, & [e_2, e_{23}] = 2e_{11} - e_{14}, \\
[e_3, e_{19}] = e_3, & [e_5, e_{11}] = e_4, & [e_5, e_{18}] = e_5 + \frac{e_{10}}{2}, \\
[e_5, e_{21}] = -e_5 - \frac{e_{10}}{2}, & [e_5, e_{23}] = -2e_1, & [e_6, e_{20}] = e_6, \\
[e_7, e_{16}] = -e_1, & [e_7, e_{18}] = e_7, & [e_7, e_{21}] = e_7, \\
[e_7, e_{22}] = 2e_{13}, & [e_8, e_{17}] = -e_1, & [e_8, e_{18}] = e_8, \\
[e_8, e_{21}] = e_8, & [e_8, e_{22}] = e_{14}, & [e_9, e_{17}] = -e_4, \\
[e_{10}, e_{16}] = -e_4, & [e_{11}, e_{13}] = -e_{10}, & [e_{11}, e_{14}] = -e_9, \\
[e_{11}, e_{17}] = -e_5, & [e_{11}, e_{18}] = -e_{11} + e_{14}, & [e_{11}, e_{21}] = e_{11} - e_{14}, \\
[e_{11}, e_{22}] = -2e_2, & [e_{11}, e_{23}] = -2e_8, & [e_{12}, e_{17}] = -e_{12}, \\
[e_{12}, e_{20}] = e_{12}, & [e_{13}, e_{16}] = -e_{10} - e_5, & [e_{13}, e_{18}] = e_{18}, \\
[e_{13}, e_{21}] = -e_{13}, & [e_{13}, e_{23}] = -2e_7, & [e_{14}, e_{16}] = -e_9, \\
[e_{14}, e_{17}] = -e_{10} - 2e_5, & [e_{14}, e_{18}] = e_{14}, & [e_{14}, e_{21}] = -e_{14}, \\
[e_{14}, e_{23}] = -4e_8, & [e_{15}, e_{16}] = -e_{15}, & [e_{15}, e_{17}] = -ae_{15},
\end{array}
$$

$$
\begin{array}{llll}
[e_{15}, e_{19}] = e_{15}, & [e_{16}, e_{18}] = \frac{e_{11}}{2} - \frac{e_{14}}{2}, & [e_{16}, e_{21}] = -\frac{e_{11}}{2} + \frac{e_{14}}{2}, \\
[e_{16}, e_{22}] = e_2, & [e_{16}, e_{23}] = e_8, & [e_{17}, e_{18}] = -\frac{e_{13}}{2}, \\
[e_{17}, e_{21}] = \frac{e_{13}}{2}, & [e_{17}, e_{23}] = e_7, & [e_{21}, e_{22}] = 2e_{22}, \\
[e_{21}, e_{23}] = -2e_{23}, & [e_{22}, e_{23}] = -4e_{21}. &
\end{array}
\tag{5.4.9}
$$

We describe the symmetry algebra by the following proposition:

Proposition 5.4.2. The symmetry Lie algebra is a twenty three dimensional Lie algebra. It is a semi direct product of twenty-dimensional solvable Lie algebra and $sl(2,\mathbb{R})$. The solvable part is $(\mathbb{R}^{15} \rtimes \mathbb{R}^5)$ a semi-direct product of $\mathbb{R}^{15}$ and $\mathbb{R}^5$. Therefore, the symmetry algebra can be identified as $(\mathbb{R}^{15} \rtimes \mathbb{R}^5) \rtimes sl(2,\mathbb{R})$.

5.5 Algebra $A^{a\epsilon}_{6,23}(a\epsilon : a \geq 0, \epsilon = 0, 1)$

The non-zero brackets for the algebra $A^{a\epsilon}_{6,23}$ are given by

$$
\begin{array}{llll}
[e_1, e_3] = e_3, & [e_1, e_4] = e_4, & [e_1, e_5] = e_6, & [e_2, e_3] = e_4, \\
[e_2, e_4] = -e_3, & [e_2, e_5] = ae_6, & [e_1, e_2] = \epsilon e_5. &
\end{array}
\tag{5.5.1}
$$

5.5.1 case 1: $A^{a,\epsilon=0}_{6,23}$

The geodesic equations when $\epsilon = 0$ are given by:

$$
\ddot{p} = \dot{p}\dot{z} - \dot{q}\dot{w}, \quad \ddot{q} = \dot{p}\dot{w} + \dot{q}\dot{z}, \quad \ddot{x} = \dot{y}(\dot{z} + a\dot{w}), \quad \ddot{y} = 0, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{5.5.2}
$$

The symmetry Lie algebra is spanned by:

$$e_1 = D_t, \quad e_2 = D_p, \quad e_3 = D_q, \quad e_4 = tD_x,$$

$$e_5 = D_x, \quad e_6 = D_y, \quad e_7 = D_w, \quad e_8 = D_z,$$

$$e_9 = tD_t, \quad e_{10} = wD_t, \quad e_{11} = yD_t, \quad e_{12} = zD_t,$$

$$e_{13} = yD_x, \quad e_{14} = zD_x, \quad e_{15} = wD_x, \quad e_{16} = pD_p + qD_q,$$

$$e_{17} = xD_x + yD_y, \qquad e_{18} = qD_p - pD_q,$$

$$e_{19} = \tfrac{t(aw+z)}{2}D_x + tD_y, \qquad e_{20} = ((aw+z)y - 2x)D_x,$$

$$e_{21} = \tfrac{w(aw+z)}{2}D_x + wD_y, \qquad e_{22} = \tfrac{z(aw+z)}{2}D_x + zD_y,$$

$$e_{23} = \tfrac{((aw+z)y-2x)}{a}D_t, \qquad e_{24} = \cos(w)e^z D_p + \sin(w)e^z D_q,$$

$$\begin{aligned} e_{25} &= \sin(w)e^z D_p - \cos(w)e^z D_q, \\ e_{26} &= \tfrac{(\frac{aw}{2}+\frac{z}{2})(awy+yz-2x)}{a}D_x + \tfrac{((aw+z)y-2x)}{a}D_y. \end{aligned} \tag{5.5.3}$$

We consider the following change of basis:

$$
\begin{array}{llll}
\overline{e}_1 = e_1, & \overline{e}_2 = e_2, & \overline{e}_3 = e_3, & \overline{e}_4 = e_5, \\
\overline{e}_5 = e_6, & \overline{e}_6 = e_{10}, & \overline{e}_7 = e_{12}, & \overline{e}_8 = e_{14}, \\
\overline{e}_9 = e_{15}, & \overline{e}_{10} = e_{21}, & \overline{e}_{11} = e_{22}, & \overline{e}_{12} = e_{24}, \\
\overline{e}_{13} = e_{25}, & \overline{e}_{14} = e_7 + \frac{ae_{13}}{2}, & \overline{e}_{15} = e_8 + \frac{e_{13}}{2}, & \overline{e}_{16} = e_9 + e_{17}, \\
\overline{e}_{17} = e_{16}, & \overline{e}_{18} = e_{18}, & \overline{e}_{19} = e_4, & \overline{e}_{20} = e_9 + \frac{e_{20}}{2}, \\
\overline{e}_{21} = e_{11}, & \overline{e}_{22} = e_{13}, & \overline{e}_{23} = e_{17} + e_{20}, & \overline{e}_{24} = e_{19}, \\
\overline{e}_{25} = e_{23}, & \overline{e}_{26} = e_{26}. & &
\end{array} \tag{5.5.4}
$$

