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Prospect Theory Preferences in Noncooperative Game Theory

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Prospect Theory Preferences in Noncooperative Game Theory

A thesis submitted in partial fulfillment of the requirements for the degree of Doctorate of Philosophy in Systems Modeling and Analysis at Virginia Commonwealth University.

by

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Doctorate of Philosophy in Systems Modeling and Analysis

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Abstract

PROSPECT THEORY PREFERENCES IN NONCOOPERATIVE GAME THEORY

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The present work seeks to incorporate a popular descriptive, empirically grounded model of human preference under risk, prospect theory, into the equilibrium theory of noncooperative games. Three primary, candidate definitions are systematically identified on the basis of classical characterizations of Nash Equilibrium; in addition, three equilibrium subtypes are defined for each primary definition, in order to enable modeling of players’ reference points as exogenous and fixed, slowly and myopically adaptive, highly flexible and non-myopically adaptive. Each primary equilibrium concept was analyzed both theoretically and empirically; for the theoretical analyses, prospect theory, game theory, and computational complexity theory were all summoned to analysis. In chapter 1, the reader is provided with background on each of these theoretical underpinnings of the current work, the scope of the project is described, and its conclusions briefly summarized. In chapters 2 and 3, each of the three equilibrium concepts is analyzed theoretically, with emphasis placed on issues of classical interest (e.g. existence, dominance, rationalizability) and computational complexity (i.e., assessing how difficult each concept is to apply in algorithmic practice, with particular focus on comparison to classical Nash Equilibrium). This theoretical analysis leads us to discard the first of our three equilibrium concepts as unacceptable. In chapter 4, our remaining two
equilibrium concepts are compared empirically, using average-level data originally aggregated from a number of studies by Camerer and Selten and Chmura; the results suggest that PT preferences may improve on the descriptive validity of NE, and pose some interesting questions about the nature of the PT weighting function (2003, Ch. 3). Chapter 5 concludes, systematically summarizes theoretical and empirical differences and similarities between the three equilibrium concepts, and offers some thoughts on future work.
Chapter One: Introduction, Motivation, and Scope

The purpose of the present work is to develop equilibrium models appropriate for incorporating prospect theory (PT) preferences into noncooperative game theory (GT). Noncooperative GT (hereafter just “GT”) is a mathematical theory of strategic interaction between multiple agents who are assumed to be incapable of making binding contracts with one another. One important objective for GT is to describe the behavior of actual human beings, both in real-world market settings and in artificial experiments in behavioral researchers’ laboratories. To the extent that GT can successfully predict the outcomes of interactions between human beings in social settings, it can serve as a useful foundation for making, for example, economic and institutional policy. The goal in the present work is to improve GT in this respect, by replacing GT’s traditional assumption of expected-utility (EU) preferences, which are often observed to be violated in single-decision maker experimental settings, with PT preferences, which are a leading alternative to EU preferences, and were specifically engineered to account for observed, experimental deviations from EU preferences.

To explain PT preferences, it is easiest to first explain EU preferences, as PT preferences strictly generalize EU preferences, are expressed in an algebraic form similar to that of EU preferences, and are defined primarily by their differences with EU preferences. EU preferences assume a set of possible lotteries (or “prospects,” anticipating the language of PT) \( L \) from which a given agent may choose; for simplicity, we assume a lottery \( l \in L \) (where \( L \) may be finite or infinite) is a probabilistic gamble over finitely many possible payoffs. For example, we may adopt a standard notation and write that \( l = (0.25, 10; 0.75, 20) \) to mean that \( l \) is a lottery in which the agent has probability one-quarter of receiving 10 USD and probability three-quarters of receiving 20 USD. With each agent we also associate a preference ordering \( \succsim \), where \( l_1 \succsim l_2 \) bears the interpretation
"The agent (weakly) prefers lottery \( l_1 \) to lottery \( l_2 \). If both \( l_1 \succeq l_2 \) and \( l_1 \succeq l_2 \) hold, we also write \( l_1 \sim l_2 \), and say the agent is indifferent between the two lotteries. A preference ordering \( \succeq \) may, in general, be quite complicated; \( \succeq \) may describe all possible ordered pairs of lotteries (in which case it is called "complete"), or it may only describe some subset of lotteries. It may be the case that \( l_1 \succeq l_2 \) and \( l_2 \succeq l_3 \) imply \( l_1 \succeq l_3 \), in which case we say \( \succeq \) is transitive, or this property may fail. EU preferences describe a single family of possible \( \succeq \); to define EU preferences, suppose for a given agent that there exist functions \( u : \mathbb{R} \rightarrow \mathbb{R} \) and \( U : L \rightarrow \mathbb{R} \) such that \( U(l) = \sum_{i=1}^{N} p_i u_i(m_i) \), where \( p_i \) is the probability under lottery \( l \) of payoff \( m_i \) occurring in USD and \( m_i \) is the payoff in \( l \)'s \( i \)th component. If functions of this type exist that satisfy the further property \( U(l_1) \geq U(l_2) \) if and only if \( l_1 \succeq l_2 \), then we say that \( U \) is an EU function that represents this agent’s preferences \( \succeq \) (also: that \( U \) is a representing function for this agent), and that this agent has EU preferences.

The EU function described in the last paragraph has quite a bit of mathematical structure, and it is this structure that PT preferences relax. First, the EU function depends only on the probabilistic and monetary structure of lotteries faced, and, implicitly, monetary outcomes indicated are understood to denote final outcomes; by contrast, PT preferences assume the existence of reference point for each agent, and make the assumption that monetary rewards below this reference point ("losses") are evaluated, ceteris paribus, with greater weight than their counterparts above the reference point ("gains"). This property we refer to as either loss aversion or reference dependence. Second, the EU function is a linear function of lottery probabilities; this restriction PT relaxes by assuming the existence of two non-linear weighting functions, \( w^+ \) and \( w^- \), which are applied to probabilities corresponding to gains and to losses, respectively. Often these probability weighting functions are assumed to be symmetric about the identity line through the origin, and/or equivalent to one another; these assumptions can be useful particularly when working small-scale illustrative

---

1To be perfectly rigorous, \( \succeq \) should be defined as a collection of ordered pairs of lotteries; we elide this development, as it seems unnecessary for clarity.

2We further note that it is typical of theorists to assume that \( u(m) \) is strictly increasing in \( m \), when \( m \) is money, or some other quantity where increasing amounts is intuitively "desirable." Strictly speaking, such assumptions are not intrinsic to EU theory, but as assumptions of this kind considerably improve the interpretability of representing functions and simultaneously improve our ability to derive theorems pertinent to real-world observations, we will generally adopt assumptions of this kind.
examples, and we often assert them when doing so. The third and final primary defining property of PT preferences under risk is a bit different from loss aversion and probability weighting, and was actually introduced in the modern version of PT (Tversky & Kahneman, 1992), but was absent from original prospect theory (OPT; Kahneman and Tversky, 1979): this property is actually intended to restore a property of EU preferences, rather than relax it. EU preferences obey first-order stochastic dominance; a lottery \( l_1 \) is said to first-order stochastically dominate another lottery \( l_2 \) if, for any monetary value \( m \), \( l_1 \) has at least as high a probability of rewarding the decision-maker with \( m \) as does \( l_2 \), with a strictly greater probability for at least one value \( m \), and to obey first-order stochastic dominance means that, if \( l_1 \) first-order stochastically dominates \( l_2 \), then \( U(l_1) > U(l_2) \). Violations of first-order stochastic dominance have been observed in experimental practice (Wakker, 2010), but are generally not observed for simple gambles where the stochastic dominance is obvious, and violations of this kind were allowed in OPT. Thus Kahneman and Tversky (1992) incorporated Quiggin’s (1993) notion of rank dependence into OPT to yield PT; this change of functional form forced PT preferences to respect first-order stochastic dominance. Algebraically, rank dependence replaces weighting probabilities \( w^+(p), w^-(p) \) that occur as factors in OPT with marginals of decumulative and cumulative probabilities, respectively; for example, \( w_i^+(\Phi(x_k)) - w_i^+(\Phi'(x_{k+1})) \) would replace \( w^+(p) \), where \( \Phi(x_k) \) is the probability under a given lottery of receiving at least payoff \( x_k \), \( \Phi'(x_{k+1}) \) is the probability under that same lottery of receiving at least \( x_{k+1} \), and we have assumed the monetary payoffs \( x_1, \ldots, x_N \) in the given lottery are in ascending order (for convenience, we make this assumption throughout this work). Altogether, the PT preference functional can be written:

\[
V_i(P, r_i) = \sum_{x_k < r_i} M^-(x_k) v_i(x_k, r_i) + \sum_{x_k > r_i} M^+(x_k) v_i(x_k, r_i) \tag{1.1}
\]

where

\[
M^-(x_k) = w_i^-(\Phi(x_k)) - w_i^-(\Phi(x_{k-1}))
\]

\[
M^+(x_k) = w_i^+(\Phi'(x_k)) - w_i^+(\Phi'(x_{k+1}))
\]

In this framework, \( P \) denotes an arbitrary prospect, \( r_i \) is agent \( i \)'s reference point, \( v_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is
agent $i$’s value function (replacing the lowercase $u$) for deterministic, degenerate “gambles”, and $V_i : L \times \mathbb{R} \to \mathbb{R}$ is agent $i$’s representing function.

PT is relevant to GT in the sense that GT assumes players have EU preferences, defined over the set of prospects induced by other players’ strategic choices. To unpack this idea requires further jargon: in GT the primary objects of analysis are games; a game is a set of players $\{1,...,N\}$, a (finite, for our purposes) set of pure strategies $S_i$ for each player $i$, and a utility function $u_i : S \to \mathbb{R}$ for each player, where $S = \prod_i S_i$. Intuitively, any given play of a game results in each player choosing a pure strategy $s_i \in S_i$, and the $N$-tuple of all of these pure strategy choices (a pure-strategy “profile”) results in a utility payoff for each player. For our purposes, it will also often be convenient to introduce a function $M_i : S \to \mathbb{R}$ for each player, mapping pure-strategy profiles to monetary outcomes, and to somewhat abusively assume that $u_i$ can also be understood as mapping the set of possible monetary rewards $M_i$ to $\mathbb{R}$, so that $u_i$ can be thought of either as directly mapping a pure-strategy profile $s = (s_1,...,s_N)$ to a utility $u_i$, or first to a monetary reward $m_i$, which in turn corresponds to the utility $u_i$. Classical GT expands this notion of a game by allowing each player to use mixed strategies, where a mixed strategy for player $i$ is a probability distribution over her pure strategies $S_i$; although some solution concepts (correlated equilibrium, in particular) allow for correlation between players’ mixed strategies, we assume joint independence of all players’ mixed strategies, as is typical of the (as yet undefined) dominant equilibrium concept, Nash Equilibrium (NE). Finally, as is natural given our exposition of EU, $u_i$ is assumed to be extended to an EU function $U_i$, which is EU with respect to the collections of mixed-strategy profiles $\sigma \in \Delta$, where $\Delta = \prod_i \Delta_i$ is the the cross-product of each player’s simplex of possible mixed strategies, $\Delta_i$; algebraically, it is assumed that $U_i(\sigma) = \sum_{s \in S} \left( \prod_{1 \leq i \leq N} \sigma_i(s_i) \right) u_i(s)$, where $s = (s_1,...,s_N)$, and $\sigma_i(s_i)$ is player $i$’s probability of playing her component $s_i$ of $s$ under her mixed-strategy $\sigma_i$ in the mixed-strategy profile $\sigma$.

Thus EU preferences in GT effectively interpret mixed strategy profiles as inducing lotteries over the space of possible outcomes or pure-strategy profiles; as a result, it seems intuitively straightforward to replace occurrences of $U_i$ with $V_i$ in this definition, and thereby arrive at an
implementation of PT preferences in GT. As we will see, a number of obstacles emerge when this is attempted, but to elucidate these challenges properly, we must first explain the notion of solution concepts and NE. A solution concept in a game $G$ is intended to pare down the space of all possible mixed-strategy profiles, $\Delta$, to a smaller space $\Delta'$ of mixed strategies that might actually be played in practice; to this end, Nash (1951) introduced the notion of NE. Nash called $\sigma \in \Delta$ an NE if

$$\sigma_i \in \arg\max_{\sigma'_i \in \Delta_i} U_i(\sigma'_i, \sigma_{-i}) \forall i$$

which is to say that $\sigma$ is an NE if no player can unilaterally deviate and improve her EU payoff. Thus an NE is an equilibrium in the sense that, if all players expect all other players to use their equilibrium strategies, then there is no EU incentive to deviate from the equilibrium mixed-strategy profile.

To incorporate PT into GT, the principal challenge is to formulate a satisfactory notion of equilibrium under PT preferences. As discussed above, a natural strategy is to simply replace each $U_i$ with $V_i$, and perhaps let the reference points $r_i$ be fixed exogenously; we could then arrive at a notion of exogenous prospect-theoretic Nash Equilibrium (ePT-NE) by proposing the definition:

$$\sigma_i \in \arg\max_{\sigma'_i \in \Delta_i} V_i(\sigma'_i, \sigma_{-i}; r_i) \forall i$$

In fact, this definition forms the basis for our first notion of PT equilibrium in Chapter 2. However, a number of theoretical difficulties are encountered in working with ePT-NE; chief among these problems is that there exists an infinite family of simple finite games that fail to have an ePT-NE, under assumptions about the preference functions $V_i$ that appear to coincide reasonably well with experimental evidence. The reasons for ePT-NE’s failure to exist in this family of games are beyond the scope of this introductory section, but suffice to say that they are connected to the possibility for $V_i$ to exist that possess multiple, isolated maxima, and to the resulting lack of convexity of the argmax sets $\arg\max_{\sigma'_i \in \Delta_i} V_i(\sigma'_i, \sigma_{-i}; r_i)$.

As a result of the theoretical pathologies discovered in ePT-NE, we were led in chapters 2-3
to consider additional definitions of equilibrium. Since our equilibrium notions are intended to
generalize NE in the sense of allowing for some or all players to have non-EU preferences, a
natural framework for defining alternative notions of PT equilibrium emerges if the many possible
characterizations of classical NE are considered. In particular, we focus on the following four
characterizations of NE:

\[(A)\sigma_i \in \arg\max_{\sigma_i' \in \Delta_i} U_i(\sigma_i', \sigma_{-i}; r_i) \forall i\]

\[\iff\]

\[(B)\sigma_i \in \text{co}\left[\arg\max_{\sigma_i' \in \Delta_i} U_i(\sigma_i', \sigma_{-i}; r_i)\right] \forall i\]

\[\iff\]

\[(C)s_i \in \arg\max_{s_i' \in S_i} U_i(s_i', \sigma_{-i}; r_i) \forall s_i \in \text{supp}[\sigma_i], \forall i\]

\[\iff\]

\[(D)s_i \in \text{co}\left[\arg\max_{s_i' \in S_i} U_i(s_i', \sigma_{-i}; r_i)\right] \forall s_i \in \text{supp}[\sigma_i], \forall i\]

These characterizations share a thematic emphasis on the convex hull operator \(\text{co}[A]\), which yields
the smallest convex set containing the set \(A\); this focus is natural, given convexity’s role in generating
pathologies of nonexistence. Furthermore, prior work by Crawford (1990) had already considered a
generalization in the spirit of definition (B), though without explicitly casting his development in
this framework. In the context of finite games, the four characterizations (A)-(D) were found to
yield three distinct concepts of PT equilibrium:

\[(A)(PT - NE)\sigma_i \in \arg\max_{\sigma_i' \in \Delta_i} V_i(\sigma_i', \sigma_{-i}; r_i) \forall i\]

\[\not\iff\]

\[(B)(PT - EB)\sigma_i \in \text{co}\left[\arg\max_{\sigma_i' \in \Delta_i} V_i(\sigma_i', \sigma_{-i}; r_i)\right] \forall i\]
We have already introduced (e)PT-NE, and mentioned that Crawford (1990) developed the second concept, which we have dubbed (exogenous) prospect-theoretic equilibrium-in-beliefs (ePT-EB), after Crawford’s interpretation of the convex hull operator as a representation of subjective player beliefs about other players’ probabilities of strategic play. Placing Crawford’s (1990) notion of non-EU equilibrium in this systematic framework of classical characterizations bore fruit for us, in that we were able to derive an additional equilibrium concept from characterizations (C)-(D): (exogenous) Combinatorially Verifiable Equilibrium (eCVE), so named owing to the combinatorial nature of characterizations (C)-(D), which in finite games require only finitely many inequalities be checked to determine whether an equilibrium has been found. In finite games, characterizations (C)-(D) remain equivalent, and eCVE is the only additional equilibrium concept gained from this investigation.5

As we explore in chapters 2-3, type (A), (B), and (C) equilibria have very different theoretical properties for PT preferences, with type (A) equilibria behaving less well than (B), and (B) behaving in turn less well than (C). We structure these investigations primarily in terms of fundamental theoretical properties, like existence, and also in terms of the algorithmic notion of computational complexity. Computational complexity theory allows us to give rigorous meaning to the computational difficulty of finding equilibria; using its tools, we are able to derive a broad equivalence in chapter 3 between the difficulty of eCVE and NE, and a more limited equivalence in chapter 2 between ePT-EB and NE, while no results could be attained for the complexity relationship between

---

5 We note, however, that the (C)-(D) equivalence breaks down infinite games; furthermore, Crawford (1990) actually used the operator $D[A]$ rather than $co[A]$, where $D[A]$ gives the set of all distributions over the elements of $A$. $D[.]$ and $co[.]$ are equivalent in finite games, justifying our usage, but also become non-equivalent in general, infinite games.
ePT-NE and NE. In terms of fundamental properties, both eCVE and ePT-EB enjoy broad existence guarantees, while the existence of PT-NE appears to depend rather sensitively on assumptions about players’ value functions. These results and a number of others, as well as a litany of additional theoretical challenges, are identified and addressed in chapters 2-3.

In addition to the primary notions of equilibrium provided by characterizations (A)-(D), we also draw on prior work by Shalev (2000), who worked with a special case (featuring only reference dependence/loss aversion) of PT preferences, to introduce two additional subtypes of each primary notion of equilibrium; each such subtype endogenizes and explicitly models reference point formation. Both of these variants interpret reference points as expected future payoffs, an interpretation defended by, for example, Koszegi and Rabin (2006, 2007); under this interpretation, it is natural to expect that, in equilibrium, players’ expectations will be consistent with reality. This idea leads to our development of myopic and non-myopic notions of each of PT-NE, PT-EB, and CVE, which are named after Shalev’s original convention. In our treatment, the key difference between myopic and non-myopic equilibria lies in the number of reference points each allows: non-myopic decision makers may have many, different expectations of payoffs, as gameplay proceeds, if information is revealed during play, while a myopic decision-maker has just one reference point. These notions of myopia and non-myopia actually differ somewhat from Shalev’s original concepts, owing primarily to additional obstacles encountered in our treatment of the additional elements of the full PT model for preferences under risk, which were absent from Shalev’s model. We thoroughly explain the motivations for our changes to myopia and non-myopia in Chapter 2.

In addition to those authors already mentioned, a number of past works have developed non-EU models of GT equilibrium, and a smaller cadre of authors have developed equilibrium models for specific use with PT preferences. We identify the central differences between our work and this past literature in detail in the chapters below; as a general rule, though, our work differs from past work on PT preferences in GT in that we treat the full model of PT preferences for preference under risk. That is, we incorporate each of the three major defining components of PT preferences—reference dependence, probability weighting, and rank dependence—while past authors have only treated one
or two of these components at a time. The inclusion of all three components simultaneously causes particularly novel interactions in the case of myopic and non-myopic equilibrium definitions, and it is in reaction to these interactions that we were led to make the changes to myopia and non-myopia discussed in chapter 2.

Having completed our theoretical development in chapters 2-3, in chapter 4 we turn to an empirical treatment of our three primary notions of PT equilibrium. To do so, we rely on Selten and Chmura’s (2008) and Camerer’s (2003, Ch. 3) experimental data on (group-average) estimates of probabilities of pure-strategy play in simultaneous games with unique mixed-strategy NE. Using this data set, we develop an intriguing set of findings: ePT-EB outperform (e,m,n)CVE in fitting equilibrium behavior, both outperform NE, and best-fitting ePT-EB and eCVE have some surprising features. Rather than agents being probabilistically insensitive, as is typically assumed of PT agents, we find that the data are fit best by agents with moderate probabilistic hypersensitivity. This stark difference suggests that it may be necessary, in general, to adopt a model with two distinct weighting functions: one for strategic uncertainty and the remaining weighting function for uncertainty generated by Nature.

We are now prepared to enter into our theoretical analyses in earnest. As explained, the following chapter begins our detailed introduction and consideration of our first two notions of PT equilibrium, PT-NE and PT-EB.
2.1 Introduction

Raiffa (1982) describes three types of game theory (GT) research—“symmetrically descriptive”, “symmetrically prescriptive,” and “asymmetrically prescriptive/descriptive.” Each of these categories has a different set of goals: to describe the behavior of all agents, as in behavioral GT (Camerer, 2003), to prescribe the normatively best behavior for each agent, and to prescribe one agent’s best action while describing the remaining agents’ expected behaviors, respectively.

Classical GT is symmetrically prescriptive. von Neumann and Morgenstern (1944) use expected-utility theory (EUT) to model human preferences for risk in their theory of games, an assumption carried forward in the concept of a Nash equilibrium (NE; 1950). EUT (Ramsey 1931, Savage 1954) is the accepted model for prescriptive research. However, the descriptive validity of EUT has been questioned, and many experimental anomalies have been identified that EUT cannot explain. In an effort to explain these anomalies, a number of alternative models of preferences under risk have been developed; Starmer (2000) provides an excellent, recent survey of work in this area. Of particular importance for our work is the prospect theory (PT) of Kahneman and Tversky (1979, 1992), a generalization of EUT aimed at accommodating three commonly observed, systematic deviations of actual human behavior from EUT: reference dependence, probabilistic insensitivity, and rank dependence. PT replaces EUT’s assumption of linearity in probabilities with a probability weighting function and replaces the utility function with a reference-dependent value function.

There is a small but growing literature that seeks to address how non-classical preferences should impact GT. Colman (2003) criticizes the dominance of traditional, EUT rationality in GT, while Kadane and Larkey (1982a, 1982b) discuss asymmetrically prescriptive/descriptive analysis.
and propose the use of an “empirically supported psychological theory making at least probabilistic predictions about the strategies people are likely to use” for descriptive analyses. In this paper, we study the inclusion of PT preferences in GT research; we first consider the natural generalization of the definition of NE to the case of PT preferences, but find that this approach generates theoretical pathologies. We document the scope and range of these pathologies, and define an alternative notion of PT equilibrium corrects each of the identified pathologies.

We are not the first to consider the inclusion of non-linear probability weighting, rank-dependence, or reference dependence in GT. Shalev (2000) developed “loss-aversion equilibrium” (LAE) to address reference dependence, but maintained EUT’s linearity in probabilities. He introduced three approaches for the treatment of the reference point: in the first, the reference point is exogenous and fixed throughout a game (eLAE). In the remaining two approaches, reference points are endogenously determined by the game’s structure and equilibrium conditions; the first of these is “myopic” in that it remains constant throughout any given game (mLAE), while the latter is “non-myopic” in that it may change throughout play of a game (nLAE) as players’ conditional expected payoffs change. Shalev showed that eLAE and mLAE exist for all finite games, but exhibits an extensive-form game where nLAE may fail to exist as a result of dynamic inconsistency.

Metzgier and Rieger (2009) developed an equilibrium concept that featured probability weighting but neither rank nor reference dependence, while Ritzberger (1994) defined an equilibrium concept that incorporated both probability weighting and rank dependence, but did not feature reference dependence. Ritzberger illustrated with a counterexample that equilibria may fail to exist if strictly convex probability weighting functions are allowed.

Berejikian (2002a, 2002b) and Butler (2007) considered PT’s impacts on GT without explicitly defining new equilibrium concepts; by contrast, we formally define novel equilibrium concepts with non-classical preferences, and in this sense our work is more nearly related to that of Ritzberger (1994) and Shalev (2000). Butler applied PT to an ultimatum game, but evaluated only the impact of reference dependence, assuming linearity in probabilities.

Although we are not the first authors to consider any of probability weighting, rank-dependence,
or reference-dependence separately in GT equilibria, we are to our knowledge the first authors
to simultaneously consider all three of these components of PT in GT. Our preliminary analysis
therefore extends the special cases of PT preferences treated by past authors in this literature.

We begin by defining a notion of equilibrium under PT preferences that naturally generalizes
the definition of classical NE; this approach gives us prospect-theoretic Nash Equilibrium (PT-
NE), and, following Shalev’s treatment of reference-point formation, we develop PT-NE in three
forms—exogenous, myopic, and non-myopic, or (e,m,n)PT-EB. We concentrate our early efforts on
documenting a number of theoretical pathologies (e.g. non-existence) that occur with (e,m,n)PT-NE,
and improving upon demonstrations of similar pathologies for special cases of PT preferences. We
then develop alternative, corresponding notions of equilibrium which correct each of the pathologies
identified in (e,m,n)PT-NE; principal among the correctives we employ is Crawford’s (1990) notion
of equilibrium-in-beliefs (EB), and so we dub these new equilibrium concepts (e,m,n)PT-EB.
In the remainder of the paper, we: demonstrate that (e,m,n)PT-EB correct the pathologies of
(e,m,n)PT-NE; develop foundational theory for (e,m,n)PT-EB, including a novel development of
complexity-theoretic ideas for non-classical GT; and illustrate interesting properties of (e,m,n)PT-EB
through a variety of small-scale examples.

2.2 Prospect-Theoretic Nash Equilibrium

An $N$-player classical game $G$ in strategic form is a $2N$-tuple $(S_1, ..., S_N, u_1, ..., u_N)$, where $S_i$ is
player $i$’s set of pure strategies, $S = \prod_{i=1}^{N} S_i$ is the set of pure strategy profiles, and $u_i : S \rightarrow \mathbb{R}$ is
player $i$’s utility function. We further define player $i$’s set of mixed strategies, $\Delta$, as the set of all
probability distributions over $S_i$, and denote by $\Delta = \prod_{i=1}^{N} \Delta_i$ the set of mixed strategy profiles; we
make no distinction between mixed and behavior strategies. For $\sigma \in \Delta$, we let $\sigma_i$ be player $i$’s
strategy and (in a conventional abuse of notation) $\sigma_{-i}$ be the tuple containing the remaining players’
mixed strategies; we further let $\sigma(s)$ be the probability under mixed strategy profile $\sigma$ of pure
strategy profile $s \in S$ occurring. Under EUT preferences, the domain of player $i$’s utility function $u_i$
is extended to arbitrary $\sigma \in \Delta$ by defining $u_i(\sigma) = \sum_{s \in S} \sigma(s)u_i(s)$. We also treat extensive-form
games of perfect recall, and introduce notation for this purpose as needed, following the exposition of Fudenberg and Tirole (1991).

In the theory of games, the primary source of uncertainty is that associated with mixed or random strategies. Given a game $G$, a Nash Equilibrium (NE) is a mixed-strategy profile $\sigma \in \Delta$ such that

$$u_i(\sigma) \geq u_i(\sigma', \sigma_{-i}) \forall \sigma' \in \Delta_i$$

(2.1)

An NE is called “mixed” if it consists of at least one non-degenerate probability distribution $\sigma_i \in \Delta_i$ for at least one player $i$ and “pure” if all probability distributions involved place unit probability on a single pure strategy. The randomization in mixed strategies is often conceived of as explicit in that a player may actually, for example, use a sequence of random numbers to determine her strategy; alternatively, randomization may be considered a convenient device for modeling each player’s beliefs or uncertainty about other players’ likely choices of pure strategies. In the classical theory these two interpretations yield the same sets of NE, and so for many purposes the choice between these two interpretations is of limited consequence to classical theory. There are, nevertheless, some arguments for the empirical relevance of distinguishing between these two perspectives; for example, Camerer (2003, p. 128) argues that data exhibiting failures of independence in players’ mixed strategy choices might be understood as consistent with equilibrium-as-beliefs, despite violating the independence assumptions of classical NE. In addition, in many situations one or the other interpretation of mixed strategies seems more plausible; for example, in many settings truly mixed strategies are regarded as unrealistic. Berejikian (2002a) illustrates this, remarking that mixed strategies “seems to cut against the very definition of real-world deterrence,” in the context of international relations, and Kadane and Larkey (1983) point out the awkwardness of “flipping coins under the table.”

In non-EUT settings like those we consider, there is a still greater need for distinguishing between these two interpretations of randomness in games, as player preferences’ non-linearities in their own probabilities can cause the sets of equilibria-as-beliefs and equilibria-as-explicit-randomization to differ. The primary reason for this is that treating equilibria as beliefs leads to a difference
in treatment by each player of her strategies from those of other players: if she is not explicitly
randomizing, then it is not clear that she experiences any uncertainty whatsoever about her own
strategic choices, though she must of course be uncertain about other players’ strategic choices; by
contrast, with equilibria-as-explicit-randomization, a player faces uncertainty about not only other
players’ strategic choices, but also her own. Thus when equilibria are just in beliefs, a player’s own
strategic probabilities are irrelevant to her decision-making, and players effectively only consider
optimization over their pure strategies, which is to say over a finite set (for finite games); we
may then take the convex hulls of players’ pure-strategy best reply sets in order to represent the
uncertainty of other players about which of these strategies will be played, and so there is a natural
sense in which the analogues of the best-reply mappings are convex. On the other hand, when
explicit randomization is present, players non-linearly weight aggregate probabilities—inclusive
of their own strategic probabilities—and this has the potential to create, among other phenomena,
non-convex best reply sets and mixed strategies that are strictly preferred to any member of their
support. The first model we consider treats mixed strategies as explicitly randomized, and yields the
notions of (e,m,n)PT-NE.

We require a formal statement of the PT model for preferences under risk. To define PT
preferences, we first denote by $O \subset \mathbb{R}$ a one-dimensional set of outcomes, which we loosely interpret
as money (as in Wakker, 2010, p. 13). We then define a prospect $P = (x_1, p_1, x_2, p_2, \ldots, x_M, p_M)$
over $O$ to be any finite tuple with $x_k \in O$ for each $k$ and $p_k \in [0, 1]$ for each $k$ with the property that
$\sum_{i=1}^{M} p_i = 1$; for convenience, we assume $x_{k+1} > x_k$ for each $k$. A typical decision-making situation involves several
such prospects, with the decision-maker choosing one from among the prospects considered. As
suggested by the formalism, a prospect $P$ may be interpreted as a lottery giving probability $p_k$ to
outcome $x_k$ for each $k$.

For each PT agent $i$ considering a decision between multiple prospects, we assume the existence
of a reference point, $r_i \in \mathbb{R}$. A formal, widely agreed upon theory of reference point determination
has yet to develop (see Baucells, Weber, & Welfens, 2010, for empirical work in this spirit), but
an agent’s reference point is often interpreted as either the agent’s status quo level of resources
(Kahneman & Tversky, 1979), or as the player’s expectations or aspirations (Koszegi & Rabin, 2006, 2007) about the outcome that will occur upon selection of some prospect. For $x_i < r_i$, we say that agent $i$ operates in a losses frame, while for $x_i > r_i$, the agent is said to operate in a gains frame. The remaining possibility, $x_i = r_i$, corresponds to a subjectively neutral outcome, which might be construed as occurring in a neutral (neither gains nor losses) frame.

We let $\Phi_P(x)$ denote the cumulative distribution function for the prospect $P$ (where $P$ is clear, we often exclude $P$ from the notation and simply write $\Phi(x)$), and similarly let $\Phi_P(x)$ be the decumulative distribution function for $P$. We denote by $w_i^+(p)$ and $w_i^−(p)$ agent $i$’s probability weighting functions when in gains or loss frames, respectively, and we denote by $v_i(x, r)$ player $i$’s value function for riskless prospects. We extend the definition of $v_i$ to risky prospects as follows:

$$v_i(P) = \sum_{x_k < r_i} M^−(x_k)v_i(x_k, r_i) + \sum_{x_k > r_i} M^+(x_k)v_i(x_k, r_i) \quad (2.2)$$

where

$$M^−(x_k) = w_i^−(\Phi(x_k)) - w_i^−(\Phi(x_k-1))$$

$$M^+(x_k) = w_i^+(\Phi'(x_k)) - w_i^+(\Phi'(x_k+1))$$

Each of the individual components employed in building $v_i(P)$ is typically assumed to have certain structural properties, motivated by a combination of intuition, mathematical convenience, and empirical evidence: $v_i(x, r)$ is assumed to be differentiable in $x$ for all $x \neq r$, to be strictly decreasing in $r$ and strictly increasing in $x$, and to exhibit loss aversion in the sense that $\frac{d^2 v_i(x-r)}{dx^2} < 0$ for all $r$. Both $w_i^+(p)$ and $w_i^−(p)$ are strictly increasing in $p$, assumed to be differentiable for all $p \in (0, 1)$ and are said to have “inverse S” shapes, in that there exists $c \in (0, 1)$ such that $\frac{d^2 w_i^+(p)}{dp^2} < 0$ for $p < c$ and $\frac{d^2 w_i^+(p)}{dp^2} > 0$ for $p > c$. Some special cases of PT make further assumptions on the weighting functions; for example, Tversky and Kahneman (1992) observe that Starmer and Sugden’s (1989) model is obtained if we set $w_i^+ = w_i^- = w_i$, and we note that this is precisely the assumption we make in many of our small-scale examples, as it considerably simplifies the resulting arithmetic. In addition, we often assume in working through small-scale examples that $w_i$
is symmetric about the 45-degree line through the origin, though we emphasize that this is a choice made primarily for expediency and clarity of presentation.

Before defining what we will mean by equilibrium under PT preferences for risk, we make one further definition, which in effect attaches several new objects to the notion of a game, allowing us the terminological flexibility to distinguish between observable objects or outcomes, utilities in the traditional EUT sense, and values in the PT sense. A finite PT Game is a 4-tuple \(( (X_i)_{i=1}^N, (S_i)_{i=1}^N, (v_i)_{i=1}^N, (u_i)_{i=1}^N) \) of \( N \)-tuples, where \( X_i \) is a function \( X_i : S \to O \) mapping pure strategy profiles into outcomes for player \( i \); as above, by outcomes we mean to indicate some directly observable, concrete quantity, such as money. Each \( S_i \) and \( u_i \) retains its original interpretation; however, we may now alternatively understand \( u_i \) as a mapping \( u_i : \text{range}(X_i) \to \mathbb{R} \), where \( u_i(X_i(s)) = u_i(s) \) for any \( s \in S \). In classical GT work there is most often no need to make explicit consideration of the outcome set \( \text{range}(X_i) \), and it is usually sufficient to deal directly with the EUT functions \( u_i \) as mappings from strategy profiles to utilities. However, we will need to carefully distinguish between \( X_i, u_i, \) and \( v_i \) in our equilibrium definitions. In particular, we require notation for \( \text{range}(X_i) \) to express the intuition that a player’s reference point \( r_i \) should be expressed in units consistent with the units of elements in \( \text{range}(X_i) \), because an individual’s status quo level of resources or expected winnings in a given game are naturally denoted in units of basic outcomes, not in subjective units of utility or value. This interpretation is consistent with, for example, the work of Palley (2012), but contrasts with that adopted by Shalev (2000); our theoretical results encompass both perspectives, but in our work with examples, we assume that reference points are expressed in the same units as outcomes. We note that in experimental work on PT preferences, reference points are always measured in the same units as some observable dimension of reward; although there may be merit in considering theories in which the reference point is expressed in units of utility, as in Shalev’s (2000) mLSE model where he assumes \( r = v_i(\sigma; r) \) in equilibrium, this kind of modeling choice represents a considerable break from the experimental literature.

We will consider situations in which all agents may have PT preferences. Thus, our definitions apply to symmetric descriptive research. However, EUT is a special case of PT, so our definitions
also apply to the asymmetrically prescriptive/descriptive setting as well. To define the \( i \)-th agent as the client decision maker with EUT preferences we simply define \( w_i(p) = p, v_i(u, r_i) = u \). We refer to this special case as hybrid EUT/PT. Although the specification of the EUT player’s preferences is straightforward, the application of PT preferences to the remaining players introduces a number of complications.

With the notion of PT value of a mixed-strategy (or behavior-strategy) profile \( v_i(\sigma, r_i) \) carefully defined, we define three notions of PT-NE, in which strategic risk is treated as “true randomness.” These three definitions are straightforward generalizations of the definitions given by Shalev (2000), who emphasized the importance of the rules determining each player’s reference point in equilibrium. The first definition closely mirrors classical NE: for given \( r_i \in \mathbb{R} \forall i \), we say a mixed-strategy (behavior strategy) profile \( \sigma \) is an \textit{exogenous PT-NE} or ePT-NE iff

\[
v_i(\sigma, r_i) \geq v_i((\sigma_i', \sigma_{-i}), r_i) \forall \sigma_i' \in \Delta_i
\] (2.3)

The trademark feature of ePT-NE is its unchanging, exogenously given reference point for each player; it makes no attempt to endogenize the reference point’s dynamics. Hence, this definition may be particularly appropriate for one-off games or other situations where the horizon over which a game is played is “too short” for agents’ reference points to adapt to circumstance.

Our second definition is inspired by Shalev’s myopic loss-aversion equilibrium; it endogenizes the behavior of \( r_i \) by assuming that, in equilibrium, a player’s reference point will equal her actual expected payoff. Formally, \( \{ \sigma, (r_i)_{i=1}^N \} \) is a \textit{myopic PT-NE} (or mPT-NE) iff (3) holds and, in addition

\[
r_i = E_i^\sigma[X_i] \forall i \in 1, \ldots, N
\] (2.4)

where \( E_i^\sigma[.] \) is the expectation of . under the distribution determined by \( \sigma \).

As alluded to above, this definition differs from the work of Shalev, in that Shalev requires \( r_i = v_i(\sigma; r_i) \), that is, that in equilibrium an agent’s reference point matches her subjective value, as opposed to the reference point matching a player’s outcome. Our mPT-NE definition shares
with Shalev’s work the intuition that, if the reference point is understood as representing a player’s expectations about a game’s outcome (Koszegi & Rabin, 2006, 2007), then in equilibrium it is reasonable to impose a consistency condition like (4): if a player’s expectations do not match the actual probabilities of outcomes, then the player’s expectations should presumably continue to change. However, we argue that a player’s actual on-average experienced outcome is more relevant to determining her expectations than is her PT value; Palley (2012) has argued similarly. Although we elect to use expected value here, it would seem equally natural to adopt a “consistency condition” of this form that refers to some other measure of central tendency of player outcome, such as the median or mode. Our consistency condition also has the advantage of simplicity, in that Shalev’s condition provides a function for \( r_i \) only implicitly, while \( r_i = E_i^\sigma [X_i] \) gives \( r_i \) explicitly.

Our third equilibrium concept is “non-myopic prospect-theory equilibrium,” or nPT-NE; this concept corresponds to Shalev’s non-myopic loss aversion equilibrium. nPT-NE allows a player’s reference point to change within a game, as expectations about outcomes may alter when a player considers some unilateral deviation in strategic choice, and/or when uncertainty about another player’s intermediate strategic choices is resolved during the transition from one information set to another in an extensive-form game. Intuitively, the motivation for non-myopic equilibrium is that both a player’s unilateral deviation and/or the resolution of information in an extensive-form game may alter a player’s expected outcome in a game, and hence may plausibly correspond to a change in reference point. Formally, we say that \((\sigma, R_i)\) —where \( R_i \) is a function mapping information set, strategy pairs into \( \mathbb{R} \) —is an nPT-NE if, for each player \( i \) and at each information set \( q_i \) for player \( i \) that is reached with non-zero probability under \( \sigma \), we have

\[
v_i(\sigma, r_i | q_i) \geq v_i((\sigma'_i, \sigma_{-i}), r_i | q_i) \forall \sigma'_i \in \Delta_i, \forall i = 1, \ldots, N
\]

\( (2.5) \)

\[
R_i(q_i) = E_i^\sigma [X_i | q_i] \forall i = 1, \ldots, N
\]

\( (2.6) \)

nPT-NE stands in relation to mPT-NE in much the same way that mPT-NE relates to ePT-NE, in that nPT-NE allows a greater degree of flexibility in the player’s reference point dynamics than
mPT-NE. We might therefore anticipate that nPT-NE may be preferable to mPT-NE in modeling games in which the play of the game consumes a large amount of time relative to the horizon over which players’ reference points are anticipated to adapt; we note that this enthusiasm must be tempered by our finding of pathologies, discussed in detail below, but a similar sentiment applies to the to-be-defined notion of nPT-EB, which corrects the primary defects of nPT-NE.

In addition to the above, general definitions, we will also require in many instances specific functional forms for the PT probability weighting and value functions, as in the analysis of specific examples it is sometimes unavoidable that greater insight can be gained from the occasional use of tractable special cases of general PT preferences. To this end we introduce a parameterized family of value functions that results from a simple modification of Shalev’s (2000) piecewise-linear value function, and a new parameterized family of probabilistic weighting functions. Our emphasis in working with and introducing each of these two families of functions is in retaining the primary qualitative features of PT weighting and value functions while providing sufficient mathematical structure that “back-of-the-envelope” calculations are feasible and retain as much as possible the simplicity of classical analysis of small-scale games.

The family of value functions that we introduce is a translation of the value function used by Shalev (2000) in his treatment of loss aversion equilibrium. Given a “basic utility” for player $i$ of $u_i$, reference point for that player of $r_i$, and loss aversion coefficient $\lambda_i \geq 0$, Shalev defines the piecewise-linear value function $v_i(u, r_i) = u$ if $u \geq r_i$ and $v_i(u, r_i) = u - \lambda_i(r_i - u)$ otherwise. This value function is continuous, is differentiable away from $r_i$, and its piecewise-linear form is algebraically straightforward. However, its reference point is subjectively non-neutral, by which we mean that, in general, $v_i(r_i, r_i) = 0$ fails. This non-neutrality of the reference point is problematic for three reasons: first, the reference point is in a psychological sense generally interpreted as subjectively neutral (Gilovich, Griffin, & Kahneman, 2002, p. 135), and so it is natural that we take $v_i(r_i, r_i) = 0$. Second, expositions of the PT model (Tversky & Kahneman, 1992; Wakker, 2010, chs. 9, 12) are often unclear about whether the reference point should be evaluated in a gains frame, losses frame, or in some other frame entirely, because the subjective neutrality
condition implies the frame used for the reference outcome is irrelevant so long as its coefficient is finite. Third, as we establish below, subjective neutrality of the reference point implies that \( v_i \) is continuous in \( r_i \). For Shalev’s (2000) work the impact of these disadvantages was limited because his notion of loss-aversion equilibrium did not incorporate probability weighting, and so any factor \( w(Pr(r_i)) \) was simply taken to be \( w(Pr(r_i)) = Pr(r_i) \), as in EUT; for value functions with subjectively non-neutral reference points, discontinuities may enter through the sudden change in the probability weighting behavior as a player switches from a losses to a gains frame, but this cannot occur if EUT weighting is assumed from the outset. We describe the relationship between continuity and subjective neutrality in more detail below; for now, we just note that, because we study PT preferences with probability weighting, we will restore the subjective neutrality condition by adopting the following value function:

\[
v_i(u, r_i) = \begin{cases} 
    u - r_i & \text{if } u \geq r_i \\
    (1 + \lambda)(u - r_i) & \text{if } u < r_i
\end{cases}
\]

Obviously we now have \( v_i(r_i, r_i) = 0 \), so the subjective neutrality condition is restored.

We also require a specific family of probability weighting functions, and here we will deviate more dramatically from prior literature and introduce a completely new family, rather than modify a preexisting one. Although many weighting function families have been introduced in the literature (see Wakker, 2010, sec. 7.2), we find that the most commonly employed parameterizations tend to be unwieldy in working small-scale illustrations. For example, a number of commonly employed families become increasingly complex under differentiation, as with the two-parameter families \( w(p) = \frac{p^a}{(p^c + (1-p)^c)^b} \) and \( w(p) = (\exp(-(-\ln(p))^a))^b \) introduced by Tversky and Kahneman (1992) and Prelec (1998), respectively. We often rely on first- and second-order conditions as a primary tool in identifying and deducing properties of potential equilibria, and so it is desirable that we use value and weighting functions for which differentiation is not especially onerous.

We are aware of one particularly tractable family of weighting functions that has been employed
heavily in the literature: the neo-additive family due to Chateauneuf, Eichberger, and Grant (2007).

The neo-additive family of weighting functions is defined by letting \( w(0) = 0 \), \( w(1) = 1 \), and \( w(p) = Ap + B \) for all \( p \in (0, 1) \) with \( A, B \in \mathbb{R} \), and (in general) jump discontinuities at 0 and 1. The neo-additive family satisfies much of our need for simplicity under differentiation in that, where its derivative exists, it is simply the constant \( A \); however, the neo-additive family’s two discontinuities can be problematic for applying classical results from analysis, and can have the unpleasant side effect of generating a large number of separate cases to be checked when solving examples.

To avoid the problems posed by points of discontinuity and non-differentiability while simultaneously retaining tractability under differentiation, we introduce a novel family of weighting functions the members of which satisfy \( w(0) = 0 \), \( w(1) = 1 \), exhibit probabilistic insensitivity, and are differentiable everywhere. Specifically, we define the 2-parameter “polynomial family”

\[
w_i(p) = \begin{cases} 
\frac{1}{1+(A-1)^{2K+1}} \left[ (Ap + (1-A))^{2K+1} + (A-1)^{2K+1} \right] & \text{if } K \in \{0, 1, 2, \ldots \} \\
\frac{1}{1+(A-1)^{|K|+1}} \left[ (Ap + (1-A))^{\frac{1}{|K|+1}} + (A-1)^{\frac{1}{|K|+1}} \right] & \text{if } K \in \{-1, -2, \ldots \}
\end{cases}
\]

where the definition is made for all \( p \in [0, 1] \), \( A \neq 0 \), \( K \in \mathbb{Z} \); one might also extend this family to all real \( K \) by taking convex combinations of \( \lfloor K \rfloor \) and \( \lceil K \rceil \), though we do not require such an extension for the present work. Positive integers \( K \) yield the “inverse S” behavior typical of PT weighting functions, with larger values of \( K \) generating greater probabilistic insensitivity and a \( w_i \) that is more nearly a step function; in particular, in the point-wise limit as \( K \to \infty \) we get \( w(p) = 0, \frac{1}{2}, 1 \) for all \( 0 < p < 1 \) when \( |A-1| < 1, A = 2, |A-1| > 1 \). This also illustrates that \( A \) controls the convexity, concavity, and symmetry properties of \( w_i \). The parameter choices \( K = 0, A = 2 \) yield EUT weighting.

At \( A = 2 \) (\( K \) unrestricted) we find a family of useful special cases, in which \( w_i \) exhibits symmetry about 45-degree line, so that \( 0.5 - w_i(p) = w_i(1-p) - 0.5 \). Symmetry of this kind implies that cumulative marginals and decumulative marginals are identical; that is, for a prospect \( P \) with possible outcomes \( o_1 \leq o_2 \leq \ldots \leq o_n \), if \( w_i \) is symmetric about the 45-degree line through the
origin, then \( w_i(\Phi(o_{k+1})) - w_i(\Phi(o_k)) = w_i(\Phi(o_{k+1})) - w_i(\Phi(o_k)) \), so that a player’s probabilistic insensitivity no longer depends on her reference point. If we further assume \( v_i(o, r_i) \) exhibits neither loss aversion nor curvature in \( o \), and specifically that \( v_i(o - r_i) = A(o - r_i) + B \), then \( i \)'s best-reply correspondence is independent of \( r_i \); we use this below to study the effects of probabilistic insensitivity independent of reference or rank dependence.

Before moving on to develop theoretical results for PT preferences in GT, and thence to the use of the polynomial weighting family in examples, we summarize the polynomial family’s desirable features: \( w_i(p) \) is continuously differentiable for all \( p \in (0, 1) \), for non-negative \( k \) (which give “inverse S” shapes) its higher-order derivatives “become simpler” in that it has a finite Taylor expansion in \( p \), it can generate both usefully symmetric and plausibly asymmetric forms, and it has just 2 parameters with reasonably simple interpretations. To our knowledge, this family has not been used in prior analyses of PT preferences; we will make extensive use of it in our work with examples, especially in the case \( A = 2, K = 1 \), which gives probabilistic insensitivity vis-a-vis a simple cubic weighting function.

2.3 Pathologies of PT-NE

In the following two sections, we explore various forms of pathology—non-existence of equilibrium, triviality of preferences, discontinuities and non-monotonicities of preference—that arise in (e,m,n)PT-NE, beginning with (e,m)PT-NE (which are simpler, and most appropriate in simultaneous games), and then considering nPT-NE. We begin by considering the issue of existence, finding that even the least restrictive of our three PT equilibrium concepts, ePT-NE, may fail to exist in finite games, and that this failure of existence occurs even even when empirically common, inverse-S shaped weighting functions are assumed (but see below for a caveat). We then document a number of other pathologies that arise from variations on the basic PT-NE concept, showing that: non-monotonicities (and even multimodal preference functions) may occur in this paradigm; if subjective neutrality is respected, then Shalev’s (2000) notion of non-myopia generates degenerate, absurd preferences; and Shalev’s notion of non-myopia generates pathologies of non-existence that
are logically distinct from those generated by ePT-NE. In the sequel, we will consider correctives for each of these problems.

2.3.1 Pathologies in Simultaneous Games

We first consider the following example due to Ritzberger (1994), which shows that ePT-NE may fail to exist under RDU preferences with strictly convex probability weighting functions:

\[
\begin{array}{c|cc}
 & \text{L} & \text{R} \\
\hline
\text{T} & 2 & 0 \\
0 & 2 & 1 \\
\text{B} & 0 & 3 \\
3 & 1 & 1 \\
\end{array}
\]

Ritzberger assumed player 1’s preferences followed RDU, specifically \( w_1(p) = p^2 \) and \( v_1(u, r_1) = u \), with \( r_1 < 0 \), while player 2’s preferences followed EUT, i.e. \( w_2(p) = p \), \( v_2(u, r_2) = u \). Given these conditions, P1 strictly prefers pure strategies to mixed strategies, but there is of course no equilibrium in pure strategies, and so this game lacks an equilibrium. Crawford (1990) provides a similar example of non-existence, assuming strictly quasi-convex preferences.

Although Ritzberger’s example shows that some games may in principle lack equilibria under RDU preferences (and so under PT preferences) if strictly convex probability weighting functions are allowed, weighting functions of this shape are at best empirically uncommon. Instead, PT preferences are usually formulated under the assumption of “inverse S” shaped weighting functions; inverse-S weighting functions are generally assumed because these appear most commonly as best fits to fit empirical data. Gonzalez and Wu (1996) and Gonzalez (1999), for example, found that inverse-S shaped functions were the most common to emerge in fitting of weighting functions to experimental data. Thus it is important that we consider whether pathologies of nonexistence can still emerge when weighting functions are required to be concave-convex, rather than strictly

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convex, despite that strictly convex preferences are logically possible in the PT model. At first, this alternative assumption on the shape of admissible weighting functions may appear to give hope for a guarantee of equilibrium existence; if we redefine \( w_1(p) = 0.5(2p - 1)^3 + 0.5 \) (which has the desired inverse-S shape), for example, then it can be shown that Ritzberger’s game possesses a mixed-strategy equilibrium (see appendix entry A.1), and so is no longer a counterexample to existence.

Unfortunately, we can improve on counterexamples that rely on strictly quasi-convex weighting or payoff functions, and show that for any smooth weighting function that is “sufficiently inverse-S shaped,” there exists a 2 × 2 game in which just one player has PT preferences, and yet there does not exist an equilibrium. We impose either of two technical conditions to capture a weighting function being “sufficiently inverse S-shaped:” we require

\[
\lim_{p \to 0} \frac{w'(p(1-2B)+B)}{p} = 0 \quad \text{or} \quad \lim_{p \to 0} w'(Bp)p = \infty.
\]

Note that \( w_1(p) = 0.5(2p - 1)^3 + 0.5 \), for example, satisfies the former of these two conditions.

**Theorem 2.1.** Suppose \( w : [0, 1] \to [0, 1] \) is twice-differentiable on \((0, 1)\), has precisely one point \( B \in (0, 1) \) at which \( w''(B) = 0 \), satisfies \( w''(p) < 0 \) for \( p < B \) and \( w''(p) > 0 \) for \( p > B \), is increasing everywhere save possibly at \( B \), and let \( r = 1 \). Further assume either \( \lim_{p \to 0} \frac{w'(p(1-2B)+B)}{p} = 0 \) or \( \lim_{p \to 0} w'(Bp)p = \infty \). Then there exists a 2 × 2 game in which just one player has PT preferences (with weighting function \( w \), reference point \( r = 1 \)), but the game lacks an ePT-NE.

Theorem 1 can be understood as a strengthening of known non-existence results for PT preferences, in the sense that it assumes empirically common, inverse-S shaped weighting functions for its PT player, as opposed to empirically rare, strictly quasi-convex functions assumed in earlier non-existence results. This strengthening should therefore give us considerable pause about the usefulness and coherence of ePT-NE; however, we note that this counter-example is still not entirely decisive, because, for the modal best-fitting parameter values (as discussed by, e.g., Wakker, p. 206-209), the most commonly used parametric families of weighting functions (e.g. Goldstein & Einhorn, 1987; Kahneman and Tversky, 1992; Prelec, 1998) do not satisfy the conditions of Theorem 1. This situation may be clarified by considering the polynomial family: polynomial-family weighting functions satisfy the conditions of Theorem 1 for all \( K = 1, 2, \ldots \), but (somewhat
obviously, as this corresponds to EUT preferences) not for $K = 0$; more importantly, if we consider a generalized polynomial family that interpolates between the integer-valued model for consecutive values of $K$ to arrive at a model that is defined for fractional values, then polynomial $w$ satisfy the conditions of Theorem 1 for all $K \geq 1$, but fail to satisfy the conditions of the theorem for all $K \in [0, 1)$, and it is this latter regime—which interpolates between EUT preferences ($K = 0$) and sufficiently inverse-S shaped preferences ($K = 1$)—in which the most commonly observed weighting functions appear to fall. Thus the most commonly fitted weighting functions appear to lie somewhere between EUT preferences and the sufficiently inverse-S shaped preferences of Theorem 1; weighting functions in this regime are “shallower” than weighting functions satisfying the conditions of Theorem 1 (see Fig. 1, illustrated using Prelec’s $w(p) = (e^{-(\ln(p))^b})^a$, with $b = 1.0467$), and this shallowness causes the example constructed in Theorem 1 to once again possess an equilibrium. Given the polynomial family’s treatment of $K \in (0, 1)$ as interpolations between a weighting function satisfying Theorem 1 and EUT weighting, we may be tempted to conjecture that the presence of any degree of EUT-linear weighting in players’ weighting functions can restore existence guarantees; in fact, further work suggests that, for modal parameter estimates, equilibria exist in all $2 \times 2$ games. We do not have a general existence guarantee, however, and so the question of existence for the most commonly occurring weighting functions therefore remains open.

Despite the caveats just discussed, Theorem 1 is disturbing, as it suggests that ePT-NE may be uninformative even in simple finite games, and that this pathology of non-existence is not the result of empirically unusual quasi-convexity in agents’ preference functions, but can arise for empirically common inverse-S shaped weighting functions as well. Thus, in the sequel, we draw on the work of Crawford (1990) to regain a guarantee of existence in finite games.

Before considering extensive-form games, we examine one further property of classical NE that fails in ePT-NE and is closely tied to the failure of existence; we call this property the “combinatorial verification property” (CVP). The CVP states that $\sigma_i \in \Delta$ is a best reply to $\sigma_{-i} \in \Delta$ iff $\forall s_i \in supp[\sigma_i]$, $s_i$ is a best reply to $\sigma_{-i}$. As a result of the CVP, the search for classical NE in finite games can
Figure 2.1: Shallowness of Best-Fit $w(p)$.

be addressed by considering all possible subsets of pure strategies for each player and, for each such combination of subsets, solving a system of equations (linear in the 2-player case, and always linear in each player’s own probabilities) to determine probabilities that make all players indifferent between their supported pure strategies. One then directly verifies that all supported pure strategies are best replies, where only deviations to other pure strategies need to be considered. The CVP’s connection to equilibrium existence is that it is equivalent to linearity of the agents’ preference functionals in their own probabilities, and this is sufficient to guarantee convexity of best-reply sets, a key condition in the most common topological proofs of existence, which rely on Kakutani’s Fixed-Point Theorem.

Although impractical in moderate-to-large examples, the CVP can be useful in solving small-scale games, and it provides a natural, alternative definition of classical NE. Unfortunately, under PT preferences we can define non-monotonic value functions, so that whether a given mixed-strategy vector is in equilibrium cannot be assessed merely by checking pure-strategy best replies. To illustrate this failure of monotonicity, consider the following example, in which P1 has PT preferences with $r_1 = 0$, $w_1(p) = 0.5(2p - 1)^3 + 0.5$, and $v_1(u; r_1) = u$:
If we fix P2’s probability of playing L at \( Pr(L) = 0.25 \), then P1’s value function is

\[
V_1(p, 0.25) = 2w(Pr(> = 3.9)) + 1.9w(Pr(> = 1.9)) = 2 \cdot (0.5 \cdot (2 \cdot p \cdot 0.25 - 1)^3 + 0.5) + 1.9 \cdot (0.5 \cdot (2 \cdot (p \cdot 0.25 + (1 - p) \cdot 0.75) - 1)^3 + 0.5),
\]

and we have, for example, \( V_1(0) < V_1(1) < V_1(0.9) \), so that we cannot test whether \( V_1(p = 0.9) \) is a best reply of P1 to \( Pr(L) = 0.25 \) by testing whether A, B are best replies. As a result, the arguments underlying the CVP fail, and so in work with small-scale examples we cannot rely on exhaustive, combinatorial search to find equilibria. We instead depend on the calculus, i.e. first- and second-order conditions.

### 2.3.2 Pathologies in Extensive-Form Games

In addition to the two pathologies of PT-NE noted for simultaneous games in the preceding section, PT preferences cause at least two further problems to emerge in extensive-form games, each connected with Shalev’s (2000) notion of non-myopia. Shalev himself called attention to the first of these pathologies, by exhibiting a single-player extensive-form game in which dynamic inconsistency caused the non-existence of an nLAE. We now reconsider his example and (similarly to our work with Ritzberger’s example for simultaneous games) find that this game possesses an nPT-NE; that nPT-NE do not inherit the pathological examples of nLAE may be surprising as nPT-NE was defined in the spirit of generalizing nLAE. The root of this difference in behavior can be traced back to a seemingly innocuous correction we made in defining \((e,m,n)\)PT-NE: enforcing the usual subjective neutrality condition, \( \nu(r; r) = 0 \), which we recall was necessary to ensure continuity of the resulting value functions, and is widely adopted throughout the literature on PT preferences.

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As it happens, non-myopia as defined by Shalev interacts badly with subjective neutrality, in the sense that it generates trivial (and absurd) preferences, and it is this triviality that causes Shalev’s nonexistence example to actually possess an nPT-NE. Despite this, we show that even the trivial preferences generated by the combination of Shalev’s non-myopia and subjective neutrality do not ensure equilibrium existence in general; as was the case with Ritzberger’s game in the preceding section, we can construct an example that shares the primary features of Shalev’s example and lacks an nPT-NE. We begin by recalling the single-player example that Shalev (2000) used to illustrate the problem of dynamic inconsistency:

![Figure 2.2: Shalev’s Example without Loss-Averse Equilibria.](image)

From the root node, Shalev’s agent strictly prefers \((L_1, L_2)\) to all other strategies. However, if the second information set is reached, then Shalev’s agent prefers \((L_1, R_2)\) to \((L_1, L_2)\), so that the agent “changes her mind.” Dynamic inconsistency of this sort is a well-known feature of non-EUT models of preference, and is often considered a defect because under reasonable conditions it allows dynamically inconsistent, non-EUT agents to be exploited by savvy competitors.

To see that Shalev’s example contains an nPT-NE, take \(v(o; r) = o - r\) for gains and \(v(o; r) = \lambda(o - r)\) for losses, with \(\lambda = 1\) and \(w(p) = p\); except that we have enforced subjective neutrality and set the reference point equal to conditional expected outcome rather than to the player’s equilibrium value, these are Shalev’s assumptions. At the first decision node (D1), \(V((L_1, L_2); E^{(L_1,L_2)}(X) = 12.5) = 0\), \(V((L_1, R_2); E^{(L_1,R_2)}(X = 10.5) = 0\), and \(V((R_1,x); E^{(R_1,x)}(X = 8) = 0\) for all strategies. In fact, at the second decision node (D2), we again find a value.
of 0 for each strategy; thus all pure strategies are nPT-NE! More generally, it is easy to see that without loss aversion, probability weighting, or value function curvature, we have $V(P,E^P[X]) = 0$ for arbitrary prospects, because $V$ may be expanded as $E^P[X] - E^P[X]$. Nevertheless, as with our work on Ritzberger’s example, we can modify the example to recover its illustration of dynamic inconsistency.

We modify Ritzberger’s example by negating the outcomes, setting $v(u; r) = (u - r)^\frac{1}{3}$ for gains and $v(u; r) = 1.1 \cdot (u - r)^\frac{1}{3}$ for losses, but maintaining $w(p) = p$; thus we have incorporated value-function curvature and loss aversion. At D1 we now have $V((L_1, L_2); -12.5) = 0.669638$, $V((L_1, R_2); -10.5) = -1.09488$, $V((R_1, x); -8) = 0\forall x$ so $(L_1, L_2)$ is strictly preferred to all other strategies, whereas, at D2, $V((x, L_2); -25) = -0.123311$ and $V((x, R_2); -21) = 0$, so $(L_1, R_2)$ is strictly preferred to $(L_1, L_2)$. Thus subjective neutrality does not cure nPT-NE of issues of dynamic inconsistency, despite generating trivial preferences in Ritzberger’s single-player game.

As this example makes clear, subjective neutrality and non-myopia taken together do not imply full triviality, in that—in the presence of loss aversion, value function curvature, or nonlinear probability weighting—there are prospects between which a subjective neutral, non-myopic decision-maker will exhibit strict preference; in fact, although we found indifference between arbitrary prospects in Shalev’s example, this required our additional assumption that equilibrium reference points equal conditional expected outcomes, as opposed to reference points equaling conditional PT values, as held in Shalev’s framework. However, under plausible assumptions about reference
point behavior in equilibrium, and even if loss aversion, value curvature, and probability weighting are present, it is true that subjectively neutral, non-myopic decision-makers exhibit indifference between an implausibly large diversity of prospects: namely, the deterministic ones. The particular condition required for this result is that, for a riskless prospect \( P \) with certain payoff \( c \), we have \( r = c \) in equilibrium; our requirement that \( r \) be the expected value gives this result, but so would the condition \( r = f(P) \) for any function \( f \) that reduces to the identity for riskless prospects, and in particular this would be true for other measures of central tendency, e.g. mode, median, or for any \( f \) with \( \text{im} f \subset \text{co} \{ \text{supp}[P] \} \), as one would expect if the reference point is interpreted as a measure of anticipated outcome(s). It is worthwhile to note that all three of the assumptions we have employed are necessary here, and in particular that equilibrium conditions like Shalev’s do not generate the same pathological behavior: Shalev imposed the non-myopic conditions \( V(P) = v(c; r) \) with \( r = v(c; r) \), and if we translate his value function to respect subjective neutrality, we get the value function \( v(x; r) = x - r \) for \( r \leq x \) and \( v(x; r) = \lambda (x - r) \) for \( r > x \). A trivial variation on an argument of Shalev’s shows that the set of consistent reference points is a singleton, and it is easily seen for \( c > 0 \) that \( r = \frac{c}{2} \) is the unique reference point satisfying these conditions, and similarly \( r = \frac{1 + \lambda}{2 + \lambda} c \) is the unique solution for \( c < 0 \), with \( r = 0 \) for \( c = 0 \); thus riskless prospects with greater payoffs would be preferred to riskless prospects with lesser payoffs. Although Shalev’s condition \( v(x; r) = r \) appears to have little basis in the experimental literature, then, it does correct this problem; in the next section, we consider an alternative solution rooted in conditions that set \( r \) to a measure of the central tendency of the prospects faced by players in equilibrium.

We have shown that non-myopia, written to require reference points to equal anticipated outcomes rather than values, combines with subjective neutrality to yield an agent with a reference point that is in a sense hyper-adaptive; such an agent carries the psychological notion of the “hedonic treadmill” to the point of caricature (Rosenbloom, 2010), and can presumably be rejected as absurd. Thus non-myopia and subjective neutrality taken together imply trivial and absurd preferences; we note again that this finding only naturally comes into view in the present work because we treat the full PT model for preferences under risk, as in the absence of probability weighting there is no
compelling reason to enforce subjective neutrality (hence Shalev’s not having done so), and so the pathological interaction that we note between subjective neutrality and non-myopia fails to reveal itself if only special cases of PT preferences are studied.

This concludes our documenting of the pathologies that occur in PT-NE. To summarize: pathologies of nonexistence occur, even for 2-player games with exogenously fixed reference points and for single-player extensive-form games when reference points are endogenized non-myopically. Prior to our work, each of these points had already been raised for models that considered players with special cases of PT preferences; our contribution to this literature is to deepen the understanding of these pathologies by: improving upon the empirical relevance of the counterexamples developed, and considering aspects of these pathologies that only reveal themselves fully in the full PT model for risk. The former we achieved most notably by providing a 2-player, simultaneous nonexistence example in which the PT player had empirically common, inverse-S shaped preferences, as opposed to the empirically rare, strictly quasi-convex weighting functions on which previously known examples rely. Our key result in the latter spirit was to exhibit a pathological interaction of subjective neutrality with Shalev’s non-myopia—an interaction that naturally became a focus of investigation because of our studying the full PT model for risk, and so requiring that subjective neutrality hold.

2.4 Prospect-Theoretic Equilibrium-in-Beliefs

We now consider means for correcting the various pathologies of PT-NE, and document a number of desirable properties of classical NE that are retained by the suitably corrected equilibrium concepts. We draw on Crawford’s (1990) notion of equilibrium in beliefs to define PT-EB as a convexified version of PT-NE; rather than require that each player $i$ best-reply to her opponents’ $\sigma_{-i}$, we instead require that $\sigma_i \in co\{\text{argmax}_{\sigma'_i \in \Delta_i} V_i(\sigma'_i, \sigma_{-i})\}$—that is, that players use strategies which lie in the convex hull of their best-reply sets; we note that Crawford only defined his equilibrium-in-beliefs concept for 2 players, and there are at least two natural generalizations of this concept to the $N$-player case, which differ in whether they assume that players agree on their beliefs about a third
player in equilibrium. We assume agreement, as this gives the resulting theory considerably greater empirical content.

Under classical von Neumann-Morgenstern preferences, the argmax set appearing within $\sigma_i \in co[\argmax_{\sigma'_i \in \Delta_i} V_i(\sigma'_i, \sigma_{-i})]$ consists of all convex combinations of pure-strategy best replies, by linearity; thus the argmax sets are either of cardinality 1 or of the cardinality of the continuum. However, PT preferences may be bimodal (or, generally, multimodal with several isolated modes), and so could generate best-reply sets with cardinality 2 (or $k$, for $k \in \mathbb{Z}^+$), for example; when preferences possess multiple, isolated modes in this fashion, the convex hull taken for EB yields a different, strictly larger set of candidate mixed strategies than that allowable for standard, best-reply definitions of equilibrium.

With the foregoing discussion in mind, we make the following definitions: $\sigma = (\sigma_1, \ldots, \sigma_n)$ is an ePT-EB for PT game $G$ if, for each player $i$, $\sigma_i \in co[\argmax_{\sigma'_i \in \Delta_i} V_i(\sigma'_i, \sigma_{-i}; r_i)]$, and $\sigma$ is an mPT-EB if these conditions hold with $r_i = E^\sigma_i [X_i]$. To define what it means for $\sigma$ to be an nPT-EB, we assume that players apply backwards-induction reasoning in equilibrium. Thus nPT-EB is defined inductively in terms of behavior strategies: given a finite game $G$, let $P$ be an enumeration of all possible subgames of $G$; containment of subgames defines a partial order on $P$ with minimal elements $M$ corresponding to the leaf nodes of the game tree for $G$. For each of these minimal elements $m \in M$, we require that the behavior strategies of $\sigma$ that appear in $m$ constitute an mPT-EB. These probabilities are then regarded as fixed; if $M \neq \{G\}$, then fixing the probabilities of behavior strategies for each $m \in M$ induces a new game tree $G'$ which itself possesses non-trivial leaf nodes, so we may repeat this process. Obviously iteration of this procedure must end, since $G$ is finite; we note that this definition of non-myopia is not as flexible as that introduced by Shalev, because we no longer allow players to update their reference points when considering hypothetical deviations, though we require updating of reference points as new information sets are reached. This will be seen to resolve the trivialities of preference that arose, in the preceding section, as a result of pathological interactions between subjective neutrality and Shalev’s full non-myopia.

With definitions fixed, we now develop foundational theory for (e,m,n)PT-EB. Specifically, we
show that each of these solution concepts A) enjoys broad existence guarantees and B) behaves well with respect to iterated dominance, but that C) ePT-EB no longer satisfy rationalizability, though ePT-NE does. Further, we show that D) the theoretical work of Shalev (2000) on the ties between (e,m,n)LAЕ can be reproduced in the broader preferential context of (e,m,n)PT-EB.

To begin, both of (e,m)PT-EB can be seen to exist in all finite games as an immediate corollary of the following result, originally due to Crawford (1990) and repeated here for convenience:

**Theorem 2.2.** Any finite matrix game of complete information whose players have continuous preference functions has an equilibrium in beliefs.

In fact, we can for simultaneous games develop a somewhat broadened version of this theorem, which also accommodates generalizations of mPT-EB:

**Theorem 2.3.** For any finite, simultaneous $N$-player game with PT preferences, complete preferences, and fixed vectors $(r_{\min}^i)_{i=1}^N, (r_{\max}^i)_{i=1}^N \subset \mathbb{R}^N$ (with $r_{\min}^i \leq r_{\max}^i$ for each $i$) and given continuous functions $(f_i : \Delta \times [r_{\min}^i, r_{\max}^i] \rightarrow [r_{\min}^i, r_{\max}^i], there exists \sigma \in \Delta$ such that for each player $i$, $\sigma_i \in \text{co} \{\arg\max_{\sigma' \in \Delta_i} V_i(\sigma'_i, \sigma_{-i}; r_i)\}$, and $r_i = f_i(\sigma; r_i)$.

This theorem applies immediately to each of (e,m)PT-EB, as well as to generalizations of mPT-EB that accommodate Shalev-like, reference-equals-value definitions of myopia. Furthermore, a simple induction shows a similar guarantee for an nPT-EB:

**Corollary 2.4.** All finite games of complete information possess an nPT-EB.

The preceding three results establish that (e,m,n)PT-EB correct the primary existence failures of (e,m,n)PT-NE; we next review a number of desirable properties of classical equilibria, and show that in each case PT-EB also enjoy these properties. Specifically, we verify that: strictly iteratively dominated strategies do not occur in equilibria, and (e,m)PT-EB are always rationalizable. We then extend several results of Shalev (2000) to the case of (e,m,n)PT-EB, establishing a number of relationships between these three solution concepts.

We begin by considering strict iterated dominance; we say that a pure strategy $s$ strictly dominates $s'$ in PT value if $v_i(X_i[s'_i, s_{-i}]) \geq v_i(X_i[s_i, s_{-i}]) \forall s_{-i},$ with strict inequality holding for at least one
iterated strict dominance is a “pruning” algorithm for games which, in an initial round, deletes all strictly dominated pure strategies, and considers the game that results with these strategies removed; in this reduced game, the same deletion of strictly dominated strategies occurs, and this is repeated until the process terminates, which it must, by the finiteness of the game. We find that, in an (e,m)PT-EB, no player will use a strategy if it would be removed by iterative deletion of strictly dominated strategies:

**Theorem 2.5.** Let $G$ be a game, $(\sigma^*, r)$ be an ePT-EB, and $s_i \in S_i$. Then, if iterated deletion of strictly dominated strategies under PT value eliminates $s_i$ after finitely many iterations, we have $s_i \notin \text{supp}[\sigma_i]$.

Since we have defined mPT-EB as a particular kind of ePT-EB satisfying the myopic consistency condition $V_i(\sigma^*, r_i) = E^{\sigma^*}[X_i]$, an immediate corollary of Theorem 5 is that mPT-EB respect the iterated deletion of strictly dominated strategies as well. Unfortunately, it does not seem that this theorem can be extended to include nPT-EB; in classical theory, one notes that every subgame-perfect equilibrium is also an NE, and so subgame-perfect equilibria must respect iterated strict dominance because NE respect iterated strict dominance. However, while we have defined mPT-EB as a special case of ePT-EB, nPT-EB may not be either of (e,m)PT-EB; indeed, this is why we were forced to adopt an algorithmic, backwards-induction definition of nPT-EB in games of imperfect information: best-reply conditions in a game may be inconsistent with best-reply conditions in properly contained subgames, and so our definition—which enforces its convexified best-reply conditions only in the leaf subgames of the game tree—may cause the convexified best-reply conditions to fail in the original game, and/or in intermediate but proper subgames.

Classical theory contains a complementary process to iterated strict dominance: rationalizability. Where strict dominance compels us to iteratively delete strategies that are unambiguously worse than at least some of their competitors, rationalizability draws attention to those strategies that could be defended as reasonable choices, under at least some specification of beliefs about other players’ likely strategies. Formally, we (generalizing the definition of Fudenberg & Tirole to the case of PT preferences) define the set of $R$ rationalizable strategies for a game $G$ by: $\Sigma_i^0 = \Delta_i$ and
\[ \Sigma_i^n = \{ \sigma_i \in \Sigma_i^{n-1} : \exists \sigma_{-i} \in \prod_{j \neq i} \text{co}[\Sigma_j^{n-1}] \text{ with } \sigma_i \in \arg\max_{\rho_i \in \Sigma_i^{n-1}} V_i(\rho_i, \sigma_{-i}; r_i) \}, \text{ with } R = \prod_i R_i \]
and \( R_i = \cap_{k=1}^{\infty} \Sigma_i^k \). We note that ePT-NE are rationalizable, but the same is not true for ePT-EB:

**Theorem 2.6.** All ePT-NE are rationalizable, for any finite game \( G \).

Theorem 6 does not extend to ePT-EB: as in the proof of Theorem 5, we may again conclude that for some \( k \) and \( i \), \( \sigma_{-i} \in \prod_{j \neq i} \text{co}[\Sigma_j^{n-1}] \), but the definition of ePT-EB only assures us that \( \sigma^*_i \in \text{co}[\arg\max_{\sigma_i \in \Delta} V_i(\sigma_i, \sigma^*_i; r_i)] \), and these two conditions are not in contradiction in any obvious fashion. In fact, it is easy to see that there are examples of 1-player games in which some ePT-EB are not rationalizable; in the 1-player case, this just means there are at least two distinct modes of the player’s preference function, separated by at least one non-mode. So, any bi-modal preference function suffices; such functions arise often in the games like those defined in the proof of Theorem 1, and Example 6.3 below provides a particular example of this phenomenon. Theorem 6 might be understood as a kind of trade-off made in accepting the EB definition as a corrective: although the EB modification allows us to salvage an existence proof, it places the resulting equilibrium concept at odds with rationalizability (and it can be seen that this is not specific to the case of PT preferences, but applies to any preferences model sufficiently rich to allow for the EB concept to be relevant in the first place).

We now leave off our consideration of the ties of (e,m)PT-NE/(e,m)PT-EB to classical theory, and focus instead on the relationships between each of (e,m,n)PT-EB. In studying the connections between each of these three treatments of the reference point’s behavior, we follow Shalev (2000); we show that several of the theorems he established for the special case of PT preferences treated by (e,m,n)LAE can be reproduced in the broader context of the full modal of PT preferences for risk, for (e,m,n)PT-EB.

We first consider pure-strategy equilibria, which are particularly important in classical games because they lend themselves readily to real-world interpretation. In the present context, pure-strategy equilibria gain further importance because under degenerate probability distributions the weighting functions of PT preferences are equivalent to EUT preferences. This vanishing of probabilistic insensitivity for pure-strategy equilibria allows us to invoke first-order stochastic
dominance of PT preferences to show that the set of pure-strategy equilibria remains unchanged in moving from NE to (e,m)PT-EB.

**Theorem 2.7.** Let $G$ be a finite, $N$-player game without nature. Then TFAE:

1. $s^* \in S$ is an NE in $G$
2. $\forall r \in \mathbb{R}^N$, $(s^*, r)$ is an ePT-NE
3. $s^*$ is an mPT-NE
4. $\forall r \in \mathbb{R}^N$, is an ePT-EB
5. $s^*$ is an mPT-EB

We note that nPT-NE/nPT-EB are excluded from the theorem statement; their relationship to (e,m)PT-NE/(e,m)PT-EB is analogous to that between subgame-perfect equilibrium and classical NE, and these of course may differ even for pure-strategy equilibria. Furthermore, there is no analogous theorem relating nPT-NE and nPT-EB, as the former may fail to exist in finite games, but the latter always exist. Theorem 7 extends a theorem of Shalev (2000) to the case of the full model for PT preferences under risk, both for (e,m)PT-NE and for their “in-beliefs” counterparts, (e,m)PT-EB.

The next theorem we consider again treats a broadening of a result originally due to Shalev; Shalev showed that, for simultaneous games, $(m, n)PT - NE$ are identical. We first note that his argument relied—and ours will rely—on the continuity of the value functions, $V_i$; thus it is critical in producing this result that the subjective neutrality condition be respected, since this assures continuity of $V_i$, as we have alluded to above. We briefly prove this:

**Lemma 2.8.** Let $P$ be a prospect. If $v_i(r_i, r_i) = 0$, then $V_i(P, r_i)$ is continuous in $r_i$.

Efforts to establish a converse for Lemma 8 are complicated by the fact that it is not clear how $V_i$ should be defined if $v_i(r_i, r_i) \neq 0$ for $r_i \in O$, because in this case it is unclear what value is to be assigned to the term in the PT value expansion corresponding to $r_i$, since it is unclear whether a gains or losses frame should be used. However, we might for example define invoke OPT to define the term associated with $o = r_i$ to be $w(Pr(o))v_i(r_i, r_i)$ for non-zero $v_i(r_i, r_i)$. This would
give $\lim_{t' \to t_i} w(Pr(o))v_i(r_i, r_i) - M_i^+(r_i)v_i(r_i, r_i') = [w(Pr(o)) - M_i^+(r_i)]v_i(r_i, r_i)$; since $w(Pr(o)) - M_i^+(r_i) = 0$ only under special conditions, continuity would depend, again, on the condition $v_i(r_i, r_i) = 0$.

Lemma 8 can be used to establish a generalization of an insightful result originally developed by Shalev (2000), which states that mPT-NE and nPT-NE are identical for simultaneous games:

**THEOREM 2.9.** In any simultaneous game $G$, the sets of (m,n)PT-NE are identical; similarly, the sets of (m,n)PT-EB are identical.

We note that the proof for the first half of Theorem 9 is due to Shalev; our contribution is to note that it can be reproduced in the broader context of the full model for PT preferences under risk, both for (m,n)PT-NE and for (m,n)PT-EB, and that enforcing subjective neutrality, unlike in Shalev’s model, is critical for developing the result in this more general setting. We note, further, that the triviality of the second argument made in Theorem 9 is unsurprising: the primary content of Theorem 9 in Shalev’s work was to show that, for simultaneous games, the ability of players to condition their reference points in nLAE on hypothetical deviations held no weight; the availability of just a single information set overwhelms this condition. Theorem 9 says that a similar condition holds for nPT-NE, and that we have removed exactly the degree of flexibility from Shalev’s non-myopia to make this also trivially true for nPT-EB.

This concludes our production of results in the spirit of classical theory; we next take a novel perspective on the issue of non-classical equilibria, and study ePT-EB through the lens of computational complexity.

2.5 Computational Complexity of ePT-EB

Although the theoretical guarantees of the preceding section are desirable and useful, a theory of equilibrium behavior cannot be regarded as complete if the equilibrium concept it proposes is intractably difficult to find in practice; in particular, if a group of interacting human beings cannot reasonably be expected to “compute” an equilibrium point through their interactions, then it can
hardly be understood as a reasonable prediction for actual behavior. It is chiefly this concern that leads us to consider the matter of the computational complexity of (e,m,n)PT-EB.

The computational complexity of GT equilibria with non-classical preferences has not been well-studied; in fact, as far as we are aware, we are the first to broach the issue of computational complexity for GT models with non-classical preferences, and so it is perhaps appropriate to consider the other pressing reasons for developing a theory of the complexity of ePT-EB. There are at least two further such reasons. First, the computational complexity of finding an equilibrium point is of intrinsic interest to the applied researcher, who must contend with this complexity in developing algorithms to find equilibria. Second, in considering the merit of any new, descriptive equilibrium concept, the descriptive merits of the solution concept must be weighted against the additional complexities it imposes upon the theorist; Morrow (1994), for example, argued against the use of PT preferences in GT, remarking: “Prospect theory, the psychologists’ alternative to utility theory, is substantially more complicated than utility theory. It requires information about individuals’ reference points and reactions to probabilities as well as their preferences over gambles.” and “Strategic logic is quite complicated, even given the simplified representation of choices in utility theory.” (p. 48) Computational complexity theory provides a unified framework in which the additional complexity (or lack thereof) introduced by a new equilibrium concept can be formally evaluated. Complexity theory has its limitations, of course; it focuses on asymptotic (and, most often, worst-case) analyses, but it should nevertheless provide a useful lens into the increased complexity of ePT-EB relative to NE.

For reasons of space, it is impossible to thoroughly review computational complexity theory here; we settle for summarizing key jargon and ideas, and referring the interested reader to comprehensive references (Garey & Johnson, 1979; Papadimitriou, 1994). Complexity theory can be briefly understood as the study of the intrinsic difficulty of solving computational problems with algorithms. Problems are understood as sets of instances, where each instance of a problem $P$ consists of an input string $x$ with length $|x|$, and an associated set of solutions $S^P_x$; the most well-known and well-developed area of the theory of computational complexity considers decision problems, where
$S_x^p \in \{0, 1\}$ ("Yes," “No”) for all input strings $x$, but we will consider the more general setting where $S_x^p$ may have arbitrary cardinality; this broader setting is often referred to as the theory of “function” or search problems to distinguish it from the more heavily studied analysis of decision problems.

A peculiarity of complexity theory is that, for most problems of interest, true guarantees of problem difficulty have not been found despite substantial effort. Complexity theorists cope with this situation by developing a theory of the reductions between problems, in the sense of showing that a given problem is no more difficult than some other problem. If both directions of reduction can be shown, then the two problems involved are said to be polynomially equivalent. Complexity theorists employ many kinds of reductions; the most widely used is the polynomial-time many-one reduction, which shows that, for some fixed, independent-of-instance polynomial $p$, an instance $a$ of a problem $A$ may be transformed in time bounded by $p(|A|)$ into an instance $b$ of a problem $B$, with the property that any solution to $b$ may in turn be transformed in polynomial time into a solution for $a$. Another widely used reduction type is the Turing reduction: problem $A$ is said to Turing-reduce to problem $B$ if, assuming the availability of a hypothetical subroutine for solving problem $B$ in polynomial time, then problem $A$ can be solved by an algorithm that calls on the subroutine for solving $B$ at most polynomially many times. Turing reductions are more general than many-one reductions, in that Turing reductions A) allow for the hypothetical subroutine for $B$ to be called many (but no more than polynomially many) times, as opposed calling this subroutine just once in the case of the many-one reduction, and B) strictly speaking, may be used to treat function problems, while the original definition of many-one reductions applied only to decision problems. In the work below, we rely on “polynomial reductions;” by this, we can be understood to mean either a Turing reduction that makes at most one call to its subroutine, or a many-one reduction that is applicable to function problems.

A considerable ambiguity in the foregoing discussion is the definition of an algorithm: there are several formal definitions available of an algorithm, but perhaps the most commonly used is the Turing machine model, in which computation is modeled by a machine that can take on a finite set of states, and can write a finite set of symbols to an infinite tape of side-by-side read-write
squares, which it may traverse one-by-one, left-to-right or right-to-left; the machine’s states include distinguished “Start” and “Halt” states. To accommodate features like hypothetically available subroutines and nondeterminism, this basic description is often augmented with additional tapes that have special properties; for our results, it is sufficient to describe the necessary reductions without immediate reference to the Turing machine model, so we do not describe this fundamental model in any greater detail. For further information on computational complexity theory, the reader is referred to Garey and Johnson (1979) for an early but exceedingly well-written treatise, or to Papadimitriou (1994) for a more recent, standard reference.

The notion of reductions motivates the ideas of *complexity class* and of *complete* problems for a complexity class. A complexity class is a set of problems all of which in some sense are no more difficult than some standard of computational complexity; for example, the famous complexity class \( P \) contains all decision problems solvable by a deterministic (non-oracle) Turing machine in polynomial-time, and the complexity class \( NP \) contains all decision problems solvable by a nondeterministic (non-oracle) Turing machine in polynomial time. Within a complexity class, there may exist problems to which every other problem in the class is reducible; if such a problem exists, it is said to be complete for the class in question, and is to be regarded as among the set of the most difficult problems in the class. All complete problems for a given class are reducible to one another, for whichever notion of reducibility is used; complete problems are therefore natural targets for attempts to either show that the class is “easy” by finding a polynomial-time algorithm for a complete problem, and so a polynomial-time algorithm for the entire class. Some complexity classes contain very many, important complete problems, all of which have resisted vigorous efforts to find polynomial-time algorithms; this is the case for \( NP \), for example, and is widely regarded as evidence that the \( NP \)-complete problems are “hard,” i.e. not polynomial-time solvable. Furthermore, showing that complete problems for classes regarded as difficult are in turn reducible to some other problem \( T \) is seen as evidence that \( T \) is itself intractable. Corresponding to \( P \) and \( NP \) are complexity classes \( FP \) and \( FNP \), whose definitions are identical but allow for the treatment of functional problems.

Although computational complexity has not, to our knowledge, been applied to the study of non-
classical equilibria, the computational complexity of finding classical NE has received impressive attention from both the mathematical economics and theoretical computer science communities. We briefly call attention to a few key achievements: it has been found that computation of an $\varepsilon$-NE (a $\sigma$ such that no player can unilaterally deviate and gain more than $\varepsilon$) in an arbitrary finite game is complete for a complexity class known as $\text{PPAD}$, which is contained in $\text{FNP}$ but contains $\text{FP}$; this is thought of as evidence of intractability, though not evidence as severe as $\text{NP}$-completeness (Daskalakis, Goldberg, & Papadimitriou, 2008). It is also known that the $\varepsilon$-approximation of an exact NE (a $\sigma$ for which there is an NE $\sigma^*$ with $||\sigma - \sigma^*|| < \varepsilon$) is complete for a class of problems known as $\text{FIXP}$, which contains $\text{FNP}$ (which in turn contains $\text{FP}$). Each of $\text{PPAD}$ and $\text{FIXP}$ contain a number of search problems that can be modeled as $\varepsilon$ or exact (respectively) fixed-point problems. We note the importance of approximation in this literature; it is in general impossible to compute equilibria exactly under a Turing machine model of computation, as irrational numbers cannot be exactly represented, and so instead theorists study the problem of approximating equilibria. It is also known that there are finite classical games in which every equilibrium contains at least one irrational component in some player’s mixed-strategy profile.

To begin building a theory of the computational complexity of ePT-EB relative to classical NE, we develop reductions from the problem of finding an approximate ePT-EB to the problem of finding an approximate NE. These arguments speak directly to the issue of whether ePT-EB complicates or simplifies the existing classical theory of GT.

We begin with the simplest case and proceed to the most complex. First, we consider pure-strategy equilibria:

**Problem PSE$_G^{PT}$**: given a PT game $G$ from a family of finite PT games $\mathcal{G}$ without nature, does $G$ possess a pure-strategy ePT-EB? If so, find it.