The non-zero brackets of the symmetry algebra are given by:

$$[e_1, e_{16}] = e_1, \qquad [e_1, e_{19}] = e_4,$$

$$[e_1, e_{20}] = e_1, \qquad [e_1, e_{24}] = \tfrac{ae_9}{2} + e_5 + \tfrac{e_8}{2},$$

$$[e_2, e_{17}] = e_2, \qquad [e_2, e_{18}] = -e_3,$$

$$[e_3, e_{17}] = e_3, \qquad [e_3, e_{18}] = e_2,$$

$$[e_4, e_{16}] = e_4, \qquad [e_4, e_{20}] = -e_4,$$

$$[e_4, e_{23}] = -e_4, \qquad [e_4, e_{25}] = -\tfrac{2e_1}{a},$$

$$[e_4, e_{26}] = -e_9 - \tfrac{e_8}{a} - \tfrac{2e_5}{a}, \qquad [e_5, e_{14}] = \tfrac{ae_4}{2},$$

$$[e_5, e_{15}] = \tfrac{e_4}{2}, \qquad [e_5, e_{16}] = e_5,$$

$$[e_5, e_{20}] = \tfrac{ae_9}{2} + \tfrac{e_8}{2}, \qquad [e_5, e_{21}] = e_1,$$

$$[e_5, e_{22}] = e_4, \qquad [e_5, e_{23}] = ae_9 + e_5 + e_8,$$

$$[e_5, e_{25}] = \tfrac{e_7}{a} + e_6, \qquad [e_5, e_{26}] = \tfrac{e_{11}}{a} + e_{10},$$

$$[e_6, e_{14}] = -e_1, \qquad [e_6, e_{16}] = e_6,$$

$$[e_6, e_{19}] = e_9, \qquad [e_6, e_{20}] = e_6,$$

$$[e_6, e_{24}] = e_{10}, \qquad [e_7, e_{15}] = -e_1,$$

$[e_7, e_{16}] = e_7,$ $[e_7, e_{19}] = e_8,$

$[e_7, e_{20}] = e_7,$ $[e_7, e_{24}] = e_{11},$

$[e_8, e_{15}] = -e_4,$ $[e_8, e_{16}] = e_8,$

$[e_8, e_{20}] = -e_8,$ $[e_8, e_{23}] = -e_8,$

$[e_8, e_{25}] = -\frac{2e_7}{a},$ $[e_8, e_{26}] = -\frac{2e_{11}}{a},$

$[e_9, e_{14}] = -e_4,$ $[e_9, e_{16}] = e_9,$

$[e_9, e_{20}] = -e_9,$ $[e_9, e_{23}] = -e_9,$

$[e_9, e_{25}] = -\frac{2e_6}{a},$ $[e_9, e_{26}] = -\frac{2e_{10}}{a},$

$[e_{10}, e_{14}] = -\frac{ae_9}{2} - e_5 - \frac{e_8}{2},$ $[e_{10}, e_{16}] = e_{10},$

$[e_{10}, e_{21}] = e_6,$ $[e_{10}, e_{22}] = e_9,$

$[e_{10}, e_{23}] = e_{10},$ $[e_{11}, e_{15}] = -\frac{ae_9}{2} - e_5 - \frac{e_8}{2},$

$[e_{11}, e_{16}] = e_{11},$ $[e_{11}, e_{21}] = e_7,$

$[e_{11}, e_{22}] = e_8,$ $[e_{11}, e_{23}] = e_{11},$

$$
\begin{array}{lll}
[e_{12}, e_{14}] = e_{13}, & [e_{12}, e_{15}] = -e_{12}, & [e_{12}, e_{17}] = e_{12}, \\
[e_{12}, e_{18}] = e_{13}, & [e_{13}, e_{14}] = -e_{12}, & [e_{13}, e_{15}] = -e_{13}, \\
[e_{13}, e_{17}] = e_{13}, & [e_{13}, e_{18}] = -e_{12}, & [e_{19}, e_{20}] = -2e_{19}, \\
[e_{19}, e_{21}] = -e_{22}, & [e_{19}, e_{23}] = -e_{19}, & [e_{19}, e_{25}] = -\frac{2e_{20}}{a}, \\
[e_{19}, e_{26}] = -\frac{2e_{24}}{a}, & [e_{20}, e_{21}] = -e_{21}, & [e_{20}, e_{22}] = e_{22}, \\
[e_{20}, e_{24}] = e_{24}, & [e_{20}, e_{25}] = -2e_{25}, & [e_{20}, e_{26}] = -e_{26}, \\
[e_{21}, e_{23}] = -e_{21}, & [e_{21}, e_{24}] = -e_{20} + e_{23}, & [e_{21}, e_{26}] = -e_{25}, \\
[e_{22}, e_{23}] = -2e_{22}, & [e_{22}, e_{24}] = -e_{19}, & [e_{22}, e_{25}] = -\frac{2e_{21}}{a}, \\
[e_{22}, e_{26}] = -\frac{2e_{23}}{a}, & [e_{23}, e_{24}] = -e_{24}, & [e_{23}, e_{25}] = -e_{25}, \\
[e_{23}, e_{26}] = -2e_{26}, & [e_{24}, e_{25}] = -e_{26}. &
\end{array}
\tag{5.5.5}
$$

We describe the symmetry algebra by the following proposition:

Proposition 5.5.1. The symmetry Lie algebra is a twenty six dimensional Lie algebra. It is a semi direct product of eighteen-dimensional solvable Lie algebra and $sl(3, \mathbb{R})$. The solvable part is $(\mathbb{R}^{13} \rtimes \mathbb{R}^5)$ a semi-direct product of $\mathbb{R}^{13}$ and $\mathbb{R}^5$. Therefore, the symmetry algebra can be identified as $(\mathbb{R}^{13} \rtimes \mathbb{R}^5) \rtimes sl(3, \mathbb{R})$.

5.5.2 case 2: $A_{6,23}^{a,\epsilon=1}$

5.5.2.1 $A_{6,23}^{a\neq 0,\epsilon=1}$

For $A_{6,23}^{a\neq 0,\epsilon=1}$, the geodesic equations are given by:

$$\ddot{p}=\dot{p}\dot{z}+\dot{w}\dot{q},\quad \ddot{q}=-\dot{p}\dot{w}+\dot{q}\dot{z},\quad \ddot{x}=\dot{y}(\dot{z}+a\dot{w}),\quad \ddot{y}=\dot{z}(\dot{z}+a\dot{w}),\quad \ddot{z}=0,\quad \ddot{w}=0. \tag{5.5.6}$$

the symmetry Lie algebra is spanned by:

$$\begin{aligned}
&e_1=D_w, && e_2=D_q, && e_3=D_p,\\
&e_4=D_y, && e_5=tD_x, && e_6=D_z,\\
&e_7=D_x, && e_8=tD_t, && e_9=D_t,\\
&e_{10}=wD_x, && e_{11}=zD_x, && e_{12}=wD_t,\\
&e_{13}=zD_t, && e_{14}=yD_x+zD_y, && e_{15}=pD_p+qD_q,\\
&e_{16}=-qD_p+pD_q, && e_{17}=\frac{t(aw+z)D_x}{2}+tD_y, && e_{18}=\frac{(awz+z^2-2y)D_x}{a},
\end{aligned}$$

$$e_{19}=\frac{(awz+z^2-2y)D_t}{a},\qquad e_{20}=\frac{(\frac{a^2w^2}{2}-\frac{z^2}{2}+y)D_x}{a}+wD_y,$$

$$e_{21}=\sin(w)e^zD_p+\cos(w)e^zD_q,\quad e_{22}=-\cos(w)e^zD_p+\sin(w)e^zD_q,$$

$$e_{23}=\frac{(\frac{aw}{2}+\frac{z}{2})(awz+z^2-2y)D_x}{a}+\frac{(awz+z^2-2y)D_y}{a}. \tag{5.5.7}$$

We consider the following change of basis:

$$
\begin{array}{llll}
\overline{e}_1 = e_4, & \overline{e}_2 = e_5, & \overline{e}_3 = e_7, & \overline{e}_4 = e_9, \\
\overline{e}_5 = e_{10}, & \overline{e}_6 = e_{11}, & \overline{e}_7 = e_{12}, & \overline{e}_8 = e_{13}, \\
\overline{e}_9 = e_{14}, & \overline{e}_{10} = e_{18}, & \overline{e}_{11} = e_{20}, & \overline{e}_{12} = e_2, \\
\overline{e}_{13} = e_3, & \overline{e}_{14} = e_{21}, & \overline{e}_{15} = e_{22}, & \overline{e}_{16} = e_1, \\
\overline{e}_{17} = e_6, & \overline{e}_{18} = e_8 - \frac{ae_{23}}{2}, & \overline{e}_{19} = e_{15}, & \overline{e}_{20} = e_{16}, \\
\overline{e}_{21} = e_8 + \frac{ae_{23}}{2}, & \overline{e}_{22} = e_{17}, & \overline{e}_{23} = e_{19}. &
\end{array}
\tag{5.5.8}
$$