We note that PSE is not just a single problem, because we have not specified $\mathcal{G}$; it could be the family of all finite games, or some far more specific family, such as the class of $N$-player Matching Pennies games or the class of congestion games. We will show that, for any $\mathcal{G}$, PSE$_G^{PT}$ is polynomially equivalent to
Problem PSE$_G$: given a game $G$ from a family of finite games $G^{PT}$ without nature, does $G$ possess a pure-strategy NE? If so, find it.

Note that, as these theorems focus on pure-strategy equilibria, we can compute the relevant equilibria exactly; we do not need to fret over approximate computation. We can easily show:

**Theorem 2.10.** Problems $PSE^{PT}_G$ and $PSE_G$ are polynomially-equivalent, i.e., each can be polynomial-time reduced to the other; thus there is a polynomial-time algorithm for Problem $P^{PT}_G$ iff there is a polynomial-time algorithm for PSE$_G$.

Although simple, Theorem 10 establishes an infinite family of equivalences for pure-strategy equilibria, and in this sense is the ideal form for results of our second type, in that it says there is no computational advantage in using either the classical or ePT-EB theories; they are equivalent, from the point of view of finding equilibria algorithmically in large-scale games. As an example, this implies in particular that finding pure-strategy ePT-EB in congestion games is a PLS-Complete problem; this follows from Theorem 10 and the PLS-Completeness of finding pure-strategy NE in congestion games, which was first established by Fabrikant, Papadimitriou, and Talwar (2004).

In view of the full content of Theorem 6, Theorem 10 could obviously be extended to include (e,m)PT-NE and mPT-EB as well.

Of course, there is a sense in which Theorem 10 also can be read as an argument against the use of (e,m,n)PT-EB: if one is only interested in pure strategy equilibria, then Theorems 6 and 10 say that there is no particularly pressing reason to use the more complicated equilibrium concept, since they are equivalent. However, as we now show, the case for mixed-strategy ePT-EB equilibria is considerably more complicated. This is so for a number of reasons: first, mixed ePT-EB may not be rational, and so may not be exactly computable within the Turing-machine model of computation. Second, the convex-hull operator appearing in the definition of ePT-EB cannot be dismissed for mixed-strategy ePT-EB in the same way that it was for pure-strategy equilibria in Theorem 6; this poses a particular obstacle for any attempts to reduce finding an ePT-EB to problems of finding a fixed point of single-valued functions, as Etessami and Yannakakis (2007) and Daskalakis et al. (2009) each did for NE. Third, the PT preference functions that arise in computing ePT-EB
may be quite complicated, considered as optimization problems, and in particular the possibility of non-trivially multimodal payoff functions again has no analogue in classical theory.

To address this potential complexity of the PT preferences, we narrow our attention to polynomial preference functions; the polynomial family of weighting functions given above identifies a member of this family, but in general any weighting function with a finite Taylor expansion will suffice. To fix a computational model, we focus on preferences that are computable by an algebraic circuit with a single output, defined over the basis \{+, −, *, /\} and the input digits 0, 1. (The restriction to polynomial preferences can be considerably loosened, by considering preference functions that are efficiently approximable by polynomial preferences, but in the interests of simplicity, we focus on polynomials.)

The importance of polynomial preference functions for our results stems from well-known results from the literature on the complexity of classical NE (Daskalakis et al., 2009), which establish that classical NE can be used to do each of the arithmetic operations, +, −, *, so long as it is known that the results of intermediate operations are confined to the interval [0, 1]. For polynomials on compact domains, this latter restriction can also be removed by a technique introduced by Etessami and Yannakakis (2007); they first double the size of the algebraic circuit representing a given polynomial, and use this doubled size to separately store the negative and positive parts of the output of each operation. This allows the algebraic circuit \(A\) computing a given polynomial to be computed by another circuit \(A'\) of size polynomial in that of \(A\), with the extra property that all save the final output is non-negative, and the only subtraction gate in \(A'\) is its final gate. Finally, Etessami and Yannakakis normalize the operations of \(A'\) by dividing its intermediate operations by a bound on the largest possible integer that can be computed by \(A'\); this normalizes the circuit so that the output of each of its operations lies in the interval [0, 1]. We state this result formally as:

**Theorem 2.11.** Suppose we are given \(M \in \mathbb{N}\) multivariate polynomials \(f_1(X_1, \ldots, X_N), \ldots, f_M(X_1, \ldots, X_N)\), each computable by an algebraic circuit \(A_i\) over basis \{+, −, *\} and input digits 0, 1, where there is a polynomial in \(N\) that bounds the time necessary to compute each \(A_i\)'s output on any binary input string; further suppose we are given a partition \(P\) of \{1, 2, ..., \(N\)\} into \(W \leq N\) sets \(P_1, P_2, \ldots P_W\).
Then:

A) there exist rational constants $K_1, K_2$ of size polynomial in $N$ such that, for each $i$, $0 \leq K_1 f_i(X_1, \ldots, X_N) + K_2 \leq 1$, and

B) there is a finite game $G$, of size polynomial in $N$, with distinguished input players $I_1, \ldots, I_W$, each of whom has number of strategies $|P_i|$, and output players $O_1, \ldots, O_M$ each of whom has 2 pure strategies, such that:

C) the probability $p_{O_i}(0)$ of $O$ playing her first strategy in any $\varepsilon$-NE of $G$ satisfies $K_1 f(p_{I_1}(0), \ldots, p_{I_1}(0)) + K_2 - p(N,D)\varepsilon \leq p_{O_i}(0) \leq K_1 f(p_{I_1}(0), \ldots, p_{I_1}(0)) + K_2 + p(N,D)\varepsilon$, where $p(N,D)$ is a bivariate polynomial in $N$ and the maximum depth taken over all circuits $A_i$, and $p_{I_k}(0)$ is player $I_k$’s probability of playing her first strategy.

Theorem 11 is a straight-forward agglomeration of results due to Daskalakis et al. (2009) and Etessami and Yannakakis (2007); we now use it to establish several results which are, to our knowledge, novel. Specifically, we will use Theorem 9’s construction to reduce finding approximate ePT-EB to finding approximate classical NE in each of the following two theorems. However, each construction treats a different set of assumptions on the PT preferences allowable and the strength of the approximation required, and these differences requiring corresponding differences in the details of the arguments made. In Theorem 12, we use the construction of Theorem 11 to build a classical game with the property that NE naturally search convex combinations players pure-strategy best-replies for an ePT-EB; given quasi-convex PT preferences (so that best replies must occur at extreme points), this will be seen to immediately give the desired reduction. Theorem 13 extends this approach still further, using it to consider general polynomial PT preferences, for which best-replies may occur even at interior points of a player’s mixed-strategy set; however, as a kind of trade-off, we must settle for a weaker notion of approximate ePT-EB in Theorem 13, and for ePT-EB with uniform bounds on the numbers of strategies per player.

We now develop Theorem 12, which addresses preferences like those used in the non-existence examples introduced by Crawford (1990) and Ritzberger (1994)—including, for example, quasi-convex preferences. If a mixed strategy $\sigma_i$ is a best-reply to some $\sigma_{-i}$, then we require that all
pure-strategies in \( \text{supp}[\sigma_i] \) be best replies as well; since classical NE already by virtue of linearity engage in a kind of search of convex of players’ pure-strategy best replies, it is perhaps unsurprising that this property is sufficient to show the reducibility to NE of ePT-EB, which differ from NE primarily in requiring a search of the convex regions between best replies. (In fact, we could abstract still further; there is nothing particularly special about focusing on pure strategies—what is crucial is that, for each player \( i \), there be an efficient subroutine that returns a no more than polynomially large set \( C_i \) containing \( i \)’s “candidate” best replies. Specifically, \( C_i \) should have the property that, if some non-extreme point \( c \) of \( C_i \)’s convex hull is a best-reply, and removing an extreme point \( x \) from \( C_i \) yields a set \( C'_i \) that does not contain \( c \), then \( x \) is also a best reply.) We have:

**THEOREM 2.12.** Define an \( \varepsilon \)-ePT-EB to be any mixed-strategy profile satisfying \( \sigma_i \in \text{co}[\arg\max_{\sigma_i} \varepsilon_i(\sigma_{-i})] \forall i \), where \( \arg\max_{\sigma_i} \varepsilon_i(\sigma_{-i}) = \{ \sigma'_i \in \Delta_i : \max_{\sigma'' \in \Delta_i} f_i(\sigma'_i, \sigma_{-i}) - f_i(\sigma'_i, \sigma_{-i}) \leq \varepsilon \} \). Suppose \( G \) is a finite PT game in which all players have polynomial preference functions \( f_i \) that satisfy \( \forall \sigma_{-i} \in \Delta_i, \sigma_i \in \arg\max_{\sigma_i} f_i(\sigma_i, \sigma_{-i}) \Rightarrow \text{supp}[\sigma_i] \subset \arg\max_{\sigma_i} \varepsilon_i(\sigma_{-i}) \). Then the problem of finding an \( \varepsilon \)-ePT-EB for an arbitrary finite PT game \( G \) and rational \( \varepsilon > 0 \) is polynomial-time reducible to the problem of finding an \( \nu \)-NE for an arbitrary finite classical game.

Theorem 12 results from setting \( C_i \) in the discussion above to player \( i \)’s set of pure strategies; this theorem (and its generalization in terms of \( C \)) is theoretically encouraging, in that it indicates that Crawford’s (1990) EB technique for correcting non-existence issues in the most commonly discussed counterexamples—those with strictly convex preferences—are not only corrected by EB, but in fact efficiently corrected. However, we have shown above that there exist non-existence counterexamples in which player preferences are more complicated, and for which both non-trivial multimodality and unique interior optima are possible. Furthermore, we argued above that these counterexamples to existence are, given the empirical evidence on shapes of weighting functions, likely to be observed in practice with greater frequency than the convex-preferences counterexamples to existence primarily discussed by past authors. It is therefore also of interest to build theory to address the complexity of ePT-EB when no fixed sets of points \( C_i \) exists as in Theorem 12, and in particular when the sets of pure strategies do not have this property. As it turns out, the approach
used to develop Theorem 12 can still be useful for this more general purpose, if we bound the number of strategies per player for the PT game. We have:

**THEOREM 2.13.** For naturals $B, K$, let $G(B, K)$ be a family of finite PT games in which every game $G \in G(B, K)$ satisfies: each player $i$ has a polynomial preference function $f_i$ of degree no greater than $B$ and which is computable by an algebraic circuit $A_i$ over basis $\{+, -, \times\}$ and binary input strings, and has no more than $K$ pure strategies. Then, given rational $\varepsilon > 0$, the problem of finding an $\varepsilon$-ePT-EB in any instance of $G(B, K)$ is polynomial-time reducible to the problem of finding an $\nu$-NE in an arbitrary finite game.

We note that both of Theorems 12-13 could be easily extended to accommodate a broader class of non-polynomial preferences. The same arguments used in these theorems can be applied to games with preference functions that can be efficiently approximated by a polynomial; one simply converts the given problem to one suitable for the arguments of these two theorems by first approximating the given preference functions with suitable polynomials. Thus Theorems 12-13 identify relatively large classes of PT games for which finding approximate ePT-EB is no more difficult than the problem of finding a classical NE in an arbitrary finite game.

Theorems 12 – 13 also provide particular ties to the class PPAD, the complexity classes for which finding an $\varepsilon$-NE is complete; in the interests of collecting these results in a single place, we also restate the connection for finding pure-strategy ePT-EB to PLS. Specifically, we have the following corollary, which summarizes the results of our complexity-theoretic work in terms of existing complexity classes used for NE:

**COROLLARY 2.14.** Problem $PSE_{G}^{PT}$ is PLS-Complete.

Denote by $PT^{C}$ the problem: given rational constant $\varepsilon > 0$ and a finite PT game $G$ in which all players have polynomial preference functions $f_i$ that satisfy $\forall \sigma_{-i} \in \Delta_i, \sigma_i \in argmax_{\sigma'_i \in \Delta_i} f(\sigma_i, \sigma) \Rightarrow supp[\sigma_i] \subset argmax_{\sigma'_i \in \Delta_i}$, find an $\varepsilon$-ePT-EB of $G$. By Theorem 10, we have $PT^{C} \in FIXP$; since it is obvious that finding an $\nu$-NE is polynomial-time reducible to $PT^{C}$, we have that $PT^{C}$ is PPAD-Complete.
Denote by $PT(G(B, K))$ the problem: given rational $\varepsilon > 0$ and an instance $F$ of $G(B, K)$, find an $\varepsilon$-ePT-EB of $G$. By Theorem 11, $PT(G(B, K)) \in PPAD$. If we also let $PT(G(\infty, \infty))$ denote the same problem as $PT(G(B, K))$, but with no restrictions on either the number of strategies per player or the Lipschitz constants of players’ preferences, then we also see immediately that $PT(G(\infty, \infty))$ is $PPAD$-Hard. Similarly, $PT(G(\infty, 1))$ is $PPAD$-Hard.

This concludes our description of foundational theory for ePT-EB. We now move on to a discussion of examples, which we use to illustrate a sampling of the theorems we have established, and to demonstrate the curiously non-monotonic relationship of player welfare to rationality of preferences.

2.6 Some Standard Game Theory Examples

We have already seen that NE, PT-NE, and PT-EB can differ. The example due to Ritzberger (1994) has a classical equilibrium; does not have an ePT-NE for quasi-convex weighting functions but does have an ePT-NE for polynomial, cubic weighting functions; and always has an ePT-EB, but this ePT-EB in general differs from its NE. In this section we show that NE and PT-EB are the same for some standard examples of games; thus, the correspondence between the equilibrium concepts is not always as complex as our development thus far may suggest. In this section’s final example, we also demonstrate the process of finding a PT-EB in a game that has no PT-NE.

2.6.1 Prisoner’s Dilemma (PD)

The PD can be defined as any game payoff-ordinally equivalent to:
The integers in the table denote basic outcomes for each player, \( o_i \in O \subset \mathbb{R} \), as contrasted with the classical description, in which these give utilities. \( D \) strictly dominates \( C \) in basic outcome for each player, so Theorem 3 implies that \((D, D)\) is the unique candidate for an ePT-NE (for any \( r \in \mathbb{R}^2 \)). Since any mPT-EB can be regarded as an ePT-EB with \( r_i = E_i^\sigma [X_i] \forall i \), \((D, D)\) is also the unique candidate for an mPT-EB, and Theorem 4 implies \((D, D)\) is the unique nPT-EB candidate. It is readily verified that \((D, D)\) in fact satisfies the best-reply conditions for each of notion of PT-EB, so \((D, D)\) is the unique strategy profile in any ePT-EB, mPT-EB, and nPT-EB. For \((m, n)\)PT-EB, \( r = (r_1, r_2) = (4, 4) \). In an ePT-EB, player \( i \) has value \( v_i(4; r) \), utility \( u_i(4) \), and receives basic outcome 4. \((m, n)\)PT-EB yield the same utilities and basic outcomes but value \( v_i(4; 4) = 0 \forall i \). This example reflects the dominance-solvability of the Prisoner’s Dilemma: since dominance eliminates all strategy profiles save one, and this is the same strategy profile for classical and PT preferences, the sets of NE and PT-EB agree. We note in passing that this analysis remains unchanged if one considers PT-NE rather than PT-EB, because Theorem 5 applies to PT-NE as well as PT-EB, as a result of PT preferences always satisfying stochastic dominance.

2.6.2 Matching Pennies (MP)

The MP game serves as another example in which the sets of classical NE and \((e, m, n)\)PT-EB are identical; here, however, dominance alone cannot solve the game:
We assume $0 \leq r_i \leq 1$ for $i = 1, 2$. If both players have PT preferences, we then have $v_1(p, q) = v_1(pq + (1 - p)(1 - q))v_1(1; r_1) + w_1(p(1 - q) + (1 - p)q)v_1(0; r_1)$ and $v_2(p, q) = v_2(p(1 - q) + (1 - p)q)v_2(1; r_2) + w_2(pq + (1 - p)(1 - q))v_2(0; r_2)$. For $p = 0.5$, player 2 is therefore indifferent between all her mixed strategies, as each $w_1(.)$ factor becomes independent of $p$; similarly, for $q = 0.5$, player 1 is indifferent between all his mixed strategies, and so $(p, q) = (0.5, 0.5)$ is an ePT-EB of this game. In this ePT-EB, both players have expected value $E_i[p, q] = 0.5$; since this is a simultaneous-game, (m,n)PT-EB are identical, so we get an (m,n)PT-EB by setting $r_i = 0.5$ in the ePT-EB. This (e,m,n)PT-EB is unique, since $\frac{d(p(1-q)+(1-p)q)}{dq} = 1 - 2p$ and $\frac{d(pq+(1-p)(1-q))}{dq} = 2p - 1$; for $p \neq 0.5$, these derivatives have opposite, non-zero sign, independent of $q$, and $w_i$ is strictly monotonic, so P2’s best-reply correspondence gives $q \in \{0, 1\}$ for $p \neq 0.5$. A symmetric analysis applies to P1. In the ePT-EB, player $i$ receives value $v_i(0.5, 0.5; r_i) = w_i(0.5) (v_i(1; r_i) + v_i(0; r_i))$, expected-utility $u_i(0.5, 0.5) = 0.5(u_i(1) + u_i(0))$, and experiences basic outcomes of 1 or 0, each with probability $\frac{1}{2}$. In the (m,n)PT-EB, the value received by player $i$ becomes $v_i(0.5, 0.5; 0.5) = w_i(0.5) (v_i(1; 0.5) + v_i(0; 0.5))$, while the utilities and basic outcomes remain unchanged.

Thus for the game MP, symmetries in the game payoffs cause the sets of PT-EB, hybrid EUT/PT, and classical NE to be identical. As with PD, we note that this conclusion would not change if we instead considered PT-NE, by virtue of the convexity of best-reply sets identified in MP.
2.6.3 Ochs’ Game (RG)

Ochs (1995) studied a game of the form that we used in the proof of Theorem 1 (he set \( C = 1, 4, 9 \)), where we showed that a PT game of this type can be generated such that the game possesses no equilibrium. We show here that Ochs’ game (with \( C = 4 \), for illustrative purposes) possesses an \((e,m,n)\)-PT-EB, but no \((e,m,n)\)-PT-NE. The game is:

<table>
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<td>4</td>
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We take \( w_1(p, q) = 0.5(2pq - 1)^3 + 0.5 \) with \( v_1(u; r_1) = u - r_1, r_1 = 1 \), and let P2 have EUT preferences. By the monotonicity properties of PT and EUT preferences, there is no equilibrium in which either player uses a pure strategy; thus P2’s preferences imply that, if there is an ePT-EB (for any \( r_1 \in (0, 4) \)), then in it P1 must play \( p = 0.5 \). To see that there exists no ePT-NE satisfying this constraint, we just consider P1’s first-order condition

\[
V_1'(0.5, q) = 12(q - 1)^2q = 0
\]

which obviously has no interior solutions. However, this game of course does possess an ePT-EB. For \((p, q)\) to be an ePT-EB, we need:

\[
p \in co[\text{argmax}_{p' \in [0,1]} V_1(p', q)]
\]

\[
q \in co[\text{argmax}_{q' \in [0,1]} V_2(p, q')]
\]
Since P2 has classical preferences, \( \text{argmax}_{q'\in[0,1]} V2(p,q') \) is always one of \{0\}, \{1\}, [0,1], corresponding to the cases \( p > 0.5, p < 0.5, p = 0.5 \) respectively; clearly we cannot have an equilibrium in which P1 plays a pure strategy. Also, if P2 plays a pure strategy \( q \in \{0,1\} \), then \( \text{argmax}_{p'\in[0,1]} V1(p',q) \) is either of \( p = 0, 1 \), corresponding to the cases \( q = 0, 1 \), respectively. Thus we cannot have an equilibrium in which P2 plays a pure strategy, so we again consider only equilibria in which \( p, q \in (0,1) \).

Since \( p \neq 0.5 \) implies pure-strategy play by P2, we must have \( p = 0.5 \); this implies P2’s inclusion is satisfied independent of \( q \). It remains, then, to find \( q \) such that \( 0.5 \in \text{co}[\text{argmax}_{p'\in[0,1]} V1(p',q)] \). The set \( \text{argmax}_{p'\in[0,1]} V1(p',q) \) is a subset of P1’s set of stationary points, which are determined by P1’s first-order condition

\[
V1'(p,q) = 3(2p(q-1) + 2(p-1)q + 1)^2(2q-1) + 12(2pq - 1)^2 q = 0
\]

If \( q > 0.5 \), this equation has no solutions; if \( q = 0.5 \), it reduces to \( 6(p - 1)^2 = 0 \), which implies \( \text{argmax}_{p'\in[0,1]} V1(p',0.5) = \{1\} \). So, we seek \( 0 < q < 0.5 \) such that \( 0.5 \in \text{argmax}_{p'\in[0,1]} V1(p',q) \). It is easy to verify that at about \( q = 0.05 \), we have \( \text{argmax}_{p'\in[0,1]} V1(p',q) \approx [0,0.742] \). Hence, \((p,q) \approx (0.5,0.05)\) is an ePT-EB of Ochs’ game with \( C = 4 \). Numerical evidence further suggests that this ePT-EB is unique; for \( 0 < q < 0.05 \) and \( 0.05 < q < 0.5 \), \( V1(p,q) \) appears to be unimodal.

This example illustrates the corrective effects of our use of Crawford’s (1990) notion of EB in PT-EB, and illustrates a natural method of solving for \((e,m,n)\text{PT-EB}\) when the details of this solution approach are non-trivially different from the classical case.

2.7 The Asymmetric Prescriptive/Descriptive Approach

Our previous examples are standard in game theory. In this section, we apply the asymmetric prescriptive/descriptive modeling approach to a modified Entrant/Incumbent game. In this game, we suppose there is a client seeking prescriptive help (the Entrant, modeled with EUT preferences) and another decision maker whose actions must be described (the Incumbent, modeled with PT...
preferences). We will apply our hybrid EUT/PT equilibrium and its variants to the following modified entrant-incumbent game (where payoffs $A, B$ give the Entrant,Incumbent payoffs, respectively):

![Figure 2.4: Entrant-Incumbent Game Exhibiting Gains To Irrationality.](image)

In this game, we are prescribing the best action to our client, the entrant, while describing the possible behavior of the incumbent. The entrant can always secure the status quo profit by not entering, and in so doing ensure the incumbent continues to enjoy its monopoly. If the entrant invades, then the incumbent may concede some portion of its market share, splitting the benefits of oligopoly (which are less than the benefits of monopoly) between the two firms. However, the incumbent firm may instead defend its stake in the market, by—for example—engaging in a costly, negative advertising campaign against the entrant. The outcome of such a campaign is uncertain: the campaign may waste firm revenue, provide publicity for the entrant, and cause backlash against the incumbent (event $E$, which occurs with probability $P_E$); alternatively, the campaign might succeed (event $I = E$) and drive the entrant once again out of the market, though at some cost. The probabilities (indicated in the game tree) with which the events $E$ and $I$ occur are
exogenously given, controlled by an artificial Nature player, and assumed known to both players. 2-player extensive-form games with a source of risk introduced by a nature player can be used to show, in simple settings, that a player may benefit from possessing EUT preferences rather than PT preferences; we demonstrate that the Entrant/Incumbent game has this property by contrasting its NE and (e,m,n)PT-EB(s) solutions.

The Entrant (P1) controls the probability \( p \) of invasion and the Incumbent (P2) controls the probability \( q \) of defending in the event of invasion; we suppose P1 has EUT preferences, while P2 has PT preferences with symmetric polynomial \( w_2(p) \) but \( v_2(u; r) = u - r \), so P2 exhibits probabilistic insensitivity and rank dependence but not reference dependence; we note that, as in our discussion when the polynomial family was introduced, these assumptions imply that P2’s best-reply correspondence is independent of \( r \), since, for all prospects \( P \) and any \( r \),

\[
V_2(P, r) = \sum [w(\geq x) - w(>x)]x - r \sum [w(\geq x) - w(>x)]x = \sum [w(\geq x) - w(>x)]x - r,
\]

so \( V_2(P, r) \geq V_2(P', r) \iff \sum [w(\geq x) - w(>x)]x \geq \sum [w(\geq x') - w(>x')]x' \). As a result, the sets of (e,m)PT-EB are identical (in strategic) probabilities, as are the sets of (e,m,n)PT-EB; furthermore, under this specification of preferences, we can perform all computations assuming \( r = 0 \) without loss of generality.

We begin with by finding the classical SPE: if P1 invades, then \( V_1(1, q) = 3 + q(26P_E - (20 + 3)) \), so he is indifferent between invasion and maintaining the status quo iff \( 3 + q(26P_E - 23) = 0 \iff q = \frac{3}{23 - 26P_E} \) (note that this is only possible for \( 0 \leq P_E < \frac{10}{13} \)), he invades if \( q < \frac{3}{23 - 26P_E} \), and maintains the status quo if \( q > \frac{3}{23 - 26P_E} \). Note that the cut-off for non-trivial P1 decision-making is \( \frac{20}{25} \approx 0.76923 \).

For P2, \( V_2(1, 1) = 2P_E + 5(1 - P_E) = 5 - 3P_E \), so an EUT P2 is indifferent between defending and not defending against an oncoming invasion iff \( 5 - 3P_E = 3 \iff P_E = \frac{2}{3} \), defends against invasions if \( P_E < \frac{2}{3} \), and does not defend if \( P_E > \frac{2}{3} \).

It is of primary interest to contrast this solution with our Entrant/Incumbent game’s (e,m)PT-EB and nPT-EB solutions. We will see, the fundamental change in the game’s solution is that P2’s PT preferences render previously incredible threats now credible, for a portion of the regime \( \frac{2}{3} < P_E < \frac{10}{13} \).
Suppose $P_E = 0.7$. Then for PT preferences, P2 has $V_2(1, 1) = 3.404$ and $V_2(1, 0) = 3$, so P2 will prefer to defend if attacked, if we consider only P2’s pure strategies. P1 might then anticipate $q = 1$ and choose not to attack, since $1 > \frac{K_1}{20 + K_1 - 26P_E} = \frac{K_1}{1.8 + K_1}$. However, this is not the full story, since P2 may prefer to play a completely mixed strategy to any of her pure strategies, as this is possible for players with PT preferences; furthermore, we must then consider whether her resulting best-reply correspondence has non-trivial convex hull. For concreteness, we set $K_1 = 3$, so P1’s cut-off value of $q$ is 0.625. At $P_E = 0.7$, P2’s payoff function $V_2(1, q)$ may be plotted:

![Figure 2.5: $V_2(1, q)$ is Unimodal.](image)

As indicated in the plot, P2’s argmax set is unimodal; applying a numerical root-finder indicates that the argmax set is the singleton containing only $q \approx 0.98$. Since this is strictly greater than P1’s threshold value of 0.625, P1 will choose not to attack—that is, P2’s threat to defend (with high probability but not certainty) if attacked is effective. By contrast, according to the SPE of this game, P2 would choose not to defend against attacks, since $P_E > \frac{2}{3}$; thus, incorporating probabilistic insensitivity and rank dependence into P2’s preferences has led P2 to successfully ward off invasion by P1. We emphasize that the only difference between the preferences of P2 in the SPE and in the nPT-EB are the incorporation of probabilistic insensitivity and rank dependence; if, for example, we
set \( w \) to be the identity rather than a cubic polynomial, then P2’s preference function would simply be EUT.

Thus, injecting a small amount of probabilistic insensitivity and departing from EUT rationality actually improves the Incumbent’s welfare in equilibrium; the Incumbent’s irrationality has enabled her to render credible a threat previously considered incredible, and to use this threat to ward off potential market invasion. We note that this outcome depends on the parameter values chosen; if \( P_E \) is chosen sufficiently small, for example, then P1 will never choose to invade, and so P2’s threat to defend or not is irrelevant. Furthermore, for \( P_E < 0.5 \) (but not arbitrarily small), probabilistic insensitivity will have the effect of making defend against invasion seem less appealing to P2 rather than more; thus for \( P_E \) in this lower range, we can arrange the reverse situation, in which an injection of probabilistic insensitivity into P2’s preferences worsens her welfare.

Before concluding, we should clarify what is meant by PT preferences being “welfare-improving” (and welfare-worsening) in this example. Although we do not provide a formal and generally applicable definition of player welfare, the contrast between the unique SPE \((p = 1, q = 0)\) and nPT-EB \((p = 0, q \approx 0.98)\) of this game is quite stark; the SPE solution induces a degenerate gamble in which the Incumbent receives an EUT payoff of 3, while the nPT-EB solution induces a degenerate gamble in which the Incumbent receives a PT payoff of 10. There is some ambiguity here, in that 3, 10 were understood as VM utilities when solving for the NE, while we interpreted them as PT values in solving for the nPT-EB, and so strictly speaking these are incomparable; we cannot simply say that the PT player receives a payoff of 10 rather than a payoff of 3, and that this is obviously welfare-improving because \( 3 < 10 \). However, both PT preferences and EUT preferences are strictly increasing in their underlying, basic outcome (e.g. money), and so the basic outcome associated with a PT payoff of 10 must be strictly greater than the basic outcome associated with a PT payoff of 3, and similarly for EUT payoffs. Thus, if we suppose the same set of basic outcomes underlies both the PT and EUT games—it should be clear that the PT and EUT preference functions can both be defined over the same set of underlying outcomes—then we can say that, in the nPT-EB, the Incumbent receives with certainty a basic outcome that is greater than the basic outcome she
receives with uncertainty in the SPE solution, and in this sense the Incumbent’s welfare is improved.

Our attribution of the Incumbent’s improved welfare to “probabilistic insensitivity” is also straightforward: if one removes probabilistic sensitivity from the PT version of our Incumbent above by setting \( w(p) = p \), the result is an EUT player, and so the nPT-EB solution becomes the SPE solution. In this sense, the only difference between the Incumbent’s preferences in the SPE and nPT-EB solutions is the introduction of distortions to the Incumbent’s probabilities.

2.8 Discussion

We have considered two general paradigms for incorporating PT preferences into GT solution concepts, depending on whether the analyst wishes to help one participant by making prescriptions, or adopt an entirely descriptive tack. If all players have PT preferences, we have a symmetric descriptive approach; if one player has EUT preferences and all other players have PT preferences, we have an asymmetric prescriptive/descriptive approach useful for decision analysis. In either of these paradigms, we must make modeling choices about how to incorporate PT preferences into individual players’ preference functions; we have considered two approaches to this problem: both approaches involved directly applying probabilistic weighting functions to the probabilities with which pure strategy profiles may occur in a game. The approaches differed in that one applied the best-reply conditions directly to players’ preference functions, while the latter approach followed Crawford (1990) in taking convex hulls of best-reply sets; it was found that the former procedure (giving PT-NE) generates a number of pathologies, including, in particular, a nonexistence-of-equilibrium example that improves on the empirical relevance of previously known non-existence results. The latter approach we used to generate PT-EB; taking Crawford’s notion of EB in conjunction with correctives for dynamic inconsistency, discontinuity, and absurdities that were discovered, we developed the notions of (e,m,n)PT-EB, a three-fold development inspired by Shalev’s (2000) three models of reference-point formation.

In addition to enjoying a guarantee of existence, PT-EB also enjoy a number of other useful properties. We have developed theoretical work documenting a large number of these properties,
including: existence in finite games, respect for iterated dominance, continuity of players’ value functions, equivalence of pure-strategy NE to pure-strategy PT-EB, and equivalence of mPT-EB and nPT-EB in simultaneous games. This work reproduces much of the foundational theory of NE for the case of PT-EB. We then also developed novel complexity-theoretic arguments for PT-EB, showing that pure-strategy ePT-EB, quasi-convex ePT-EB, and—for games satisfying bounds on pure-strategy counts and preferences’ Lipschitz constants—general $\varepsilon$-ePT-EB are no more difficult than finding an NE in an arbitrary game. We also developed an upper bound on the complexity of finding ePT-EB in general games.

While complexity-theoretic results are primarily reassuring for large-scale examples, it is also important to study small-scale, illustrative examples, and that these small-scale examples both have interesting properties and not be intractably difficult to understand. To allay concerns of this sort, we have introduced a polynomial family of weighting functions that is convenient for working with small-scale examples, and behaves particularly nicely under differentiation. Together with our basic theoretical results for PT-EB, this family of weighting functions allowed us to solve a variety of examples; of particular interest in these examples was the final one, a modified Entrant/Incumbent game which demonstrated that irrationality in the sense of PT, non-EUT preferences may either benefit or worsen player welfare. This suggests a number of interesting topic for further study: if PT players actually outperform their EUT counterparts, there may in suitable settings not only be a lack of pressure for PT players to behave rationally, but actual reason to expect PT preferences to propagate, whether by imitation or increased acquisition of resources. Furthermore, the possibility of imitation of PT preferences also suggests that models of “bluffing” might be useful for studying the appearance of PT preferences; it may be that (both EUT and non-EUT) players at times have incentives to feign non-rational preferences in order to secure greater payoffs in equilibrium, and this idea could be modeled using a suitable notion of Bayes-Nash equilibrium. In addition to the issue of bluffing, exceptions to and limitations of the results we have developed may also suggest avenues for future work. We call explicit attention to two of these possibilities: first, we found that, while ePT-EB largely possess “nice” theoretical properties, they fail to respect rationalizability, despite
that ePT-NE respect rationalizability. The importance of this loss of rationalizability deserves further exploration; specifically, one wonders whether interpreting mixed strategies as beliefs of players rather than genuine randomizations suggests that rationalizability is of lesser importance than in models with true randomization. Second, while we were able to identify reasonably large classes of games that are no more difficult than the full problem of finding a classical NE in a finite game, the reductions employed forced us to restrict either the types of equilibria sought, the kinds of preferences functions admitted, the number of strategies allowed per player, or some combination of these criteria. It would be useful to explore complexity reductions for non-classical preferences in specific, important examples of games (e.g. congestion games, graphical games), and/or to develop intractability results, in order to further map out the complexity relationship between non-classical and classical equilibria. We are, to our knowledge, the first authors to investigate this relationship, but it seems a particularly fruitful and important area for future work, if we are to understand the increased cost (or lack thereof) to GT researchers of working with non-classical preferences.

We have extended prior work in this by use of the full PT model (modulo ambiguity aversion) of preferences, and in doing so have developed a number of novel equilibrium concepts. We have documented various virtues and vices of these equilibria, offered correctives where problems appeared resolvable, and provided numerous examples illustrating both the structure and natural implications that arise from the use of these new notions of equilibrium. We hope that this work will help to stimulate further study of the use of hybrid EUT/PT preferences for asymmetric prescriptive/descriptive research and the full model of PT preferences in GT.

This concludes our theoretical investigation of PT-NE and PT-EB. In the next chapter, we place these two generalizations of NE in a systematic framework, use this framework to identify CVE as a third possible notion of equilibrium, and investigate CVE’s theoretical properties.
Chapter Three: Theoretical Analysis of CVE

3.1 Introduction and Motivation

In modeling strategic interactions between multiple agents, classical noncooperative game theory (GT) defines a finite $N$-player game by assuming the existence of pure-strategy sets $S_1, S_2, \ldots, S_N$ and utility functions $\bar{u}_i : S = \prod_{i=1}^N S_i \to \mathbb{R}$. Most often it is further assumed that players may generate jointly independent “random” strategies $\sigma \in \Delta = \prod_{i=1}^N D[S_i]$, where $D[A]$ is the set of all probability distributions over $A$ (Nash, 1951). Finally, it is supposed that players’ utility functions $\bar{u}_i$ may be extended to von Neumann-Morgenstern (VM) expected utility (EU) functions $u_i : \Delta \to \mathbb{R}$ which agree with $\bar{u}$ as appropriate for the point-mass distributions and otherwise depend linearly on the probability $\sigma(s)$ of $s = (s_1, s_2, \ldots, s_N) \in S$ occurring if all players play consistent with $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ (von Neumann & Morgenstern, 1944). The extension is usually given as $u_i(\sigma) = \sum_{s \in S} \sigma(s) \bar{u}_i(s)$. In a conventional abuse we denote by $\sigma_{-i}$ all components of $\sigma$ save the $i$th one, and we say $\sigma = (\sigma_i, \sigma_{-i})$ is an NE iff $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}) \forall i \in \{1, \ldots, N\}, \sigma'_i \in D[S_i] := \Delta_i$; this is the best-reply condition, indicating that $\sigma_i$ is a unilateral best reply of player $i$ to the NE strategies of all other players. GT revolves largely around the solution concept of NE, which in turn relies upon the notion of VM preferences and the associated EU preference functions; in this work, we propose an alternative to NE that is appropriate for use with non-EU preference functions.

Our primary motivation for considering alternatives to the classical framework is that a range of human behaviors consistently observed in single decision-maker settings cannot be accommodated by VM preferences and that there exist mathematically tractable generalizations of EU that naturally account for many such behaviors. We emphasize “human” because game theory is also often used to model non-human agents, such as companies or genotypes; we restrict attention to human agents.
It is not our purpose here to discuss in detail the empirical challenges to EU models; we assume these challenges are recognized as motivation for non-EU models of preference, and we refer the interested reader to Starmer (2000) for an excellent, accessible review relevant work.

We develop non-EU equilibrium concepts that are plausible, mathematically well-behaved, and tractable. Our theoretical results are general, in that no particular non-EU preference model is specified, but when dealing with examples, we apply our approach to the case of PT preferences (Kahneman & Tversky, 1979; Tversky & Kahneman, 1992; Wakker, 2010), which are a leading contender among EU alternatives. Although our theory is agnostic about players’ underlying preference function, several of our results stem from a particular interest in PT preferences; notably, we follow Shalev (2000) and build on the work of Chapter 2 to define three distinct versions of equilibrium, each corresponding to a different behavior of players’ “reference points.” In general, we refer to all variants of the equilibrium concept advanced as “combinatorially verifiable equilibria” (CVE).

Our central finding is that CVE retain the most desirable theoretical properties of classical NE. We separate theoretical desiderata into three areas: fundamental theory (e.g. existence), psychological plausibility, and computational complexity. The remainder of the paper investigates CVE in each of these areas, appealing for the study of CVE on the basis of this evidence. In section 2, we follow Shalev (2000) and define three variants of CVE—exogenous (eCVE), myopic (mCVE), and non-myopic (nCVE). We discuss the meaning of these three variants, and show that they each enjoy existence guarantees in all finite games, among other properties. In section 3, we discuss the psychological meaning of CVE; this discussion focuses whether random strategies are best understood as modeling subjective uncertainty, or essentially objective randomness, as associated with “coin-flipping.” In this section we also argue that CVE is closer to existing experimental evidence than obvious alternatives, and we show that CVE preserves the epistemic properties established for NE by Aumann and Brandenberger (1995). In section 4, we study the computational complexity of eCVE, and show that eCVE have identical complexity class characterizations to NE, when considered over all finite games, regardless of whether pure equilibria, $\varepsilon$-equilibria, or
exact equilibria are sought. Section 5 applies (e,m,n)CVE to a variety of small-scale games using PT preferences, illustrating typical solution methods for CVE. Section 6 develops two elaborate examples, demonstrating mechanisms by which “irrational” preferences can increase player welfare and cause rational players to imitate irrational players. Section 7 closes with a brief consideration of extensions for the current work.

3.2 Appeal 1: Fundamental Theory

Our primary purpose in this first of our three appeals is to show that (e,m,n)CVE behave well with respect to three important properties of classical NE: existence (in finite games), respect for iterated dominance, and respect for rationalizability. In addition, we also show that a number of results developed by Shalev (2000) for a special case of PT-NE hold for CVE; these results identify connections between (e,m,n)CVE.

To motivate the definition of CVE, we consider four characterizations of classical NE for finite games. Letting $co[A]$ denote the convex hull of the set $A$, $supp[\sigma_i]$ be the set of pure-strategies in the support of $\sigma_i$, and identifying mixed strategies with their vector representations, we have:

**Observation 3.2.1.** For finite games, we have the following equivalences:

\[
\begin{align*}
(A) \quad \sigma_i &\in \arg\max_{\sigma'_i \in \Delta_i} U_i(\sigma'_i, \sigma_{-i}; r_i) \quad \forall i \\
\leftrightarrow \\
(B) \quad \sigma_i &\in co[\arg\max_{\sigma'_i \in \Delta_i} U_i(\sigma'_i, \sigma_{-i}; r_i)] \quad \forall i \\
\leftrightarrow \\
(C) \quad s_i &\in \arg\max_{s'_i \in S_i} U_i(s'_i, \sigma_{-i}; r_i) \quad \forall s_i \in supp[\sigma_i], \forall i
\end{align*}
\]

PT preferences have three primary components: reference dependence, non-linear probability weighting, and rank dependence; of these, Shalev modeled only reference dependence. In addition, his value function was piecewise-linear, and so disallowed much of the curvature commonly associated with risk preferences for classical EU preferences; in this sense, Shalev developed his model to focus as exclusively on reference dependence as is possible.
\[ (D) s_i \in \text{co} \{ \max_{s'_i \in S_i} U_i(s_i', \sigma_{-i}; r_i) \} \forall s_i \in \text{supp}[\sigma], \forall i \]

While each of these characterizations could serve equally well as the definition for classical NE, when we admit non-EU preferences into consideration for finite games, two of these if and only if’s breakdown; we now have the relationships:

**Observation 3.2.2.** For finite games, where \( V_i : \prod_i \Delta_i \to \mathbb{R} \) is a general non-EU preference functional for player \( i \), we have:

(A) \((PT - NE)\) \quad \sigma_i \in \max_{\sigma'_i \in \Delta_i} V_i(\sigma'_i; \sigma_{-i}; r_i) \forall i

\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n}\n
We note that (A) and (D) are also non-equivalent; we excluded this from the statement of the observation (but not from the proof) to simplify presentation. Furthermore, the equivalence of (C), (D) degenerates as well, if one considers infinite games, because in infinite games pure strategies may no longer correspond to extreme points of the convex hulls of their “vector” (sequence or function) representations. We do not consider infinite games, but this distinction between (C) and (D) may pose an interesting topic for future work.

In Observation 1.2, we have supplied the additional labels PT-NE, PT-EB, and CVE to the four definitions given; these labels reflect our interest in PT preferences, and correspond to “prospect-
theoretic Nash equilibrium," "prospect-theoretic equilibrium-in-beliefs," and, of course, CVE. In Chapter 2, we investigated PT-NE and PT-EB; PT-NE is the simplest of these equilibria, in that it straightforwardly generalizes the best-reply definition of classical NE. PT-EB builds primarily on the work of Crawford (1990), convexifying the relevant argmax set$^2$; mathematically, this is useful because it corrects the pathology of non-existence caused by the use of certain non-quasiconvex $V_i$ for PT-NE. We further note that each of PT-NE, PT-EB, and CVE appear to emphasize different sources of uncertainty; PT-NE appears to assume that players face uncertainty about their own strategies as well as those of other players, in that $\sigma_i$ appears as an argument to the function $V_i$. By contrast, CVE only models each player’s uncertainty about what other players might do; $\sigma_{-i}$ appears as an argument to $V_i$, but $\sigma_i$ is conspicuously absent. PT-EB appears to incorporate both kinds of uncertainty. As uncertainty is central to both GT and PT, we expand on this discussion in the sequel.

We now define (m,n)CVE. Each of (m,n)CVE are concerned with an additional parameter in each $V_i$; we now write $V_i(\sigma, r_i)$ to reflect that $V_i$ depends not only on the mixed strategies of all players, but also on a reference point associated with player $i$. In PT, this reference point corresponds to a subjectively neutral state; outcomes below this point are regarded as gains and those below it as losses, with the slope of $V_i$ for losses roughly twice that for gains. One common interpretation of $r_i$ (originating with Koszegi & Rabin, 2006, 2007) treats $r_i$ as player $i$’s anticipated outcome. Consistent with this interpretation, Shalev (2000) proposed that, in equilibrium, $r_i$ should match the actual (on-the-average) experienced outcome of player $i$; otherwise, one would expect player $i$’s $r_i$ to change, since expected and actual outcomes differ. This motivates the following definition of mCVE:

**Definition 3.1.** Given a game $G$ with non-EU preference functionals $V_i(\sigma, r_i) : \Delta \times \mathbb{R} \to \mathbb{R}$ for

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$^2$We note that Crawford actually used the distributional operator $D[\cdot]$ rather than the convex hull operator $\text{co}[\cdot]$; distinguishing between these two possibilities could give rise to at least two further definitions for non-EU equilibria, but we ignore this possibility, because the $D[\cdot]$ and $\text{co}[\cdot]$ operators behave equivalently in finite games, and even when infinite games are considered, $D[\cdot]$ behaves identically with definition (C) in Observation 1.2. We prefer to use $\text{co}[\cdot]$ primarily because it is more immediately tied to the mathematical purpose of Crawford’s definition, and, secondarily, because it allows a distinction between type-(C) and type-(D) equilibria in infinite games.
each $i$, we say $\sigma$ is an mCVE if $s_i \in \arg\max_{s_i' \in S_i} V_i(s_i', \sigma_{-i}; r_i) \forall s_i \in \text{supp}[\sigma_i], \forall i$, where $r_i = EV_i[\sigma]$, the expected value (in the underlying monetary payoffs of $G$) to player $i$.