The non-zero brackets of the symmetry algebra are given by:

$$[e_1, e_9] = e_3, \qquad [e_1, e_{10}] = -\tfrac{2e_3}{a},$$

$$[e_1, e_{11}] = \tfrac{e_3}{a}, \qquad [e_1, e_{18}] = \tfrac{e_6}{2} + \tfrac{ae_5}{2} + e_1,$$

$$[e_1, e_{21}] = -\tfrac{e_6}{2} - \tfrac{ae_5}{2} - e_1, \qquad [e_1, e_{23}] = -\tfrac{2e_4}{2},$$

$$[e_2, e_4] = -e_3, \qquad [e_2, e_7] = -e_5,$$

$$[e_2, e_8] = -e_6, \qquad [e_2, e_{18}] = -e_2,$$

$$[e_2, e_{21}] = -e_2, \qquad [e_2, e_{23}] = -e_{10},$$

$$[e_4, e_{18}] = e_4, \qquad [e_4, e_{21}] = e_4,$$

$$[e_4, e_{22}] = \tfrac{e_6}{2} + \tfrac{ae_5}{2} + e_1, \qquad [e_5, e_{16}] = -e_3,$$

$$[e_6, e_{17}] = -e_3, \qquad [e_7, e_{16}] = -e_4,$$

$$[e_7, e_{18}] = e_7, \qquad [e_7, e_{21}] = e_7,$$

$$[e_7, e_{22}] = \tfrac{e_{10}}{2} + e_{11}, \qquad [e_8, e_{17}] = -e_4,$$

$$[e_8, e_{18}] = e_8, \qquad [e_8, e_{21}] = e_8,$$

$$[e_8, e_{22}] = \tfrac{ae_{10}}{2} + e_9, \qquad [e_9, e_{10}] = -\tfrac{2e_6}{a},$$

$$[e_9, e_{11}] = \tfrac{e_6}{a} - e_5, \qquad [e_9, e_{17}] = -e_1,$$

$$[e_9, e_{18}] = ae_{10} + e_9, \qquad [e_9, e_{21}] = -ae_{10} - e_9,$$

$[e_9, e_{22}] = -e_2,$ $[e_9, e_{23}] = -\frac{2e_8}{a},$

$[e_{10}, e_{11}] = \frac{2e_5}{a},$ $[e_{10}, e_{16}] = -e_6,$

$[e_{10}, e_{17}] = -\frac{2e_6}{a} - e_5,$ $[e_{10}, e_{18}] = -e_{10},$

$[e_{10}, e_{21}] = e_{10},$ $[e_{10}, e_{22}] = \frac{2e_2}{a},$

$[e_{11}, e_{16}] = -ae_5 - e_1,$ $[e_{11}, e_{17}] = \frac{e_6}{a},$

$[e_{11}, e_{18}] = e_{10} + e_{11},$ $[e_{11}, e_{21}] = -e_{10} - e_{11},$

$[e_{11}, e_{22}] = -\frac{e_2}{a},$ $[e_{11}, e_{23}] = -\frac{2e_7}{a},$

$[e_{12}, e_{19}] = e_{12},$ $[e_{12}, e_{20}] = -e_{13},$

$[e_{13}, e_{19}] = e_{13},$ $[e_{13}, e_{20}] = e_{12},$

$[e_{14}, e_{16}] = e_{15},$ $[e_{14}, e_{17}] = -e_{14},$

$[e_{14}, e_{19}] = e_{14},$ $[e_{14}, e_{20}] = e_{15},$

$[e_{15}, e_{16}] = -e_{14},$ $[e_{15}, e_{17}] = -e_{15},$

$[e_{15}, e_{19}] = e_{15},$ $[e_{15}, e_{20}] = -e_{14},$

$[e_{16}, e_{18}] = -\frac{a^2e_{10}}{2} - \frac{ae_9}{2},$ $[e_{16}, e_{21}] = \frac{a^2e_{10}}{2} + \frac{ae_9}{2},$

$$
\begin{aligned}
&[e_{16}, e_{22}] = \tfrac{ae_2}{2}, && [e_{16}, e_{23}] = e_8 \\
&[e_{17}, e_{18}] = -\tfrac{ae_{11}}{2} - ae_{10} - e_9, && [e_{17}, e_{21}] = \tfrac{ae_{11}}{2} + ae_{10} + e_9, \\
&[e_{17}, e_{22}] = \tfrac{e_2}{2}, && [e_{17}, e_{23}] = \tfrac{2e_8}{a} + e_7, \\
&[e_{21}, e_{22}] = 2e_{22}, && [e_{21}, e_{23}] = -2e_{23}, \\
&[e_{22}, e_{23}] = -\tfrac{2e_{21}}{a}.
\end{aligned}
\tag{5.5.9}
$$

We describe the symmetry algebra by the following proposition:

Proposition 5.5.2. The symmetry Lie algebra is a twenty three dimensional semi direct product of twenty dimensional solvable Lie algebra $S_{1,20}$ and $sl(2,\mathbb{R})$. The nilradical a fifteen dimensional nilpotant Lie algebra $N_{1,11} \oplus \mathbb{R}^4$ which a direct sum of $N_{1,11}$ an eleven dimensional nilpotent Lie algebra and a four dimensional abelian Lie algebra $\mathbb{R}^4$. The complement to the nilradical is a four dimensional non-abelian. Therefore, the symmetry Lie algebra can be identified as $S_{1,20} \rtimes sl(2,\mathbb{R})$.

5.6 Algebra $A_{6,24}$

The non-zero brackets for the algebra $A_{6,24}$ are given by

$$
[e_1, e_5] = e_5 + e_6, \quad [e_1, e_6] = e_6, \quad [e_2, e_4] = e_4, \quad [e_1, e_2] = e_3. \tag{5.6.1}
$$

The geodesic equations are given by:

$$
\ddot{p} = \dot{p}\dot{z}, \quad \ddot{q} = \dot{w}(\dot{q} + \dot{x}), \quad \ddot{x} = \dot{x}\dot{w}, \quad \ddot{y} = \dot{z}\dot{w}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{5.6.2}
$$

The symmetry Lie algebra is spanned by:

$$
\begin{aligned}
&e_1 = D_t, && e_2 = tD_y, && e_3 = D_y,\\
&e_4 = D_p, && e_5 = D_q, && e_6 = D_x,\\
&e_7 = D_z, && e_8 = D_w, && e_9 = tD_t,\\
&e_{10} = pD_p, && e_{11} = wD_t, && e_{12} = zD_t,\\
&e_{13} = wD_y, && e_{14} = zD_y, && e_{15} = xD_q,\\
&e_{16} = qD_q + xD_x, && e_{17} = e^w D_q, && e_{18} = e^z D_p,\\
&e_{19} = (wz-2y)D_t, && e_{20} = (wz-2y)D_y, && e_{21} = (w-1)e^w D_q + e^w D_x.
\end{aligned}
\tag{5.6.3}
$$

We consider the following change of basis:

$$
\begin{aligned}
&\overline{e}_1 = e_5, && \overline{e}_2 = e_{17}, && \overline{e}_3 = e_6, && \overline{e}_4 = e_{21},\\
&\overline{e}_5 = e_{15}, && \overline{e}_6 = e_1, && \overline{e}_7 = e_3, && \overline{e}_8 = e_4,\\
&\overline{e}_9 = e_{11}, && \overline{e}_{10} = e_{12}, && \overline{e}_{11} = e_{13}, && \overline{e}_{12} = e_{14},\\
&\overline{e}_{13} = e_{18}, && \overline{e}_{14} = e_7 + \tfrac{e_{13}}{2}, && \overline{e}_{15} = e_8 + \tfrac{e_{14}}{2}, && \overline{e}_{16} = e_9 - \tfrac{e_{20}}{2},\\
&\overline{e}_{17} = e_{10}, && \overline{e}_{18} = e_{16}, && \overline{e}_{19} = e_2, && \overline{e}_{20} = e_9 + \tfrac{e_{20}}{2},\\
&\overline{e}_{21} = e_{19}.
\end{aligned}
\tag{5.6.4}
$$

and the non-zero brackets of the symmetry algebra are given by:

$$
\begin{array}{llll}
[e_1, e_{18}] = e_1, & [e_2, e_{15}] = -e_2, & [e_2, e_{18}] = e_2, & [e_3, e_5] = e_1, \\
[e_3, e_{18}] = e_3, & [e_4, e_5] = e_2, & [e_4, e_{15}] = -e_2 - e_4, & [e_4, e_{18}] = e_4, \\
[e_6, e_{16}] = e_6, & [e_6, e_{19}] = e_7, & [e_6, e_{20}] = e_6, & [e_7, e_{16}] = e_7, \\
[e_7, e_{20}] = -e_7, & [e_7, e_{21}] = -2e_6, & [e_8, e_{17}] = e_8, & [e_9, e_{15}] = -e_6, \\
[e_9, e_{16}] = e_9, & [e_9, e_{19}] = e_{11}, & [e_9, e_{20}] = e_9, & [e_{10}, e_{14}] = -e_6, \\
[e_{10}, e_{16}] = e_{10}, & [e_{10}, e_{19}] = e_{12}, & [e_{10}, e_{20}] = e_{10}, & [e_{11}, e_{15}] = -e_7, \\
[e_{11}, e_{16}] = e_{11}, & [e_{11}, e_{20}] = -e_{11}, & [e_{11}, e_{21}] = -2e_9, & [e_{12}, e_{14}] = -e_7, \\
[e_{12}, e_{16}] = e_{12}, & [e_{12}, e_{20}] = -e_{12}, & [e_{12}, e_{21}] = -2e_{10}, & [e_{13}, e_{14}] = -e_{13}, \\
[e_{13}, e_{17}] = e_{13}, & [e_{19}, e_{20}] = -2e_{19}, & [e_{19}, e_{21}] = -2e_{20}, & [e_{20}, e_{21}] = -2e_{21}.
\end{array}
\tag{5.6.5}
$$

We describe the symmetry algebra by the following proposition:

Proposition 5.6.1. The symmetry Lie algebra is a twenty one dimensional Lie algebra. It is a semi direct product of an eighteen-dimensional solvable Lie algebra and $sl(2,\mathbb{R})$. The nilradical is thirteen-dimensional decomposable Lie algebra. In fact, the nilradical is a direct sum of $A_{5,1}$ in Winternitz [27] and $\mathbb{R}^8$. The nilradical has a five-dimensional abelian complement. Therefore, the symmetry algebra can be identified as $((A_{5,1} \oplus \mathbb{R}^8) \rtimes \mathbb{R}^5) \rtimes sl(2,\mathbb{R})$, where the non-zero brackets of $A_{5.1}$ are given by:

$$[e_3, e_5] = e_1, [e_4, e_5] = e_2. \tag{5.6.6}$$

5.7 Algebra $A_{6,25}^{ab}(ab : a^2 + b^2 \neq 0)$

The non-zero brackets for the algebra $A_{6,25}^{ab}$ are given by

$$\begin{aligned} &[e_1, e_4] = ae_4, \quad [e_1, e_5] = e_6, \quad [e_1, e_6] = -e_5, \quad [e_2, e_4] = be_4, \\ &[e_2, e_5] = e_5, \quad [e_2, e_6] = e_6, \quad [e_1, e_2] = e_3. \end{aligned} \tag{5.7.1}$$

The geodesic equations are given by:

$$\ddot{p} = \dot{p}(b\dot{w} + a\dot{z}), \quad \ddot{q} = -\dot{w}\dot{z}, \quad \ddot{x} = \dot{x}\dot{z} - \dot{y}\dot{w}, \quad \ddot{y} = \dot{x}\dot{w} - \dot{y}\dot{z}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{5.7.2}$$

For the general case $A_{6,25}^{a\neq 0, b\neq 0}$, the symmetry Lie algebra is spanned by:

$$\begin{aligned} &e_1 = D_t, && e_2 = tD_q, && e_3 = D_q, \\ &e_4 = D_y, && e_5 = D_x, && e_6 = D_p, \\ &e_7 = D_w, && e_8 = D_z, && e_9 = tD_t, \\ &e_{10} = pD_p, && e_{11} = wD_q, && e_{12} = zD_q, \\ &e_{13} = wD_t, && e_{14} = zD_t, && e_{15} = xD_x + yD_y, \\ &e_{16} = (wz + 2q)D_q, && e_{17} = (wz + 2q)D_t, && e_{18} = e^{bw}e^{az}D_p. \end{aligned} \tag{5.7.3}$$

We implement the following change of basis:

$$\begin{array}{llll}
\overline{e}_1 = e_1, & \overline{e}_2 = e_3, & \overline{e}_3 = e_{13}, & \overline{e}_4 = e_{11}, \\
\overline{e}_5 = -e_7 + \frac{be_8}{a}, & \overline{e}_6 = e_4, & \overline{e}_7 = e_5, & \overline{e}_8 = e_6, \\
\overline{e}_9 = e_{11} + \frac{ae_{12}}{b}, & \overline{e}_{10} = e_{13} + \frac{ae_{14}}{b}, & \overline{e}_{11} = e_{18}, & \overline{e}_{12} = e_8 - \frac{e_{11}}{2}, \\
\overline{e}_{13} = e_9 + \frac{e_{16}}{2}, & \overline{e}_{14} = e_{10}, & \overline{e}_{15} = e_{15}, & \overline{e}_{16} = e_2, \\
\overline{e}_{17} = e_9 - \frac{e_{16}}{2}, & \overline{e}_{18} = e_{17}. & &
\end{array} \tag{5.7.4}$$

and the non-zero brackets of the symmetry algebra are given by:

$$
\begin{array}{lll}
[e_1, e_{13}] = e_1, & [e_1, e_{16}] = e_2, & [e_1, e_{17}] = e_1, \\
[e_2, e_{13}] = e_2, & [e_2, e_{17}] = -e_2, & [e_2, e_{18}] = 2e_1, \\
[e_3, e_5] = e_1, & [e_3, e_{13}] = e_3, & [e_3, e_{16}] = e_4, \\
[e_3, e_{17}] = e_3, & [e_4, e_5] = e_2, & [e_4, e_{13}] = e_4, \\
[e_4, e_{17}] = -e_4, & [e_4, e_{18}] = 2e_3, & [e_5, e_{12}] = \frac{e_2}{2}, \\
[e_5, e_{13}] = -\frac{be_9}{2a} + \frac{be_4}{a}, & [e_5, e_{17}] = \frac{be_9}{2a} - \frac{be_4}{a}, & [e_5, e_{18}] = -\frac{be_{10}}{a} + \frac{2be_3}{a}, \\
[e_6, e_{15}] = e_6, & [e_7, e_{15}] = e_7, & [e_8, e_{14}] = e_8, \\
[e_9, e_{12}] = -\frac{ae_2}{b}, & [e_9, e_{13}] = e_9, & [e_9, e_{17}] = -e_9, \\
[e_9, e_{18}] = 2e_{10}, & [e_{10}, e_{12}] = -\frac{ae_1}{b}, & [e_{10}, e_{13}] = e_{10}, \\
[e_{10}, e_{16}] = e_9, & [e_{10}, e_{17}] = e_{10}, & [e_{11}, e_{12}] = -ae_{11}, \\
[e_{11}, e_{14}] = e_{11}, & [e_{16}, e_{17}] = -2e_{16}, & [e_{16}, e_{18}] = 2e_{17}, \\
[e_{17}, e_{18}] = -2e_{18}. & &
\end{array}
\tag{5.7.5}
$$

We describe the symmetry algebra by the following proposition:.