In defining nCVE, we take our inspiration from Shalev (2000), but depart from the detail of his definition. Non-myopia is intended as a solution concept that accounts for dynamic structure in a game—that is, for games in which players may have multiple, non-empty information sets. In this sense non-myopic equilibria are akin to classical theory’s subgame perfect equilibrium (SPE). Just as some $\sigma$ is defined to be SPE for a game $G$ iff $\sigma$ is an NE in every subgame of $G$, Shalev imposed a best-reply condition (though not expressed explicitly in the language of best replies) on $\sigma$ in at every information set; however, Shalev did not simply require that a non-myopic equilibrium $\sigma$ be a myopic equilibrium in every subgame. Instead, he allowed for an extra degree of flexibility: in evaluating non-myopic players’ best replies, Shalev allowed for players to update their reference points when considering hypothetical strategic deviations as well as changing their reference points when information sets change. These changes have clear motivations: a change in one’s own strategy and a change in information set will both alter player expectations of payoff, in general; thus, depending on the nature of changes in the reference point, it may be that the reference point should depend on these properties of equilibrium play. In Chapter 2, in studying PT-EB, we found two problems with this approach: first, directly generalizing SPE’s best-reply (in every subgame) condition when dealing with non-EU preferences may cause dynamic inconsistencies and so pathologies of non-existence in finite games (Shalev also noted this problem). Second, if preferences update in response to hypothetical deviations from equilibrium play, and do so according to $r_i = EV_i[\sigma]$, and if the PT reference point is assumed to be subjective neutral (i.e. $V_i(r_i, r_i) = 0$, where the first $r_i$ is a deterministic prospect yielding $r_i$ with certainty), as is standard and necessary to ensure continuity of PT preferences, then absurdities result. Specifically, these assumptions yield player indifference between all deterministic prospects. We address each of these problems in our definition of nCVE: first, we do not allow players to condition their reference points on hypothetical deviations; the only dependence admitted for $r_i$ is on the equilibrium mixed strategy profile. Second, we continue in the spirit of considering alternative classical characterizations as the basis for CVE;
since classical SPE could equivalently be defined as any $\sigma$ in a game $G$ which results from some application of the backwards-induction algorithm to $G$, we define $\sigma$ to be an nCVE if it could result from some application of the backwards induction algorithm to $G$, with the requirement that $G$ yield in the game induced by each iteration of the backwards induction algorithm.

**Definition 3.2.** For a game $G$ with non-EU preference functionals $V_i(\sigma; r_i)$ for each $i$, we say $\sigma \in \Delta$ is an nCVE if $\sigma$ could result from an application of the backwards induction algorithm to $G$, i.e., if $\sigma$ induces an mCVE in the “lowest” subgames of the game tree, and if at iteration $i$ of the algorithm, holding strategic probabilities fixed in all subtrees already considered (taking these now as “plays by nature”), $\sigma$ induces an mCVE in each of the subgames under consideration.

This definition has several rather nice properties: the relationship between nCVE and mCVE is analogous to that between SPE and NE (in the sense of defining SPE by backwards induction, not as an NE in all subgames), existence results for nCVE in finite games now follow by an easy induction from existence results for mCVE, and the definition of nCVE may be substantially motivated by an appeal to classical equivalences, in the same manner as our original definition of eCVE. Furthermore, both characterization (C) in Observation 1.1 and the characterization of SPE in terms of backwards-induction are combinatorial or algorithmic in character; for the backwards-induction definition of nCVE this is obvious, but the combinatorial character of characterization (C) is also clear (and has been noted by other authors; see e.g. Papadimitriou, 2007 for a forceful argument to this effect). Characterization (C) reduces the search among continuum-many mixed-strategy profiles to a search among all possible supports, and exactly this sort of combinatorial search-among-supports has been exploited in a number of algorithms for finding NE (e.g. the Lemke-Howson algorithm, Lemke & Howson, 1964). Of course, some structure must be lost in defining nCVE in terms of backwards-induction rather than as an mCVE in all subgames; in particular, we note that nCVE are not a special case of either mCVE or eCVE. Instead these sets are incomparable; there are games with nCVE that are neither mCVE nor eCVE, and in which all mCVE and eCVE are not nCVE. In addition, players in nCVE may have a negative value for information, in some conditions; this is for reasons identical to those lain out by Machina (1989) in considering the causes of and
correctives for dynamic inconsistency in non-EU decision makers.

We make one final note before developing the theory in earnest: although we use the consistency condition \( r_i = EV_i(\sigma) \) to define mCVE, Shalev (2000) used \( r_i = V_i(\sigma, r_i) \). This yields an implicit equation which is mathematically more interesting than the explicit condition \( r_i = EV_i(\sigma) \), but, as we have argued in Chapter 2, is out-of-synch with the experimental literature, in which \( r_i \) has units of some observable outcome measure, e.g. dollars. Nevertheless, m,n)CVE are sufficiently well-behaved that we can formulate our theoretical results to accommodate Shalev-like conditions, and, in fact, still more general conditions of the form \( r_i = f_i(\sigma, r_i) \) for any continuous function \( f_i \).

We now begin our theoretical developments. First, we highlight the singular property that distinguishes CVE from PT-NE and PT-EB, and from which follows its inheritance of most of classical NE’s properties: player preferences behave as-if they are linear in each player’s own strategic probabilities. We state this formally:

**Theorem 3.3.** In any finite game \( G \), the condition \( \sigma_i \in co[\arg\max_{s \in S_i} V_i(s, \sigma_{-i}; r_i)] \forall i \) holds iff \( \sigma_i \in \arg\max_{\sigma'_i \in \Delta_i} V_i(\sigma'_i, \sigma_{-i}; r_i) \), where \( V_i(\sigma'_i, \sigma_{-i}; r_i) = \sum_{s' \in S_i} \sigma'_i(s') V_i(s', \sigma_{-i}; r_i) \).

We note that the proof of Theorem 1 is essentially identical to (though more general than) standard proofs of the (A)-(C) equivalence in Observation 1.1 (as in Theorem 2.1 of Papadimitriou, 2007). We have simply reversed the usual roles of linearity and definition 1.1.(C). Linearity is derived directly from the VM axioms in classical theory, and the (A)-(C) equivalence is classically proven as a consequence of VM linearity; we have instead taken (C) as an axiom, and shown that linearity follows as a consequence. We note that Theorem 1 does not imply that eCVE are identical to their classical counterparts; although Theorem 1 shows that the dependence of players’ payoffs on their own strategic probabilities can be assumed to hold linearly for CVP-NE, the dependence of each player’s payoffs on her opponents strategic probabilities will in general be nonlinear.

Linearity of player payoffs in players’ own strategic probabilities immediately restores a range of important classical properties. We easily attain an existence guarantee for all finite games, and do so using a standard application of Kakutani’s Fixed Point Theorem:
THEOREM 3.4. All finite games with continuous $V_i(\sigma, r_i)$ for each $i$ possess an (e,m,n)CVP-NE (where $r_i$ is arbitrary for each player, when considering eCVE). This also holds if, given continuous functions $f_i$ with non-empty, compact, convex domain and range for each $i$, we replace the condition $r_i = EV_i[\sigma]$ in the definition of mCVE by the condition $r_i = f_i(\sigma, r_i)$.

(e,m,n)CVE also respect iterated dominance, in the sense that in any (e,m,n)CVE, all players place 0 probability on any pure strategy that can be eliminated by iterated strict dominance. This is reassuring, as iterated dominance is often considered a baseline for any reasonable equilibrium concept. We have the following theorem:

THEOREM 3.5. Given a finite game $G$, if pure strategy $s_i$ of player $i$ is deleted by iterated strict dominance, then in any (e,m)CVE $\sigma$, $s_i \not\in supp[\sigma]$.

Although we have noted that iterated dominance is often upheld as a baseline requirement of equilibrium concepts, it is has also come under considerable descriptive criticism from the behavioral game theoretic community. In empirical practice, players appear to exhibit a limit on the number of iterations they consider when applying dominance; this idea has motivated the development of alternative equilibrium concepts by some authors, which relax the “infinite” inductive reasoning employed by iterated dominance. In this sense, the full power of Theorem 2.3 may not be regarded as descriptively crucial; however, Theorem 2.3 could be understood as a step in the direction of the descriptive ideal, just as the initial classical work on iterated dominance served as a precursor to the developments of behavioral models with limited numbers of iterated deletions. We discuss the possibility of merging (e,m,n)CVE with behavioral models featuring limited iterative deletions in the closing section of this paper.

THEOREM 3.6. Let $G$ be a finite game, and fix $r_i$. If $\sigma_i$ is non-rationalizable with respect to $V_i$, then it is not played by $i$ in any eCVE. This result also holds for any mCVE $\sigma$, if the non-rationalizability is with respect to $V_i$ evaluated at $r_i = EV_i[\sigma]$.

We note that Theorem 2.4 does not hold if rationalizability is required with respect to $V_i$ rather than $V_i$. By focusing on pure-strategy best replies to mixed strategies, CVE effectively remove
information about $V_i$ that governs its behavior as $\sigma_i$ varies; it is as if $V_i$ is not player $i$'s preference function at all unless we fix $\sigma_i$ to a pure strategy, and so it is not clear that rationalizability with respect to $V_i$ is a meaningful requirement.

We close by producing two results in imitation of similar work for a special case of PT preferences by Shalev (2000). We first identify conditions (satisfied by PT preferences) under which (e,m)CVE and NE are identical for pure-strategy equilibria. Secondly we note that (m,n)CVE are identical for simultaneous games. Our first result is:

**Theorem 3.7.** Let $G$ be a finite game without nature. Suppose $V_i(\sigma, r_i)$ is strictly decreasing in $r_i$, and that each $V_i$ and $u_i$ is strictly increasing in the underlying monetary outcome of the game. Then any pure-strategy profile $s \in S$ is an eCVE iff it is an mCVE iff it is an NE.

This concludes our “mathematical” advocacy for CVE. In the next section, we argue that CVE not only behaves well in terms of the usual mathematical desiderata of GT, but in fact accords more closely with both psychological intuition and experimental evidence than does its natural “best reply”-based alternatives (exemplified by PT-NE and PT-EB).

### 3.3 Appeal 2: Psychologizing, Experiment, and Epistemics

In this section we consider the psychology behind CVE, its fidelity to experimental evidence, and we show that the seminal epistemic results of Aumann and Brandenberger (1995) for NE can be reproduced for eCVE. We first consider the substantive interpretation of strategic randomness in games, identifying each of PT-NE, PT-EB, and CVE as allowing for distinct forms of randomness, whether subjective or objective or both. We argue that subjective uncertainty is the form with greatest relevance in most modeling situations, and that this suggests CVE as a natural default modeling tool. Our second argument considers the relevance of the experimental psychology literature to each of PT-NE, PT-EB, and CVE; we argue that, in contrast to CVE, each of the former equilibrium concepts artificially introduce continuum-many prospects into choice situations that involve only finitely many prospects. As the bulk of the experimental evidence considers finite-choice situations,
we further argue that this ties CVE more closely to the experimental literature than either PT-NE or PT-EB. We also relate this argument to the well-known problem of ex post commitment under risk-loving preferences.

In classical GT, even very simple finite games lack equilibria in pure strategies; this disadvantage of pure-strategy equilibria motivates the consideration of models in which players may choose random distributions over pure strategies. This procedure is natural mathematically and grants us immediately a notion of mixed-strategy equilibrium (MSE), which, as Nash (1951) famously demonstrated, exists in all finite games. Further evidence for the appeal of MSE comes from such games as rock-paper-scissors, where it hardly seems natural to call anything other than uniform randomization by both players an “equilibrium.” Thus MSE fit in a natural fashion into GT, and solve fundamental problems of nonexistence in simple games; however, as is often the case, just as MSE addressed important questions, it also poses new ones. Perhaps the most significant of these questions is: what is the substantive meaning of strategic randomness?

GT provides us with two ready answers to this question. The first is: we literally mean that players commit ex ante to the use of particular random distributions for choosing their pure strategies, and then play of the game proceeds after single draws are made from each player’s distribution. The second interpretation is more subtle but perhaps even more closely related to the pioneering work in decision theory of VM, Savage (1954), and de Finetti: random distributions may not be models of “truly random” processes at all, but rather just representations of each player’s uncertainty about the pure strategies that will be used by her opponents.

Both interpretations of randomness carry with them difficult questions. If randomness is objective, then how is it that player actually achieving random draws from the distribution she uses? Doubtless there are situations where players approximate objective randomness by the explicit use of computerized pseudo-random number generators or by consulting tables of pseudo-random numbers, but these situations must surely be exceptional; we do not in practice observe people “flipping coins” to make everyday strategic decisions. Of course, it may be that human agents, at least, do not need explicit tools to generate random numbers; maybe we have evolved the ability to generate
random sequences naturally. However, this hypothesis appears to be rejected by experimental data (Towes & Neil, 1998; Wagenaar, 1972). We conclude that objective strategic randomness is at best an unusual phenomenon in the world; presumably it is most plausible as a description of strategic interaction when agents are highly motivated to be unpredictable, have ample time to consult and deploy a randomizing device, and have suitable technology for producing pseudo-random numbers.

Given our conclusion that objective GT randomness is a rare phenomenon, we feel it is natural to focus our analytic and modeling efforts on subjective GT randomness. In classical GT, these two models are largely in predictive agreement: whether uncertainty is viewed as subjective or objective is irrelevant when solving for the NE of a Matching Pennies or Prisoner’s Dilemma game. The objective-subjective distinction may motivate subtle interpretive differences in evaluating empirical data, but by-and-large it will have few empirical consequences, and so decisively settling on one or the other interpretation is not a first-order concern. If this were the case for non-EU preferences, we could shelve this issue and not worry over much about the meaning of randomness in games; for non-EU preferences, however, we believe that subjective randomness and objective randomness imply the use of very different equilibrium models—of CVE and PT-NE, respectively, with PT-EB serving as a third model that incorporates both subjective and objective randomness.

To illustrate this idea, it is helpful to consider the case of PT preferences, as randomness is critical to their use. As with EU preferences, PT preferences have a representing function, typically denoted by $V_i$ for individual $i$. $V_i$ is defined over prospects, or probabilistic gambles; the original definition (Tversky & Kahneman, 1992) treated only gambles with finitely many outcomes, and we focus on that definition. $V_i$ is defined by

$$V_i(P, r_i) = \sum_{x_k < r_i} M^-(x_k)v_i(x_k, r_i) + \sum_{x_k > r_i} M^+(x_k)v_i(x_k, r_i)$$

(3.1)

where

$$M^-(x_k) = w_i^-(\Phi(x_k)) - w_i^-(\Phi(x_{k-1}))$$

$$M^+(x_k) = w_i^+(\Phi'(x_k)) - w_i^+(\Phi'(x_{k+1}))$$
This definition requires considerable unpacking: $r_i$ is player $i$’s reference point, $w^-$ her probability weighting function for losses, $w^+$ her weighting function for gains, and $v_i$ her value function for deterministic (probabilistically degenerate or certain) prospects. Possible outcomes $x_k$ under prospect $P$ are placed in ascending order by index $k$; this is important because the unusual form of $V_i$ treats gains and losses differently. As mentioned above, $v_i$ has greater slope for losses than gains; in addition, $M^-(x_k)$ uses the cumulative probability distribution function $\Phi$ while $M^+(x_k)$ uses the decumulative probability distribution function; this perhaps esoteric-seeming algebraic form is designed to ensure that a player maximizing $V_i$ will satisfy first-order stochastic dominance.

The most interesting elements of PT for our purposes are the non-linear weighting functions, $w^-_i, w^+_i$. These distort the probabilities faced by players; we can attain an EU-like treatment of probabilities if these are both set to the identity function, as the marginals then reduce to underlying probabilities. However, PT preferences instead most commonly assume inverse-S shaped weighting functions, which overweight low probabilities and underweight large ones.

To apply PT preferences to GT, we need to know what probabilities to place within $\Phi(x_k), \Phi'(x_k)$. We can regard the mixed strategy profile $\sigma \in \Delta$ as inducing a probability distribution (“prospect”) over the outcome space of a game, and the partial mixed-strategy profile $\sigma_{-i}$ as inducing a set of prospects for player $i$. However, $\sigma_{-i}$ also induces a smaller, finite set of prospects, one associated with each $s_i \in S_i$. Now, we consider the ramifications of uncertainty: if it is objective, then all players are (e.g. by using a computer to pseudo-randomize, or flipping literal coins) choosing distributions from their $\Delta_i$ with which to play their pure strategies, and, once this choice is made (and supposing they can commit to play any pure-strategy chosen), then ex ante they do not know the outcome of their draw or the specific pure strategy that they will play any more than do their opponents. So, they face an aggregate uncertainty that encompasses both the uncertainty generated by other players’ explicit choices to randomize as well as that generated by their own distribution over pure strategies; it is therefore natural within a probability-weighting framework to apply probability-weighting functions to the aggregate probabilities induced over outcomes by the entirety of a mixed-strategy profile $\sigma$. Of course, this is exactly what we do in the PT-NE framework: we suppose players’
choose best replies from among their continuum-large set of mixed strategies, where each mixed strategy induces its own prospect to be considered as a separate, possible choice to be made by the player, its value compared against all the remaining continuum-many mixed strategies.

If uncertainty is subjective, representing uncertainty about what other players will do, and if all players have reached epistemic agreement about the appropriate probabilistic description of other players’ likely choices of pure strategy, a very different conclusion emerges: players are no longer choosing between continuum-many prospects, each induced by a distinct, explicit mixed strategy. Instead, each player has only finitely many (in a finite game) pure strategies from which to choose, one for each \( s \in S \), and associated with each of these is probabilistically describable uncertainty induced by a partial mixed-strategy profile \( \sigma_{-i} \). Thus, if uncertainty is merely a device for modeling subjective uncertainty, then we are naturally led to consider a best-reply condition that holds only over each player’s pure strategy set, rather than defining equilibrium in terms of best replies over players’ mixed-strategy sets. This reasoning naturally leads to the definition of CVE\(^3\).

PT-EB present still a third option: there is objective randomness, and optimization by players is done over their set of mixed strategies, indicating that they face uncertainty about their own choice of pure strategy, as well as the pure strategies to be played by other players. The defining condition \( \sigma_i \in co[\arg\max_{\sigma_i' \in \Delta} V_i(\sigma_i', \sigma_{-i}; r_i)] \) for PT-EB takes the convex hull of the resulting argmax sets; this convex hull may be interpreted as a representation by all players other than player \( i \) about the specific mixed strategy that \( i \) will play from within the set \( \arg\max_{\sigma_i' \in \Delta} V_i(\sigma_i', \sigma_{-i}; r_i) \). Comparing this reasoning to that given for CVE, the \( co[\cdot] \) operator appears to be a representation of subjective uncertainty, while the appearance or non-appearance of objective uncertainty determines whether optimization is done over \( S_i \) or \( \Delta_i \).

Thus, in finite games, subjective uncertainty leads naturally to a definition of equilibrium in which best replies are taken over finite sets, while objective uncertainty leads to definitions in which best replies are taken over sets with the cardinality of the continuum. As we have argued that

\(^3\)But we note that this was emphatically not how we arrived at this definition; as explained in the introduction, it was identified by a systematic search of characterizations of NE.

\(^4\)This interpretation of the \( co[\cdot] \) may be clearer if we recall that, in finite games, it is equivalent to the \( D[\cdot] \) operator, which is more naturally understood as a representation of uncertainty.
subjective uncertainty is the more important of these two modeling choices in practice, we think this strongly suggests the use of the finite best-reply condition, as formalized in CVE. However, the difference in cardinality remarked upon here is important in its own right: assuming uncertainty is objective, we argued that players consider choosing from a continuum-large set of probability distributions, and compare all of these in turn. This is of course not a plausible description of the actual calculations people perform in reality, but models are generally “as if” approximations of the world and not intended as literal descriptions of the mechanics of human decision-making, so this objection is not decisive. Nevertheless, relating the cardinality distinction back to the experimental literature is informative.

The prototypical experiment in the behavioral decision-making, behavioral economics, and heuristics and biases literature, from which PT preferences were born, consists in presenting human participants with a sequence of choices between finitely many prospects. Allais (1953), in his famous paradox, poses just two choices each, with each choice between only two prospects; sequences of choices between finitely many prospects also formed the basis for the researches of Tversky and Kahneman (1992). We emphasize this point: the experimental literature relies primarily on observations of choices by a single decision-maker between finitely many prospects and uses these observations to make inferences about participants’ binary preference relations between the presented prospects. Furthermore, at no point in the process of observing player choices between prospects $A$ and $B$ is it supposed that the player, in actuality, generates a continuum of random distributions between these two prospects, and chooses the most-preferred of these distributions. Of course, there is no apparent incentive for this kind of randomization by players in single decision-maker settings, and this would be a gratuitous increase in complexity of the resulting models with little apparent motivation. However, equilibrium concepts developed in the vein of PT-NE and PT-EB model precisely this sort of process: if we take PT-NE or PT-EB seriously as GT solution concepts, then, when applied to single-player finite games, we find exactly that players—rather than merely choosing between the most-preferred of finitely many available prospects—instead generate a continuum of distributions over the available options, and choose the most preferred of
these distributions. This property of type (A) and (B) non-EU equilibria often produces puzzling quirks, such as players choosing to play fully randomized strategies in single-player games, or when playing a multiplayer game in which every information set is a singleton. In short, players choose to randomize for the love of randomizing, and not because of any desire to remain unpredictable to competitors, as is the standard GT motivation for randomization.

We reiterate the distinction between the interpretations of choice implied by PT-NE and PT-EB and the standard single-decision maker experiment: PT-NE and PT-EB suggest that what should be inferred in observing a specific choice in an experiment between finitely many prospects is not that there is a binary preference for prospect A over prospect B, but rather that the participant stopped, generated a continuum-large set of possible distributions from which she could make draws in order to select A or B, chose the most desirable of those possible distributions, drew from it, and then chose either A or B. The seeming implausibility of this procedure aside, it also suggests that for experimental work to bear on the validity of PT-NE and PT-EB at all, we would need to estimate the preferred distributions rather than the preferred, directly observable prospects, again undermining the connection between experimental evidence and PT-NE or PT-EB.

As we have argued, absent a GT-like motivation for the strategic desirability of being unpredictable, an interpretation like this—in which we suppose participants artificially construct and consider their preferences over a non-apparent continuum of choices rather than just the finitely many ones with which they were immediately presented—sounds contrived and strange. At the very least it seems to violate preferences for parsimony in model-building: where only a finite number of choices are actually observed, we have introduced an artificial infinity of mixed strategies from which we suppose the participants to have chosen. These mixed strategies are artificially introduced coin flips: we suppose players are capable of generating their own random strategies, and so, in fact, the choice sets they face are never as simple as those with which they are presented, but in fact consist of all possible distributions over their choice sets. Thus we have introduced artificial infinities and coin flips where neither is needed to understand our observations.

We do not take away from the above discussion that humans are incapable of choosing from
among continuum-many choices, of course, nor even that PT-NE or PT-EB are poor solution concepts; we just argue that the standard kinds of experiments carried out in the behavioral decision-making literature do not help to inform us about how the kinds of choices modeled in PT-NE and PT-EB would be made in practice. In short, the standard experimental evidence bears directly on the finite best-reply condition that we have formalized in CVE, but what that same evidence may have to say about an infinite best-reply condition, as in Observation 1.2.(A) (PT-NE) or 1.2.(B) (PT-EB) models, is unclear.

We close this section by providing an epistemic analysis of CVE—that is, on identifying conditions on the knowledge and beliefs of the players in a game that are sufficient to imply that their beliefs form an equilibrium. Specifically, we show that the two central results of Aumann and Brandenberger (1995) can be reproduced for CVE. The first of these results shows that, for 2-player games, mutual knowledge (of payoffs and one another’s beliefs) is sufficient to imply that beliefs form a CVE. The second of these results applies to the general case of \( n \geq 3 \) players, and shows that mutual knowledge of game structure and rationality together with common knowledge of beliefs is again a sufficient condition. We give both results together:

**Theorem 3.8.** Consider a 2-player game in which both players have possibly non-EU preference functionals, \( V_1, V_2 \), reference points \( r_1, r_2 \), and the two players have conjectures \( \sigma_1, \sigma_2 \) about their opponent’s probabilities of play. If \( V_1, V_2, r_1, r_2 \) are mutual knowledge, it is mutual knowledge that each player will optimize with respect over pure strategies with respect to \( V_i \) (holding her opponent’s strategy fixed), and the \( \sigma_1, \sigma_2 \) are mutually known, then \((\sigma_1, \sigma_2)\) is an eCVE.

We emphasize that the proof of Theorem 3.1 is essentially that of Aumann and Brandenberger; our contribution is merely to note that these arguments can be employed with minimal change in the eCVE framework. Theorem 3.1 establishes that in 2-player games, as with NE, mutual knowledge of game structure (including preference functions), (CVE-)rationality, and players’ conjectures about probabilities of play is sufficient to imply that those conjectures form an eCVE. Thus the epistemic requirements for eCVE in 2-player games are no more burdensome than those for NE. Similarly, Aumann and Brandenberger’s \( N \)-player argument can be applied readily to eCVE:

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THEOREM 3.9. Consider a finite $N$-player game, and each player has a conjecture about the mixed strategy of every other player. If the players have a common prior, their preference functions $V_i$ and reference points $r_i$ are mutually known, it is mutually known that each player $i$ will optimize (over pure strategies) with respect to $V_i$ (holding her opponents’ strategies fixed), and the players’ conjectures are commonly known. Then for each player $j$, the remaining players agree on their conjecture about $j$’s mixed strategy, and these conjectures constitute an eCVE.

To summarize this appeal: experimental evidence appears to speak most clearly to non-EU models of GT equilibrium fashioned in the mold of CVE, and has very little to say about the artificially introduced infinite sets that emerge in PT-NE or PT-EB models. Although there remain important psychological and experimental questions about CVE, consideration of the differences between PT-NE, PT-EB, and CVE seems to lead us to natural avenues for addressing some of the most pressing questions about CVE. Furthermore, the seminal epistemic results of Aumann and Brandenberger (1995), which allow us to formally probe the psychological meaning of subjective randomness in equilibrium, can be reproduced for CVE. This connection between CVE and the work of Aumann and Brandenberger further reinforces our belief in the close connection between CVE and subjective randomness; we note that the proofs rely critically on Theorem 2.1, which is not enjoyed by either of PT-NE or PT-EB.

In the next section, we consider one of the costs of using CVE relative to NE: its greater complexity. We show that, in a rigorous complexity-theoretic sense, key computational problems associated with finding CVE are no more difficult than for finding NE. In this fashion we hope to partially address concerns that the additional complexity of CVE may be too burdensome to justify their use.

3.4 Appeal 3: Algorithms and Complexity

In this section we address the computational complexity of eCVE. Specifically, we show that A) finding pure-strategy eCVE in network games is PLS-complete (under preferential assumptions identical to those of Theorem 2.5), B) finding $\epsilon$-eCVE in general finite games is PPAD-complete,
and C) approximating exact eCVE in general finite games is FIXP-complete. We emphasize that these completeness results are precisely identical to those for pure-strategy NE in network games, for $\varepsilon$-NE in general finite games, and for approximating exact NE in general finite games. Thus eCVE appear to be no more difficult to compute than NE, at least in the asymptotic, worst-case sense of computational complexity theory.

Two main approaches have been used to apply the complexity theory of theoretical computer science to game theory; these two approaches differ primarily in how they map the notion of an NE of a family of games into a problem to be solved. The first and standard approach to classifying NE complexity considers a collection of games $C$ (such as the family of all finite games, for example), defines some appropriate measure of size $s : C \rightarrow \mathbb{N}$ on these games, and asks: does there exist an algorithm $A$ and a fixed polynomial $p$ such that, for any $G \in C$, $T(s(G)) \leq p(s(G))$ and with a guarantee that the algorithm outputs an NE of $G$ on terminating? The key observation here is that the game, including its utility functions, are regarded as input for $A$; as such, the utility functions are treated as fixed objects describing some instance of the problem, and any satisfactory algorithm must be prepared to contend with an essentially arbitrary set of utility functions within the class $C$.

The second, "revealed preference" approach to describing the complexity of finding an NE differs from the first approach in that, rather than feed $A$ a full description of some game $G \in C$, it gives $A$ some limited control over what kinds of utility functions must be addressed. In particular, in this approach it is assumed that utility functions are just "as if" tools employed by a theoretician to explain observed data, as opposed to their being concrete, observable, and objective features of agents out in the world. For simplicity, and because it is better developed in the classical case, we focus on the standard algorithmic approach to classifying equilibrium complexity.

At this point we should mention that, for expository simplicity, we have somewhat overstated the distinction between the standard and revealed-preference approaches: although they do not go as far as to allow candidate algorithms to choose utility functions, theorists in the standard approach do argue that the set of utility functions an algorithm must be prepared for is not truly "arbitrary." Rather it is argued that only utility functions corresponding to succinctly representable games are
relevant; in an $N$-player game $G \in C$ where all players have $|S|$ strategies, and in which the game’s payoff matrix has no noticeable structure, describing even a single utility function $U : \mathbb{R}^N \rightarrow \mathbb{R}$ may take $|S|^N$ individual datum. This figure grows exponentially in $N$, and so for truly arbitrary games, the input size $s(G)$ may become enormous even for games with only a modest number of players; this unfortunate situation also implies that generating “polynomial-time algorithms” is artificially easy, because an algorithm that is “polynomially time in $s(G)$” may nevertheless be exponential in, for example, $N$, if $s(G)$ grows exponentially in $N$, as the figure $|S|^N$ suggests. So, theorists working in the standard approach, like those working in the revealed-preference tradition, embrace a proviso on the utility functions that are computationally relevant: we should only consider games with succinct representations, in the sense that we should only be interested in games with description lengths that grow polynomially in the game’s number of players and in the maximum number of strategies per player.

In our view, both the standard and revealed-preference approaches are relevant to the study of the computational complexity of finding NE. The choice between the two depends primarily on the hypotheses a researcher is willing or unwilling to make about the payoff functions of the agents she is describing in some particular instance of game play; in the revealed-preference approach, the set of hypotheses asserted is quite limited: all that is said to be known of the agents’ payoff functions is that they are of VM (or Savage) type, and that the observed strategic choices are consistent with best replies under the payoff functions. By contrast, the standard approach makes very strong hypotheses about the structure of agents’ payoff functions: an instance of a game $G$ is identified with a complete description of all players’ payoff functions. Considering intermediate approaches that lie between the two extremes of the standard and revealed-preference approaches may also be fruitful; certainly, theoreticians often make monotonicity and concavity assumptions on utility functions, for example, but do not go so far as to suppose the utility functions themselves are known. Although we acknowledge the relevance of both approaches, we focus our work on the standard approach; in this connection, we present three results on the computational complexity of eCVE.

Our first result states that, under the monotonicity hypotheses of Theorem 2.5, finding a pure-
strategy (e,m)CVE in an arbitrary network game is PLS-complete. Formally, we have:

**Theorem 3.10.** Consider the computational problem $N(t)$, where $t \in \{e, m\}$: given a network game $N$ with polynomial-time computable payoff functions $V_i$ and rational-valued reference parameters $r_i$, find a pure-strategy tCVE for $N$. Suppose that each $V_i$ satisfies the hypotheses of Theorem 2.5. Then this problem is PLS-complete.

Theorem 4.1 is of precisely the sort that reassures us that eCVE (and mCVE, in this case) are no more complicated to find than NE, but it is a trivial consequence of Theorem 2.5, and Theorem 2.5 obviates the need for (e,m)CVE in this case. Since NE are identical to (e,m)CVE, one may as well study NE if your interest is in pure-strategy equilibria for network games (or indeed pure-strategy equilibria in any family of games, since the equivalence of Theorem 2.5 applies much more broadly than in network games). Our next theorem provides a similar, but non-trivial and useful, guarantee, showing that $\epsilon$-eCVE are PPAD-complete to compute:

**Theorem 3.11.** Consider the problem: given a finite $N$-player game with multivariate polynomial and polynomially computable payoff functions $V_i$ (represented by a polynomial-large algebraic circuit, $C_i$, say), rational reference points $r_i$, and rational $\epsilon > 0$, find $\sigma$ such that no player can unilaterally deviate from $\sigma$ and gain more than $\epsilon$ (an $\epsilon$-eCVE). This problem is PPAD-complete.

Finally, we show that approximating an exact eCVE is also equivalently difficult, in a complexity-theoretic sense, to approximating an exact NE: both problems are FIXP-complete. For NE, this was established by Etessami and Yannakakis (2010); as many of our results have, the following theorem depends crucially on Theorem 2.5:

**Theorem 3.12.** Consider the problem: given a finite game $G$ with payoff functions $V_i$, rational reference points $r_i$, and rational $\epsilon > 0$, find $\sigma$ such that there is an exact eCVE, $\sigma^*$, no more than $\epsilon$ from $\sigma$ (in the Euclidean norm, regarding mixed strategies as vectors in Euclidean space, say). This problem is FIXP-complete.

These results show that, for sufficiently restricted classes of equilibria and games (pure-strategy equilibria in network games), eCVE are no more difficult to compute than NE, and similarly for
sufficiently broad classes of games, as with the preceding two results on general finite games. It may still be the case that the complexity of these two kinds of equilibria diverges for intermediate cases; the proof techniques of Theorems 4.2 and 4.3 rely heavily on the availability of arithmetical-gadget games, and so for any class of games not rich enough to contain these games, a divergence of complexity might be found. However, Theorems 4.1-4.3 provide a comforting basis from which to begin exploring the complexity of eCVE. We hope that reductions of this kind will help to address and elaborate on concerns about the complexities of non-EU GT. Morrow (1993), in his text on GT for political scientists, highlights a concern of this sort: "Prospect theory, the psychologists’ alternative to utility theory, is substantially more complicated than utility theory... Explicating political theory by using choice theory is difficult as it stands; strategic logic is complex. We should know what we can learn about politics with game theory before discarding it for an uncertain alternative." Computational complexity results cannot entirely characterize the complexity of a modeling strategy, but we hope that they can serve as one significant tool in helping theorists to understand whether, and by how much, a non-EU GT equilibrium solution increases the complexity of GT.

3.5 Examples

In this section we introduce a number of small-scale examples. These examples serve four purposes: first, they illustrate typical solution methods for (e,m,n)CVE. Second, they demonstrate the use of theorems developed in the earlier sections of this work. Third, they show that non-EU preferences can improve player welfare in GT, and that these welfare gains can amplify the impact of non-EU preferences on equilibrium solutions of games. Lastly, they demonstrate a natural definition of Bayesian equilibrium in the context of non-EU preferences and nCVE.

In a few of these examples, we find it useful to specifically make reference to PT preferences, as opposed to non-EU preferences generally. When we do so, we make particular use of the following non-linear probability, parametric family of weighting functions, which was introduced in Chapter
2 for use in small-scale examples:

\[
\begin{align*}
\omega(p) &= \begin{cases}
\frac{1}{1+(A-1)^{2K+1}} \left[ (Ap + (1-A))^{2K+1} + (A - 1)^{2K+1} \right] & \text{if } K \in \{0, 1, 2, \ldots\} \\
\frac{1}{1+(A-1)^{2|K|+1}} \left[ (Ap + (1-A))^{2|K|+1} + (A - 1)^{2|K|+1} \right] & \text{if } K \in \{-1, -2, \ldots\} 
\end{cases}
\end{align*}
\]

When we use this family, we always assume \(A = 2\); this gives a weighting function that is symmetric about the 45-degree line through the origin. We choose \(K\) as convenient, noting that it can be interpreted as an index of the strength of an agent’s probabilistic insensitivity. Throughout this section, we also find it useful to assume that \(V_i\) is strictly increasing in monetary payoffs and strictly decreasing in \(r_i\).

3.5.1 Matching Pennies (MP)

The canonical Matching Pennies game is given by the matrix of player utilities:

\[
\begin{array}{cc}
& L & R \\
T & 0 & 1 \\
1 & 0 & \\
B & 1 & 0 \\
0 & 1 & \\
\end{array}
\]

The interpretation of the MP matrix is marginally different in our usage than in the classical case, in that its entries usually denote utilities received from the relevant pure-strategy profiles, while we use entries to denote observable payoffs (e.g. USD). This change is necessary in order to represent changes in value as players’ reference points change. We let \((p, q)\) be the vector containing the row player’s probability of playing \(T\) and the column player’s probability of playing \(L\), respectively; we identify the row player as P1 and the column player as P2. We assume \((r_1, r_2) \in (0, 1) \times (0, 1);\)
it seems natural that player reference points should lie within the convex hull of achievable pure-strategy payoffs. (The analysis is easily carried out for \( r_i < 0 \) or \( r_i > 1 \) as well, of course.)

By hypothesis on the monotonicity properties of \( V_1, V_2 \), no player can use any pure strategy in equilibrium, regardless of \( r_1, r_2 \); thus, with given reference vector \((r_1, r_2)\), we identify MP’s eCVE by seeking \((p, q) \in (0, 1) \times (0, 1)\) such that

\[
V_1((1, q), r_1) = V_1((0, q), r_1)
\]

and

\[
V_2((p, 1), r_2) = V_1((p, 0), r_2)
\]

For PT preferences, the row player’s conditions yield

\[
w_1(q)v_1(1, r_1) + w_1(1 - q)v_1(0, r_1) = w_1(q)v_1(0, r_1) + w_1(1 - q)v_1(1, r_1) \Rightarrow
\]

\[
w_1(q)(v_1(1, r_1) - v_1(0, r_1)) = w_1(1 - q)(v_1(1, r_1) - v_1(0, r_1)) \Rightarrow
\]

\[
w_1(q) = w_1(1 - q)
\]

which is uniquely satisfied at \( q = 0.5 \), assuming only that \( w_1 \) is strictly monotonic. Identical reasoning gives \( p = 0.5 \); thus the eCVE solution (for PT preferences) and the classical solution agree. Given the overwhelming symmetry in the structure of the MP game, we find this reassuring; it is difficult to imagine a persuasive argument for any other solution to MP. As the argument for \((p, q) = (0.5, 0.5)\) was independent of \((r_1, r_2)\), this also identifies MP’s unique mCVE (in which \( r_1 = r_2 = 0.5 \)), and this in turn gives MP’s unique nCVE, since MP is a simultaneous game.

### 3.5.2 Beauty Contest (BC)

Keynes’ beauty contest game illustrates the power of iterated dominance; with the same caveats of interpretation as in MP, we define the BC by the following: there are \( N \) players, for a positive integer
and each player has pure-strategy space \{0, 1, \ldots, 100\}. The player(s) who guesses closest to $\frac{2}{3}$ of the unweighted arithmetic of all players’ pure strategies receives payoff of 1; all other players receive 0. Each player’s preference function $V_i$ is assumed to be strictly increasing in her monetary payoff, and strictly decreasing in her reference point $r_i$.

The BC is solvable by strict iterated dominance, and yields a guess of 0 by all players; one simply reasons as follows: no player should ever choose any number greater than 66, since $\frac{2}{3}$ of the average can never exceed 66. On deleting these pure strategies for each player, we may now iterate and delete all strategies greater than $\frac{2}{3} \times 66$; repeating this process leaves only the pure strategy 0 undeleted from each player’s pure strategy set. The BC is also a simultaneous game, and so "Guess 0." for all players is the BC’s (e,m,n)CVE. The BC thus illustrates two principles: first, (e,m,n)CVE are sufficiently well-behaved that baseline classical principles of rationality like strict iterated dominance need not be abandoned when using (e,m,n)CVE, regardless of the underlying preference structures of the players over gambles. Second, for descriptive purposes, it is well-known that players do not solve the BC “correctly;” rather than apply iterated dominance infinitely, actual players appear to only apply iterated dominance a finite number of times each, and to assume that other players will behave similarly. Thus, in descriptive applications where iterated dominance is a major force in weening out solution methods, it may be necessary to merge (e,m,n)CVE with models of limited iteration by players, such as the Cognitive Hierarchy Theory of Camerer (2003). We pursue this idea further in discussion.

3.5.3 Crawford’s Counterexample (CC)

To illustrate failures of existence for non-classical preferences that fail the VM independence axiom, Crawford (1990) provided the following 2-player zero-sum game, where tabular entries indicate monetary payoffs:
Crawford assumed $a, b > 0$, that both players prefer first-order stochastically dominant gambles to their dominated counterparts, and that at least one player has strictly quasi-convex preferences; in our formalism, these assumptions apply to the preference functions $V_1, V_2$, defined over probability distributions on monetary payoffs. In a type (A) generalization of classical NE, such as PT-NE, this sort of first-order stochastic dominance assumption implies that, in equilibrium, if one player uses a pure strategy then both do; strict quasi-convexity of one player’s preferences then implies that player will use a pure strategy, but there is obviously no pure-strategy equilibrium in any CC game, so under these preferential assumptions the CC game lacks a type (A) non-classical equilibrium, and in particular lacks a PT-NE.

For eCVE, however, the assumptions made on $V_1, V_2$ over all probabilistic gambles apply only to gambles determined by one’s opponent, and this restores equilibrium existence. Let $p, q$ and $P_1, P_2$ have the same interpretations as in MP. Then we require:

$$V_1((1, q), r_1) = V_1((0, q), r_1)$$

and

$$V_2((p, 1), r_2) = V_1((p, 0), r_2)$$

Equivalently, we need $\overline{V}_1((1, q), r_1) = \overline{V}_1((0, q), r_1)$ and similarly for the column player; $\overline{V}_1$ depends continuously on $q$, so that if $V_1((1, 1), r_1) \geq V_1((1, 1), r_1)$ and $V_1((1, 0), r_1) \geq V_1((1, 0), r_1)$, then some $q^*$ exists for which $\overline{V}_1((1, q), r_1) = \overline{V}_1((0, q), r_1)$. The conditions $V_1((1, 1), r_1) \geq V_1((1, 1), r_1)$
and \( V_1((1,0), r_1) \geq V_1((1,0), r_1) \) are implied by \( V_1 \)'s respect for first-order stochastic dominance, so we conclude that the desired \( q^* \) exists. We can argue similarly for P2, so that CC has an eCVE for any \( r_1, r_2 \) and any \( a, b > 0 \), assuming only that both players’ preference functionals \( V_i \) respect first-order stochastic dominance.

3.6 Bayesian nCVE & Propagation of Irrationality

In Chapter 2, we developed the following, modified entrant/incumbent game (EI), with payoffs again to be understood as monetary rewards:

Let \( p \) be the Entrant’s probability of invading the market and \( q \) be the Incumbent’s probability of defending against an attack, if invaded. With these assumptions in place, it can be shown that this game has a unique nPT-NE at \((p = 0, q \approx 0.98)\), as well as a unique SPE of \((p = 1, q = 0)\); the incumbent therefore benefits if the nPT-NE solution prevails, as opposed to the SPE solution, in
the sense that the gamble faced by the incumbent in the nPT-NE solution first-order stochastically dominates the gamble faced by the incumbent in the SPE solution. Furthermore, in the nPT-NE, the incumbent’s preferences differ from classical preferences only in their incorporation of probability weighting—setting \( w_2(p) = p \) gives P2 classical preferences. In this sense, an injection of PT “irrationality” alone was enough to improve the incumbent’s welfare.

It is easy to show that qualitatively identical conclusions apply for nCVE: we let the Incumbent have cubic polynomial probability weighting function \( w_2(p) = \frac{1}{2} (2p - 1)^3 + \frac{1}{2} \), value function \( v_2(m) = m - r_2 \), and reference point \( r_2 = 3 \). Letting \( q \) be the Incumbent’s probability of defending if invaded and \( p \) the Entrant’s probability of invasion, we have:

\[
V_2(1, 0) = 0 \\
V_2(1, 1) = -\left( \frac{1}{2} (2 \cdot 0.7 - 1)^3 + \frac{1}{2} \right) + \left( \frac{1}{2} (2 \cdot 0.3 - 1)^3 + \frac{1}{2} \right) 2 \approx 0.532 \\
V_2(0, q) = 7 \forall q
\]

Since 0.532 > 0, the Incumbent always defends if invaded. We contrast this with the EU solution: setting \( w_2(p) = p \) and \( v_2(m) = m - 3 \) gives the Incumbent EU preference function \( U_2(p, q) \) with

\[
U_2(1, 0) = 0 \\
U_2(1, 1) = 0.7 \cdot -1 + 2 \cdot 0.3 = -0.1 \\
U_2(0, q) = 7 \forall q
\]

so that an EU Incumbent—differing only from our PT Incumbent by the incorporation of non-linear weighting—would choose not to defend if invaded. To complete both solutions, we suppose the Entrant has EU preferences with \( v_1(m) = m \), for simplicity; this gives \( U_1(1, 0) = 3, U_1(1, 1) = -1.8, U_1(0, q) = 0 \forall q \), so that the Entrant chooses to invade iff the Incumbent does not choose to defend invaded. More to the point, the Entrant will choose to invade against our EU Incumbent, but
not against our PT Incumbent; thus the PT Incumbent earns, in the nCVE of this game, a monetary payoff of 10 with certainty, while the EU Incumbent is forced to concede market share, earning a monetary payoff of only 3. We attribute this difference in welfare for the Incumbent player to the change from EU to PT preferences, as this was the only change that was made in the two analyses of this game; we emphasize, further, that the players’ preference functions were identical, except for the incorporation of probabilistic insensitivity, vis-a-vis non-linear probability weighting, into the PT Incumbent’s preferences.

This example illustrates that, somewhat counterintuitively, deviations from EU rationality can benefit players: the PT Incumbent was able to secure additional monetary reward only because the non-EU aspects of her preferences rendered credible a threat that would have been, for an analogous EU player, an incredible threat. The intuition behind this phenomenon is not unfamiliar in the classic GT literature; Fudenberg and Tirole (1991, p. 327) discuss a signaling game in which there are two kinds of firms, “sane" and “crazy," where sane firms exhibit standard and reasonable EU preferences, but crazy firms always behave aggressively, in all circumstances; they find that sane firms may attempt to pass themselves off as crazy in order to secure additional payoff in equilibrium. In both the signaling game with sane-crazy firms and in our EU-PT Entrant/Incumbent game, players may benefit from actual (in the case of our PT Incumbent) or apparent (as with sane firms, feigning insanity) non-rationalities. Both games tell a story about the sometimes advantages to using a reputation for recklessness as a threat; seen in this light, our EU-PT Entrant/Incumbent game relates a familiar intuition, but makes an additional contribution to this literature in rooting its “elements of non-rationality" in rigorously developed, experimentally rooted models of non-EU preference.

We can carry this analogy further: we can effectively mimic the structure of the reputation game considered by Fudenberg and Tirole but substituting our PT Incumbent firm for their “crazy" firm; in the sequel, we refer to this example as Bayesian Entrant/Incumbent (BEI). To do so, we first develop a notion of equilibrium suitable for working with extensive-form games of

\footnote{Fudenberg and Tirole’s reputation game was, in turn, a simplified variant of a reputation model due to Kreps and Wilson (1982) and Milgrom and Roberts (1982).}
incomplete information, in which private types may indicate a player’s preference type, whether EU or PT. The relevant classical concept is perfect Bayesian equilibrium (PBE); to define a notion of Bayesian nCVE (B-nCVE) appropriate for signaling games, we modify the development of PBE by Fudenberg and Tirole (1991, p. 325). Fudenberg and Tirole call a mixed-strategy profile \( \sigma \) and a set of posterior beliefs \( \mu(.|s) \) a PBE of a signaling game if:

\[
\forall \theta, \sigma_1^*(.|\theta) \in \arg\max_{\alpha_1} \sum_{a_1} \mu(\theta|a_1)u_1(\alpha_1, \sigma_2^*, \theta)
\]

\[
\forall a_1, \sigma_2^*(.|a_1) \in \arg\max_{\alpha_2} \sum_{\theta} \mu(\theta|a_1)u_2(a_1, \alpha_2, \theta)
\]

\[
\sum_{\theta' \in \Theta} p(\theta') \sigma_1^*(a_1|\theta') > 0 \Rightarrow \mu(\theta|a_1) = \frac{p(\theta') \sigma_1^*(a_1|\theta)}{\sum_{\theta'' \in \Theta} p(\theta'') \sigma_1^*(a_1|\theta''})
\]

where \( p(.) \) is P2’s prior over P1’s type, \( \Theta \) is P1’s set of possible types, and \( \mu(.|s) \) is P2’s posterior over P1’s type given an observed pure strategy.

As in our treatment of NE, it is our goal to develop an analogue of PBE that incorporates PT preferences, and which respects linearity of player preferences’ in their own strategic probabilities; following our developments above, we propose to call \( \sigma, \mu(.|s) \) a B-nCVE if:

\[
\forall \theta, \forall s_1 \in \text{supp}[\sigma_1^*(.|\theta)], s_1 \in \arg\max_{s_1} V_1(s_1, \sigma_2^*, \theta)
\]

\[
\forall s_1, \forall s_2 \in \text{supp}[\sigma_2^*(.|s_1)], s_2 \in \arg\max_{s_2} V_2(s_1, s_2, \mu(\theta|s_1))
\]

\[
\sum_{\theta' \in \Theta} p(\theta') \sigma_1^*(s_1|\theta') > 0 \Rightarrow \mu(\theta|s_1) = \frac{p(\theta') \sigma_1^*(s_1|\theta)}{\sum_{\theta'' \in \Theta} p(\theta'') \sigma_1^*(s_1|\theta'')}
\]

We note the use of characterization (C) of NE in inspiring this definition, as all optimization originally performed over mixed strategies is now done over pure strategies. The third condition is, as in PBE, just the requirement that Bayes’ rule be respected; this may be descriptively questionable.

---

6 We exclude “perfection” from the naming convention, as non-myopia already addresses the incredible-threats-in-subgames problem to which perfection normally refers.

7 Other authors, e.g. Wang (1999) and Epstein, Noor, & Sandroni (2008), have advanced more general updating rules for use with non-EU preferences.
but it represents an incremental incorporation of PT preferences into existing theory, and so should help to isolate the impacts of PT preferences on equilibria. One other major change is apparent: we have moved the posterior \( \mu(\theta|s_1) \) inside the value function \( V_2 \); it is now an argument, rather than a factor to be multiplied by \( V_2 \). If we consider the case of PT preferences and bear in mind characterization (C)’s close relationship to probabilities as a representation of subjective uncertainty, the rationale for doing so is clear: \( \mu \) expresses \( P_2 \)’s subjective uncertainty about \( P_1 \)’s type, given the action taken by \( P_1 \), and so \( \mu \) is an element of \( P_2 \)’s subjective uncertainty, just as \( \sigma_i^* \) is an element of \( P_2 \)’s subjective uncertainty in the simultaneous games treated by the basic definitions of (e,m,n)CVE. As a result, \( P_2 \)’s preferences, if of PT type, must incorporate \( \mu \), so as to apply non-linear probability weighting to her aggregate uncertainty (which is just \( \mu \) itself, since she observes \( P_1 \)’s pure strategy before choosing \( \sigma_2^* \)).

Although we have developed the definition of B-nCVE for the general signaling-games case where \( P_1, P_2 \) may both have non-EU preferences, in the sequel we assume \( P_2 \) has EU preferences. This choice allows us to isolate the effect of interest, which is the pooling or separating behavior of EU and non-EU signalers—that is, of \( P_1 \). In addition, this modeling choice should ameliorate concerns over the use of normative, Bayesian updating in the definition of B-nCVE for signaling games; since \( P_2 \) is an EU/rational player, it is natural that she apply Bayesian updating in her treatment of uncertainty, and this issue does not arise for \( P_1 \), who has possibly non-EU preferences but does not face any non-singleton information sets.

We now return to our development of a signaling game in which EU players may behave as if they are PT players, in order to secure additional payoff, as suggested by the welfare increases observed in the Entrant/Incumbent game. To ease the algebraic complexity of the example, we will allow ourselves somewhat more freedom in specifying the Incumbent firm’s preferences than we did in the original Entrant/Incumbent example; we now assume \( w_2(p) = \frac{1}{2}(2p - 1)^{2K+1} + \frac{1}{2} \) for some non-negative integer \( K \). An EU Incumbent has \( K = 0 \), but we now assume the PT Incumbent has \( K > 0 \). We further assume that, regardless of type, both firms have identity utility/value functions (i.e., \( v_i(m) = m - r_i \) with \( r_i = 0 \forall i \)). We use \( U_1 \) to denote the Entrant’s utility function, \( U_2 \) for the
Incumbent’s utility function if she is of EU type, and $V_2$ for a PT Incumbent’s value function. Note that, for any non-degenerate ($(0 < p < 1)$) gamble $P = (p, A; 1 - p, B)$ with $A \geq 0 \geq B$ any $\varepsilon > 0$, we may choose $K$ sufficiently large so that $|V_2(P) - \frac{A + B}{2}| < \varepsilon$. Similarly, we can always choose $p$ so that $|U_2(P) - B| < \varepsilon$. As a result, for any real $D > 0$, we can first choose $B = -D$, $A = 3D$, and $p$ sufficiently small that $|U_2(P) - (-D)| < \varepsilon$, and then $K$ sufficiently large that $|V_2(P) - D| < \varepsilon$, so that $|U_2(P) - V_2(P)| \approx 2D$. Thus the “magnitude of disagreement” between our EU and PT Incumbent on the attractiveness of attacking can be made arbitrarily large, $U_2(P) < 0 < V_2(P)$, and $|U_2(P)|, |V_2(P)|$ may also both be made arbitrarily large.

The game itself has three periods. In the first period, the Incumbent’s type (which is private information) is determined, and only the Incumbent moves; she may either publicly invest in resources purportedly to be used in a period-2 attack on the Entrant, or concede market share. Both public investment and concession of market share are costly to the Incumbent; concession of market share is beneficial to the Entrant, while the investment of resources to prepare an attack has no period-1 cost to the Entrant. In the second period, only the Entrant moves: the Entrant may exit the market, in which case the Incumbent enjoys period-2 monopoly rights, or she may remain and contest the market. In period 3, only the Incumbent moves: if the Entrant has not exited, then the Incumbent may complete her investment and carry out an attack, if she also invested appropriately in period 1 to prepare for an attack. If the Incumbent did not invest in the possibility of an attack, then she can only concede market share to the Entrant.

All monetary payoffs are received at the end of period 3. If the Incumbent attacks and the Entrant remains, then the Incumbent experiences prospect $P$, for which a PT Incumbent has value $\approx D$ and an EU Incumbent has value $\approx -D$, using the process proposed above for determining the prospect’s detailed structure and the PT Incumbent’s degree of probabilistic insensitivity; intuitively, the attack has a definite cost $D$ associated with it, and a possibility of direct benefit $3D$ if the attack is successful, but the attack is successful with small probability. The PT player’s probabilistic insensitivity causes the small probability of success to loom large, relative to the EU player’s linear treatment of this probability. When attacked, if she has remained in the market, the Entrant
experiences a loss with certainty; her reward in this case is \( L < 0 \). If the Entrant exits and the Incumbent invested in period 1, then the Incumbent receives \( M - \frac{D}{2} \); \( M \) is the benefit of period-2 monopoly, while \( \frac{D}{2} \) is the per-period cost of investing in an attack; in this case, the Entrant earns 0. If the Incumbent invests in an attack in period 1 but does not carry out the attack in period 3 (and the Entrant remained), the Incumbent earns \( C - \frac{D}{2} \), where the concession benefit \( C \) satisfies \( D > C > 0 \); in this case the Entrant receives payoff of \( C \). If the Incumbent did not invest in an attack in period 1, then both players receive \( C \). The only pure-strategy profile for which the EU and PT Incumbent’s preference functions disagree is that ending in the prospect \( P \); the above therefore completes the specification of payoffs. Lastly, we specify \( p \) as the Entrant’s prior probability of belief that the Incumbent is of PT type.

We now consider the B-nCVE of this game. First, we note that, regardless of type or priors, if the Incumbent does not invest in period 1, then the Entrant will remain in the market, and the Incumbent will be forced to concede market share in period 3, so that both players receive a payoff of \( C \). Recognizing this, a PT Incumbent will always invest in period 1, because if the Entrant exits, she gets \( M - \frac{D}{2} > C \), and if the Entrant remains, she gets \( \approx D > C \). In this sense, our PT Incumbent behaves like Fudenberg and Tirole’s “crazy” firm; the PT Incumbent always chooses to invest in attacking the Entrant firm. As in Fudenberg and Tirole’s reputation game, it is now helpful to consider B-nCVE by type: separating, pooling, and hybrid.

In a separating equilibrium, by definition, an EU Incumbent chooses not to invest in period 1, in contrast to her PT kindred. Since in equilibrium play the two types have deterministically distinct strategies, the Entrant now updates her posterior beliefs: \( p \to 1 \) if she observes investment, and \( p \to 0 \) if she observes no investment. In the case of investment, the Entrant is (correctly) certain that she faces a PT Incumbent, and knows this PT Incumbent will carry out her threat if the Entrant remains, so the Entrant exits the market, because her loss in the event of an attack satisfies \( L < 0 \). Thus, if the Incumbent is of PT type, the Incumbent invests, the Entrant correctly infers the Incumbent is PT, the Entrant exits the market, the Entrant earns 0 for exiting, and the Incumbent earns \( M - \frac{D}{2} \) for investing in period 1 and driving off the competition. If the Incumbent is of EU
type, the Incumbent does not invest, the Entrant correctly infers the Incumbent is EU, the Entrant remains in the market, and both firms earn \( C \). In a separating equilibrium, an EU Entrant earns \( C \), but an EU Entrant could always choose to invest in period 1, attempting to pass itself off as a PT Incumbent. Given the Entrant’s posterior beliefs in a separating equilibrium, an EU Incumbent that deviates in this way could earn \( M - \frac{D}{2} \), so we must have \( M - \frac{D}{2} \leq C \), if separating equilibrium is to prevail; this condition is also clearly sufficient for a B-eCVE to exist.

In a pooling equilibrium, an EU Incumbent always invests in period 1, in an attempt to pass herself off as a PT Incumbent, and so secure a period-3 monopoly. In a pooling equilibrium, EU and PT players have perfectly correlated strategies, so the Entrant’s posteriors are identical to her priors. Our EU Incumbent behaves like Fudenberg and Tirole’s sane incumbent, in that for a pooling equilibrium to prevail, there must be a positive probability of the Entrant exiting on observing investment in period 1; this follows because, if the EU Incumbent simply conceded share in period 1, she could earn \( C \), but if she invests and the Entrant remains, the EU Incumbent can earn only \( \approx -D < 0 < C \) or \( C- \approx D < C \). We further note that, if the Incumbent is EU, and she invests in period 1 but the Entrant remains, then the Incumbent will conceder, since \( \approx -D < C- \approx D \). The EU Entrant earns 0 if she exits, and her posterior beliefs remain unchanged in a pooling equilibrium, so for her to consider exit, she must believe “strongly enough” that the Entrant is in fact of PT type; algebraically, we need \( pL + (1-p)C \leq 0 \). If this condition holds, then there is a B-nCVE in which both PT and EU Incumbents invest, the Entrant’s posteriors remain unchanged (but in the probability-0 case of observing no-investment, they set \( p \to 0 \)), and the Entrant exits (though she would not exit if no-investment were observed). In a pooling equilibrium, both PT and EU Incumbents earn \( M - \frac{D}{2} \), while the Entrant earns 0.

As in Fudenberg and Tirole’s analysis, if both conditions identified here are violated, we also find hybrid equilibria in which non-degenerate mixed strategies are played by the Entrant and EU Incumbent, and continua of equilibria occur if the latter of the two conditions is satisfied at equality. We emphasize that both the analysis of this game and the game’s structure largely imitate Fudenberg and Tirole’s (1991, p.327-328) treatment of a simple reputation game; our unique contribution is
to show that non-EU preferences can serve in the role of the “crazy” player in such a game. The use of experimentally grounded, formal models of non-EU preferences has several advantages over the use of a crazy EU player: first, a crazy EU player’s preferences may bear no resemblance to real-world observables, as they are by definition specified so that the EU player will prefer to take a single action, regardless of monetary reward; in particular, a crazy EU player’s utility function has limited interpretive value, as assumptions cannot be made regarding the monotonicity of the player’s utility in observable payoffs, e.g. USD. Second, a crazy EU player cannot be said to be genuinely non-rational in the rigorous sense in which that term is typically used by economists and decision theorists; she is not vulnerable to exploitation by savvy players aware of inconsistencies in her preferences. Our example, combined with an empirically well-supported non-EU model of preferences, removes both of these limitations of reputation games like the one analyzed by Fudenberg and Tirole. As a result, we believe this example suggests that B-nCVE and non-EU models of preference grounded in experimental evidence can be useful for studying bluffing, and for investigating the amplification or stamping out of non-rational player preferences on strategic outcomes. Lastly, we note that, although we have developed this example to illustrate that EU players may imitate non-EU players in attempt to develop a reputation for non-rationality, the same principles we have employed here could be used to construct an analogous example in which non-EU players would imitate EU players.

3.7 Conclusions

In this work we have identified a non-EU generalization of NE, CVE, that emerges naturally from a systematic consideration of characterizations for NE. Our primary interest is in PT preferences, and so, following Shalev (2000), we developed three variants of CVE, allowing for reference points to be treated as exogenously fixed, myopically determined by a game’s structure, or non-myopically determined by a game’s structure. We also developed a notion of Bayesian exogenous CVE suitable for use in signaling games which allow the type of player preferences to be private information.

We appealed for the use of CVE as a solution concept by showing that (e,m,n)CVE inherit
We treated these properties in three categories: classical/fundamental, psychological/epistemic, and computational complexity. In the classical/fundamental category, we showed that (e,m,n)CVE enjoy an existence guarantee in all finite games, that (e,m)CVE respect iterated dominance and rationalizability, and that (m,n)CVE are identical in simultaneous games. In the psychological/epistemic category, we argued that CVE treat strategic uncertainty as a representation of subjective uncertainty, that this interpretation corresponds better to evidence in the single-decision maker literature than the objective or hybrid alternative interpretations, and we showed that Aumann and Brandenberger’s (1995) formalization of and seminal results on epistemic considerations for the in-beliefs interpretation of NE also hold for eCVE. In the final category of computational complexity, we showed that three major computational CVE search problems are equivalently hard to their classical NE counterparts; in particular, we proved that finding pure-strategy eCVE in general network games is PLS-Complete (under monotonicity assumptions on the preference functionals that are satisfied by, for example, PT preferences), that finding an $\varepsilon$-eCVE in a general finite game is PPAD-Complete, and that approximating an exact eCVE in a general finite game is FIXP-Complete.

We also illustrated the use of CVE in a number of small-scale examples. The first four of these examples (MP, BC, CC, EI) concentrated on (e,m,n)CVE, and demonstrated theorems proved in the theoretical sections of the paper and/or illustrated a typical, “paper-and-pen” solution approach for finding CVE. The EI example also illustrated a fascinating property of CVE: players with non-EU preferences, despite being “non-rational,” may have greater welfare in equilibrium than EU players who are identical in every respect save for the non-EU player’s incorporation of non-rational elements of preference, such as PT non-linear probability weighting. Our final example, BEI, built on the welfare-improving property of non-EU preferences observed in the EI game in order to show that, in some settings, EU players may choose to imitate non-EU players in order to improve their welfare; thus the behavioral effects of non-EU preferences may not be stamped out by market competition, but could in fact be amplified, as rational players may seek to pass themselves off as non-rational players when developing such a reputation is advantageous.
We note that reputation effects are well-known and studied in the classic literature already, as in the sane-crazy firms reputation game analyzed by Fudenberg and Tirole (1991); our marginal contribution to this discussion is to illustrate that reputation effects may be observed entirely due to the non-rational elements of non-EU preferences, and to connect this analysis to the importance of non-EU preferences in heterogeneous markets.

We have developed much of the foundational theory concerning (e,m,n)CVE, and shown how this might be extended to a notion of Bayesian CVE, but there are many pressing questions we were unable to address, and which provide natural avenues for future work. First, non-EU preferences in general and PT preferences specifically capture some deviations from rationality, but fail to reflect many others, such as limits on the number of iterations players will use in analyzing dominance-solvable games; thus future work should seek to combine models that account for these other known issues, such as Camerer’s (2003) Cognitive Hierarchy Model, with models like PT preferences, which primarily address the form of players’ representing functions over lotteries. Second, although the framework we have developed is sufficient for addressing non-EU models of preferences over risk, we have not provided a means for addressing Knightian uncertainty, ambiguity aversion, and anomalies in the spirit of the Ellsberg (1961) paradox; given that in real-world interactions players often do not have ready access to probabilistic descriptions of uncertainty, ambiguity/Knightian uncertainty is undoubtedly important, and generalizing the present work to accommodate ambiguity would represent a natural, incremental improvement on our knowledge of CVE. Third, it would be useful to further extend our work in each of the major areas identified above: for fundamental theory, an existence result for infinite games is needed, as is work connecting the existence of NE in infinite games to CVE in those games, and work on the existence of Bayesian CVE (suitably defined for general games). It would, for example, be of special interest to identify infinite games lacking NE but possessing CVE, and of games possessing CVE but lacking NE, and to connect these kinds of examples to particular families of non-EU preference functions. Similarly, it would be useful to extend our reproductions of Aumann and Brandenberger’s (1995) epistemic results to infinite games. In addition, it would be helpful to consider whether Harsanyi’s (1973) purification theorem—the
other major classical result heavily associated with the interpretation of equilibria as subjective and “in beliefs”—can be reproduced for CVE. For computational complexity, we have produced results for eCVE, but this perspective is relevant for (m,n)CVE and Bayesian CVE as well, and results extending our work in these directions would be useful. In addition, complexity results should also be sought from the revealed preference approach to complexity of game-theoretic problems (Echenique, Golovin, & Wierman, 2011), although this theory is considerably less well-developed in the classical case than is the standard approach. Fifth, if infinite games are considered, the equivalence of type C and D characterizations for non-EU preferences breaks down; similarly, the equivalence of the distributional $D[.]$ and convex hull $co[.]$ operators breaks down in this setting, and so it would be of interest to consider the additional, distinct notions of non-EU equilibrium that these alternatives may motivate. Some novel issues also arise in defining nCVE for infinite games, since backwards-induction requires a “final node” at which to begin, and there may be no such node in an infinite game; it would be of interest to study various resolutions to this problem, as, for example, by defining an nCVE of an infinite game as a suitable limit of nCVE for subsets of the original game tree.

We close by briefly developing two conjectures related to this work. First, we note that the complexity results in this paper are broader than those that were achieved in Chapter 2 for PT-NE and PT-EB, where no results were obtained for PT-NE, and results for PT-EB required bounds on the number of strategies per player. Although we do not have negative results showing that the restrictions in these results are necessary, we have reason to believe that PT-NE and PT-EB are provably more difficult than CVE: our primary reason for this belief is that finding each of PT-NE and PT-EB appears to imply the ability to find global optima of effectively arbitrary preference functions, and global optimization is known to be hard, even for quadratic functions. Secondarily, we note that the natural proof techniques used for our CVE PPAD-Completeness and FIXP-Completeness cannot be applied for PT-NE or PT-EB: our argument underlying the PPAD result relies on the linearity of player preferences between the extreme points of players’ probability simplices, but this property is entirely lost in PT-NE and largely lost in PT-EB. The FIXP result depends again on this
linearity property, but indirectly, in the sense that linearity is used in Nash’s (1951) proof that, for any finite game, there exists an efficiently computable function whose fixed points correspond to the game’s equilibria. Thus we conjecture that PT-NE and PT-EB are provably complete for classes of problems strictly larger than PPAD and FIXP, for the relevant \( \epsilon \) and approximation problems; unfortunately, we do not have a particular class in mind for these characterizations, though it may be possible to establish NP-Hardness of approximating exact PT-NE by reduction from quadratic optimization, and it would seem useful to consider whether these two problems can be shown to be in PSPACE.
Chapter Four: Empirical Analysis of PT-EB & CVE

In the first three chapters of this work, we introduced our 3 primary theoretical constructs—prospect-theoretic Nash Equilibrium (PT-NE), prospect-theoretic equilibrium in beliefs (PT-EB), and combinatorially verifiable equilibrium (CVE)—and conducted comprehensive theoretical analyses of each of them. Three different versions of these equilibria were defined: exogenous (e), myopic (m), and non-myopic (n), each variant corresponding to a different model of players’ reference points or expectations about payoffs. Particular attention was directed to preservation of key properties: existence in finite games, respect for iterated dominance, computational reductions to Nash Equilibrium (NE), and so on. A number of problems were found with PT-NE, most notably its failure to exist in general finite games; however, theoretical analysis was less decisive in contrasting PT-EB with CVE. Although CVE appear to be somewhat more well-behaved than PT-EB, both concepts enjoy broad existence guarantees, respect iterated dominance, and allow for some forms of computational reduction to NE. In the present chapter, we further contrast CVE and PT-EB by conducting preliminary investigations on each equilibrium’s ability to fit empirical data. Given the preliminary nature of this chapter, we focus on the exogenous variants of these solution concepts, ePT-EB and eCVE. Specifically, we study the goodness-of-fit for ePT-EB and eCVE to data taken from 24 simultaneous games reported on in Selten and Chmura (2008) and Camerer (2003, Ch. 3).

For the reader’s convenience, we very briefly recapitulate, at a high level, our development of ePT-EB and eCVE. These two equilibrium concepts were developed to accommodate non-expected-utility (EU) preferences in game theory (GT), and specifically to rectify theoretical problems discovered with incorporating PT preferences into GT under our original notion of non-EU equilibrium, ePT-NE. PT preferences are an empirically motivated generalization of EU preferences that allow for the three distinct non-EU effects of probability weighting, rank dependence, and reference
dependence (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992; Wakker, 2010). These deviations from EU preferences, and non-linear probability weighting especially, cause a number of theoretical problems to emerge when developing GT equilibrium concepts with players that have PT preferences. Pathologies of non-existence, trivial preferences, and dynamic inconsistency develop if the standard definition for NE is naturally generalized to the case of PT preferences; furthermore, crucial combinatorial structure is lost, suggesting an increase in computational intractability. In chapter 2, we showed that this considerable list of theoretical problems plagues our notion of ePT-NE.

Recognizing the many problems generated by generalizing the usual definition of NE to yield PT-NE, we sought alternative definitions for NE that would not yield the many problems witnessed for ePT-NE. Our first victory in this respect was to discover the work of Crawford (1990), who proposed to convexify the usual best-reply condition given in NE’s definition using the distributional operator $D[A]$, which gives the set of all distributions over the set $A$; this led to our development of PT-EB. On further considering Crawford’s approach, it was recognized that the distributional operator could be replaced by the convex-hull operator $co[\cdot]$, and that combining this approach with a well-known combinatorial characterization of NE could help us to identify still other definitions of non-EU equilibrium. These considerations led us to consider the following four-fold characterization of NE:

**Observation 4.0.1.** For finite games, we have the following equivalences:

\[
\begin{align*}
(A) & \quad \sigma_i \in \arg\max_{\sigma'_i \in \Delta_i} U_i(\sigma'_i, \sigma_{-i}; r_i) \forall i \\
\Leftrightarrow \\
(B) & \quad \sigma_i \in co[\arg\max_{\sigma'_i \in \Delta_i} U_i(\sigma'_i, \sigma_{-i}; r_i)] \forall i \\
\Leftrightarrow \\
(C) & \quad s_i \in \arg\max_{s'_i \in S_i} U_i(s'_i, \sigma_{-i}; r_i) \forall s_i \in \text{supp}(\sigma_i) \forall i
\end{align*}
\]
\[ (D) s_i \in \text{co}[\text{argmax}_{s_i' \in S_i} U_i(s_i', \sigma_{-i}; r_i)] \forall s_i \in \text{supp}[\sigma_i], \forall i \]

Generalizing in the obvious fashion each of the NE definitions (A) and (B) gives the notions of PT-NE and PT-EB, respectively. Definitions (C)-(D), however, gave us a new notion of equilibrium, CVE. CVE was found to be better behaved than both PT-EB and PT-NE, and in particular to enjoy a broad existence guarantee, to respect iterated dominance and rationalizability, to obey epistemic conditions analogous to those for NE, and for general finite games to be computationally no more difficult to find than NE.

Thus our movement from PT-NE to PT-EB and, finally, CVE, exhibits increasing structure and desirable theoretical behavior: PT-NE do not exist even in some $2 \times 2$ games and appear to behave quite different from NE computationally\(^1\) although they do respect both iterated dominance and rationalizability. The convexification procedure that leads from PT-NE to PT-EB improves on this situation by restoring existence in all finite games and enabling computational reductions in games with a bounded number of strategies per player, although there is some cost to this gain, in that PT-EB no longer respect rationalizability. Finally, the switch from requiring best replies over mixed strategies to requiring them only over pure strategies, which is the crucial difference between definitions (A)-(B) and (C)-(D) and gives CVE, yields considerable fruit, as CVE respect each of these four important theoretical properties, and enjoy epistemic guarantees comparable to NE as well.

On the basis of theoretical differences alone, it seems reasonable to reject PT-NE as a solution concept: failure to exist even in simple $2 \times 2$ games is a considerable disadvantage, and often leaves PT-NE with no predictive value. The contrast between PT-EB and CVE, however, is not quite so stark; while CVE unambiguously behave better by our four theoretical criteria, it is conceivable that PT-EB may prove the more useful of the two concepts, especially as PT-EB allow for both

\(^1\)Although we have rigorously provided no lower bounds on the complexity of PT-NE, it is easy to see that the problem of finding a type-(A) non-EU equilibrium contains quadratic programming as a special case; quadratic programming is known to be NP-Hard (Sahni, 1974), so the problem of finding a type-A non-EU equilibrium must also be NP-Hard.
subjective and objective uncertainty, while CVE capture only subjective uncertainty.

Our theoretical analyses have thus led us to two candidate equilibrium concepts: PT-EB and CVE. It is desirable to further examine these two concepts, in an effort to identify a singular notion of equilibrium as most suitable for descriptive analysis. This desire to contrast PT-EB and CVE leads naturally to our project in this chapter, which is to conduct a preliminary, comparative empirical analysis of these two notions of equilibrium. As this phase of our work is preliminary, we concentrate on the exogenous notions of ePT-EB and CVE, rather than contend with the full range of 6 available equilibrium concepts.

At a high-level, our approach is simple: we consider a conveniently available sample of experimental, group-average games data of modest size, develop simple algorithms for identifying ePT-EB and eCVE, and examine the seek lower bounds on the goodness-of-fit of these two models to the available data, considered over a range of parameter values. We collect data on 24 games reported on in Selten and Chmura (2008—12 games) and Camerer (2003, Ch. 3—12 games), develop and apply straightforward, exhaustive algorithms for finding eCVE and ePT-EB in these games with various parameter settings, and examine the congruence of these equilibria to the observed group-average data. For each equilibrium concept, our analysis has two phases: in the first phase, we examine the best-fitting parameter settings for each model to each individual game; the purpose of this phase is to determine whether each solution concept has the ability to fit the available data, for some specification of player preferences. This test is a rather weak challenge for a parametric model, but it is useful for gaining a sense of any fundamental limitations on the ability of our models to explain certain patterns in the data. In the second phase, we identify the best-fitting models for each equilibrium concept to the entire set of data and identify common patterns in the parameter settings of these best-fitting models. Having analyzed each equilibrium concept’s fit to the data separately, we conclude by comparing and contrasting their fits to one another and to NE’s fit to the data.

A number of interesting findings emerge from the empirical analyses in this chapter. First, ePT-EB appear to fit the data substantially better than either eCVE or NE, although best-fitting
eCVE models also outperform NE by a substantial margin; the significance of ePT-EB’s success in this respect is further reinforced by the fact that, due to computational burden, we were only able to assess 4 distinct ePT-EB models, while, for eCVE, we were able to consider some 3993 distinct models. Second, best-fitting ePT-EB and eCVE models share some common, interesting properties: in these models, players tend to exhibit probabilistic hypersensitivity, rather than the probabilistic insensitivity typically found for PT preferences in single-decision maker experiments. In addition, in these models the unique point in players’ weighting functions where concavity gives way to convexity occurs in the region (0.5, 1.0), rather than at approximately 0.42, as Wakker (2010) indicates has been found in modal fits of parametric weighting functions for single-decision maker experiments. Third, best-fitting eCVE models were distinguished primarily by variations in weighting function parameters (for probabilistic sensitivity and asymmetry), with variations in the degree of loss aversion contributing relatively little to explanatory power.

The layout of this chapter is as follows: in section 2, we describe our methodological approach in detail. In section 3, we present the results of Phases 1 and 2 of our analysis for ePT-EB. In section 4, we present the results of Phases 1 and 2 of our analysis for eCVE. In section 5, we compare and contrast the results of sections 3 and 4, compare our findings to those for NE, and discuss the potential implications of our results.

4.1 Methodology

To examine the validity of ePT-EB and eCVE empirically, we draw primarily on the study of Selten and Chmura (2008) and Camerer’s (2003, Ch. 3) review of studies to gather a collection of group-average data on 2x2 and binary\(^2\) simultaneous, strategic-form games with unique NE. As this description suggests, our empirical analysis is restricted in a number of ways: first, we focus on exogenous equilibria, rather than myopic or non-myopic equilibria. Second, we consider only 2x2 or binary simultaneous games, to the exclusion of larger games and games with non-trivial information structure. Third, all of the games we study feature unique, fully mixed NE, and so

\(^2\)A binary game is a game with no more than 2 possible payoffs per player, considered over all pure-strategy profiles.
in particular we do not study games ordinally equivalent to, for example, the Battle of the Sexes (which has three, isolated NE) or Prisoner’s Dilemma (which is dominance-solvable for a unique NE). Fourth, strictly speaking, we identify \( \varepsilon \)-ePT-EB rather than exact eCVE. Fifth, we suppose all players have PT preferences with identical coefficients of loss aversion, probability weighting asymmetry, and probabilistic insensitivity; put simply, we assume symmetry of player preferences (but not symmetry of monetary payoffs in pure-strategy profiles). 24 games in Selten and Chmura and Camerer satisfy our criteria, giving 17 simultaneous, non-binary 2x2 games and 6 simultaneous, binary games of varying size.

The restrictions we have imposed on this chapter’s scope are motivated by a combination of simplicity, pragmatism, and theoretical interest. Narrowing focus to exogenous, unique, fully mixed equilibria in 2x2 games draws attention to the role of nonlinear probability weighting in analysis, a key feature of interest our study of the full model of PT preferences for risk. Focusing on exogenous, unique equilibria in 2x2 games also helps to keep our analysis simple and of manageable size; this is important especially for ePT-EB, because our exhaustive algorithm in this case is computationally quite burdensome, as we describe in more detail below. Binary games of arbitrary size are considered as well, because these provide a benchmark case in which our equilibria agree with NE, and for which we can therefore easily find the game’s ePT-EB and ePT-NE. Our assumption of symmetry in PT preferences has two purposes: first, it serves our interest in maintaining a manageable project scope, effectively halving the required number of computational trials per parameter in each model. Second, we are interested in descriptive applications, and the data sets we draw on afford only inter-person group averages; we cannot, therefore, fit these parameters to accommodate individual players’ behaviors.

Throughout our analysis we use the polynomial probability weighting family and the family of piece-wise linear value functions, both originally introduced in Chapter 2. Both families are analytically simple, contain relatively few parameters, and contain parameters with considerable interpretive content. Furthermore, this study can serve, in a secondary role, as an initial investigation into the empirical flexibility of the polynomial family. For the reader’s reference we give the value
function family as
\[
v_i(u, r_i) = \begin{cases} 
  u - r_i & \text{if } u \geq r_i \\
  (1 + \lambda)(u - r_i) & \text{if } u < r_i
\end{cases}
\]

and the polynomial weighting function family as:
\[
w_i(p) = \begin{cases} 
  \frac{1}{1+(A-1)^{2K+1}} \left[ (Ap + (1-A))^{2K+1} + (A-1)^{2K+1} \right] & \text{if } K \in \{0, 1, 2, \ldots\} \\
  \frac{-1}{1+(A-1)^{2K+1}} \left[ (Ap + (1-A))^{2K+1} + (A-1)^{2K+1} \right] & \text{if } K \in \{-1, -2, \ldots\}
\end{cases}
\]

We also recall that the typical form of a PT value function (with certain convenient symmetries assumed—see chapters 2-3 for greater detail) is:
\[
V_i(P) = \sum_{x_k < r_i} M^-(x_k) v_i(x_k, r_i) + \sum_{x_k > r_i} M^+(x_k) v_i(x_k, r_i) \quad (4.1)
\]

where
\[
M^-(x_k) = w_i(\Phi(x_k)) - w_i(\Phi(x_{k-1}))
\]
\[
M^+(x_k) = w_i(\Phi'(x_k)) - w_i(\Phi'(x_{k+1}))
\]

where \( w_i \) is player i’s weighting function, \( r_i \) is her reference point, \( \Phi \) is her cumulative distribution function (defined relative to each prospect under consideration), and \( \Phi' \) is her decumulative distribution function.

We study data drawn from 24 games: the first 17 consist of 12 2\times2 games from Selten and Chmura (2008) and 5 2\times2 games reviewed by Camerer (2003, Ch. 3). From Camerer’s review, the non-binary 2\times2 games we consider were originally studied by Lieberman (1962), Malcolm and Lieberman (1965), Binmore, Swierzbinski, and Proulx (2001), Bloomfield (1994), and Ochs (1995). The 6 binary games we use are from Camerer’s review as well: O’Neill (1987), Rapoport and Boebel (1992), Mookerjhee and Sopher (1997), Binmore, Swierzbinski, and Proulx (2001),
and Ochs (1995). Our primary focus is on the 17 non-binary 2x2 games, because our equilibrium concepts agree with NE for general binary games; this can be shown as an easy observation:

**Observation 4.1.1.** Let $G$ be a finite binary game. Then the sets of NE, (e,m)PT-NE, (e,m)PT-EB, and eCVE for $G$ are identical. Furthermore, the sets of subgame-perfect equilibria, nPT-NE, nPT-EB, and nCVE are identical for $G$.

Table 4.1 depicts the 17 2x2 games on which our analyses are concentrated, with monetary payoffs in the format (row player, column player):

---

3 Some studies, e.g. Ochs (1995), appear in both lists, because they studied both binary games and non-binary 2x2 games.

4 A rough variant of this observation was conjectured to the author by an anonymous reviewer of an early variant of Chapter 2; we extend our thanks to that reviewer.
Table 4.1: 17 2x2 Games and NE

| Game 1 |  | Game 2 |  | Game 3 |  |
|--------|  |--------|  |--------|  |
| 10,8   | 0,18 | U_{NE}=0.091 | 9,4   | 0,13, U_{NE}=0.182 | 8,6   | 0,14 | U_{NE}=0.273 |
| 9,9    | 10,8 | L_{NE}=0.909 | 6,7   | 8,5 | L_{NE}=0.727 | 7,7   | 10,4 | L_{NE}=0.909 |
| Game 4 |  | Game 5 |  | Game 6 |  |
| 7,4    | 0,11 | U_{NE}=0.364 | 7,2   | 0,9 | U_{NE}=0.364 | 7,1   | 1,7 | U_{NE}=0.455 |
| 5,6    | 9,2  | L_{NE}=0.818 | 4,5   | 8,1 | L_{NE}=0.727 | 3,5   | 8,0 | L_{NE}=0.636 |
| Game 7 |  | Game 8 |  | Game 9 |  |
| 10,12  | 4,22 | U_{NE}=0.091 | 9,7   | 3,16 | U_{NE}=0.182 | 8,9   | 3,17 | U_{NE}=0.273 |
| 9,9    | 14,8 | L_{NE}=0.909 | 6,7   | 11,5 | L_{NE}=0.727 | 7,7   | 13,4 | L_{NE}=0.909 |
| Game 10 |  | Game 11 |  | Game 12 |  |
| 7,6    | 2,13 | U_{NE}=0.364 | 7,4   | 2,11 | U_{NE}=0.364 | 7,3   | 3,9 | U_{NE}=0.455 |
| 5,6    | 11,2 | L_{NE}=0.818 | 4,5   | 10,1 | L_{NE}=0.727 | 3,5   | 10,0 | L_{NE}=0.636 |
| Game 13 |  | Game 14 |  | Game 15 |  |
| 3,-3   | -1,1 | U_{NE}=0.750 | -2,2  | 3,-3 | U_{NE}=0.167 | 80,40 | 40,100 | U_{NE}=0.400 |
| -9,9   | 3,-3 | L_{NE}=0.250 | -1,1  | -2,2 | L_{NE}=0.833 | 40,80 | 100,40 | L_{NE}=0.600 |
| Game 16 |  | Game 17 |  |
| 9,0    | 0,1  | U_{NE}=0.5  | 4,0   | 0,1  | U_{NE}=0.54 |
| 0,1    | 1,0  | L_{NE}=0.1  | 0,1   | 1,0  | L_{NE}=0.20 |
We refer to the row player’s upper strategy as $U$, and the column player’s left strategy as $L$; the NE probabilities of play for each of these quantities are also indicated in Table 4.1. Games 1 – 12 are from Selten and Chmura (2008), while Games 13 – 17 come from the various sources specified above, and collected in Camerer (2003, Ch. 3). The remaining 7 binary games, Games 18 – 23, are depicted in the appendix.

Games 1 – 17 games all have unique, fully mixed NE, and a number of them are ordinally equivalent. We leave the data for ordinally equivalent games disaggregated, as both eCVE and ePT-EB are sensitive to quantitative variations in the details (e.g. ratios, differences, scales) of game payoffs. Table 4.2 gives the observed group-average rates of choice for $U$ ($U_{obs}$) and $L$ ($L_{obs}$) each game, repeats the NE values, and gives the sum-of-squared deviations for NE:
Table 4.2: 17 Games and NE: Goodness of Fit

<table>
<thead>
<tr>
<th>Game</th>
<th>$U_{obs}$</th>
<th>$L_{obs}$</th>
<th>$U_{NE}$</th>
<th>$L_{NE}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Game 1</td>
<td>0.079</td>
<td>0.690</td>
<td>0.091</td>
<td>0.909</td>
</tr>
<tr>
<td>Game 2</td>
<td>0.217</td>
<td>0.527</td>
<td>0.182</td>
<td>0.727</td>
</tr>
<tr>
<td>Game 3</td>
<td>0.163</td>
<td>0.793</td>
<td>0.273</td>
<td>0.909</td>
</tr>
<tr>
<td>Game 4</td>
<td>0.286</td>
<td>0.736</td>
<td>0.364</td>
<td>0.818</td>
</tr>
<tr>
<td>Game 5</td>
<td>0.327</td>
<td>0.664</td>
<td>0.364</td>
<td>0.727</td>
</tr>
<tr>
<td>Game 6</td>
<td>0.445</td>
<td>0.596</td>
<td>0.455</td>
<td>0.636</td>
</tr>
<tr>
<td>Game 7</td>
<td>0.141</td>
<td>0.564</td>
<td>0.091</td>
<td>0.909</td>
</tr>
<tr>
<td>Game 8</td>
<td>0.250</td>
<td>0.586</td>
<td>0.182</td>
<td>0.727</td>
</tr>
<tr>
<td>Game 9</td>
<td>0.254</td>
<td>0.827</td>
<td>0.273</td>
<td>0.909</td>
</tr>
<tr>
<td>Game 10</td>
<td>0.366</td>
<td>0.699</td>
<td>0.364</td>
<td>0.818</td>
</tr>
<tr>
<td>Game 11</td>
<td>0.331</td>
<td>0.652</td>
<td>0.364</td>
<td>0.727</td>
</tr>
<tr>
<td>Game 12</td>
<td>0.439</td>
<td>0.604</td>
<td>0.455</td>
<td>0.636</td>
</tr>
<tr>
<td>Game 13</td>
<td>0.571</td>
<td>0.306</td>
<td>0.750</td>
<td>0.250</td>
</tr>
<tr>
<td>Game 14</td>
<td>0.251</td>
<td>0.915</td>
<td>0.167</td>
<td>0.833</td>
</tr>
<tr>
<td>Game 15</td>
<td>0.416</td>
<td>0.545</td>
<td>0.400</td>
<td>0.600</td>
</tr>
<tr>
<td>Game 16</td>
<td>0.600</td>
<td>0.300</td>
<td>0.500</td>
<td>0.100</td>
</tr>
<tr>
<td>Game 17</td>
<td>0.540</td>
<td>0.340</td>
<td>0.500</td>
<td>0.200</td>
</tr>
<tr>
<td>Sum of Squared Deviations</td>
<td>0.080809</td>
<td>0.352635</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The column-wise total sum-of-squares give a total sum-of-squared deviations, for NE, of 0.433444; we will later compare this value to those for ePT-EB and eCVE. The fit of classical NE to these 17 games can be seen visually in the Fig. 4.1.