Proposition 5.7.1. The symmetry Lie algebra is an eighteen dimensional Lie algebra. It is a semi direct product of fifteen dimensional solvable Lie algebra and $sl(2,\mathbb{R})$. The nilradical is an eleven dimensional decomposable Lie algebra. In fact, the nilradical is a direct sum of $A_{5,1}$ in Winternitz [27] and $\mathbb{R}^6$. The nilradical has a four dimensional abelian complement. Therefore, the symmetry algebra can be identified as $((A_{5,1} \oplus \mathbb{R}^4) \rtimes \mathbb{R}^5) \rtimes sl(2, \mathbb{R})$, where the non-zero brackets of $A_{5.1}$ are given by:

$$[e_3, e_5] = e_1, [e_4, e_5] = e_2. \tag{5.7.6}$$

5.8 Algebra $A^a_{6,26}$

The non-zero brackets for the algebra $A^a_{6,26}$ are given by

$$[e_1, e_5] = ae_5 + e_6, \quad [e_1, e_6] = ae_6 - e_5, \quad [e_2, e_4] = e_4, \quad [e_1, e_2] = e_3. \tag{5.8.1}$$

The geodesic equations are given by:

$$\ddot{p} = \dot{p}\dot{z}, \quad \ddot{q} = \dot{w}\dot{z}, \quad \ddot{x} = \dot{w}(a\dot{x} - \dot{y}), \quad \ddot{y} = \dot{w}(\dot{x} + a\dot{y}), \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{5.8.2}$$

For the general case $A^{a\neq 0}_{6,26}$, the symmetry Lie algebra is spanned by:

$$\begin{array}{llll}
e_1 = D_t, & e_2 = tD_q, & e_3 = D_q, & e_4 = D_x, \\
e_5 = D_p, & e_6 = D_y, & e_7 = D_z, & e_8 = D_w, \\
e_9 = tD_t, & e_{10} = pD_p, & e_{11} = wD_q, & e_{12} = zD_q, \\
e_{13} = wD_t, & e_{14} = zD_t, & e_{15} = xD_x + yD_y, & e_{16} = e^z D_p, \\
e_{17} = yD_x - xD_y, & e_{18} = (wz + 2q)D_q, & e_{19} = (wz + 2q)D_t, &
\end{array}$$

$$e_{20} = e^{aw}\cos(w)D_x + e^{aw}\sin(w)D_y, \quad e_{21} = e^{aw}\sin(w)D_x - e^{aw}\cos(w)D_y. \tag{5.8.3}$$

We consider the following change of basis:

$$\begin{array}{llll}
\overline{e}_1 = e_1, & \overline{e}_2 = e_3, & \overline{e}_3 = e_4, & \overline{e}_4 = e_5, \\
\overline{e}_5 = e_6, & \overline{e}_6 = e_{11}, & \overline{e}_7 = e_{12}, & \overline{e}_8 = e_{13}, \\
\overline{e}_9 = e_{14}, & \overline{e}_{10} = e_{16}, & \overline{e}_{11} = e_{20}, & \overline{e}_{12} = e_{21}, \\
\overline{e}_{13} = e_7 + \frac{e_{11}}{2}, & \overline{e}_{14} = e_8 + \frac{e_{12}}{2}, & \overline{e}_{15} = e_9 - \frac{e_{18}}{2}, & \overline{e}_{16} = e_{10}, \\
\overline{e}_{17} = e_{15}, & \overline{e}_{18} = e_{17}, & \overline{e}_{19} = e_2, & \overline{e}_{20} = e_9 + \frac{e_{18}}{2}, \\
\overline{e}_{21} = e_{19}. & & &
\end{array} \tag{5.8.4}$$

The non-zero brackets of the symmetry algebra are given by:

$$
\begin{aligned}
&[e_1, e_{15}] = e_1, && [e_1, e_{19}] = e_2, && [e_1, e_{20}] = e_1,\\
&[e_2, e_{15}] = e_2, && [e_2, e_{20}] = -e_2, && [e_2, e_{21}] = -2e_1,\\
&[e_3, e_{17}] = e_3, && [e_3, e_{18}] = -e_5, && [e_4, e_{16}] = e_4,\\
&[e_5, e_{17}] = e_5, && [e_5, e_{18}] = e_3, && [e_6, e_{14}] = -e_2,\\
&[e_6, e_{15}] = e_6, && [e_6, e_{20}] = -e_6, && [e_6, e_{21}] = -2e_8,\\
&[e_7, e_{13}] = -e_2, && [e_7, e_{15}] = e_7, && [e_7, e_{20}] = -e_7,\\
&[e_7, e_{21}] = -2e_9, && [e_8, e_{14}] = -e_1, && [e_8, e_{15}] = e_8,\\
&[e_8, e_{19}] = e_6, && [e_8, e_{20}] = e_8, && [e_9, e_{13}] = -e_1,\\
&[e_9, e_{15}] = e_9, && [e_9, e_{19}] = e_7, && [e_9, e_{20}] = e_9,\\
&[e_{10}, e_{13}] = -e_{10}, && [e_{10}, e_{16}] = e_{10}, && [e_{11}, e_{14}] = -ae_{11} + e_{12},\\
&[e_{11}, e_{17}] = e_{11}, && [e_{11}, e_{18}] = e_{12}, && [e_{12}, e_{14}] = -ae_{12} - e_{11},\\
&[e_{12}, e_{17}] = e_{12}, && [e_{12}, e_{18}] = -e_{11}, && [e_{19}, e_{20}] = -2e_{19},\\
&[e_{19}, e_{21}] = -2e_{20}, && [e_{20}, e_{21}] = -2e_{21}.
\end{aligned}
\tag{5.8.5}
$$

We describe the symmetry algebra by the following proposition:

Proposition 5.8.1. The symmetry Lie algebra is a twenty one dimensional Lie algebra. It is a semi direct product of an eighteen-dimensional solvable Lie algebra and $sl(2,\mathbb{R})$. The nilradical is twelve-dimensional abelian Lie algebra and has a six-dimensional abelian complement. Therefore, the symmetry algebra can be identified as: $(\mathbb{R}^{12} \rtimes \mathbb{R}^6) \rtimes sl(2,\mathbb{R})$.

5.8.1 case 1:$A_{6,26}^{a=0}$

The symmetry Lie algebra is spanned by:

$$
\begin{array}{lll}
e_1 = D_t, & e_2 = tD_q, & e_3 = D_q, \\
e_4 = D_p, & e_5 = D_x, & e_6 = D_y, \\
e_7 = D_z, & e_8 = D_w, & e_9 = tD_t, \\
e_{10} = pD_p, & e_{11} = wD_q, & e_{12} = zD_p, \\
e_{13} = wD_t, & e_{14} = zD_t, & e_{15} = xD_x + yD_y, \\
e_{16} = e^z D_p, & e_{17} = yD_x - xD_y, & e_{18} = \cos(w)D_x + \sin(w)D_y, \\
e_{19} = (wz - 2q)D_q, & e_{20} = (wz - 2q)D_t, & e_{21} = \sin(w)D_x - \cos(w)D_y,
\end{array}
$$

$$
\begin{aligned}
e_{22} &= (-\cos(w)y + x\sin(w))D_x + (-\cos(w)x - y\sin(w))D_y, \\
e_{23} &= (\cos(w)x + y\sin(w))D_x + (-\cos(w)y + x\sin(w))D_y.
\end{aligned}
\tag{5.8.6}
$$

We implement the following change of basis:

$$
\begin{array}{llll}
\overline{e}_1 = e_1, & \overline{e}_2 = e_3, & \overline{e}_3 = e_4, & \overline{e}_4 = e_5, \\
\overline{e}_5 = e_6, & \overline{e}_6 = e_{11}, & \overline{e}_7 = e_{12}, & \overline{e}_8 = e_{13}, \\
\overline{e}_9 = e_{14}, & \overline{e}_{10} = e_{16}, & \overline{e}_{11} = e_{18}, & \overline{e}_{12} = e_{21}, \\
\overline{e}_{13} = e_7 + \frac{e_{11}}{2}, & \overline{e}_{14} = e_8 - \frac{e_{17}}{2} + \frac{e_{12}}{2}, & \overline{e}_{15} = e_9 - \frac{e_{19}}{2}, & \overline{e}_{16} = e_{10}, \\
\overline{e}_{17} = e_{15}, & \overline{e}_{18} = e_2, & \overline{e}_{19} = e_9 + \frac{e_{19}}{2}, & \overline{e}_{20} = e_{20}, \\
\overline{e}_{21} = e_{17}, & \overline{e}_{22} = e_{22}, & \overline{e}_{23} = e_{23}. &
\end{array}
\tag{5.8.7}
$$

and the non-zero brackets of the symmetry algebra are given by:

$$[e_1, e_{15}] = e_1, \quad [e_1, e_{18}] = e_2, \quad [e_1, e_{19}] = -e_1,$$

$$[e_2, e_{15}] = e_2, \quad [e_2, e_{19}] = -e_2, \quad [e_2, e_{20}] = -2e_1,$$