![Figure 4.1: NE (x) v. Actual (y)](image)

Pearson’s $r$ for this relationship is quite large: $r = 0.928234$ with estimated p-value $p = 2.665e - 15$; thus NE appears to perform quite well in predicting group-average rates of strategic play in 2x2 simultaneous games. Nevertheless, we hope that ePT-EB and eCVE may improve on this result.

As a quick aside, we note that, throughout the foregoing analysis, it has been assumed that player preferences are risk neutral, i.e., that players’ utility functions are positive linear transformations of their payoffs. This assumption was also imposed by Camerer, for example (except in studies of binary games, where risk-neutrality is irrelevant), and we maintain it as far as possible for analytic simplicity.

---

5 All scatterplots were produced using Sage v. 5.11.5 (Stein et al., 2013).
6 Pearson’s $r$, partial correlations, and associated diagnostics were computed using R v. 3.1.1 (R Core Team, 2014). In particular, R’s lm, fit, and plot commands were used, as was the R package ppcor (Kim, 2012).
7 Our piece-wise linear value function may be understood as introducing an element of risk non-neutrality; this is an unavoidable consequence of incorporating loss aversion. Nevertheless, the piecewise-linear value function eliminates the use of value function curvature as a model of risk aversion/seeking; it only allows non-neutrality via the kink at its reference point.

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To compare the empirical fit of NE to the empirical fit of ePT-EB and eCVE, we must specify particular values for the parameters $K_i, A_i, \lambda_i,$ and $r_i,$ for each $i$. For each equilibrium concept, we do this by generating a number of sample points from the possible parameter space, and then searching from within these sampled points for the best-fitting model. We do this in two phases for each equilibrium concept, where in the first phase best-fitting models are found for each individual game, and in the second phase a single best-fitting model is found over all games, using the usual sum-of-squared deviations as our measure of fit.

For ePT-EB, for each game and player $i$, we set $r_i$ to the average possible payoff for player $i$, counted with multiplicity, and we set $\lambda_i$ to 1, so there is no effect of loss aversion. We primarily vary $A_i$ and $K_i$: $A_i$ is set (for both players) to 2 (symmetric weighting function) or 3, and $K_i$ to 1 (probabilistically insensitive, inverse-S shaped weighting function) or $-1$ (probabilistically hypersensitive, S-shaped weighting function), generating 4 different ePT-EB predictions per game. For ePT-EB, we study a larger set of models: the $r_i$ are set to the minimum, average, or maximum of each player’s payoffs, the $\lambda_i$ are varied from 1.0 to 3.0 in increments of 0.2, the $K_i$ are varied from $-2$ to 2 in increments of 0.4, and the $A_i$ are varied from 2 to 4 in increments of 0.2. Thus we examine 3993 possible eCVE models.

We study very different sets of parameter values for ePT-EB and eCVE because the algorithmic approaches necessary for finding these two solution concepts differ greatly. Both algorithms are exhaustive in nature, and both treat discretizations in which each player’s 1-dimensional probability simplex is approximated by a large number of point values. 100 points are used for the ePT-EB algorithm, and 250 points for the eCVE algorithm; thus, for each player, we separately examine equilibria where they may play their first pure strategy with probability 0.0, 0.01, 0.02, ..., 1.0 for ePT-EB and with probability 0.0, 0.004, 0.008, ..., 1.0 for eCVE. The primary reason that we are able to use a much finer discretization and study many more possible models for eCVE is that, when finding an eCVE, each player’s equilibrium mixed strategy is entirely determined by the requirement that her opponent be indifferent between her two available pure strategies. We therefore only need to exhaustively step through, for example, the column player’s possible probabilities, in
order to find the two consecutive mixed strategies of the column player for which the rank-ordering
of the row player’s pure-strategy payoffs reverses\(^8\); this means that an individual implementation of
the algorithm requires only \(O(N)\) calls to a subroutine for computing PT payoffs, where \(N\) is the
number of grid points used in the discretization of each player’s simplex. By contrast, the exhaustive
search used for ePT-EB must examine every possible mixed-strategy profile in the discretization,
and for each of these, check whether an approximate ePT-EB has been found; this requires \(O(N^2)\)
calls to a subroutine for checking whether an ePT-EB has been found. Since our subroutine for
checking whether an ePT-EB has been found in turn makes \(O(N)\) calls to a PT value function
subroutine, we make in this case at least \(O(N^3)\) calls to a PT-value subroutine, a figure which
grows considerably faster in \(N\) than \(O(N)\). The details of our subroutine for determining if an
approximate ePT-EB has been found involve additional computational burdens as well. Assessing
whether some mixed-strategy profile constitutes an ePT-EB requires: A) finding both players’
(approximate, i.e. within some \(\varepsilon\)) argmax sets, holding the other player’s mixed strategy fixed at its
candidate equilibrium value, and B) checking whether the candidate equilibrium mixed strategy
for each player lies in the convex hull of that player’s argmax set. We have already described the
computational cost for handling A): we simply iterate, for each player, over their possible mixed
strategies (within our discretized set), and identify those in this set that deliver within some \(\varepsilon\) of
the maximum possible value. This introduces \(2N\), or \(O(N)\), additional PT-value function calls.
Addressing B) is somewhat more involved: we are faced with the problem of determining whether
a vector \(v\) (player 1’s mixed-strategy, written as a vector) is in the convex hull of a finite set of
such vectors, \(co[v_1,v_2,...,v_M]\). This can be dealt with by continuous linear programming, and in
particular is equivalent to solving the linear program:

\[
\max \sum_{i=1}^{N} w_i
\]

\(^8\)We note that seeking a point of reversal in this manner presupposes a lack of pure-strategy equilibria; this condition
is satisfied for us, since games without pure-strategy NE must also lack pure-strategy ePT-EB and eCVE, by the results
of Chapters 2-3.
subject to

\[ 0 \leq \sum_{i=1}^{N} w_i \leq 1 \]

\[ \sum_{i=1}^{N} w_i v_{ji} = q_j \forall j = 1, \ldots, H \]

where the variables \( w_i \) correspond to weights to be placed on the vectors, and \( v_{ji} \) is the \( j \)th component of vector \( i \). In our work, we have made use of an implementation of the simplex algorithm in Python2.7 (Python, 2014; simplex implementation originally due to Wietsma, 2012, with one minor bug fix performed by the present author) for use in our code, and applied it to solve a suitable linear program for each convex-hull inclusion to be checked. Although the simplex algorithm is quite efficient in practice, this use of the simplex algorithm twice for each of the \( O(N^2) \) mixed-strategy profiles to be checked introduces considerable additional computational burden.

Thus identifying ePT-EB for Games 1 – 17 is a non-trivial matter, primarily due to the complexity of assessing the players’ best-reply inclusions. In fact, there is one additional problem in finding ePT-EB: as described above, we must find \( \varepsilon \)-ePT-EB (see Chapter 2 for a definition), rather than exact ePT-EB. This situation is no different from our approach with eCVE; however, where with eCVE we are guaranteed (in 2x2 games without a pure-strategy NE) that our algorithm will terminate having found at least one approximate eCVE, no such guarantee exists for our ePT-EB algorithm. Two properties contribute to this difference: first, since we only need to consider a single player’s discretized simplex for eCVE, we know there must exist at least one point of preference reversal between pure strategies for each player, as the remaining player’s probability of play is varied. This is not so for ePT-EB, since we must consider discretizations for both players’ simplices simultaneously. Second, we have no elementary closed-form expression, at a given mixed-strategy profile \( \sigma \), for the degree to which each player’s mixed strategy is out-of-equilibrium; by contrast, for eCVE, we could take the absolute value of each player’s difference in their pure-strategy payoffs as a suitable measure, and minimize this quantity for each player—such an approach would clearly converge for any specified degree of precision. We have no corresponding value for ePT-EB, where even assessing whether an approximate ePT-EB has been found requires the use of the simplex
Thus we introduce a relaxation parameter $\epsilon$ into our subroutine (indicating the degree to which “best replies” are merely $\epsilon$ best-replies) for checking whether an $\epsilon$-ePT-EB has been found, but this introduces a considerable complication: a suitable value for $\epsilon$ must be chosen. Clearly we need $\epsilon \geq 0$, but $\epsilon$ can be neither too small (which will tend to generate a singleton best-reply set and an empty ePT-EB set) nor too large (which will tend to generate needlessly large best-reply sets and sets of $\epsilon$-ePT-EB). It is possible to estimate an upper bound on the necessary size of $\epsilon$ (to guarantee a non-empty ePT-EB set is returned), using suitable Lipschitz constants for the weighting functions to determine how much the players’ value functions may vary in between any two discretization grid points; however, we find that in practice these upper bounds are much too large to be useful. As a result, our technique for finding suitable $\epsilon$ is itself exhaustive: we begin by identifying, for each game, $\epsilon \geq 0$ such that there are no $\epsilon$-ePT-EB ($\epsilon = 0$ typically satisfies this criterion), and increase the value of $\epsilon$ by small increments (we generally increment $\epsilon$ by hundredths, thousands, or millionths of a unit, depending on our past success in finding $\epsilon$-ePT-EB for a game) until the set of $\epsilon$-ePT-EB found is non-empty. Often this procedure yields a set of consecutive equilibria. In these cases, we take the average of the set of $\epsilon$-ePT-EB found as our estimate of the true $\epsilon$-ePT-EB. Although this procedure for choosing suitable $\epsilon$ in order to estimate exact ePT-EB is somewhat heuristic, we know that, for any $d > 0$, if we choose a sufficiently small $\epsilon$ and a sufficiently fine discretization, then our set of $\epsilon$-ePT-EB found will be non-empty, and all $\epsilon$-ePT-EB found will lie within distance $d$ of a true ePT-EB; this is a straightforward consequence of the Bolzano-Weierstrass Theorem, compactness of the simplex, and continuity of our players’ value functions.\footnote{To show this formally: suppose it does not hold; this allows us to derive an infinite set of points in the cross-product of the players’ simplices, which must have a convergent subsequence by Bolzano-Weierstrass. Continuity implies that the limit of this subsequence is the desired ePT-EB.}

In practice, our algorithm for finding all eCVE of a 2x2 game runs in less than 1 second, while our algorithm for finding all $\epsilon$-ePT-EB of a 2x2 game requires at least 10 minutes, and in many cases consumes more than 1 hour of run-time. This dramatic difference in run-speed explains our restricted sampling of parameter space for ePT-EB, and our thorough search of parameter space for
eCVE. We note, despite our algorithm for eCVE performing dramatically faster than our ePT-EB algorithm for 2x2 games, both algorithms scale poorly with game size; adding additional players or numbers of strategies per player entails an exponential increase in run-time for both algorithms. The primary virtues of our code are its simplicity and its ability to exhaustively search (a discretization of) the entire space of mixed-strategy profiles.

Full code for each of our two algorithms is available in the appendix to this chapter. We now describe the results of both algorithms.

4.2 ePT-EB: Computational Evidence

In this section, we describe our computational evidence for ePT-EB—both the estimates arrived at for each of our models of ePT-EB and each of our games, and the degree of fit this implies for each model. As previously explained, we separate this section into two phases: in the first phase, we consider fit of each model to each individual game. In the second phase, we consider the fit of each model over all games.

4.2.1 ePT-EB: Phase One

Since only four models were for for ePT-EB to each game, it is possible for us to reasonably display our ePT-EB estimates for each model and each game. Tables 4.3-4.6 depict this data:
Table 4.3: ePT-EB Estimated Equilibria Goodness-of-Fit, $K = -1, A = 2$

<table>
<thead>
<tr>
<th>Game</th>
<th>$U_{obs}$</th>
<th>$L_{obs}$</th>
<th>$U_{PT - EB}$</th>
<th>$L_{PT - EB}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Game 1</td>
<td>0.079</td>
<td>0.690</td>
<td>0.170</td>
<td>0.710</td>
</tr>
<tr>
<td>Game 2</td>
<td>0.217</td>
<td>0.527</td>
<td>0.357</td>
<td>0.525</td>
</tr>
<tr>
<td>Game 3</td>
<td>0.163</td>
<td>0.793</td>
<td>0.355</td>
<td>0.720</td>
</tr>
<tr>
<td>Game 4</td>
<td>0.286</td>
<td>0.736</td>
<td>0.470</td>
<td>0.580</td>
</tr>
<tr>
<td>Game 5</td>
<td>0.327</td>
<td>0.664</td>
<td>0.488</td>
<td>0.528</td>
</tr>
<tr>
<td>Game 6</td>
<td>0.445</td>
<td>0.596</td>
<td>0.490</td>
<td>0.535</td>
</tr>
<tr>
<td>Game 7</td>
<td>0.141</td>
<td>0.564</td>
<td>0.125</td>
<td>0.730</td>
</tr>
<tr>
<td>Game 8</td>
<td>0.250</td>
<td>0.586</td>
<td>0.275</td>
<td>0.520</td>
</tr>
<tr>
<td>Game 9</td>
<td>0.254</td>
<td>0.827</td>
<td>0.310</td>
<td>0.730</td>
</tr>
<tr>
<td>Game 10</td>
<td>0.366</td>
<td>0.699</td>
<td>0.405</td>
<td>0.590</td>
</tr>
<tr>
<td>Game 11</td>
<td>0.331</td>
<td>0.652</td>
<td>0.465</td>
<td>0.530</td>
</tr>
<tr>
<td>Game 12</td>
<td>0.439</td>
<td>0.604</td>
<td>0.510</td>
<td>0.490</td>
</tr>
<tr>
<td>Game 13</td>
<td>0.571</td>
<td>0.306</td>
<td>0.482</td>
<td>0.582</td>
</tr>
<tr>
<td>Game 14</td>
<td>0.251</td>
<td>0.915</td>
<td>0.650</td>
<td>0.430</td>
</tr>
<tr>
<td>Game 15</td>
<td>0.416</td>
<td>0.545</td>
<td>0.514</td>
<td>0.482</td>
</tr>
<tr>
<td>Game 16</td>
<td>0.600</td>
<td>0.300</td>
<td>0.267</td>
<td>0.509</td>
</tr>
<tr>
<td>Game 17</td>
<td>0.540</td>
<td>0.340</td>
<td>0.395</td>
<td>0.500</td>
</tr>
</tbody>
</table>

| Sum of Squared Deviations | 0.2356975382 | 0.1989763214 |
Table 4.4: ePT-EB Estimated Equilibria Goodness-of-Fit, $K = -1, A = 3$

<table>
<thead>
<tr>
<th>Game</th>
<th>$U_{obs}$</th>
<th>$L_{obs}$</th>
<th>$U_{PT-EB}$</th>
<th>$L_{PT-EB}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Game 1</td>
<td>0.079</td>
<td>0.690</td>
<td>0.094</td>
<td>0.755</td>
</tr>
<tr>
<td>Game 2</td>
<td>0.217</td>
<td>0.527</td>
<td>0.228</td>
<td>0.622</td>
</tr>
<tr>
<td>Game 3</td>
<td>0.163</td>
<td>0.793</td>
<td>0.295</td>
<td>0.750</td>
</tr>
<tr>
<td>Game 4</td>
<td>0.286</td>
<td>0.736</td>
<td>0.368</td>
<td>0.656</td>
</tr>
<tr>
<td>Game 5</td>
<td>0.327</td>
<td>0.664</td>
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<td>0.630</td>
</tr>
<tr>
<td>Game 6</td>
<td>0.445</td>
<td>0.596</td>
<td>0.470</td>
<td>0.575</td>
</tr>
<tr>
<td>Game 7</td>
<td>0.141</td>
<td>0.564</td>
<td>0.111</td>
<td>0.720</td>
</tr>
<tr>
<td>Game 8</td>
<td>0.250</td>
<td>0.586</td>
<td>0.214</td>
<td>0.630</td>
</tr>
<tr>
<td>Game 9</td>
<td>0.254</td>
<td>0.827</td>
<td>0.291</td>
<td>0.767</td>
</tr>
<tr>
<td>Game 10</td>
<td>0.366</td>
<td>0.699</td>
<td>0.350</td>
<td>0.670</td>
</tr>
<tr>
<td>Game 11</td>
<td>0.331</td>
<td>0.652</td>
<td>0.380</td>
<td>0.635</td>
</tr>
<tr>
<td>Game 12</td>
<td>0.439</td>
<td>0.604</td>
<td>0.470</td>
<td>0.580</td>
</tr>
<tr>
<td>Game 13</td>
<td>0.571</td>
<td>0.306</td>
<td>0.680</td>
<td>0.360</td>
</tr>
<tr>
<td>Game 14</td>
<td>0.251</td>
<td>0.915</td>
<td>0.339</td>
<td>0.817</td>
</tr>
<tr>
<td>Game 15</td>
<td>0.416</td>
<td>0.545</td>
<td>0.440</td>
<td>0.560</td>
</tr>
<tr>
<td>Game 16</td>
<td>0.600</td>
<td>0.300</td>
<td>0.517</td>
<td>0.171</td>
</tr>
<tr>
<td>Game 17</td>
<td>0.540</td>
<td>0.340</td>
<td>0.507</td>
<td>0.287</td>
</tr>
<tr>
<td>Sum of Squared Deviations</td>
<td>0.0865788971</td>
<td>0.0682092171</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 4.5: ePT-EB Estimated Equilibria Goodness-of-Fit, $K = 1, A = 2$

<table>
<thead>
<tr>
<th>Game</th>
<th>$U_{obs}$</th>
<th>$L_{obs}$</th>
<th>$U_{PT-EB}$</th>
<th>$L_{PT-EB}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Game 1</td>
<td>0.079</td>
<td>0.690</td>
<td>0.030</td>
<td>0.240</td>
</tr>
<tr>
<td>Game 2</td>
<td>0.217</td>
<td>0.527</td>
<td>0.060</td>
<td>0.270</td>
</tr>
<tr>
<td>Game 3</td>
<td>0.163</td>
<td>0.793</td>
<td>0.010</td>
<td>0.945</td>
</tr>
<tr>
<td>Game 4</td>
<td>0.286</td>
<td>0.736</td>
<td>0.155</td>
<td>0.915</td>
</tr>
<tr>
<td>Game 5</td>
<td>0.327</td>
<td>0.664</td>
<td>0.150</td>
<td>0.865</td>
</tr>
<tr>
<td>Game 6</td>
<td>0.445</td>
<td>0.596</td>
<td>0.255</td>
<td>0.820</td>
</tr>
<tr>
<td>Game 7</td>
<td>0.141</td>
<td>0.564</td>
<td>0.030</td>
<td>0.225</td>
</tr>
<tr>
<td>Game 8</td>
<td>0.250</td>
<td>0.586</td>
<td>0.060</td>
<td>0.785</td>
</tr>
<tr>
<td>Game 9</td>
<td>0.254</td>
<td>0.827</td>
<td>0.100</td>
<td>0.945</td>
</tr>
<tr>
<td>Game 10</td>
<td>0.366</td>
<td>0.699</td>
<td>0.160</td>
<td>0.910</td>
</tr>
<tr>
<td>Game 11</td>
<td>0.331</td>
<td>0.652</td>
<td>0.160</td>
<td>0.855</td>
</tr>
<tr>
<td>Game 12</td>
<td>0.439</td>
<td>0.604</td>
<td>0.260</td>
<td>0.643</td>
</tr>
<tr>
<td>Game 13</td>
<td>0.571</td>
<td>0.306</td>
<td>0.920</td>
<td>0.155</td>
</tr>
<tr>
<td>Game 14</td>
<td>0.251</td>
<td>0.915</td>
<td>0.100</td>
<td>0.950</td>
</tr>
<tr>
<td>Game 15</td>
<td>0.416</td>
<td>0.545</td>
<td>0.204</td>
<td>0.363</td>
</tr>
<tr>
<td>Game 16</td>
<td>0.600</td>
<td>0.300</td>
<td>0.710</td>
<td>0.030</td>
</tr>
<tr>
<td>Game 17</td>
<td>0.540</td>
<td>0.340</td>
<td>0.690</td>
<td>0.070</td>
</tr>
</tbody>
</table>

Sum of Squared Deviations | 0.8726296723 | 0.515679429
Table 4.6: ePT-EB Estimated Equilibria Goodness-of-Fit, $K = 1, A = 3$

<table>
<thead>
<tr>
<th>Game</th>
<th>$U_{obs}$</th>
<th>$L_{obs}$</th>
<th>$U_{PT-EB}$</th>
<th>$L_{PT-EB}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Game 1</td>
<td>0.079</td>
<td>0.690</td>
<td>0.030</td>
<td>0.900</td>
</tr>
<tr>
<td>Game 2</td>
<td>0.217</td>
<td>0.527</td>
<td>0.060</td>
<td>0.790</td>
</tr>
<tr>
<td>Game 3</td>
<td>0.163</td>
<td>0.793</td>
<td>0.110</td>
<td>0.935</td>
</tr>
<tr>
<td>Game 4</td>
<td>0.286</td>
<td>0.736</td>
<td>0.170</td>
<td>0.900</td>
</tr>
<tr>
<td>Game 5</td>
<td>0.327</td>
<td>0.664</td>
<td>0.170</td>
<td>0.865</td>
</tr>
<tr>
<td>Game 6</td>
<td>0.445</td>
<td>0.596</td>
<td>0.300</td>
<td>0.845</td>
</tr>
<tr>
<td>Game 7</td>
<td>0.141</td>
<td>0.564</td>
<td>0.030</td>
<td>0.045</td>
</tr>
<tr>
<td>Game 8</td>
<td>0.250</td>
<td>0.586</td>
<td>0.070</td>
<td>0.805</td>
</tr>
<tr>
<td>Game 9</td>
<td>0.254</td>
<td>0.827</td>
<td>0.050</td>
<td>0.960</td>
</tr>
<tr>
<td>Game 10</td>
<td>0.366</td>
<td>0.699</td>
<td>0.190</td>
<td>0.935</td>
</tr>
<tr>
<td>Game 11</td>
<td>0.331</td>
<td>0.652</td>
<td>0.163</td>
<td>0.878</td>
</tr>
<tr>
<td>Game 12</td>
<td>0.439</td>
<td>0.604</td>
<td>0.300</td>
<td>0.840</td>
</tr>
<tr>
<td>Game 13</td>
<td>0.571</td>
<td>0.306</td>
<td>0.910</td>
<td>0.165</td>
</tr>
<tr>
<td>Game 14</td>
<td>0.251</td>
<td>0.915</td>
<td>0.900</td>
<td>0.940</td>
</tr>
<tr>
<td>Game 15</td>
<td>0.416</td>
<td>0.545</td>
<td>0.197</td>
<td>0.385</td>
</tr>
<tr>
<td>Game 16</td>
<td>0.600</td>
<td>0.300</td>
<td>0.910</td>
<td>0.040</td>
</tr>
<tr>
<td>Game 17</td>
<td>0.540</td>
<td>0.340</td>
<td>0.585</td>
<td>0.070</td>
</tr>
<tr>
<td>Sum of Squared Deviations</td>
<td>0.94662852</td>
<td>0.9362716299</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In this phase of analysis, the primary question we would hope to address is: can ePT-EB explain the data available from each game at all? Although our weighting functions appear to have sufficiently many parameters to fit an arbitrary pair of data points (the number produced by each individual game), their behavior is non-linear in these parameter values, and in many games exhibits finite asymptotes as parameters are varied from $-\infty$ to $\infty$. Thus it is possible that ePT-EB may not be able to explain the data in several of the games we have considered; however, to address this question satisfactorily, we require that many models be tested. 4 models is certainly insufficient for understanding the global limits of ePT-EB’s behavior as we traverse its parameter space.

Nevertheless, we can answer a related question: for each of the games in question, which model performs best? Answering this question may give us some insight into whether different games require different behavior of players’ weighting functions to model adequately. Again we use the sum of squared deviations (over each game’s two data points) as our goodness-of-fit criterion: as it happens, the answer to this question is rather lop-sided: $K = -1, A = 3$ performs best in Games 1, 2, 3, 4*, 5*, 6, 7, 8*, 9, 10, 11, 12*, 13, 14*, 15, and 17, where an asterisk indicates that, although this model was best among the 4 ePT-EB models studied, it was outperformed by NE (though NE, too, is of course a special case of ePT-EB). Only in Game 16 did another of our 4 models outperform $K = -1, A = 3$; there, $K = -1, A = 2$ was the best-performing model. Thus the data in Tables 4.3-4.6 is rather well-behaved, in that negative values of $K$ seem to decisively outperform positive ones, to most often outperform a 0 value of $K$ (which corresponds to NE), and most often it appears advantageous for a model to involve some degree of asymmetry ($A = 3$ rather than $A = 2$), though, of the two phenomena, $K$’s modeling of probabilistic hypersensitivity appears to be the more important.

A number of data points in Tables 4.3 – 4.6 should be noted as problematic, in that the sets of $\varepsilon$-equilibria from which they were computed were unusually large. Specifically, the following models were derived from $\varepsilon$-ePT-EB sets that ranged over a set of discrete grid points with range in one of its two mixed strategies larger than 0.15: for $A = 2, K = 1$, Game 1 (L range: 0.22), Game 2 (0.26), Game 7 (0.21), Game 12 (0.43), Game 15 (0.66), for $A = 3, K = 1$, Game 15 (L range: 119
0.70) and Game 17 (U range: 0.23), and for $K = -1, A = 2$, Game 8 (U range: 0.19). Of these, Game 12’s $A = 2, K = 1$ L estimate is especially curious, in that its set of $\epsilon$-ePT-EB appears to be non-convex, which is to say that it “skips” intermediate grid points, rather than containing all possible mixed strategies between two extremes; Game 12’s $A = 2, K = 1$ L estimate is the only one of this kind. In addition to these estimates of great variation, several additional estimates were derived from sets with ranges between 0.1 and 0.15: for $K = -1, A = 3$, Game 2 (U range: 0.14) and Game 14 (L range: 0.13), for $K = -1, A = 2$, Game 8 (L range: 0.19), for $K = 1, A = 2$, Game 8 (0.15) and Game 15 (U range: 0.13), and for $K = 1, A = 3$, Game 8 (0.15) and Game 15 (U range: 0.13).

Estimates of great variation should be regarded with some suspicion, and as a result, we infer that our ePT-EB data for $K = 1, A = 2$ and $K = 1, A = 3$ may contain considerable variation due entirely to algorithmic failures. Fortunately, these two models seem to perform rather poorly in any event, and the best-performing model, $K = -1, A = 3$, has just two points derived from sets with range greater than 0.09. In Phase 2 of this analysis, we single out this best-performing model for further consideration.

4.2.2 ePT-EB: Phase Two

To examine the fit of our $K = 1, A = 3$ model to the data, we first show the relationship between this model’s predictions and the observed data visually:
Relative to the corresponding Fig. 4.1 for NE, there appears to be some noticeable degree of improvement in the fit; there is less dispersion about the 45-degree line than in Fig. 4.1, as expected. In this case, Pearson’s $r$ is estimated as $r = 0.94793$, with an estimated $p$-value less than $2.2e-16$. Although this point estimate for the correlation between observed, group-average mixed strategy use and the $K = -1, A = 3$ ePT-EB model is higher than the estimated correlation of $0.9143076$ for NE and the observed data, we note that 90% confidence intervals for each of these correlations contains the other. For the ePT-EB Pearson’s $r$, for example, a 90% confidence interval is given by $(0.9079177, 0.9708218)$, while a 90% confidence interval for the estimated NE Pearson’s $r$ is $(0.8740654, 0.9596053)$. We note that, in either case, the inclusion is slight, and our data set is limited in both size and quality, so that follow-up work to determine if this is a true difference may be merited; however, we have also optimized, in at least a very limited fashion (given that we only explored 4 possible ePT-EB models), over the parameter space of ePT-EB, and in this sense we might expect that some degree of what we have found is due to overfitting. In any event, follow-up work should help to discern whether $K = -1, A = 3$ truly outperforms NE as a description of typical player behavior in 2x2 simultaneous games.

Before moving on to consider the evidence for eCVE, we make a few observations about Tables
4.3-4.6: first, it is notable that our two best-performing models, $K = -1, A = 3$ and $K = -1, A = 2$, both feature a negative value of $K$. This property is interesting because, in single-decision maker settings, modal parameter fits tend to give inverse-S-shaped weighting functions (Wakker, 2010), but negative $K$ correspond, in the polynomial family, to S-shaped weighting functions; this leads us to the intriguing suggestion that players may react to strategic risk differently from exogenous risk generated by nature. Second, our best-fitting model features a non-symmetric weighting function; this is less surprising, as in the single decision-maker literature, modal fits tend to produce asymmetric weighting functions with the break between concavity and convexity occurring at approximately $p = 0.42$. However, considered in tandem with a negative value of $K$, an asymmetry parameter with value $A > 2$ generates a break between conavity and convexity at values of $p > 0.5$, and in particular our best-fitting model has a weighting function with break between concavity and convexity occurring at approximately $p = 0.65$. As 0.5 is roughly equidistant from 0.65 and 0.42, we are tempted to view the “strategic-risk weighting function” as equivalent to the standard, modally-fit weighting functions in single-decision maker settings, but reflected about the 45-degree line.

We now develop a similar analysis for eCVE; as we will see, similar conclusions apply for eCVE as for ePT-EB.

4.3 eCVE: Computational Evidence

For eCVE, we have a considerable body of data: some 3993 individual models were evaluated on each game, yielding 67881 individual predictions. As a result, displaying the full set of data in table form would be unwieldy; instead, we rely on a Python script to select the best-performing models, and we focus our analyses on identifying common characteristics of these models. As in studying ePT-EB, we separate our analysis into two phases: one for studying fit to individual games, and the latter for studying fit to all games considered simultaneously.
4.3.1 eCVE: Phase One

In this phase, we seek to address whether eCVE can, even in principle, explain the data in each of our 17 games. Our approach to this question is straightforward: since we have discretized each player’s probability simplex into hundredths, we consider a game “explainable” if the players’ strategies in the best-fitting eCVE model both lie within 0.005 of their observed strategic probabilities. Thus in this section we consider only the single best-fitting eCVE model, considered within each game. Table 4.7 displays the relevant data:
Table 4.7: eCVE Best-Fitting Models, by Game

<table>
<thead>
<tr>
<th>Game</th>
<th>$U_{obs}$</th>
<th>$L_{obs}$</th>
<th>$U_{CVE}$</th>
<th>$L_{CVE}$</th>
<th>$A_{Best}$</th>
<th>$K_{Best}$</th>
<th>$r_{Best}$</th>
<th>$\lambda_{Best}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Game 1</td>
<td>0.079</td>
<td>0.690</td>
<td>0.264</td>
<td>0.816</td>
<td>3.8</td>
<td>2.0</td>
<td>Pess</td>
<td>1.0</td>
</tr>
<tr>
<td>Game 2</td>
<td>0.217</td>
<td>0.527</td>
<td>0.236</td>
<td>0.516</td>
<td>4.0</td>
<td>-1.6</td>
<td>Opt</td>
<td>1.0</td>
</tr>
<tr>
<td>Game 3</td>
<td>0.163</td>
<td>0.793</td>
<td>0.248</td>
<td>0.780</td>
<td>4.0</td>
<td>-1.6</td>
<td>Opt</td>
<td>1.0</td>
</tr>
<tr>
<td>Game 4</td>
<td>0.286</td>
<td>0.736</td>
<td>0.296</td>
<td>0.744</td>
<td>3.4</td>
<td>-0.4</td>
<td>Opt</td>
<td>1.0</td>
</tr>
<tr>
<td>Game 5</td>
<td>0.327</td>
<td>0.664</td>
<td>0.336</td>
<td>0.664</td>
<td>3.0</td>
<td>-1.6</td>
<td>Opt</td>
<td>1.0</td>
</tr>
<tr>
<td>Game 6</td>
<td>0.445</td>
<td>0.596</td>
<td>0.448</td>
<td>0.604</td>
<td>2.4</td>
<td>-0.8</td>
<td>Opt</td>
<td>1.0</td>
</tr>
<tr>
<td>Game 7</td>
<td>0.141</td>
<td>0.564</td>
<td>0.252</td>
<td>0.704</td>
<td>2.6</td>
<td>-2.0</td>
<td>Opt</td>
<td>1.0</td>
</tr>
<tr>
<td>Game 8</td>
<td>0.250</td>
<td>0.586</td>
<td>0.268</td>
<td>0.616</td>
<td>2.2</td>
<td>-0.4</td>
<td>Opt</td>
<td>1.0</td>
</tr>
<tr>
<td>Game 9</td>
<td>0.254</td>
<td>0.827</td>
<td>0.248</td>
<td>0.816</td>
<td>4.0</td>
<td>-1.6</td>
<td>Opt</td>
<td>1.0</td>
</tr>
<tr>
<td>Game 10</td>
<td>0.366</td>
<td>0.699</td>
<td>0.368</td>
<td>0.700</td>
<td>3.6</td>
<td>-1.6</td>
<td>Avg</td>
<td>1.4</td>
</tr>
<tr>
<td>Game 11</td>
<td>0.331</td>
<td>0.652</td>
<td>0.336</td>
<td>0.668</td>
<td>3.0</td>
<td>-1.2</td>
<td>Opt</td>
<td>1.0</td>
</tr>
<tr>
<td>Game 12</td>
<td>0.439</td>
<td>0.604</td>
<td>0.432</td>
<td>0.608</td>
<td>2.4</td>
<td>-0.8</td>
<td>Opt</td>
<td>1.0</td>
</tr>
<tr>
<td>Game 13</td>
<td>0.571</td>
<td>0.306</td>
<td>0.576</td>
<td>0.300</td>
<td>2.2</td>
<td>-0.4</td>
<td>Avg</td>
<td>1.6</td>
</tr>
<tr>
<td>Game 14</td>
<td>0.251</td>
<td>0.915</td>
<td>0.180</td>
<td>0.812</td>
<td>4.0</td>
<td>0.8</td>
<td>Pess</td>
<td>1.0</td>
</tr>
<tr>
<td>Game 15</td>
<td>0.416</td>
<td>0.545</td>
<td>0.436</td>
<td>0.568</td>
<td>3.8</td>
<td>-1.6</td>
<td>Pess</td>
<td>1.0</td>
</tr>
<tr>
<td>Game 16</td>
<td>0.600</td>
<td>0.300</td>
<td>0.504</td>
<td>0.300</td>
<td>2.4</td>
<td>-1.6</td>
<td>Pess</td>
<td>1.0</td>
</tr>
<tr>
<td>Game 17</td>
<td>0.540</td>
<td>0.340</td>
<td>0.504</td>
<td>0.340</td>
<td>3.0</td>
<td>-1.6</td>
<td>Pess</td>
<td>1.0</td>
</tr>
</tbody>
</table>

In Table 4.7, “Pess,” “Avg,” and “Opt” denote our three versions of reference-point determination: Pessimistic (minimum), Average, and Optimistic (maximum). $A$, $K$, and $\lambda$ bear their usual interpretations.

In terms of ability to explain the data by the test criterion we defined, Table 4.7 makes clear that eCVE performs abysmally: over the parameter ranges we considered, eCVE fails to explain even a single one of the 17 games listed. Although in several games one of the two players’ strategies lie within 0.005 of their observed value, there is not a single game in which this holds true for both strategies. It may be that a finer or broader sampling of the parameter space would correct this problem, but it could also be that eCVE has fundamental limitations in its ability to explain this data set.

Table 4.7 also enables us to make a number of interesting inferences about the best-performing parameter values, in the same spirit as in the preceding section for ePT-EB. Negative values of $K$ seem to be important, optimistic and pessimistic reference-setting appear almost exclusively, the loss
aversion coefficient $\lambda$ appears to be relatively unimportant, and the symmetry parameter, $A$, varies widely, with perhaps some tendency to take on large values ($A \geq 3.0$). The importance of negative $K$ is consistent with our findings for eCVE; this reinforces our interest in probabilistic hypersensitivity for strategic risk. The role of optimism and pessimism stands in contrast to our choice to use the average as reference point in our ePT-EB work, and suggests that exploring optimistic and pessimistic reference-setting may be worthwhile there as well. The relative unimportance of $\lambda$ is perhaps surprising given its significance in single decision-maker PT, although it should be noted that, with pessimistic reference-setting, $\lambda$ is by definition irrelevant, and with optimistic reference-setting, $\lambda$ induces an identical scaling and shift of all payoffs; thus it is unsurprising that it is exactly the two cases with average reference-setting that yield a best-fitting $\lambda > 1.0$. Thus, while our results are consistent with $\lambda$ being largely irrelevant, they are also consistent with the influence of the reference point on probability weighting (vis-a-vis rank dependence) simply being more important than the effects of the reference point on loss aversion.

In summary, the computational results of this section pose a challenge for eCVE’s empirical validity, but generally reinforce the themes of our earlier analysis of ePT-EB, especially as relates to the importance of probabilistic insensitivity and the relative unimportance of $\lambda$. Table 4.7’s results also pose interesting questions about the variation in $K$ and players’ reference-setting. We bear these insights in mind as we move into our final computational section; we will find similar themes emerge in fitting a single eCVE model to all 17 of our games simultaneously.

4.3.2 eCVE: Phase Two

In this section, we study the best-fitting eCVE models to our 17 games. As we were able to evaluate a very large number of eCVE models, we are able to examine commonalities among the top 20 best-fitting models, in order to gain deeper insight into the importance or non-importance of each parameter to eCVE’s goodness-of-fit. Table 4.8 summarizes these top 20 models and their sum-of-squared deviations:
Table 4.8: eCVE Best-Fitting Models, by Game

<table>
<thead>
<tr>
<th>$r_{Best}$</th>
<th>$K_{Best}$</th>
<th>$A_{Best}$</th>
<th>$\lambda_{Best}$</th>
<th>Sum-of-Squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>Avg</td>
<td>-1.6</td>
<td>4.0</td>
<td>1.0</td>
<td>0.21866</td>
</tr>
<tr>
<td>Avg</td>
<td>-1.2</td>
<td>4.0</td>
<td>1.0</td>
<td>0.22402</td>
</tr>
<tr>
<td>Avg</td>
<td>-1.6</td>
<td>3.8</td>
<td>1.0</td>
<td>0.225748</td>
</tr>
<tr>
<td>Avg</td>
<td>-2.0</td>
<td>4.0</td>
<td>1.0</td>
<td>0.226316</td>
</tr>
<tr>
<td>Avg</td>
<td>-1.2</td>
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We first assess the fit of the best-performing eCVE model \((K = -1.6, A = 4.0, \lambda = 1.0, r = \text{Avg})\) to our group-average data in Games 1-17. Visually, the relationship between these two appears as:

![Figure 4.3: eCVE (K = -1.6, A = 4.0, \lambda = 1.0, r = \text{Avg}) (y) v. Actual (x)](image)

Pearson’s \(r\) for the eCVE-data relationship is computed to be \(r = 0.9327207\), with estimated \(p\)-value of \(8.882e-16\) and 90% confidence interval \((0.8817307, 0.9621692)\). We thus find ourselves in a position similar to that for the best-fitting ePT-EB model: both a 90% confidence interval for the eCVE Pearson’s \(r\) and a 90% confidence interval for the NE Pearson’s \(r\) contain one another. In fact, the largest confidence interval for the NE Pearson’s \(r\) that does not include 0.9327207 occurs for a 14.3% confidence interval—not very confident indeed! A similar calculation for ePT-EB and NE shows that the largest such confidence interval corresponds approximately a 63% confidence level; this, too, falls well short of conventional standards, but is more convincing than the evidence for eCVE. As before, of course, limitations on the size and quality of our data should recommend some caution in these interpretations; furthermore, our analysis for eCVE is far more likely to suffer from problems of overfitting than our ePT-EB analysis, given the large number of eCVE models through which we sifted.

Table 4.8 contains a number of additional insights. In contrast to the case for fitting individual games in Table 4.7, we now see that models with average-level reference-setting perform quite well,
and in fact account for all of the top 11 models; we note, also, the disappearance of pessimistic reference-setting from the list of best-fitting models, and remark that in our full data set the first occurrence of a pessimistic reference-setting model is in the 569th place, with sum-of-squared deviations 0.377204 (full table not included in Appendices, due to length; available from author on request). However, Table 4.7’s other commonalities are reinforced: the best-performing models tend to have negative values of $K$ with large magnitude, large values of $A$, and $\lambda$ is comparatively unimportant. With just three exceptions, the only models in Table 4.8 that incorporate $\lambda > 1$ do so with Optimistic reference-setting, in which all outcomes are regarded as gains and so there is no loss aversion; furthermore, the three Average reference-setting models that have $\lambda > 1$ set $\lambda = 1.2$, the $\lambda$ we tested that was greater than unity. In fact, these three patterns for $K, A, \lambda$ persist at least through the top 50 best-performing models, and thus seem to be relatively robust features of the best-performing models.

Thus we again find that models incorporating hypersensitivity, substantial weighting function asymmetry, and relatively little loss aversion outperform their competitors. Given that the top 10 best-performing models all have $A \geq 3.6$, and all save one have $A \geq 3.8$, we are tempted in follow-up work to investigate still larger values of $A$; it may be that $A = 4.0$ is insufficiently large to find the best-fitting region in eCVE parameter space.

4.3.3 Binary Games & Comparative Tests

In the foregoing sections, we described our algorithmic approaches to computational testing for ePT-EB and eCVE in our 17 2x2 games for which these equilibria may differ from NE. We also summarized the resulting evidence for these 17 games. In this section, we complete our analytics by examining the behavior of Pearson’s $r$ for each of NE and our best-performing ePT-EB and eCVE models, using our full data set of 23 games. Since the three equilibria agree (for any model) on these 5 games, the rank-order of their Pearson’s $r$ values will not change, but the increased sample size and extent of agreement or disagreement of predictions with data in these games may either increase or decrease all 3 Pearson’s $r$ values, and may impact calculated levels of statistical significance.
Having completed this analysis, we then close this section by examining the correlation of our best-performing ePT-EB and eCVE models with the observed data, after partialing out the variation already accounted for by NE predictions.

All 6 of Games 18-23 (depicted in the Appendix) are taken from Camerer’s (2003, Ch. 3) review. These binary games vary in size: Games 18 (O’Neill, 1987) and 22 (Binmore et al., 2001) are $4 \times 4$ zero-sum games, Games 19-20 (Rapoport & Boebel, 1992—2 games) are $5 \times 5$, Games 22 (Mookerjee & Sopher, 1997—2 games averaged; games equivalent under first-order stochastic dominance) is a $6 \times 6$ zero-sum game, and Game 23 (Ochs, 1995) is a $2 \times 2$ zero-sum game. Given the sizes of these games, we now have an additional 40 data points—6 each from the $4 \times 4$ games, 8 from each of the $5 \times 5$ games, 10 from the $6 \times 6$ game, and 2 from the $2 \times 2$ game. Figures 4.4-4.6 depict in scatterplots the relationships between our data and NE, ePT-EB, and eCVE, respectively, inclusive of our full set of 23 games.

Figure 4.4: NE (y) v. Actual (x)

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10We note that, while Games 18-22 are all zero-sum, the equivalence of ePT-EB, eCVE, and NE holds for binary games that are non-zero sum as well.
Figure 4.5: ePT-EB ($K = -1, A = 3$) (y) v. Actual (x)

Figure 4.6: eCVE ($K = -1.6, A = 4.0, \lambda = 1.0, r = Avg$) (y) v. Actual (x)
Corresponding to each of these figures we have the Pearson’s $r$ estimates $0.9457662$ ($p < 2.2e - 16$, 90% confidence interval of $(0.9208889, 0.9629714)$), $0.961124$ ($p < 2.2e - 16$, 90% confidence interval of $(0.9430854, 0.9735234)$), and $0.9522924$ ($p < 2.2e - 16$, 90% confidence interval of $(0.9303015, 0.9674618)$), respectively. In short, all three equilibrium concepts perform extremely well in the set of binary games studied, and this effect improves each of their correlations in our full sample relative to their (already quite good) performance in the original set of 17 games. All 3 Pearson’s $r$ estimates now lie quite comfortably in one another’s 90% confidence intervals; of course, since all 3 equilibrium concepts agree on these additional 40 data points, this increasing similarity in estimated correlations is not surprising.

Before concluding this chapter, we close with two final tests of the following question: is the relationship between our best-performing ePT-EB and eCVE predictions and our observed data statistically significant after partialing out the variation already accounted for by NE? This question is well-suited for the use of linear partial correlation, as the relationship between our data and predictions appears linear in all cases, residual plots for the relevant relationships bear no discernible patterns, and Q-Q plots appear consistent with normality. The relevant statistical test data are: for ePT-EB, a partial correlation of $0.527938$, statistically significant (at all conventional levels: 0.9, 0.95, 0.99) with an estimated $p$ value of $1.62388e - 07$, and for eCVE, a partial correlation of $0.3783262$, also statistically significant at all conventional levels with an estimated $p$ value of 0.0005735937. Furthermore, the relationship for NE to the observed data fails to be significant after controlling for either variation explained by eCVE (partial correlation of $0.1707919$ with $p$-value $0.1441312$) or ePT-EB (partial correlation of $0.04073029$ with $p$-value of $0.731234$). At least according to this criterion, it appears that the improvement attained by our ePT-EB and eCVE models is statistically significant; for ePT-EB, this is moderate evidence of its utility over classical NE, although it must be stressed that the variance explained in moving from NE to ePT-EB

\[\text{[11]}\text{See the Appendix to this chapter for these graphs. The graphs do identify a few observations that may be outliers; we reproduce these computations in the Appendix with these observations excluded. The results with these points excluded are qualitatively similar, with the exception that the partial correlation between the data and NE after controlling for ePT-EB becomes significant at the 0.05 level (but not at the 0.01 level). Both partial correlation conclusions for eCVE and NE are strengthened, while the data’s partial correlation with ePT-EB’s after controlling for NE is slightly weakened.}\]
increases only from 89.44737% for NE to 92.37593% for ePT-EB, so the practical significance of this improvement might be questioned. For eCVE, the corresponding change in variance explained is from 89.44737% for NE to 90.68608%—just over a single percentage point increase. Instead, we cautiously suggest that both of eCVE and ePT-EB may improve on NE’s predictive accuracy in simultaneous games, but that further work is needed to decisively settle this question; the present work may best be used as a guide for regions of parameter space to examine.

4.4 Conclusions

In this chapter, we have complemented our theoretical analysis of PT equilibrium concepts for GT by conducting an exploratory empirical analysis of ePT-EB and eCVE on a set of group-average data collected from Selten and Chmura (2008) and Camerer (2003, Ch. 3). This data set contained 24 games and 74 data points; 12 of these games were non-binary, 2x2 simultaneous games (giving 34 data points), and the remaining 12 were binary, simultaneous, zero-sum games of varying size (giving 40 data points).

All three of NE and our best-performing eCVE and ePT-EB models performed extremely well on our group-average data set. For the full set of 74 data points, all three yielded Pearson’s $r$ estimates in excess of 0.94, with estimates of explained variance ranging from roughly 89% to 92%. Despite this small variation in explained variances and Pearson’s $r$ estimates, the relationship of our best-performing eCVE and ePT-EB models’ predictions to the observed group-average data was significant at all conventional levels after controlling for the relationship of NE to the data using partial correlations, and the converse was not true of NE after controlling for either eCVE or ePT-EB.

Despite empirical the success of eCVE and ePT-EB over classical NE as evaluated by appropriate partial correlations, given NE’s high rate of success in capturing the variations in our data set, these slight differences may be of limited practical significance. In addition, several limitations of our approach are significant and recommend caution in making strong inferences from this study: first, we lacked access to individual-level data, and so could not fit our models to heterogeneity among
players. Second, our data was limited in size (with just 34 data points on which NE and ePT-EB, eCVE might disagree), drew on work performed by many different experimenters with differences in experimental implementation, and was limited to simultaneous 2x2 and simultaneous, binary, zero-sum games.

It bears mention that our study’s focus on small simultaneous games may actually have worked against the empirical success of our models over and above NE, because NE appears to perform relatively well in this realm, and because these games are very different from the kinds of single-decision maker situations in which we know that VM preferences are outperformed by non-EU models of preference (like PT preferences). A natural follow-up to our study should incorporate analysis of both larger simultaneous games and games with simple extensive-form, perfect-information structure, ending in nodes controlled by Nature. Perfect-information extensive-form games ending in Nature nodes should closely replicate the situations typical of single decision-maker PT experiments, since, at the final node of a game, the active player’s only uncertainty is exogenous and generated by Nature, just as in single decision-maker experiments.

In addition to caveats about our experimental data, we also must add caveats about our computational data. Two cautions are appropriate, each with opposite implications: first, our algorithm for finding ePT-EB contains a heuristic element, in that it is not clear what choice of the relaxation parameter $\epsilon$ is ideal, and this necessitates a trial-and-error process for identifying the smallest possible $\epsilon$ for which the algorithm yields a non-empty $\epsilon$-ePT-EB set. Even when such an $\epsilon$ is (approximately) identified computationally, it is often the case that the resulting set is not a singleton, even for games that have unique ePT-EB; thus we are forced to approximate the true ePT-EB by taking measures of central tendency for the algorithm’s output, and error in this process may cause our estimated ePT-EB to differ from each game’s true ePT-EB. This may cast some degree of doubt on our results for ePT-EB, especially for ePT-EB predictions with large variation (though, for our best-fitting ePT-EB model, none of the sets used to generate predictions had range of more than 0.15). Second, our algorithm for finding ePT-EB is a very slow exhaustive algorithm, owing primarily to its discretization of the players’ product of simplices. Secondary factors affecting its
speed were its need to run the algorithm repeatedly for distinct $\epsilon$ values, its need to find argmax sets for every discretized grid point pair, its need to solve a new linear program for every point of the discretization\footnote{Although the simplex algorithm used for this purpose is in practice efficient, it nevertheless contains many elementary floating-point operations; additionally, simplex is known to be inefficient in a worst-case sense.} and its need for finding real-valued roots of negative reals (for which we used the mpmath, Python2.7 library due to Johansson et al., 2010). The large computational cost of running our ePT-EB algorithm severely limited our ability to study the full extent of ePT-EB’s parameter space; in the face of this limitation, it is perhaps quite impressive that ePT-EB yielded the best-performing of our models.

Some of the algorithmic challenges we faced might be overcome in follow-up work. One very basic change could be made: rather than use the polynomial weighting family, we might use any of a range of alternative families (surveyed in, e.g., Wakker, 2010) which rely on exponentiation, logarithms, and fractions of positive powers of reals rather than taking real-valued roots of negative reals. More sophisticated changes may be suitable as well; for example, it may be possible to adapt a form of fictitious play (Brown, 1951) to the case of ePT-EB. However, some significant changes would be required: where fictitious play maps histories into suitable best-replies, an analogous process for ePT-EB would need to map histories of play into sets, or into the convex hulls of argmax sets. These changes are substantial and may considerably alter the computational complexity and convergence properties of the resulting algorithm. Such an approach would also most likely forgo the ability to exhaustively find all ePT-EB of a game, although, if successful, it could displace the need for solving linear programs, finding argmax sets, or iteratively searching a discretization of the product simplex, all of which are key bottlenecks in the current algorithm’s run-time. We might also implement a parallel-programming version of each of our algorithms; this approach has the advantage of allowing us to employ the massively parallel architecture of modern graphics-processing units and takes natural advantage of the “embarrassingly parallel” nature of our exhaustive algorithms. This change to a parallel algorithm seems quite promising; in preliminary work of this sort, we have achieved speed increases of just over 2 orders of magnitude for a variant of our eCVE algorithm.
Despite the many challenges faced in this chapter, our results seem to make recommend some important avenues future work: first, probabilistic insensitivity and probabilistic asymmetry seem to dominate both ePT-EB and eCVE model performance, with polynomial-family values of $K \approx -1.8$ and large values of $A$ performing best. The mode of reference-setting also appears to be important, but whether pessimistic, average, or optimistic reference-setting is descriptively best is not clear; the top 10 best-fitting eCVE models featured average reference-setting, and pessimistic reference setting did not occur until the 569th best model, but considerable variations in reference-setting were found among our best-fitting eCVE models, and even pessimistic reference-setting performed quite well in fits to individual games. By contrast with these first three parameters, small variations in loss aversion vis-a-vis $\lambda$ seemed to be relatively unimportant, with its influence swamped by changes in the first three parameters, and reference-setting often set to pessimistic or optimistic, both of which directly undermine loss aversion’s importance. In summary, it seems that in future work, the parameter space should be further explored, emphasizing the possibility of significantly larger degrees of asymmetry, modestly larger degrees of probabilistic hypersensitivity, and allowing for a wider range of reference-setting modes. In addition, it may be useful to evaluate a much wider, more dispersed range of loss aversion values; it may be that changes in $\lambda$ simply need to be very large relative to changes in the remaining parameters, although for $\lambda$ very much larger than 2.0 we begin to depart from the individual decision-making evidence on this parameter value.

The results for our parameters $A$ and $K$ of probabilistic asymmetry and probabilistic hypersensitivity were especially consistent and provocative. We consistently found modestly large, negative values of $K$ and very large, positive values for $A$, implying an S-shaped weighting function that exhibits probabilistic hypersensitivity and a separating point of about $p = 0.65$ between the weighting function’s concave and convex regions. Both of these properties stand in contrast to modal fits for parametric families in the single decision-maker literature, which are generally inverse-S shaped and feature a separating point of about $p = 0.42$. Intuitively, where single decision-makers appear to treat all intermediate probabilities as if they are more nearly equivalent than is in fact the case, our parametric fits suggest decision makers who behave as if betting that their opponents
will almost certainly play some particular pure strategy(ies)—those with probabilities above the separating cut-off of $p = 0.65$. It would be particularly interesting if our results for $A$ and $K$ were found to be robust in follow-up work, because this would suggest the need for a dual-weighting model, with different weighting functions used for strategic and non-strategic uncertainty. This also leaves unclear which of these two functions should be used for uncertainty that combines strategic and non-strategic elements, though our intuition here is that non-strategic uncertainty may be a kind of default mode, with strategic uncertainty dominating whenever it is present.
Chapter Five: Summary of Findings & Concluding Thoughts

In this chapter, we briefly recapitulate our findings in Chapters 2-4, outline concluding thoughts, and propose follow-up work. At the outset of this work, it was our goal to systematically propose and investigate suitable notions of GT equilibrium under PT preferences. In pursuing this goal, we initially drew on the work of Shalev (2000) to develop the three-fold notions of (e,m,n)PT-NE, which combine Shalev’s endogenizations of the reference point with the natural generalization of NE’s classical definition. Problems soon came to light with this approach, however, including nonexistence in simple finite games, lack of respect for rationalizability, absurd indifference in preferences, and difficulty in deriving computational complexity guarantees. As a result, we turned to the work of Crawford (1990) to formulate the concepts of (e,m,n)PT-EB, which use the convex-hull (or distributional) operator to correct some of the pathologies found in (e,m,n)PT-NE; the remaining pathologies were corrected by modifying Shalev’s notion of non-myopia to more nearly resemble algorithmic characterizations of SPE, rather than relying on SPE’s standard “NE over all subgames” definition.

Consideration of our development of PT-EB and our use of an alternative characterization for SPE to correct problems encountered with Shalev’s non-myopia then led us to systematically map out those alternative characterizations of NE possible using just convex-hull operators and varying best-reply conditions between pure and mixed strategies. Taking this perspective on the problem of non-EU equilibrium led us to two new definitions of PT equilibrium which happen to agree in finite games, thus giving form to our notion of CVE. Like PT-EB, CVE was found to correct many of the problems originally noticed in PT-NE; ultimately, CVE was found to be more tractable even than PT-EB, and in particular allowed us to fully reproduce (for its exogenous variant) a number of the epistemic and computational complexity results known for classical NE.
Having completed our theoretical analyses, we could largely dismiss the use of PT-NE as an equilibrium concept, but it remained unclear whether PT-EB or CVE would be the more useful model empirically, and so in our penultimate chapter, we set out to compare and contrast these two models against simultaneous-games group-average data available in Selten and Chmura (2008) and Camerer (2003, Ch. 3). These analyses focused on ePT-EB, eCVE, and contrasting these two models with NE; we developed separate algorithms for each concept and sampled a number of possible models from the space of all possible ePT-EB and eCVE models, concentrating on the ranges of parameter space that correspond best to empirical findings for PT preferences in single decision-maker settings. The results of this work delivered a number of intriguing insights: first, all three models performed extremely well in explaining our binary, simultaneous and 2x2, simultaneous games data, with all three models achieving Pearson’s $r$ estimates in excess of 0.9. Second, best-fitting variants of each of eCVE and ePT-EB outperformed NE, both in the sense of achieving a higher Pearson’s $r$, and in the sense of maintaining a statistically significant relationship with the observed data after controlling for NE’s relationship to the data. Third, our best-fitting eCVE and ePT-EB models feature a curious mix of parameter values: $\lambda$ is most often set to 1.0 (or just above it), reflecting no (or little) loss aversion, while the method of reference-setting appears important to model performance, but varies widely. Most notably, best-fitting models consistently yielded weighting-function estimates that exhibited probabilistic hypersensitivity and significant weighting-function asymmetry, giving a weighting function estimate that differs from the modal single decision-maker weighting functions in both shape and the point at which the function’s concave and convex regions are separated.

Although the present work has made considerable inroads into understanding the three notions of PT-NE, PT-EB, and CVE, a great many questions remain unanswered, and these motivate natural follow-up work. We identify just a few of the most pressing of these follow-up problems here: first, our models employ the full PT model for uncertainty under risk, and this considerably advances the use of PT preferences in GT over past models, but this still leaves the issue of ambiguity (situations where explicit probabilities are unknown and decision-makers do not posit any particular
subjective probabilities) unaddressed. As exposited by, for example, Wakker (2010), the full formulation of PT is intended to accommodate ambiguity; considered together with our emphasis on distinguishing subjective from objective risk, this may lead to yet another distinction necessary in the GT lexicon on uncertainty. Second, while we have provided significant evidence that CVE are theoretically preferable to PT-EB, in empirical practice, PT-EB modestly outperformed CVE, and this is so despite that we were able to investigate a great many more CVE models than PT-EB models. However, we should note that the data against which we tested CVE, PT-EB, and NE largely favored NE and disfavored CVE: CVE is the only one of our three models that reduces to conventional PT when applied to trivial single decision-maker “games,” or in the penultimate nodes of games of perfect information with a Nature player controlling the final nodes. Thus, in a sense, the single decision-maker literature can be expected to weigh heavily in favor of CVE, and games of perfect information with Nature may provide an ideal setting for contrasting these three equilibrium concepts. Differences were slight in the set of small, simultaneous games we considered, but in games of perfect information with Nature, we can expect larger differences, if the single decision-maker literature is to be a guide. This may in turn lead us to still further questions, and possibly to formulating models in which separate weighting functions are used to address strategic and non-strategic uncertainty. Third, our theoretical results address most of the immediately pressing questions about PT-NE, PT-EB, and CVE, but several additional issues must be addressed: our complexity-theoretic results should be extensible from ePT-EB and eCVE to (m,n)PT-EB and (m,n)CVE, and lower bounds (in the usual modulo $P \neq NP$ sense) seem likely and desirable for PT-NE. Additionally, our theoretical results are concentrated on finite games, but forms of PT have been developed that are suitable for continuous distributions (Kothiyal et al., 2011), there are many interesting games with infinitely many players or strategies per player, and it seems clear that our algorithmic definition of non-myopia can be adapted to games of infinite time-horizon with some suitable, limit-based adaptations.

To briefly summarize: we have contributed three novel notions of GT equilibrium under PT preferences, and provided both theoretical and empirical analyses of these three concepts. In this
work, we were able to largely dismiss one of these three concepts, PT-NE, but theoretical and empirical work appear to point in opposite directions in contrasting PT-EB and CVE, though both equilibrium concepts appear to outperform NE for simultaneous games. Still, there is considerable reason to believe CVE will prove the superior equilibrium concept in general, and we have proposed follow-up work suitable for both exploring this conjecture and generally improving upon our understanding of GT equilibria with PT preferences. It is hoped that this work will stimulate further interest in research on how the full model of PT preferences may be fruitfully integrated with GT.
5.1 Acknowledgements

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Bibliography
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Appendix A
(Proofs for Chapter 2)

Throughout these proofs, we use $R_i(\cdot)$ to denote player $i$’s best reply set.

A.1: Ritzberger’s game has an equilibrium with polynomial $w$:

Proof. Let $w_1(p) = 0.5(2p - 1)^3 + 0.5$. Then in Ritzberger’s (1994) $2 \times 2$ game:

\[ v_1(p, q) = -0.5(pq)^3 + 0.5(p + q - 2pq - 1)^3 + 0.5(q - pq - 1)^3 + 1.5 \]

\[ v_2(p, q) = 2pq + (1 - p)(1 - q) = 3pq + 1 - p - q \]

Hence,

\[ \frac{dv_1}{dp} = -1.5(pq)^2q + 1.5(p + q - 2pq - 1)^2(1 - 2q) - 1.5(q - pq - 1)^2q \]

\[ \frac{d^2v_1}{dp^2} = -6pq^3 + 1.5(p + q - 2pq - 1)(1 - 2q)^2 + 3q^3 - 3q^2 \]

\[ \frac{dv_2}{dq} = 3p - 1 \]

This gives $R_2(p, q) = \{0\}$ if $p < \frac{1}{3}$, $R_2(p, q) = \{1\}$ if $p > \frac{1}{3}$, and $R_2(p, q) = [0, 1]$ if $p = \frac{1}{3}$. We can use this fact to simplify our analysis by only considering $\frac{dv_1}{dp}$ under the assumption that $q = R_2(p, q)$ is satisfied. To begin, suppose $p < \frac{1}{3}$; then $q = 0$, so $\frac{dv_1}{dp} = 1.5(p - 1)^2$, which implies that $p^* = 1 = r_1(p, 0)$, but then $p^* > \frac{1}{3}$, contrary to assumption, so there are no equilibria for which $q = 0$ and $p < \frac{1}{3}$. Now suppose $p > \frac{1}{3}$; then $\frac{dv_1}{dp} = -6p^2$ which implies $p^* = 0 = r_1(p, 1)$, contrary to $p > \frac{1}{3}$. The remaining case to consider has $q$ unrestricted but $p^* = \frac{1}{3}$; so, we seek $q$ such that
\[
\frac{dv_1}{dp} = -\frac{2}{27}q^3 + 1.5\left(\frac{1}{3} + q - \frac{2}{3}q - 1\right)^2(1 - 2q) - 1.5(q - \frac{1}{3}q - 1)^2q = 0,
\]
which holds iff
\[
\left(\frac{1}{3}q - \frac{2}{3}\right)^2(1 - 2q) - \left(\frac{2}{3}q - 1\right)^2q = \frac{1}{9}q^3
\]

We use the computer algebra system SAGE (Stein et al., 2013) to find \(q^* \approx 0.24605255887084812\). At \((p, q) = (\frac{1}{3}, 0.2461)\), we have \(\frac{d^2v_1}{dp^2} = -3pq^3 + 1.5(p + q - 2pq - 1)(1 - 2q)^2 + 3(q - pq - 1)q^2 \approx -0.39295\), which implies that, with respect to \(p\), the \((p^*, q^*)\) candidate identified here is a local (in fact, global, since \(\frac{d^2v_1}{dp^2} < 0 \forall p \in [0, 1]\)) maximum. So, Ritzberger’s (1994) example does have an equilibrium. \(\square\)

A.2: Proof of Theorem 1

Proof. Consider games with the following form:

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with \(C\) large and positive, P1 has PT preferences with \(r_1 = 1, v_1(u; r) = u - r\) and \(w_1\) satisfying the conditions given in the hypothesis; P2 has EUT preferences. First, we note there is no equilibrium in which either player uses a pure strategy; furthermore, it is possible for any \(p^* \in (0, 1)\) to choose P2’s payoffs so that, in any MSE, P1 must play her first pure strategy with probability \(p^*\). We set \(p^* = B\). This leaves us with the first-order conditions for P1:

\[
V1(p, q) = (C - 1)w(pq) - w((1 - p)q + (1 - q)p)
\]

\[
V1'(p^*, q) = (C - 1)w'(p^*q)q - w'((1 - p^*)q + (1 - q)p^*)(1 - 2q) = 0
\]
where we need \( V1' > 0 \) for \( p < p^* \) and \( V1' < 0 \) for \( p > p^* \). Thus, for there to exist an MSE \((B, q)\), we must be able to find \( q \in (0, 1) \) such that

\[
(C - 1)w'(Bq)q = w'(q(1 - 2B) + B)(1 - 2q)
\]

To complete the proof, we separate into cases, corresponding to our two alternative technical conditions. First, we consider Case 1: suppose \( \lim_{p \to 0} w'(Bp)p = \infty \). Then the left-hand side in

\[
(C - 1)w'(Bq)q = w'(q(1 - 2B) + B)(1 - 2q)
\]

grows arbitrarily large as \( q \to 0 \), and has limit \((C - 1)w'(B) > 0 \) as \( q \) approaches 1. The left-hand side has no roots, and has value \((C - 1)w'(0.5B)0.5 > 0 \) at \( q = 0.5 \). The right-hand side varies continuously from \( w'(B) \geq 0 \) at \( q = 0 \) to \(-w'(1 - B) < 0 \) at \( q = 1 \), with a single root at \( q = 0.5 \). By continuity, we have \( w'(q(1 - 2B) + B)(1 - 2q) \leq 0 \forall q \in [0.5, 1] \), and \((C - 1)w'(Bq)q > 0 \forall q \in [0.5, 1] \), so if equality holds, it must hold for some \( q \in (0, 0.5) \) (note that we are not interested in the possibility \( q = 0 \), since there are no equilibria with pure strategies).

\( w'(q(1 - 2B) + B)(1 - 2q) \) is a continuous function on \([0, 0.5] \), so it has a well-defined maximum \( U_{RHS} \) and minimum \( U_{LHS} \) on this interval; similarly, \( w'(Bq)q \) is a continuous function on \([\varepsilon, 0.5] \) for any \( \varepsilon > 0 \), so it attains a maximum \( U^e_{LHS} \) and minimum \( L^e_{LHS} \) values on this interval. Since \( \lim_{p \to 0} w'(Bp)p = \infty \), we can choose \( \varepsilon > 0 \) sufficiently small so that \( w'(Bp)p > U_{RHS} \) on \([0, \varepsilon] \); with such a choice of \( \varepsilon \), the only remaining possibilities for the first-order condition to hold have \( q \in [\varepsilon, 0.5] \). This can occur only if \((C - 1)U^e_{LHS} < U_{RHS} \), but since we know \( U^e_{LHS} > 0 \) by our earlier remarks, we can make a choice of \( C \) sufficiently large so that this is impossible.

We close by addressing Case 2: assume \( \lim_{p \to 0} \frac{w'(p(1 - 2B) + B)}{p} = 0 \), and rearrange the stationarity condition (recalling \( 0 < q < 1 \)) to

\[
(C - 1)w'(Bq)q = \frac{w'(q(1 - 2B) + B)(1 - 2q)}{q}
\]

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The left-hand side varies continuously in $q$ from $(C - 1)w'(0) > 0$ at $q = 0$ to $(C - 1)w'(B) \geq 0$ at $q = 1$; it has no interior roots, and so by continuity is strictly positive for all interior $q$. The right-hand side also varies continuously in $q$ on $(0, 1)$; it has limit 0 as $q \to 0$ by hypothesis, and takes value $-w'(1 - B) \leq 0$ at $q = 1$. It has one interior root at $q = 0.5$, so—by continuity—the right-hand side is negative for all $0.5 < q < 1$, hence the only candidates for interior solutions have $q \in (0, 0.5)$. Since the right-hand side is continuous on $[0, 0.5]$, it achieves a well-defined maximum value, $M_R$, on this interval; similarly, $w'(Bq)$ is continuous on $[0, 0.5]$, so it attains a well-defined minimum $M_L > 0$—a strict inequality follows because $w'(Bq) > 0$ for all $q \neq 1$. The right-hand side is independent of $C$, so if $C$ is chosen large enough that $CM_L > M_R$, then the stationarity condition cannot be satisfied for any $q \in (0, 1)$; hence, there is no ePT-NE. □

A.3: Proof of Theorem 3

Proof. The proof is by Kakutani’s Fixed Point Theorem. We apply Kakutani’s to the correspondence $F : \Delta \times (\prod_{i=1}^{N}[r_{\min}^i, r_{\max}^i]) \to P(\Delta \times (\prod_{i=1}^{N}[r_{\min}^i, r_{\max}^i]))$ which maps $(\sigma, r)$ to the set 

\[
\{(b_1, ..., b_N, f_1(\sigma, r_1), ..., f_N(\sigma, r_N)) : b_i \in \text{co}[\text{argmax}_{\sigma'_i \in \Delta} \nu_i(\sigma'_i, \sigma_{-i}; r_i)]\}.
\]

The condition for being a fixed point of $F$ is identical to the equilibrium conditions we have provided, so the theorem is equivalent to showing that $F$ has a fixed point.

The domain of $F$, $D = \Delta \times (\prod_{i=1}^{N}[r_{\min}^i, r_{\max}^i])$, is non-empty by definition of $\Delta$ and since $r_{\min}^i \leq r_{\max}^i$ for each $i$; it is compact and convex because it is the product of the compact, convex sets $\Delta$ (in turn the product of the compact, convex simplices $(\Delta_i)^N$) and $(\prod_{i=1}^{N}[r_{\min}^i, r_{\max}^i])^N$. For any $(\sigma, r) \in \Delta \times (\prod_{i=1}^{N}[r_{\min}^i, r_{\max}^i])$, $F(\sigma, r)$ is non-empty because each $f_i$ is well-defined and the set $\text{argmax}_{\sigma'_i \in \Delta} \nu_i(\sigma'_i, \sigma_{-i}; r_i)$ is non-empty by the continuity of $\nu_i$. $F(\sigma, r)$ is convex because $\text{co}[\text{argmax}_{\sigma'_i \in \Delta} \nu_i(\sigma'_i, \sigma_{-i}; r_i)]$ is convex and $f(\sigma, r_1)$ is a singleton (and so convex).

It remains to show that $F$ is upper hemicontinuous. $F$ is upper hemicontinuous at $(\sigma, r)$ if, for any sequence $(\sigma^i, r^i)_{i=1}^{\infty}$ with points in $D$ and any sequence $((b'_1, ..., b'_N, f^i_1(\sigma^i, r_1), ..., f^i_N(\sigma^i, r^i_N)))_{i=1}^{\infty}$ with $(b'_1, ..., b'_N, f^i_1(\sigma^i, r_1), ..., f^i_N(\sigma^i, r^i_N))$ in $F(\sigma^i, r^i)$ for each $i$, we have that $\lim_{i \to \infty}(\sigma^i, r^i) = (\sigma, r)$ and $\lim_{i \to \infty}(b'_1, ..., b'_N, f^i_1(\sigma, r_1), ..., f^i_N(\sigma, r^i_N)) = (b, \bar{f})$ imply $(b, \bar{f}) \in F(\sigma, r)$.
Suppose this is not the case; first, we note that, for any \( j \), \( \sigma^i \rightarrow \sigma \), \( r^i_j \rightarrow r_j \) and \( f^i_j (\sigma^i, r^i_j) \rightarrow f_j^i \) imply \( r_j = f_j^i (\sigma_j, \sigma_{-j}, r_j) \), by continuity of \( f \). So, if upper hemicontinuity fails, then some component \( b_j \) of \( b \) is not a convex combination of best replies for player \( j \) to \( \sigma_{-j} \) with her reference point fixed to \( f_j \), and so \( b_j \) lies at least some distance \( \varepsilon \) from the set \( C_j (\sigma_{-j}, f) \) of player \( j \)'s convex hull of best replies. By hypothesis and continuity of \( v_j, f_j \), we can for any \( \nu_1, \nu_2 > 0 \) choose \( M \) sufficiently large that, for all \( i \geq M \), \( b^i_j \) lies no more than distance \( \nu_1 \) from \( C_j (\sigma_{-j}, f) \) and no more than distance \( \nu_2 \) from \( b^i_j \). Choosing \( \nu_1, \nu_2 \) sufficiently small and applying the triangle inequality, we have a contradiction to \( b_j \) lying at least distance \( \varepsilon \) from \( C_j (\sigma_{-j}, f) \). □

A.4: Proof of Corollary 4

*Proof.* The proof is a simple induction; we begin applying the preceding theorem to the last subgame. We then fix all probabilities in that subgame to their equilibrium values, and use these fixed values and the preceding theorem to infer the existence of suitable values for the next-largest subgame, defined in the obvious fashion. Repeating this procedure yields an nPT-EB. □

A.5: Proof of Theorem 5

*Proof.* We proceed by induction: if \( s_i \) is eliminated in the initial round of deletions, then \( \exists s'_i \) such that \( v_i (X_i [s'_i, s_{-i}]) \geq v_i (X_i [s_i, s_{-i}]) \forall s_{-i} \), with the equality holding strictly for at least one \( s_{-i} \). If there is a player \( i \) with \( s_i \in \text{supp}[\sigma^i_\ast] \), then since \( \sigma^i_\ast \in \text{co}[\text{argmax}_{\sigma_i \in \Delta} V_i (\sigma_i, \sigma^i_{-i}; r)] \) by defn of ePT-EB, we have \( s_i \in \text{supp}[\rho_i] \) for some \( \rho_i \in \text{argmax}_{\sigma_i \in \Delta} V_i (\sigma_i, \sigma^i_{-i}; r) \). But the gamble induced by \( \rho_i \) is strictly, first-order stochastically dominated by any gamble that shifts the probability placed on \( s_i \) by \( \rho_i \) to \( s'_i \) instead; since PT preferences respect first-order stochastic dominance, \( \rho_i \) cannot be in \( \text{argmax}_{\sigma_i \in \Delta} V_i (\sigma_i, \sigma^i_{-i}; r) \) if \( s_i \in \text{supp}[\rho_i] \), a contradiction.

For the inductive step: if \( s_i \) is eliminated in round \( k \), then it was not eliminated in round \( k - 1 \), but there remains some strategy \( s'_i \), which in the reduced game considered in round \( k \), strictly dominates \( s_i \). By inductive hypothesis, all strategies eliminated in round \( k - 1 \) are played with probability 0. Thus, if \( s_i \in \text{supp}[\sigma^i_\ast] \), then, as in our argument for the initial step, there is some \( \rho_i \) with \( s_i \in \text{supp}[\rho_i] \) and such that \( \rho_i \) is a best-reply to \( \sigma_{-i} \), but reallocation of probability assigned
by $\rho_i$ to $s_i$ instead to $s'_i$ must induce a gamble that strictly, first-order stochastically dominates $\rho_i$, since $s'_i$ strictly dominates $s_i$ when played against $\sigma_{-i}$, by inductive hypothesis. Thus $\rho_i$ cannot be a best-reply, since PT preferences respect first-order stochastic dominance—a contradiction. □

A.6: Proof of Theorem 6

Proof. Let $R$ be the set of rationalizable mixed strategies for $G$, and suppose $\sigma^*$ is an ePT-NE. If $\sigma^*$ is not in $R$, then there is some minimal iteration $k$ at which a component $i$ of $\sigma^*$ is deleted by the iterative process defining $R$; this means $\sigma^*_i$ is not a best reply to any $\sigma_{-i} \in \prod_{j \neq i} \text{co}[\Sigma_j^{k-1}]$. But, by the minimality of $k$, $\sigma^*_{-i} \in \prod_{j \neq i} \text{co}[\Sigma_j^{k-1}]$, and by definition of ePT-NE $\sigma^*_i$ is a best reply to $\sigma^*_{-i}$.

□

A.7: Proof of Theorem 7

Proof. The proof that ii), iii) imply i) proceeds as follows: for $s^* \in S$, there is no risk and each player $i$ receives some particular outcome $o_i \in \mathbb{R}$ with certainty, so $V_i(X_i[\sigma^*]; r_i)$ reduces to $v_i(o_i; r_i)$; the best-reply condition (for ePT-NE, mPT-NE) and monotonicity of $v_i$ in $r_i$ then implies $o_i \geq o'_i$ for each $o'_i$ attainable by a unilateral, pure-strategy deviation. But $u_i$ is also monotonic in $o_i$, so $u_i(X_i[\sigma^*]) \geq u_i(X_i[s'_i, \sigma^*_{-i}])$ for $s'_i \in S_i$, hence $s^*$ is an NE.

The converse is similar but also depends on $V_i$ respecting first-order stochastic dominance. $s^* \in S$ an NE in $G$ implies each player $i$ receives some $o_i$ with certainty under $s^*$. The best-reply condition and monotonicity of $u_i$ in $o_i$ implies that $o_i \geq o'_i$, where $o'_i$ is any outcome achievable by unilateral deviation of player $i$ from $S^*_i$ to $\sigma_i \in \Delta_i$. But this means the distribution over outcomes induced by $(\sigma_i, S^*_{-i})$ can be constructed by shifting probability mass from $o_i$ to alternative outcomes $o'_i$ to which $o_i$ is weakly preferred. Since $V_i$ respects first-order stochastic dominance, we have $V_i(s^*; r_i|q_i) \geq V_i((\sigma_i, s^*_{-i}); r_i|q_i)$ for any $\sigma_i \in \Delta_i$, any $r$ with $i$th component $r_i$, and any information set $q_i$.

ii), iii) imply iv), v) respectively, because any equilibrium is also an EB. That iv) implies imply ii) follows because in any pure-strategy ePT-EB $(s^*, r)$, in a finite game, we have for any player $i$ that $s^*_i \in \text{co} [\arg\max_{\sigma_i \in \Delta_i} V_i(\sigma_i, \sigma^*_{-i}; r_i)]$ implies $s^*_i$ is an extreme point of this set, hence in particular
\( s^*_i \in \arg\max_{\sigma_i \in \Delta_i} V_i(\sigma_i, \sigma^*_i; r_i) \), which implies that \((s^*, r) \) is an ePT-NE. This contains the proof that v) implies iii) as a special case, in which \( r_i = E^{\sigma^*}[X_i] \forall i \).

**A.8: Proof of Lemma 8**

**Proof.** In general, \( V_i(P; r_i) = \sum_{x_k < r_i} M^+(x_k) v_i(x_k, r_i) + \sum_{x_k > r_i} M^-(x_k) v_i(x_k, r_i) \); similarly, for \( r'_i = r_i - \varepsilon = o - \varepsilon \), we have \( V_i(P; r'_i) = \sum_{x_k < r'_i} M^+(x_k) v_i(x_k, r'_i) + \sum_{x_k > r'_i} M^-(x_k) v_i(x_k, r'_i) \), and so the difference \( V_i(P; r_i) - V_i(P; r'_i) \) may be expressed as a sum of differences, all save one of which are expressible in one of the forms: \( M^+(x_k) v_i(x_k, r_i) - M^+(x_k) v_i(x_k, r'_i) \) or \( M^-(x_k) v_i(x_k, r_i) - M^-(x_k) v_i(x_k, r'_i) \).

If \( r_i \in O \) (\( r_i \) corresponds to a realizable outcome of \( P \)) then there is also a term of the form \( 0 - M^+(r_i) v_i(r_i, r'_i) \), where \( 0 \) is the value of the term containing \( v_i(r_i, r_i) \). \( v_i \) is continuous in its second component, so \( \lim_{r'_i \to r_i} v_i(r_i, r'_i) = v_i(r_i, r_i) \); hence, as \( r'_i \to r_i \), each term becomes 0, except perhaps \( 0 - M^+(r_i) v_i(r_i, r'_i) \), which satisfies \( \lim_{r'_i \to r_i} [0 - M^+(r_i) v_i(r_i, r'_i)] = M^+(r_i) v_i(r_i, r_i) \).

**A.9: Proof of Theorem 9**

**Proof.** The proof for (m,n)PT-NE is identical to that given by Shalev (2000), where Shalev’s function \( r_i(\sigma) \) is replaced by \( E^\sigma_i[X_i] \). The tools used in Shalev’s proof are continuity of \( V_i \) in each of its components and monotonicity of \( V_i \) in its second component; these are provided by Lemma 1 (the subjective neutrality condition) and the basic properties of \( v_i, w_i \). The proof for (m,n)PT-EB is trivial: \( G \) has just a single subgame, itself, and the definition of nPT-NE says exactly that \( \sigma^* \) is an nPT-EB iff it gives an mPT-NE in this subgame.

**A.10: Proof of Theorem 10**

**Proof.** The proof follows immediately from Theorem 7. Given \( G \in \mathbb{G} \), we can in polynomial time describe the obvious corresponding PT game \( G' \in \mathbb{G}' \), and using a subroutine to solve the latter problem, we immediately discover the solution to the former. Likewise for the former problem, but with \( G \) a PT game and \( G' \) the corresponding classical game.

**A.11: Proof of Theorem 12**
Proof.} Given a PT game $P$ with polynomial preference functions $f_i, i = 1, \ldots, N$ with pure strategies $s_{ij}, i = 1, \ldots, N, j = 1, \ldots, s_{\lceil s_i \rceil}$, we use Theorem 9 to choose normalizing constants $K_1 \geq 1, K_2 \in \mathbb{R}$, and then to form a game $G$ which has distinguished input players $1, \ldots, N$ with strategy sets (but not payoffs) identical to the strategy sets of their analogues in the given PT game. For each player $i$ and strategy $j$ among these core players, there is another player, $O_{ij}$, who has just two strategies, $S_0_{ij}, S_1_{ij}$. $G$ is constructed so that in any $\delta$-NE in which players $i = 1, \ldots, N$ play the partial-strategy profile $\sigma = (\sigma_1, \ldots, \sigma_N)$, $O_{ij}$ plays $S_0_{ij}$ with probability $\sigma_0(O_{ij})$ that is related to player $i$’s PT payoff $f_i(s_{ij}, \sigma_{-i})$ for deviating from $\sigma_i$ to playing the pure strategy $s_{ij}$ by the inequality $K_1 f_i(s_{ij}, \sigma_{-i}) + K_2 - p(N, D) \delta \leq \sigma_0(O_{ij}) \leq K_1 f_i(s_{ij}, \sigma_{-i}) + K_2 + p(N, D) \delta$; this is possible because, for fixed reference points and polynomial preferences, deviation payoffs are also multivariate polynomials in the strategic probabilities of players 1, ..., $N$. To complete the description of $G$, we pay each player $i = 1, \ldots, N$ 1 if they play $s_{ij}$ when $O_{ij}$ plays $S_0$, and 0 otherwise.

We claim that if we set $\delta = \frac{\epsilon}{3p(N, D)}$, then for any $\delta$-NE of $G$, the mixed-strategy profile $\sigma = (\sigma_1, \ldots, \sigma_N)$ played by players $i = 1, \ldots, N$ constitutes an $\epsilon$-PT-NE in $P$. By construction of $G$, the PT payoff $f_i(s_{ij}, \sigma_{-i})$ to player $i$ in $P$ for deviating from $\sigma_i$ to $s_{ij}$, against $\sigma_{-i}$, satisfies $|\sigma_0(O_{ij}) - (K_1 f_i(s_{ij}, \sigma_{-i}) + K_2)| \leq \frac{\epsilon}{3}$. Since this is a $\delta$-NE of $G$, we have $\sigma_0(O_{ij'}) \leq \sigma_0(O_{ij}) + \delta$ for all $s_{ij} \in supp[\sigma_i], s_{ij'} \in S_i$. Thus $K_1 f_i(s_{ij'}, \sigma_{-i}) + K_2 - \frac{\epsilon}{3} \leq K_1 f_i(s_{ij'}, \sigma_{-i}) + K_2 + \frac{\epsilon}{3} + \delta$, so $f_i(s_{ij'}, \sigma_{-i}) \leq f_i(s_{ij}, \sigma_{-i}) + \delta$. By the hypothesis on $f_i$, this implies $s_{ij} \in co[\arg\max_{\sigma_{-i}} f_i(\sigma_{-i})]$, where $\arg\max_{\sigma_{-i}} f_i(\sigma_{-i}) = \{\sigma'_i \in \Delta_i : \max_{\sigma'_{-i} \in \Delta_{-i}} f_i(\sigma'_{-i}, \sigma_{-i}) - f_i(\sigma'_i, \sigma_{-i}) \leq \epsilon\}$, as needed. Thus $\sigma$ is an $\epsilon$-PT-EB in $P$.

By Theorem 11, it is clear that this transformation is polynomial in $N$ and $\frac{1}{\epsilon}$, so we are done. 

\hfill $\square$

A.12: Proof of Theorem 13

\textbf{Proof.} As in the proof of Theorem 12, we construct a graphical $G'$ game which has constants $K_1 \geq 1, K_2$ and $N$ distinguished input players as its starting point; since the number of strategies per player satisfies a uniform bound for all possible games and all players’ preferences satisfy some Lipschitz constant $M$, we can, for any $\rho > 0$, choose a constant $K'$ number of mixed strategies...
(\sigma_1',...,\sigma_{K'}') per player with the property that, for any \sigma, there exists k \in \{1,..,K'\} such that |f_i(\sigma_k),\sigma_{-i}) - f_i(\sigma_i',\sigma_{-i})| < \rho. Each root player i has pure-strategy set given by (\sigma_1',...,\sigma_{K'}'); this set is assumed to include player i’s pure strategies in the given PT game. The construction proceeds by extending the graphical game such that there are distinguished players P_{ij} with two pure strategies each and whose preferences have the property that, in any \delta-NE, P_{ij} plays her first strategy with probability P_{ij}(0) related to player i’s PT payoff f_i(\sigma_j',\sigma_{-i}) for playing mixed strategy \sigma_j' against \sigma_{-i} by the inequality |P_{ij}(0) - (K_1 f_i(\sigma_j',\sigma_{-i}) + K_2)| \leq p(N,D)\delta. As in Theorem 10, we again specify the payoffs of the input players by assigning each player i payoff of 1 if she plays \sigma_j' when P_{ij} plays his first strategy, and payoff of 0 otherwise.

By the same argument used in the proof of Theorem 12 (but dividing the \delta used there by 2), we can choose \delta so that f_i(\sigma_j',\sigma_{-i}) \leq f_i(\sigma_j',\sigma_{-i}) + \frac{\epsilon}{2} for any j' and any \sigma_j' played with positive probability by the ith input player. If we also set \rho = \frac{\epsilon}{2}, then by our choice of (\sigma_1',...,\sigma_{K'}) we have, for any \sigma_j' played with positive probability by the ith input player in a \delta-NE and any \sigma_i \in \Delta_i, that there exists k' with |f_i(\sigma_j',\sigma_{-i}) - f_i(\sigma_i,\sigma_{-i})| \leq |f_i(\sigma_j',\sigma_{-i}) - f_i(\sigma_{k'},\sigma_{-i})| + |f_i(\sigma_{k'},\sigma_{-i}) - f_i(\sigma_i,\sigma_{-i})| \leq \epsilon. We claim that \sigma^*, the mixed-strategy profile over pure strategies in G induced by the partial-strategy profile of input players’ probabilities for \sigma_j' in G', is an \epsilon-ePT-EB in G, if the input players’ probabilities are taken from a \delta-NE in G', with \delta and \rho chosen as indicated.