$$[e_3, e_{16}] = e_3, \quad [e_4, e_{14}] = \frac{e_5}{2}, \quad [e_4, e_{17}] = e_4,$$

$$[e_4, e_{21}] = -e_5, \quad [e_4, e_{22}] = e_{12}, \quad [e_4, e_{23}] = e_{11},$$

$$[e_5, e_{14}] = -\frac{e_4}{2}, \quad [e_5, e_{17}] = e_5, \quad [e_5, e_{21}] = e_4,$$

$$[e_5, e_{22}] = -e_{11}, \quad [e_5, e_{23}] = e_{12}, \quad [e_6, e_{14}] = -e_2,$$

$$[e_6, e_{15}] = e_6, \quad [e_6, e_{19}] = -e_6, \quad [e_6, e_{20}] = -2e_8,$$

$$[e_7, e_{13}] = -e_2, \quad [e_7, e_{15}] = e_7, \quad [e_7, e_{19}] = -e_7,$$

$$[e_7, e_{20}] = -2e_9, \quad [e_8, e_{14}] = -e_1, \quad [e_8, e_{15}] = e_8,$$

$$[e_8, e_{18}] = e_6, \quad [e_8, e_{19}] = e_8, \quad [e_9, e_{13}] = -e_1,$$

$$[e_9, e_{15}] = e_9, \quad [e_9, e_{18}] = e_7, \quad [e_9, e_{19}] = e_9,$$

$$[e_{10}, e_{13}] = -e_{10}, \quad [e_{10}, e_{16}] = e_{10}, \quad [e_{11}, e_{14}] = \frac{e_{12}}{2},$$

$$[e_{11}, e_{17}] = e_{11}, \quad [e_{11}, e_{21}] = e_{12}, \quad [e_{11}, e_{22}] = -e_5,$$

$$[e_{11}, e_{23}] = e_4, \quad [e_{12}, e_{14}] = -\frac{e_{11}}{2}, \quad [e_{12}, e_{17}] = e_{12},$$

$$
\begin{aligned}
&[e_{12}, e_{21}] = -e_{11}, \quad [e_{12}, e_{22}] = e_4, \qquad [e_{12}, e_{23}] = e_5, \\
&[e_{18}, e_{19}] = -e_{18}, \quad [e_{18}, e_{20}] = -2e_{19}, \quad [e_{19}, e_{20}] = -2e_{20}, \\
&[e_{21}, e_{22}] = 2e_{23}, \quad [e_{21}, e_{23}] = -2e_{22}, \quad [e_{22}, e_{23}] = -2e_{21}.
\end{aligned} \tag{5.8.8}
$$

We describe the symmetry algebra by the following proposition:

Proposition 5.8.2. The symmetry Lie algebra is a twenty three -dimensional Lie algebra. It is a semi direct product of a seventeen-dimensional solvable Lie algebra and two copies of $sl(2,\mathbb{R})$. Furthermore, the symmetry Lie algebra has a twelve-dimensional abelian nilradical and five-dimensional abelian complement.. Therefore, the symmetry algebra can be identified as: $(\mathbb{R}^{12} \rtimes \mathbb{R}^5) \rtimes (sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}))$.

5.9 Algebra $A^{\epsilon}_{6,27} : (\epsilon = 0, 1)$

The non-zero brackets for the algebra $A^{\epsilon}_{6,27}$ are given by

$$
\begin{aligned}
&[e_1, e_3] = e_4, \quad [e_1, e_5] = e_6, \quad [e_1, e_6] = -e_5, \quad [e_2, e_5] = e_5, \\
&[e_2, e_6] = e_6, \quad [e_1, e_2] = \epsilon e_3.
\end{aligned} \tag{5.9.1}
$$

5.9.1 $A^{\epsilon=0}_{6,27}$

The geodesic equations where $\epsilon = 0$ are given by:

$$
\ddot{p} = \dot{p}\dot{w} - \dot{q}\dot{z}, \quad \ddot{q} = \dot{p}\dot{z} + \dot{q}\dot{w}, \quad \ddot{x} = 0, \quad \ddot{y} = \dot{x}\dot{z}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{5.9.2}
$$

The symmetry Lie algebra is spanned by:

$$
\begin{array}{lll}
e_1 = D_t, & e_2 = tD_y, & e_3 = D_y, \\
e_4 = D_p, & e_5 = D_q, & e_6 = D_x, \\
e_7 = D_w, & e_8 = D_z, & e_9 = tD_t, \\
e_{10} = wD_t, & e_{11} = xD_t, & e_{12} = zD_t, \\
e_{13} = wD_y, & e_{14} = xD_y, & e_{15} = zD_y, \\
e_{16} = pD_p + qD_q, & e_{17} = xD_x + yD_y, & e_{18} = qD_p - pD_q, \\
e_{19} = tD_x + \frac{tz}{2}D_y, & e_{20} = zD_x + \frac{z^2}{2}D_y, & e_{21} = (xz - 2y)D_t, \\
e_{22} = (xz - 2y)D_y, & e_{23} = wD_x + \frac{wz}{2}D_y, &
\end{array}
$$

$$
\begin{aligned}
&e_{24} = e^w \cos(z)D_p + e^w \sin(z)D_q, \qquad e_{25} = e^w \sin(z)D_p - e^w \cos(z)D_q, \\
&e_{26} = (xz - 2y)D_x + (\frac{xz^2}{2} - yz)D_y.
\end{aligned} \tag{5.9.3}
$$

We implement the following change of basis:

$$\begin{aligned}
&\overline{e}_1 = e_1, && \overline{e}_2 = e_3, && \overline{e}_3 = e_4, && \overline{e}_4 = e_5, \\
&\overline{e}_5 = e_6, && \overline{e}_6 = e_{10}, && \overline{e}_7 = e_{12}, && \overline{e}_8 = e_{13}, \\
&\overline{e}_9 = e_{15}, && \overline{e}_{10} = e_{20}, && \overline{e}_{11} = e_{23}, && \overline{e}_{12} = e_{24}, \\
&\overline{e}_{13} = e_{25}, && \overline{e}_{14} = e_7, && \overline{e}_{15} = e_8 + \tfrac{e_{14}}{2}, && \overline{e}_{16} = e_9 + e_{17}, \\
&\overline{e}_{17} = e_{16}, && \overline{e}_{18} = e_{18}, && \overline{e}_{19} = e_2, && \overline{e}_{20} = e_9 + \tfrac{e_{22}}{2}, \\
&\overline{e}_{21} = e_{11}, && \overline{e}_{22} = e_{14}, && \overline{e}_{23} = e_{17} + e_{22}, && \overline{e}_{24} = e_{19}, \\
&\overline{e}_{25} = e_{21}, && \overline{e}_{26} = e_{26}.
\end{aligned} \tag{5.9.4}$$

and the non-zero brackets of the symmetry algebra are given by:

$$
\begin{array}{lll}
[e_1, e_{16}] = e_1, & [e_1, e_{19}] = e_2, & [e_1, e_{20}] = e_1, \\
[e_1, e_{24}] = e_5 + \frac{e_9}{2}, & [e_2, e_{16}] = e_2, & [e_2, e_{20}] = -e_2, \\
[e_2, e_{23}] = -e_2, & [e_2, e_{25}] = -2e_1, & [e_2, e_{26}] = -2e_5 - e_9, \\
[e_3, e_{17}] = e_3, & [e_3, e_{18}] = -e_4, & [e_4, e_{17}] = e_4, \\
[e_4, e_{18}] = e_3, & [e_5, e_{15}] = \frac{e_2}{2}, & [e_5, e_{16}] = e_5, \\
[e_5, e_{20}] = \frac{e_9}{2}, & [e_5, e_{21}] = e_1, & [e_5, e_{22}] = e_2, \\
[e_5, e_{23}] = e_5 + e_9, & [e_5, e_{25}] = e_7, & [e_5, e_{26}] = e_{10}, \\
[e_6, e_{14}] = -e_1, & [e_6, e_{16}] = e_6, & [e_6, e_{19}] = e_8, \\
[e_6, e_{20}] = e_6, & [e_6, e_{24}] = e_{11}, & [e_7, e_{15}] = -e_1, \\
[e_7, e_{16}] = e_7, & [e_7, e_{19}] = e_9, & [e_7, e_{20}] = e_7, \\
[e_7, e_{24}] = e_{10}, & [e_8, e_{14}] = -e_2, & [e_8, e_{16}] = e_8, \\
[e_8, e_{20}] = -e_8, & [e_8, e_{23}] = -e_8, & [e_8, e_{25}] = -2e_6, \\
[e_8, e_{26}] = -2e_{11}, & [e_9, e_{15}] = -e_2, & [e_9, e_{16}] = e_9, \\
[e_9, e_{20}] = -e_9, & [e_9, e_{23}] = -e_9, & [e_9, e_{25}] = -2e_7,
\end{array}
$$

$$
\begin{aligned}
&[e_9, e_{26}] = -2e_{10}, && [e_{10}, e_{15}] = -e_5 - \tfrac{e_9}{2}, && [e_{10}, e_{16}] = e_{10},\\
&[e_{10}, e_{21}] = e_7, && [e_{10}, e_{22}] = e_9, && [e_{10}, e_{23}] = e_{10},\\
&[e_{11}, e_{14}] = -e_5 - \tfrac{e_9}{2}, && [e_{11}, e_{16}] = e_{11}, && [e_{11}, e_{21}] = e_6,\\
&[e_{11}, e_{22}] = e_8, && [e_{11}, e_{23}] = e_{11}, && [e_{12}, e_{14}] = -e_{12},\\
&[e_{12}, e_{15}] = e_{13}, && [e_{12}, e_{17}] = e_{12}, && [e_{12}, e_{18}] = e_{13},\\
&[e_{13}, e_{14}] = -e_{13}, && [e_{13}, e_{15}] = -e_{12}, && [e_{13}, e_{17}] = e_{13},\\
&[e_{13}, e_{18}] = -e_{12}, && [e_{19}, e_{20}] = -2e_{19}, && [e_{19}, e_{21}] = -e_{22},\\
&[e_{19}, e_{23}] = -e_{19}, && [e_{19}, e_{25}] = -2e_{20}, && [e_{19}, e_{26}] = -2e_{24},\\
&[e_{20}, e_{21}] = -e_{21}, && [e_{20}, e_{22}] = e_{22}, && [e_{20}, e_{24}] = e_{24},\\
&[e_{20}, e_{25}] = -2e_{25}, && [e_{20}, e_{26}] = -e_{26}, && [e_{21}, e_{23}] = -e_{21},\\
&[e_{21}, e_{24}] = -e_{20} + e_{23}, && [e_{21}, e_{26}] = -e_{25}, && [e_{22}, e_{23}] = -2e_{22},\\
&[e_{22}, e_{24}] = -e_{19}, && [e_{22}, e_{25}] = -2e_{21}, && [e_{22}, e_{26}] = -2e_{23},\\
&[e_{23}, e_{24}] = -e_{24}, && [e_{23}, e_{25}] = -e_{25}, && [e_{23}, e_{26}] = -2e_{26},\\
&[e_{24}, e_{25}] = -e_{26}.
\end{aligned}
\tag{5.9.5}
$$

We describe the symmetry algebra by the following proposition:

Proposition 5.9.1. The symmetry Lie algebra is a twenty six-dimensional semi direct product of an eighteen solvable Lie algebra and eight-dimensional semi-simple

$sl(3,\mathbb{R})$. Furthermore, the symmetry Lie algebra has a thirteen-dimensional abelian nilradical. Therefore, the symmetry algebra can be identified as: $(\mathbb{R}^{13} \rtimes \mathbb{R}^5) \rtimes sl(3,\mathbb{R})$.

5.9.2 $A_{6,27}^{\epsilon=1}$

The geodesic equations where $\epsilon = 1$ are given by:

$$\ddot{p} = \dot{q}\dot{w}, \quad \ddot{q} = \dot{z}\dot{w}, \quad \ddot{x} = \dot{z}\dot{x} - \dot{w}\dot{y}, \quad \ddot{y} = \dot{z}\dot{y} + \dot{w}\dot{x}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{5.9.6}$$

The symmetry Lie algebra is spanned by:

$$\begin{array}{lll}
e_1 = D_z, & e_2 = D_p, & e_3 = D_x, \\
e_4 = D_w, & e_5 = D_y, & e_6 = D_q, \\
e_7 = tD_t, & e_8 = D_t, & e_9 = tD_p, \\
e_{10} = zD_p, & e_{11} = wD_p, & e_{12} = wD_t, \\
e_{13} = zD_t, & e_{14} = qD_p + zD_q, & e_{15} = xD_x + yD_y, \\
e_{16} = yD_x - xD_y, & e_{17} = twD_p + 2tD_q, & e_{18} = \frac{w^2}{2}D_p + wD_q, \\
e_{19} = wzD_p + 2zD_q, & e_{20} = (wz - 2q)D_t, &
\end{array}$$

$$\begin{aligned}
&e_{21} = e^z\cos(w)D_x + e^z\sin(w)D_y, \qquad e_{22} = e^z\sin(w)D_x - e^z\cos(w)D_y, \\
&e_{23} = (qw - \tfrac{zw^2}{2})D_p + (-wz + 2q)D_q.
\end{aligned} \tag{5.9.7}$$

We implement the following change of basis:

$$
\begin{array}{llll}
\overline{e}_1 = e_2, & \overline{e}_2 = e_6, & \overline{e}_3 = e_8, & \overline{e}_4 = e_9, \\
\overline{e}_5 = e_{10}, & \overline{e}_6 = e_{11}, & \overline{e}_7 = e_{12}, & \overline{e}_8 = e_{13}, \\
\overline{e}_9 = e_{14}, & \overline{e}_{10} = e_{18}, & \overline{e}_{11} = e_{19}, & \overline{e}_{12} = e_3, \\
\overline{e}_{13} = e_5, & \overline{e}_{14} = e_{21}, & \overline{e}_{15} = e_{22}, & \overline{e}_{16} = e_1, \\
\overline{e}_{17} = e_4, & \overline{e}_{18} = e_7 + \frac{e_{23}}{2}, & \overline{e}_{19} = e_{15}, & \overline{e}_{20} = e_{16}, \\
\overline{e}_{21} = e_7 - \frac{e_{23}}{2}, & \overline{e}_{22} = e_{17}, & \overline{e}_{23} = e_{20}. &
\end{array}
\tag{5.9.8}
$$

and the non-zero brackets of the symmetry algebra are given by:

$$
\begin{array}{lll}
[e_2, e_9] = e_1, & [e_2, e_{18}] = e_2 + \frac{e_6}{2}, & [e_2, e_{21}] = -e_2 - \frac{e_6}{2}, \\
[e_2, e_{23}] = -2e_3, & [e_3, e_4] = e_1, & [e_3, e_{18}] = e_3, \\
[e_3, e_{21}] = e_3, & [e_3, e_{22}] = 2e_2 + e_6, & [e_4, e_7] = -e_6, \\
[e_4, e_8] = -e_5, & [e_4, e_{18}] = -e_4, & [e_4, e_{21}] = -e_4, \\
[e_4, e_{23}] = -e_{11} + 2e_9, & [e_5, e_{16}] = -e_1, & [e_6, e_{17}] = -e_1, \\
[e_7, e_{17}] = -e_3, & [e_7, e_{18}] = e_7, & [e_7, e_{21}] = e_7, \\
[e_7, e_{22}] = 2e_{10}, & [e_8, e_{16}] = -e_3, & [e_8, e_{18}] = e_8, \\
[e_8, e_{21}] = e_8, & [e_8, e_{22}] = e_{11}, & [e_9, e_{10}] = -e_6, \\
[e_9, e_{11}] = -2e_5, & [e_9, e_{16}] = -e_2, & [e_9, e_{18}] = e_{11} - e_9, \\
[e_9, e_{21}] = -e_{11} + e_9, & [e_9, e_{22}] = -2e_4, & [e_9, e_{23}] = -2e_8, \\
[e_{10}, e_{17}] = -e_2 - e_6, & [e_{10}, e_{18}] = e_{10}, & [e_{10}, e_{21}] = -e_{10}, \\
[e_{10}, e_{23}] = -2e_7, & [e_{11}, e_{16}] = -2e_2 - e_6, & [e_{11}, e_{17}] = -e_5, \\
[e_{11}, e_{18}] = e_{11}, & [e_{11}, e_{21}] = -e_{11}, & [e_{11}, e_{23}] = -4e_8, \\
[e_{12}, e_{19}] = e_{12}, & [e_{12}, e_{20}] = -e_{13}, & [e_{13}, e_{19}] = e_{13}, \\
[e_{13}, e_{20}] = e_{12}, & [e_{14}, e_{16}] = -e_{14}, & [e_{14}, e_{17}] = e_{15},
\end{array}
$$

$$
\begin{array}{lll}
[e_{14}, e_{19}] = e_{14}, & [e_{14}, e_{20}] = e_{15}, & [e_{15}, e_{16}] = -e_{15}, \\
[e_{15}, e_{17}] = -e_{14}, & [e_{15}, e_{19}] = e_{15}, & [e_{15}, e_{20}] = -e_{14}, \\
[e_{16}, e_{18}] = -\frac{e_{10}}{2}, & [e_{16}, e_{21}] = \frac{e_{10}}{2}, & [e_{16}, e_{23}] = e_{7}, \\
[e_{17}, e_{18}] = -\frac{e_{11}}{2} + \frac{e_{9}}{2}, & [e_{17}, e_{21}] = \frac{e_{11}}{2} - \frac{e_{9}}{2}, & [e_{17}, e_{22}] = e_{4}, \\
[e_{17}, e_{23}] = e_{8}, & [e_{21}, e_{22}] = 2e_{22}, & [e_{21}, e_{23}] = -2e_{23}, \\
[e_{22}, e_{23}] = -4e_{21}. & &
\end{array}
\tag{5.9.9}
$$

We describe the symmetry algebra by the following proposition:

Proposition 5.9.2. The symmetry Lie algebra is a twenty three dimensional semi direct product of twenty dimensional solvable Lie algebra $S_{2,20}$ and $sl(2, \mathbb{R})$. The nilradical is a fifteen dimensional nilpotant Lie algebra $N_{2,11} \oplus \mathbb{R}^4$ which is a direct sum of $N_{2,11}$ an eleven dimensional nilpotent Lie algebra and a four dimensional abelian Lie algebra $\mathbb{R}^4$. The complement of the nilradical is a four dimensional non-abelian set. Therefore, the symmetry Lie algebra can be identified as $S_{2,20} \rtimes sl(2, \mathbb{R})$.

5.10 Conclusions

In this work, we have investigated the symmetry Lie algebra of the geodesic equations of the canonical connection on a Lie group corresponding to the eight classes of Lie algebra $A_{6,20} - A_{6,27}$ in [13]. In each case, we list the non-zero brackets of the given Lie algebra, the geodesic equations and a basis for the symmetry Lie algebra in terms of vector fields. For every symmetry Lie algebra, we identify its nilradical, solvable complement and semi-simple factor is there is one: a summary of our results is given in Table 3.

Six-dimensional Lie Algebras	Dimension	Identification
$A_{6,20}^{ab}(ab : a^2 + b^2 \neq 0)$	21	$(\mathbb{R}^{12} \rtimes \mathbb{R}^6) \rtimes sl(2, \mathbb{R})$
$A_{6,21}^{a}$	21	$((A_{5,1} \oplus \mathbb{R}^8) \rtimes \mathbb{R}^5) \rtimes sl(2, \mathbb{R})$
$A_{6,22}^{\epsilon=0}$	26	$(\mathbb{R}^{13} \rtimes \mathbb{R}^5) \rtimes sl(3, \mathbb{R})$
$A_{6,22}^{\epsilon=1}$	23	$(R^{15} \rtimes \mathbb{R}^5) \rtimes sl(2, \mathbb{R})$
$A_{6,23}^{a,\epsilon=0}$	26	$(\mathbb{R}^{13} \rtimes \mathbb{R}^5) \rtimes sl(3, \mathbb{R})$
$A_{6,23}^{a,\epsilon=1}$	23	$S_{1,20} \rtimes sl(2, \mathbb{R})$
$A_{6,24}$	21	$((A_{5.1} \oplus \mathbb{R}^8) \rtimes \mathbb{R}^5) \rtimes sl(2, \mathbb{R})$
$A_{6,25}^{ab}(ab : a^2 + b^2 \neq 0)$	18	$((A_{5,1} \oplus \mathbb{R}^4) \rtimes \mathbb{R}^5) \rtimes sl(2, \mathbb{R})$
$A_{6,26}^{a}$	21	$(\mathbb{R}^{12} \rtimes \mathbb{R}^6) \rtimes sl(2, \mathbb{R})$
$A_{6,26}^{a=0}$	23	$(\mathbb{R}^{12} \rtimes \mathbb{R}^5) \rtimes (sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R}))$
$A_{6,27}^{\epsilon=0}$	26	$(\mathbb{R}^{13} \rtimes \mathbb{R}^5) \rtimes sl(3, \mathbb{R})$
$A_{6,27}^{\epsilon=1}$	23	$S_{2,20} \rtimes sl(2, \mathbb{R})$

Table 3. Six Dimensional Lie Algebras with Non Abelian Complement and Identification of the Symmetry Algebra.

CHAPTER 6

CONCLUSION AND FUTURE WORK

In this dissertation we focus on identifying the symmetry Lie algebras of the geodesic equations of the canonical connection on Lie groups. In particular, we consider the class of solvable indecomposable Lie algebras with co-dimension two nilradical that have an abelian and non-abelian complement. In my dissertation we considered the following:

1. In dimension four , there is only one Lie algebra to consider, namely $A_{4,12}$. We calculated the geodesic equations by developing a Maple code. The input will be a vector field representation of the Lie algebra, and the output would be the connection components, and hence the geodesic equations. For this algebra, we calculated the symmetries and identified the Lie algebra. Moreover, we used these symmetries to find explicit transformations that leave the system of ODE's invariant. In dimesnion five, there are three Lie algebras with co-dimension two abelian nilradical. Namely $A_{5,33}$, $A_{5,34}$,and $A_{5,35}$. For each case, we identified the symmetry Lie Algebra and presented our results in Chapter 3.

2. In dimension six, we considered the solvable indecomposable Lie algebras with co-dimension two. There are two cases here:

    - The case where the Lie algebra has a four-dimensional abelian nilradical and a two-dimensional abelian complement. In thie case, we have nineteen Lie algebras to consider, namely, $A_{6,1}$-$A_{6,19}$ in Turkowski's list [13]. We identified the symmetry Lie algebra for each of the nineteen algebras and our results are summarized in Table 2.

- The case where the Lie algebra has a four dimensional abelian niradical and a tow-dimensional non-abelian complement. In thie case, we have eight Lie algebras to consider, namely, $A_{6,20}$-$A_{6,27}$ in Turkowski's list [13]. We identified the symmetry Lie algebra for each of the nineteen algebras and our results are summarized in Table 3.

For future work, we plan on considering the following problems:

1. The six-dimensional case where the nilradical is not abelian and the complement is not abelian as well. This covers the rest of Lie algebras in Turkowski's list [13]. These Algebras are $A_{6,28} - A_{6,40}$.

2. Consider the problem for the n-dimensional case: Given an n dimensional Lie algebra with an abelian nilradical and an abelian two-dimensional complement. In this case, we would like to construct the geodesic equations in dimension n, the Lie invariance conditons and solve the system of partial differential equations as much as we can. The goal would be to identify the symmetry Lie algebra in general dimension.

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Throughout my Ph.D. journey, I have had the honor of contributing to the field through both scholarly publications and active participation in significant academic gatherings. Among my published works is "Lie Symmetries of the Canonical Connection: Codimension Two Abelian Nilradical," which appeared in the Journal of Generalized Lie Theory and Applications in November 2022, and is related to the content of Chapter 3 of my dissertation. Additionally, "Classification of the symmetry Lie algebras for six-dimensional co-dimension two Abelian nilradical Lie algebras," is related to the content of Chapter 4 of my dissertation, was featured in the AIMS Mathematics Journal in December 2023, with Dr. Ryad Ghanam as my co-author for both papers. Moreover, I had the opportunity to share our research at the Joint Mathematics Meetings (JMM) in Boston, USA, in January 2023.

These endeavors have profoundly deepened my appreciation for and dedication to mathematics, opening doors to significant professional growth and establishing connections within the scholarly community.