We have already shown that each \sigma_j' played with positive probability by player i is an \epsilon best-reply to \sigma_{-i} in G. By definition, the mixed strategy \sigma_i^* over player i’s pure strategies in G induced by (\sigma_j')_{j=1}^{K'} must lie in the convex hull of these \sigma_j'; since these have each been shown to be \epsilon best replies in G', we have that that \sigma_i^* is in the set co[\argmax_{i} f_i(\sigma_{-i}^*)], where \argmax_{i} f_i(\sigma_{-i}^*) = \{\sigma_i' \in \Delta_i : max_{\sigma'' \in \Delta_i} f_i(\sigma_i'',\sigma_{-i}^*) - f_i(\sigma_i',\sigma_{-i}^*) \leq \epsilon\}, as needed.
Appendix B
(Proofs for Chapter 3)

Observation 2.1:

Proof. By linearity of VM preferences, the argmax sets occurring in (A) and (B) are already convex; thus \( \text{co} [\text{argmax}_{\sigma_i' \in \Delta_i} U_i(\sigma_i', \sigma_{-i}; r_i)] = \text{argmax}_{\sigma_i' \in \Delta_i} U_i(\sigma_i', \sigma_{-i}; r_i) \), so (A) and (B) are equivalent. The equivalence of (A) and (C) is a well-known observation in classical GT, and its proof depends again on the linearity of VM preferences; see Theorem 2.1 in Papadimitriou (2007), for example. The equivalence of (C) and (D) follows by linearity and the finiteness of the games considered; linearity implies that all pure-strategy best replies appear in \( \text{argmax}_{s_i' \in S_i} U_i(s_i', \sigma_{-i}; r_i) \), and so \( \text{co} [\text{argmax}_{s_i' \in S_i} U_i(s_i', \sigma_{-i}; r_i)] \) cannot contain any pure strategy not already in \( \text{argmax}_{s_i' \in S_i} U_i(s_i', \sigma_{-i}; r_i) \), since these are extreme points of the convex hull considered. □.

Observation 2.2:

Proof. To see that (A) is not equivalent to (B), consider any finite two-player game in which \( V_1, V_2 \) are both strictly convex, but for which there is no equilibrium in which any player uses a pure strategy; the pure-strategy values might be specified as in Matching Pennies, for example. Then there is no \( \sigma \) satisfying (A), since each player strictly prefers her pure strategies to her mixed strategies, but no equilibrium exists in which any player uses a pure strategy. By contrast, a theorem of Crawford’s (1990) establishes that an equilibrium of type (B) exists; note, however, that in such a \( \sigma \), all players must use fully mixed strategies. We note that Crawford (1990) himself used an example of this type to demonstrate that equilibria defined by the “best-reply” condition—our type (A) equilibria—may fail to exist in finite games.
To show \((A) \not\Leftrightarrow (C)\) and \((B) \not\Leftrightarrow (C)\), consider a one-player game, with \(V_i\) strictly concave and having a unique, fully mixed strategy as its global maximum. This one-player game has a unique type-(A) and type-(B) equilibrium—the maximizing point—but a continuum of type-(C) equilibria (namely any mixed strategy whatsoever).

**Theorem 2.1:**

Proof. \(\sigma_i \in co[\arg\max_{s \in S_i} V_i(s, \sigma_{-i})]\forall i\) means that \(\sigma_i\) is a convex combination of pure-strategy best replies; this implies equality of the payoffs (in the amount \(K\), say) for each \(s \in supp[\sigma_i]\), so that any linear combination of their payoffs is also \(K\). Conversely, if a linear combination of payoffs achieves \(K\), all its component pure-strategies must be best replies; otherwise, the payoff could be improved by a shifting of probability from a non-best-reply to a best-reply. □

**Theorem 2.2:**

Proof. For eCVE, the proof is a straightforward application of Theorem 1 and Kakutani’s Fixed Point Theorem: we apply Kakutani’s to the mapping \(F : \Delta \rightarrow S\), where \(S\) contains all mixed-strategy profiles of the form \((BR_i)_i\), \(BR_i\) being any best reply to \(\sigma_{-i}\) with respect to the preference functional \(\bar{V}_i\) of Theorem 2.1. \(\Delta\) is obviously non-empty, and is compact and convex because it is a product of finite-dimensional unit simplices \(\Delta_i\), each of which is both compact and convex. \(F\) is non-empty and closed for any \(\sigma \in Delta\) by continuity of \(V_i\) and compactness of \(\Delta\); \(F\) is convex because \(\bar{V}_i\) is linear in \(\sigma_i\), and \(F\) is upper hemicontinuous by Berge’s Maximum Theorem and continuity of \(V_i\). By Kakutani’s Fixed Point Theorem, \(F\) has a fixed point; it is obvious that this is an eCVE.

The proof for mCVE is identical; we just consider \(V_i(\sigma, EV_i(\sigma)) = V_i(\sigma)\) rather than \(V_i(\sigma, r_i)\). Regarding \(V_i(\sigma, EV_i(\sigma))\) as a function of \(\sigma\) alone, all the properties used in the proof for eCVE are satisfied for case of mCVE as well.

The existence of nCVE is follows from the existence of mCVE and a straightforward induction: mCVE exist for the “lowest” subgames of the game tree by the result for mCVE and (collapsing this subgame’s mCVE into a play by nature in the next highest subgame), at any iteration of the backwards-induction algorithm, existence of an mCVE is again guaranteed in the level of the tree.
currently considered. By finiteness of the game, this procedure must terminate, and clearly yields (existence of) an nCVE.

For general continuous functionals \( f_i \) and the generalized Shalev-like consistency condition, the argument is moderately more involved. We consider the mapping \( G : \Delta \times \prod_i D_i \rightarrow S \times \mathbb{P}(\prod_i D_i) \), where \( D_i \) is the domain for \( f_i \). \( G \) maps \( (\sigma, r_1, \ldots, r_N) \) to the set of all vectors of the form \( (BR_1, \ldots, BR_N, f_1(\sigma, r_1), \ldots, f_N(\sigma, r_N)) \), where \( BR_i \) is a best-reply (again under \( V_i \)) for player \( i \) to \( \sigma_{-i} \) with reference parameter \( r_i \). \( \Delta \times \prod_i D_i \) is non-empty, compact, and convex because \( \Delta \) is non-empty, compact, and convex (by reasoning identical to that given for eCVE) and each \( D_i \) is non-empty, compact, and convex by hypothesis. \( G \) is non-empty and closed because each \( BR_i \) is non-empty and closed by continuity of \( V_i \) and compactness of \( \Delta \), and because each \( f_i(\sigma, r_i) \) is a singleton, hence non-empty and compact. \( G \) is convex-valued because each \( BR_i \) is convex-valued by linearity of \( V_i \), and singletons are convex. Thus \( G \) has a fixed point; that any such fixed point is equilibrium satisfying \( r_i = f_i(\sigma, r_i) \) is clear. Myopic and non-myopic existence proofs for the case with \( r_i = f_i(\sigma, r_i) \) are identical to the expected-value case (except they depend on the existence proof for exogenous equilibrium with \( r_i = f_i(\sigma, r_i) \) rather than with expected-value, of course). □

**Theorem 2.3:**

**Proof.** Suppose \( \sigma \) is an eCVE. By Theorem 2.1, we may treat players as if their best-reply conditions are with respect to \( V_i \); since \( V_i \) is linear in \( i \)’s own probabilities, it respects first-order stochastic dominance, and so respects strict (deterministic) dominance. We now proceed by induction: if \( s_i \) is deleted in the initial round of deletions, then it is strictly dominated, and so it cannot be a best reply under \( V_i \). If \( s_i \) is deleted in a later round then all strategies deleted in earlier rounds are (by inductive hypothesis) played with probability 0, and we know that with these strategies removed from the game there is some \( s'_i \) that strictly dominates \( s_i \). Thus, under the inductive hypothesis, any gamble induced by shifting probability from \( s_i \) to \( s'_i \) first-order stochastically dominates the gamble induced by remaining at the original probability of playing \( s_i \), and so player \( i \) must play \( s_i \) with probability 0.
If \( \sigma \) is an mCVE, then we may simply fix each player’s \( r_i \) to her expected value during equilibrium play and repeat the argument for eCVE. □

**Theorem 2.4:**

*Proof.* Since \( \sigma_i \) is non-rationalizable, there is some iteration at which it is deleted. If it is deleted in the initial round, then \( \sigma_i \) is not a best reply to any \( \sigma_{-i} \), and the result is a trivial consequence of Theorem 2.1. If \( \sigma_i \) is deleted at some later iteration, then \( \sigma_i \) is not a best reply to any \( \sigma_{-i} \) in the convex hull of the undeleted, mixed strategies \( \sigma_{-i} \) for her opponents, and by inductive hypothesis, her opponents do not play any mixed strategy deleted prior to this iteration. Again the result follows directly from Theorem 2.1. As in Theorem 2.3, the result for mCVE follows by the same argument if we fix \( r_i = EV_i[\sigma] \). □

**Theorem 2.5:**

*Proof.* Suppose \( \sigma \) is an eCVE and fix \( r_i \). By Theorem 2.1, each player \( i \) plays a pure-strategy best reply \( s_i \) to \( s_{-i} \) with respect to \( V_i \), having fixed \( r_i \). This implies that the monetary outcome \( M \) associated with \( s_i \) is at least as large as that associated with \( (s'_i, s_{-i}) \) for any \( s'_i \in S_i \). Since \( V_i \) is strictly increasing in \( M \), \( s_i \) will still be a best reply if we set \( r_i = EV_i[s] \), so \( s \) is an mCVE, and similarly \( s \) is an NE by strict monotonicity of \( u_i \). This argument is reversible, establishing the converse. □

**Theorem 3.1:**

*Proof.* This proof is an obvious adaptation of the 2-player result (“Theorem A”) in Aumann and Brandenberger’s paper. We repeat their proof here for convenience with the necessary changes. As Aumann and Brandenberger’s formal representation of player beliefs requires considerable additional machinery, for the purposes of this proof, we adopt Aumann and Brandenberger’s notation where possible; we note that this means, for the scope of this proof (and the proof of the very next theorem), the symbols \( s_i, S, s_i \) bear a different meaning from their meaning in the rest of this paper.
We present the full $N$-player notation, as this is important for our next theorem, though we only need notation for 2 players presently. Aumann and Brandenberger let a strategic-form game $G$ be given by a finite set $\{1, \ldots, n\}$ of players and a set $\{A_1, \ldots, A_N\}$ of action sets, one one per player. They let $A$ be the cross-product of $A_1, \ldots, A_N$, and define an interactive belief system for $G$ to be a set $S_i$ of types for each player $i$, and, for each $s_i \in S_i$, a probability distribution on the set $S^{-i}$ of all possible combinations of other players’ types, an action $a_i \in A_i$, and a payoff function $g_i : A \to \mathbb{R}$. In addition, we suppose there is a real reference point $r_i$ given for each player, and define $g_i(a) = V_i(a, r_i)$ for each $a \in A$. We set $S$ to be the cross product of the sets $S_i$, and refer to its members $s \in S$ as states; these correspond to $N$-tuples of types, one per player; events are subsets of $S$. The types $s_i$ may be interpreted as probability distributions with domain $S^{-i}$; we extend these distributions to $S$ by setting $Pr(E; s_i)$ to be the probability of the event $\{s^{-i} \in S^{-i} : (s_i, s^{-i}) \in E\}$. $A^{-i}$ has an analogous meaning to $S'$, and has generic element $a^{-i}$. If $x$ is a function on $S$, then $[x]$ is understood as the event $\{s \in S : x(s) = x\}$. We define a conjecture $\phi^i$ for player $i$ to be a probability distribution on $A^{-i}$, and we call the marginal of $\phi^i$ on $A_j$ (for $j \neq i$) player $i$’s conjecture about $j$ under $\phi^i$. As in Aumann and Brandenberger’s development, player $i$’s theory in state $s$ can be understood to generate a conjecture $\phi^i(s)(a^{-i}) := p([a^{-i}]; s_i)$. $\Phi(s)$ denotes the $N$-tuple of conjectures induced by $s$ for each player.

Aumann and Brandenberger called the $i$th player rational at a state $s$ if $a_i(s) \in \operatorname{argmax}_{a'_i \in A_i} U_i(a'_i, a^{-i}(s)|s_i)$, where $|s_i$ indicates that this EU function is computed with respect to probability distributions determined by player $i$’s beliefs as given in state $s_i$, i.e., according to his conjecture $\phi^i(s)$. We make the obvious change, calling $i$ a value optimizer (or “CVE-rational”) if $a_i(s) \in \operatorname{argmax}_{a'_i \in A_i} V_i(a'_i, a^{-i}(s), r_i|s_i)$; we note that $a_i(s)$ is required to be a best reply over pure strategies, as in eCVE, and that this choice is consistent with Aumann and Brandenberger’s original definition of rationality.

We say a player $i$ knows an event $E$ at a state $s$ if $i$ assigns probability 1 to $E$ the occurrence of $E$, we define $K_i E$ to be the set of all $s$ in which $i$ knows $E$, define $K^1 E := K_1 E \cap \ldots \cap K_N E$, and $\operatorname{CKE} := \prod_{i=1}^n (K^1)^i E$. As in Aumann and Brandenberger’s work, if $s \in K^1 E$ then we say $E$ is mutually known at $s$, and if $s \in \operatorname{CKE}$, we say $E$ is commonly known at $s$. Finally, we say a
probability distribution $P$ over $S$ is a common prior if for all players $i$ and types $s_i$, $P$ conditioned on $s_i$ is $p(\cdot; s_i)$—that is, if the conditional distribution of $P$ on $s_i$ is the probability distribution given by $s_i$'s theory.

With these preliminaries, Aumann and Brandenberger prove the following fact: Player $i$ knows that he attributes probability $\pi$ to $E$ iff he in fact assigns probability $\pi$ to $E$; that is, players do not know falsehoods. This proof depends in no way on the changes we have made to incorporate PT preferences into Aumann and Brandenberger’s framework, so we state it without proof; as they further note, this fact implies that if, for an $N$-tuple of conjectures $\phi$ and a state $s$, it is mutually known that $\mathbf{E} = \phi$, then $\mathbf{E}(s) = \phi$, or that mutual knowledge of the conjectures implies that the conjectures are correct. This last observation we call “Fact A.”

To establish this theorem, Aumann and Brandenberger prove a single additional lemma: if $g$ is a game, $\phi$ is a $N$-tuple of conjectures, and at a state $s$ is it mutually known that $g = g$, that the players are CVE-rational, and that $\mathbf{E} = \phi$. Then if the conjecture $\phi^i$ of player $i$ gives probability probability to action $a_j$ of player $j$, it follows that $a_j$ maximizes $V_j$, taken over pure strategies, with probabilities determined by the conjecture $\phi^j$.

Aumann and Brandenberger establish this lemma as follows: by Fact A, $\phi^i$ is player $i$’s conjecture at state $s$, so at $s$ player $i$ assigns positive probability to the event $[a_j]$, i.e., to player $j$ playing pure strategy $a_j$, and by the lemma on mutually knowledge, player $i$ gives unit probability to $j$ being CVE-rational, $[\phi^j]$, and $[V_j]$. The intersection of these four events is therefore nonempty, so we can find a state in which $j$ is CVE-rational, she uses pure strategy $a_j$, her conjecture is $\phi^j$, and her payoff function is $V^j$, hence $a_j(s) \in \arg\max_{a'_j \in A_j} V_j(a'_j, a^{-j}(s), r_j|s_j). \square \square$

**Theorem 3.2:**

**Proof.** As with Theorem 3.1, this proof is a straightforward adaptation of the $N$-player result (“Theorem B”) in Aumann and Brandenberger’s paper; for this proof, we use the same formalism introduced in Theorem 3.1. Aumann and Brandenberger first produce a number of additional lemmata which depend in no way on the specification of player preferences, so we restate those results here without proof. The first two results are that $K_i(\prod_{j=1}^{\infty} E_j) = \prod_{j=1}^{\infty} K_i E_j$ and $CKE \subset
$K_CKE$; the third result requires a bit more prose: if $P$ is a common prior, $K_H \subset H$, and $p(E; s_i) = \pi \forall s \in H$, then $P(E \cap H) = \pi P(H)$. The final lemma states that, if $Q$ is a probability distribution on $A$ with $Q(a) = Q(a_i)Q(a^{-i}) \forall a \in A, \forall i$, then $Q(a) = Q(a_1)\ldots Q(a_n) \forall a$.

Aumann and Brandenberger first use these lemmata to show that, if we define $F := CK[\phi]$, take $P$ as common prior with $P(F) > 0$, and define $Q(a) := P([a]F)$, then $Q(a) = Q(a_i)Q(a^{-i})$ and, if a mixed strategy for player $j$ is defined by $\sigma_j(a_j) := Q(a_j)$, it can be shown that $\phi^i(a^{-i})\prod_{j \neq i} \sigma_j(a_j)$. Finally, we reach a point in their proof where the structure of eCVE is important: by the hypotheses of the theorem and the preceding lemmata, there exists a state $s \in S$ in which it is mutually known that $g = g$, that the players are CVE-rational, and that $\mathbf{OE} = \phi$, so that (again invoking a lemma) each $a_j$ in the support of $\phi^i$ for any $j \neq i$ must satisfy $a_j(s) \in \text{argmax}_{a_j \in A_j} V_j(a'_j, a^{-j}(s), r_j|s_j)$, where $s_j$ induces $\phi^j$. This is of course exactly the criterion for $\sigma$ to constitute an eCVE of the given game. As with Theorem 3.1, we emphasize this argument is essentially due to Aumann and Brandenberger; our contribution is to note that the maximization logic employed here, which is cast in terms of best replies taken over pure strategy sets, carries over almost without change to the case of eCVE. □

**Theorem 4.1:**

**Proof.** This is an immediate corollary of Theorem 2.5 and prior work on NE of network games; (e,m)CVE and NE are identical in these games, and so finding any one of the three is equivalent to finding any of the others. □

**Theorem 4.2:**

**Proof.** The proof of this result is essentially identical to that of Theorem 10 in Chapter 2. We outline the proof briefly: first, that the given problem is PPAD-Hard follows because finding $\nu$-NE is known to be PPAD-Complete (Daskalakis, Goldberg, and Papadimitriou, 2009), and the given problem contains finding $\varepsilon$-NE as a special case. We also know from Daskalakis et al. and Etessami and Yannakakis (2010) that $\nu$-NE of graphical games can be used to efficiently approximate arbitrary polynomials on finite-dimensional compact domains. The Daskalakis et al. constructions can be used to build a graphical game with $N$ distinguished core players and a separate subtree of graphical
subgames for each pure strategy $s^i_k$ of each core player $i$. Each such subtree has a distinguished, last player $P_{i,k}$ with two pure strategies, and has the property that, in $\frac{\varepsilon}{D}$-NE, where $D$ is the size of $C_i$, this player plays her first strategy with probability no more than $\varepsilon$ from player $i$'s deviation payoff (in $V_i$) for switching from her current mixed strategy to $s^i_k$, when played against the mixed strategies currently played by all other players; that $V_i$ is a polynomial implies that each of these deviation payoffs is a polynomial, so this approximation can be performed in the usual way. The final step is to make the core players’ VM payoffs equivalent to a non-weighted linear sum of their subtree players’ probabilities of playing their first strategies; this is achieved by paying the core players’ 1 for choosing one of their subtree players, when that player uses her first strategy, and 0 otherwise.

An $\frac{\varepsilon}{D}$-NE of this game corresponds to an $\varepsilon$-eCVE of the original game in the obvious way. That this is so follows from the linearity of each $\overline{V}_i$: since these are linear in each players’ own strategic probabilities, the linearity of VM preferences naturally causes the core players’ payoffs in the game just defined to maximize $\overline{V}_i$. □

**Theorem 4.3:**

**Proof.** Both directions of this proof are adaptations of similar arguments developed by Etessami and Yannakakis (2010) for the corresponding classical theory, and of Nash’s (1951) original proof that NE exist in all finite games. First, we show that finding an eCVE reduces to the problem of finding a fixed point for a function defined by a computational circuit with basis over $\{+, -, \ast\}$. This is actually quite easy: it can be seen readily by using the obvious modification of Nash’s function, defined by $F_{ij}(x) = \frac{x_{ij} + \max \{0, g_{ij}(x)\}}{1 + \sum_{i=1}^{n} \max \{0, g_{ij}(x)\}}$, where $g_{ij}$ is player $i$'s change in value (of $\overline{V}_i$, with $r_i$ fixed) for switching from the strategy dictated by $x$ to her $j$th strategy. So long as the value functions $\overline{V}_i$ are efficiently representable by an algebraic circuit, so too is $F$.

The proof that fixed points of $F$ are eCVE is identical to Nash’s (1951) original proof; the critical property necessary for this proof is linearity of each player’s preferences in her own payoffs, as provided by Theorem 2.5. If we sum $F_{ij}(x)$ over $j$, it is clear that $F_{ij}(x)$ maps $\Delta$ to $\Delta$, i.e., that $F_{ij}(\sigma)$ taken over all pure strategies $j$ for player $i$ defines a mixed strategy for each $i$. 163
Suppose $\sigma^*$ is a fixed point of $F$, but that it is not an eCVE. Then $\sum_j F_{ij}(\sigma^*) = 1 + \sum_j \max\{0, g_{ij}(x)\} > 1$; call this sum $S_i$. Since $\sigma^*$ is a fixed point, we have $F_i(\sigma^*) = \sigma^*_i$; this implies $\sigma^*_i = \frac{\sigma^*_i + \max\{0, g_{ij}(x)\}}{S_i}$, which in turn gives $\sigma^*_i = \frac{\max\{0, g_{ij}(x)\}}{S_i - 1}$. It follows that $\sigma^*_i(s_j)(V_i(s_j, \sigma^*_{-i}; r_i) - V_i(\sigma^*; r_i)) = \sigma^*_i(s_j)\max\{0, g_{ij}(\sigma^*)\}$; if $\max\{0, g_{ij}(\sigma^*)\} > 0$, this is obvious, while if $\max\{0, g_{ij}(\sigma^*)\} = 0$, then $\sigma^*_i(s_j) = \frac{1}{S_i - 1}\max\{0, g_{ij}(\sigma^*)\} = 0$. This allows us to complete Nash’s argument as follows

\[
0 = V_i(\sigma^*_i, \sigma^*_{-i}) - V_i(\sigma^*_i, \sigma^*_{-i})
\]

\[
= \left( \sum_j \sigma^*_i(s_j)V_i(s_j, \sigma^*_{-i}) \right) - V_i(\sigma^*_i, \sigma^*_{-i})
\]

\[
= \sum_j \sigma^*_i(s_j)\max\{0, g_{ij}(x)\}
\]

\[
= \sum_j (S_i - 1) (\sigma^*_i(s_j))^2
\]

a contradiction, since this last term is strictly positive. Note that the critical juncture in this argument for our purposes is the inference that $V_i(\sigma^*_i, \sigma^*_{-i}) - V_i(\sigma^*_i, \sigma^*_{-i})$ implies $\left( \sum_j \sigma^*_i(s_j)V_i(s_j, \sigma^*_{-i}) \right) - V_i(\sigma^*_i, \sigma^*_{-i})$. This inference is not possible without Theorem 2.5; thus the structure provided by CVE allows Nash’s original argument to carry over to the present, more general context.

The converse reduction is trivial: approximating an eCVE contains the approximation of NE, and since the latter is FIXP-complete, the former must be FIXP-hard. □
Appendix C

(Proofs and Graphs for Chapter 4)

Observation 4.0.2:

Proof. This follows from two facts: A) PT preferences (and the representing functional $\overline{V}_i$ of chapter 2, for any underlying preferences) obey first-order stochastic dominance, and B) in a binary game, first-order stochastic dominance yields a complete preference ordering (though with the possibility of indifference) on the set of mixed-strategy profiles. Since EU preferences are a special case of PT preferences, this shows that, independent of the details of preference, underlying monetary payoffs determine the same preference ordering among mixed-strategy profiles for EU and PT players, and so the argmax conditions appearing in the definitions of ePT-EB and eCVE are identical to those in the definition of NE. This implies that each player’s argmax set (if taken over mixed strategies) is already convex (because this always the case for NE, which has convex best-reply sets as a result of the linearity of EU preferences), so the convex-hull operators are superfluous, and all three definitions agree. Since every myopic equilibrium is an exogenous equilibrium of the appropriate type, the result for myopic equilibria follows trivially. The result for non-myopic equilibria follows from a straightforward induction. We note that finiteness has not been invoked at any point in this argument; the restriction to finite games is necessary only because we have not provided definitions of (e,m,n)PT-NE, (e,m,n)PT-EB, or (e,m,n)CVE for general games, as the use of convex hulls in infinite-dimensional spaces is nettlesome, and defining non-myopia in games with infinitely deep game trees is complicated by the lack of a set of “final” game nodes.
Table C.1: 2x2 Binary Game(s):

<table>
<thead>
<tr>
<th></th>
<th>Game 18 (Ochs, 1995)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1, 0</td>
</tr>
<tr>
<td></td>
<td>0, 1</td>
</tr>
</tbody>
</table>

Table C.2: 4x4 Binary Game(s)

<table>
<thead>
<tr>
<th>Game 19 (O’Neill, 1987)</th>
<th>Game 20 (Binmore et al., 2001)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5, 5</td>
<td>-5, 5</td>
</tr>
<tr>
<td>5, -5</td>
<td>5, -5</td>
</tr>
<tr>
<td>-5, 5</td>
<td>-5, 5</td>
</tr>
<tr>
<td>5, -5</td>
<td>5, -5</td>
</tr>
</tbody>
</table>

Table C.3: 5x5 Binary Game(s)

<table>
<thead>
<tr>
<th>Game 21 (Rapoport &amp; Boebel, 1992)</th>
<th>Game 22 (Rapoport &amp; Boebel, 1992)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10, -6</td>
<td>15, -1</td>
</tr>
<tr>
<td>-6, 10</td>
<td>-1, 15</td>
</tr>
<tr>
<td>-6, 10</td>
<td>-1, 15</td>
</tr>
<tr>
<td>-6, 10</td>
<td>-1, 15</td>
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<tr>
<td>-6, 10</td>
<td>-1, 15</td>
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<tr>
<td>-6, 10</td>
<td>-1, 15</td>
</tr>
<tr>
<td>-6, 10</td>
<td>-1, 15</td>
</tr>
</tbody>
</table>

Table C.4: 6x6 Binary Game(s)

<table>
<thead>
<tr>
<th>Game 23 (Mookerjee &amp; Sopher, 1997)</th>
</tr>
</thead>
<tbody>
<tr>
<td>W</td>
</tr>
<tr>
<td>L</td>
</tr>
<tr>
<td>L</td>
</tr>
<tr>
<td>L</td>
</tr>
<tr>
<td>L</td>
</tr>
<tr>
<td>W</td>
</tr>
</tbody>
</table>

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NOTE: Game 23 is actually an archetype for two first-order stochastically equivalent games, whose results were averaged in our data set. The game is zero-sum; L always denotes a payoff of 0 rupees, while W was 10 rupees in one game and 0 in the remaining game.

**Python2.7 eCVE Algorithm:**

```python
#!/ C:\Python27\'
import sys
import time
import math
import mpmath
from copy import deepcopy

eps = 0.02
equil_p = []
equil_q = []
equils = []
bigN = 250
m_set1 = set([])
m_set2 = set([])
m1_sorted = []
m2_sorted = []
m_dict1 = {}
m_dict2 = {}
num_strat1 = 2
num_strat2 = 2
lam1 = 2.0
lam2 = 10.0
K1 = 1.0
```

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K2 = 1.0
A1 = 2.0
A2 = 2.0
r1 = 5
r2 = 5

# takes real b, nat e as inputs, computes unique real root
def r1 rt(b, e):
    rt_val = 0.0
    # e = int(e)
    roots = [mpmath.root(b, e, k) for k in range(e)]
    # for r in roots:
    #     if float(mpmath.im(r)) == 0.0:
    #         return float(mpmath.re(r))
    e = int(e)
    # print "e = " + str(e) + " and e/2 = " + str(e/2)
    # print "base = " + str(b)
    if b < 0.0:
        return float(mpmath.root(b, e, e/2))
    else:
        return float(mpmath.root(b, e))

# fills m_set1, m_set2 with monetary payoff possibilities, m_dict1, m_dict2 w/ (s1, s2) -> m mappings
def fill_payoffs():
    for s1 in xrange(1, num_strat1+1):
        for s2 in xrange(1, num_strat2+1):
m_set1.update([M1(s1, s2)])
m_set2.update([M2(s1, s2)])
m_dict1[(s1, s2)] = M1(s1, s2)
m_dict2[(s1, s2)] = M2(s1, s2)

def sort_payoffs(p_num):
    if p_num == 1:
        return sorted(m_set1)
    elif p_num == 2:
        return sorted(m_set2)

# Cum/Decum Prob Fxns of monetary payoffs for each player
# Take as inputs p, q lists and relevant monetary reward
def CDF1(p, q, m1):
    tot_prob = 0.0
    for i in xrange(0, len(p)):
        for j in xrange(0, len(q)):
            if M1(i+1, j+1) <= m1:
                tot_prob += p[i] * q[j]
    return tot_prob

def DDF1(p, q, m1):
    tot_prob = 0.0
    for i in xrange(0, len(p)):
        for j in xrange(0, len(q)):
            if M1(i+1, j+1) >= m1:
                tot_prob += p[i] * q[j]
return tot_prob

def CDF2(p, q, m2):
    tot_prob = 0.0
    for i in xrange(0, len(p)):
        for j in xrange(0, len(q)):
            if M2(i+1, j+1) <= m2:
                tot_prob += p[i] * q[j]
    return tot_prob

def DDF2(p, q, m2):
    tot_prob = 0.0
    for i in xrange(0, len(p)):
        for j in xrange(0, len(q)):
            if M2(i+1, j+1) >= m2:
                tot_prob += p[i] * q[j]
    return tot_prob

# dec takes a real r, returns its decimal part
def dec(r):
    d_part = abs(r - int(r))
    return d_part

# w1/2 takes a probability (& real K1/K2, real A1/2), returns weighted probability
def w1(p):
    K_dwn = math.floor(K1)
\( w_{\text{dwn}} = 1.0 - \text{dec}(K_1) \)
\( K_{\text{up}} = K_{\text{dwn}} + 1.0 \)
\( w_{\text{up}} = 1.0 - w_{\text{dwn}} \)
\( w_{\text{tot}} = 0.0 \)

if \( K_{\text{dwn}} \geq 0.0 \):
    \[
    A_{\text{fac}} = \frac{1.0}{1.0 + \text{math.pow}(A1 - 1.0, 2.0 \times K_{\text{dwn}} + 1.0)}
    \]
    \[
    p_{\text{fac}} = \text{math.pow}(A1 \times p + 1.0 - A1, 2.0 \times K_{\text{dwn}} + 1.0) + \text{math.pow}(A1 - 1.0, 2.0 \times K_{\text{dwn}} + 1.0)
    \]
    \( w_{\text{tot}} += w_{\text{dwn}} \times A_{\text{fac}} \times p_{\text{fac}} \)
else:
    \[
    A_{\text{fac}} = \frac{1.0}{1.0 + \text{rl} \_ \text{rt}(A1 - 1.0, 2.0 \times \text{abs}(K_{\text{dwn}}) + 1.0)}
    \]
    \[
    p_{\text{fac}} = \text{rl} \_ \text{rt}(A1 \times p + 1.0 - A1, 2.0 \times \text{abs}(K_{\text{dwn}}) + 1.0) + \text{rl} \_ \text{rt}(A1 - 1.0, 2.0 \times \text{abs}(K_{\text{dwn}}) + 1.0)
    \]
    \( w_{\text{tot}} += w_{\text{dwn}} \times A_{\text{fac}} \times p_{\text{fac}} \)

if \( K_{\text{up}} \geq 0.0 \):
    \[
    A_{\text{fac}} = \frac{1.0}{1.0 + \text{math.pow}(A1 - 1.0, 2.0 \times K_{\text{up}} + 1.0)}
    \]
    \[
    p_{\text{fac}} = \text{math.pow}(A1 \times p + 1.0 - A1, 2.0 \times K_{\text{up}} + 1.0) + \text{math.pow}(A1 - 1.0, 2.0 \times K_{\text{up}} + 1.0)
    \]
    \( w_{\text{tot}} += w_{\text{up}} \times A_{\text{fac}} \times p_{\text{fac}} \)
else:
    \[
    A_{\text{fac}} = \frac{1.0}{1.0 + \text{rl} \_ \text{rt}(A1 - 1.0, 2.0 \times \text{abs}(K_{\text{up}}) + 1.0)}
    \]
\[ K_{\text{up}} + 1.0 \] 

\[ p_{\text{fac}} = r1\_rt(A1 \times p + 1.0 - A1, 2.0 \times \text{abs}(K_{\text{up}}) + 1.0) + r1\_rt(A1 - 1.0, 2.0 \times \text{abs}(K_{\text{up}}) + 1.0) \]

\[ w_{\text{tot}} \text{ += } w_{\text{up}} \times A_{\text{fac}} \times p_{\text{fac}} \]

\[ \text{return } w_{\text{tot}} \]

def w2(q):
    K_dwn = \text{math.floor}(K2)
    w_dwn = 1.0 - \text{dec}(K2)
    K_up = K_dwn + 1.0
    w_up = 1.0 - w_dwn
    w_tot = 0.0

    if K_dwn >= 0.0:
        A_fac = 1.0 / (1.0 + \text{math.pow}(A2 - 1.0, 2.0 \times \text{abs}(K_dwn) + 1.0))
        q_fac = \text{math.pow}(A2 \times q + 1.0 - A2, 2.0 \times K_dwn + 1.0) + \text{math.pow}(A2 - 1.0, 2.0 \times K_dwn + 1.0)
        w_tot += w_dwn \times A_{\text{fac}} \times q_{\text{fac}}
    else:
        A_fac = 1.0 / (1.0 + r1\_rt(A2 - 1.0, 2.0 \times \text{abs}(K_dwn) + 1.0))
        q_fac = r1\_rt(A2 \times q + 1.0 - A2, 2.0 \times \text{abs}(K_dwn) + 1.0) + r1\_rt(A2 - 1.0, 2.0 \times \text{abs}(K_dwn) + 1.0)
\[ w_{\text{tot}} += w_{\text{dwn}} \times A_{\text{fac}} \times q_{\text{fac}} \]

if \( K_{\text{up}} \geq 0.0 \):

\[ A_{\text{fac}} = \frac{1.0}{1.0 + \text{math.pow}(A2 - 1.0, 2.0 \times K_{\text{up}} + 1.0)} \]
\[ q_{\text{fac}} = \text{math.pow}(A2 \times q + 1.0 - A2, 2.0 \times K_{\text{up}} + 1.0) + \text{math.pow}(A2 - 1.0, 2.0 \times K_{\text{up}} + 1.0) \]
\[ w_{\text{tot}} += w_{\text{up}} \times A_{\text{fac}} \times q_{\text{fac}} \]

else:

\[ A_{\text{fac}} = \frac{1.0}{1.0 + \text{rl}_{\text{rt}}(A2 - 1.0, 1.0/\left(2.0 \times \text{abs}(K_{\text{up}}) + 1.0\right))} \]
\[ q_{\text{fac}} = \text{rl}_{\text{rt}}(A2 \times q + 1.0 - A2, 2.0 \times \text{abs}(K_{\text{up}}) + 1.0) + \text{rl}_{\text{rt}}(A2 - 1.0, 2.0 \times \text{abs}(K_{\text{up}}) + 1.0) \]
\[ w_{\text{tot}} += w_{\text{up}} \times A_{\text{fac}} \times q_{\text{fac}} \]

return \( w_{\text{tot}} \)

# M1/2 takes two ints, returns corresponding monetary payoff;
# these are given by the game's structure

def M1(s1, s2):
    if s1 == 1 and s2 == 1:
        return 7.0
    elif s1 == 1 and s2 == 2:
        return 0.0
    elif s1 == 2 and s2 == 1:
        return 4.0
elif s1 == 2 and s2 == 2:
    return 8.0

def M2(s1, s2):
    if s1 == 1 and s2 == 1:
        return 2.0
    elif s1 == 1 and s2 == 2:
        return 9.0
    elif s1 == 2 and s2 == 1:
        return 5.0
    elif s1 == 2 and s2 == 2:
        return 1.0

# l1lV1/2 takes ml/2 & ref point, returns corresponding (deterministic) PT value
def l1lV1(ml, r1):
    if ml >= r1:
        return ml - r1
    else:
        return lam1 * (ml - r1)

def l1lV2(m2, r2):
    if m2 >= r2:
        return m2 - r2
    else:
        return lam2 * (m2 - r2)
# bigV1/2 takes two probability lists & ref point, returns corresponding (stochastic) PT value

def bigV1(p, q, r1):
    total_val = 0.0
    for i in xrange(0, len(m1_sorted)):
        if i > 0 and i < len(m1_sorted) - 1:
            if m1_sorted[i] >= r1:
                total_val += (w1(DDF1(p, q, m1_sorted[i])) - w1(DDF1(p, q, m1_sorted[i+1]))) * liiV1(m1_sorted[i], r1)
            else:
                total_val += (w1(CDF1(p, q, m1_sorted[i])) - 0.0) * liiV1(m1_sorted[i], r1)
        elif i == 0:
            if m1_sorted[i] >= r1:
                total_val += (w1(DDF1(p, q, m1_sorted[i])) - w1(DDF1(p, q, m1_sorted[i+1]))) * liiV1(m1_sorted[i], r1)
            else:
                total_val += (w1(CDF1(p, q, m1_sorted[i])) - 0.0) * liiV1(m1_sorted[i], r1)
        elif i == len(m1_sorted) - 1:
            if m1_sorted[i] >= r1:
                total_val += (w1(DDF1(p, q, m1_sorted[i])) - w1(DDF1(p, q, m1_sorted[i+1]))) * liiV1(m1_sorted[i], r1)
            else:
                total_val += (w1(CDF1(p, q, m1_sorted[i])) - 0.0) * liiV1(m1_sorted[i], r1)
if m1_sorted[i] >= r1:
    total_val += (w1(DDF1(p, q, m1_sorted[i])) - 0.0) * li1V1(m1_sorted[i], r1)
else:
    total_val += (w1(CDF1(p, q, m1_sorted[i])) - w1(CDF1(p, q, m1_sorted[i-1]))) * li1V1(m1_sorted[i], r1)

return total_val

def bigV2(p, q, r2):
    total_val = 0.0
    for i in xrange(0, len(m2_sorted)):
        if i > 0 and i < len(m2_sorted) - 1:
            if m2_sorted[i] >= r2:
                total_val += (w2(DDF2(p, q, m2_sorted[i])) - w2(DDF2(p, q, m2_sorted[i+1]))) * li1V2(m2_sorted[i], r2)
            else:
                total_val += (w2(CDF2(p, q, m2_sorted[i])) - w2(CDF2(p, q, m2_sorted[i-1]))) * li1V2(m2_sorted[i], r2)
        elif i == 0:
            if m2_sorted[i] >= r2:

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total_val += ( w2(DDF2(p, q,
    m2_sorted[i])) − w2(DDF2(p, q,
    m2_sorted[i+1])) ) * li1V2(
    m2_sorted[i], r2)

else:
    total_val += ( w2(CDF2(p, q,
    m2_sorted[i])) − 0.0 ) * li1V2
    (m2_sorted[i], r2)

elif i == len(m2_sorted) − 1:
    if m2_sorted[i] >= r2:
        total_val += ( w2(DDF2(p, q,
            m2_sorted[i])) − 0.0 ) * li1V2
        (m2_sorted[i], r2)
    else:
        total_val += ( w2(CDF2(p, q,
            m2_sorted[i])) − w1(CDF2(p, q,
            m2_sorted[i−1])) ) * li1V1(
            m2_sorted[i], r2)

return total_val

# given q, r1, maximizes V1 over p
# currently set up for P1 having 2 strats
def max_V1(q, r1):
    vals = [bigV1([1.0,0.0],q,r1), bigV1([0.0,1.0],q,r1)]
    return max(vals)

# given p, r2, maximizes V2 over q

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# currently set up for P2 having 2 strats

def max_V2(p, r2):
    vals = [bigV2(p,[1.0,0.0],r2), bigV2(p,[0.0,1.0],r2)]
    return max(vals)

# checks that pure strategies in support of p are optimal

def P1_supp_chk(p,q,r1):
    if p[0] > 0.0 and bigV1([1.0,0.0],q,r1) <= bigV1([0.0,1.0],q,r1) - eps:
        return False
    elif p[1] > 0.0 and bigV1([0.0,1.0],q,r1) <= bigV1([1.0,0.0],q,r1) - eps:
        return False
    return True

# checks that pure strategies in support of q are optimal

def P2_supp_chk(p,q,r2):
    if q[0] > 0.0 and bigV2(p,[1.0,0.0],r2) <= bigV2(p,[0.0,1.0],r2) - eps:
        return False
    elif q[1] > 0.0 and bigV2(p,[0.0,1.0],r2) <= bigV2(p,[1.0,0.0],r2) - eps:
        return False
    return True

# finds sample points in simplices, tests them; needs to be
# written differently for each M x N game; M/N change numbers of

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loops

# currently set up for 2 x 2 games
# also currently set with exogenous r1 = 0.5, r2 = 0.5; could
easily set them each to EV_i

def find_equil():
    p = [1.0, 0.0]
    q = [1.0, 0.0]
    tru_p = [-1.0, -1.0]
    tru_q = [-1.0, -1.0]

    last_p_diff = 0.0
    last_q_diff = 0.0

    for i in xrange(0, bigN+1):
        p[0] = float(i) / float(bigN)
        p[1] = 1.0 - p[0]
        p_diff = bigV2(p, [0.0, 1.0], r2) - bigV2(p,
               [1.0, 0.0], r2)

        if i > 0:
            if (last_p_diff <= 0 and p_diff > 0) or (last_p_diff >= 0 and p_diff < 0):
                tru_p = deepcopy(p)
                equil_p.append(deepcopy([tru_p]))
            last_p_diff = deepcopy(p_diff)

        q[0] = float(i) / float(bigN)
        q[1] = 1.0 - q[0]
q_diff = bigV1([0.0, 1.0], q, r1) - bigV1([1.0, 0.0], q, r1)

if i > 0:
    if (last_q_diff <= 0 and q_diff > 0) or (last_q_diff >= 0 and q_diff < 0):
        tru_q = deepcopy(q)
        equil_q.append(deepcopy([tru_q]))

last_q_diff = deepcopy(q_diff)

def set_ref(r, p):
    ref_pt = 0.0

    if r == 0:
        # set r to min of payoffs for player p
        if p == 1:
            ref_pt = min(m1_sorted)
        elif p == 2:
            ref_pt = min(m2_sorted)
    elif r == 1:
        # set r to mean of payoffs for player p
        if p == 1:
            for s1 in xrange(1, num_strat1+1):
                for s2 in xrange(1, num_strat2+1):
                    ref_pt += M1(s1, s2)
            ref_pt = ref_pt / float(num_strat1 * num_strat2)
        elif p == 2:
            180
for s1 in xrange(1, num_strat1 + 1):
    for s2 in xrange(1, num_strat2 + 1):
        ref_pt += M2(s1, s2)
        ref_pt = ref_pt / float(num_strat1 * num_strat2)

elif r == 2:  # set r to max of payoffs for player p
    if p == 1:
        ref_pt = max(m1_sorted)
    elif p == 2:
        ref_pt = max(m2_sorted)

return ref_pt

fill_payoffs()

m1_sorted = sort_payoffs(1)
m2_sorted = sort_payoffs(2)

start_time = time.time()

for e in xrange(1, 2):
    eps = float(e) / 100.0
    for r in xrange(1, 2):
        r1 = set_ref(r, 1)
        r2 = set_ref(r, 2)
        for lam in xrange(0, 2):
            lam1 = float(lam) * 0.2 + 1.0
            lam2 = float(lam) * 0.2 + 1.0
            for k in xrange(0, 1):
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\[ K_1 = \text{float}(k) \times 0.4 - 2.0 \]

\[ K_2 = \text{float}(k) \times 0.4 - 2.0 \]

for A in xrange(0, 1):
    \[ A_1 = \text{float}(A) \times 0.2 + 2.0 \]
    \[ A_2 = \text{float}(A) \times 0.2 + 2.0 \]
    equils = []
    equil_p = []
    equil_q = []
    f = file('SC_G5_CVE_Min_Mean_Fix.txt', 'a')
    sys.stdout = f
    print("Selten & Chmura CVE Game 12")
    print("bigN = " + str(bigN) + " eps = " + str(eps))
    find_equil()
    data = ""
    for ms in xrange(0, len(equil_p)):
        data += str(equil_p[ms][0][0])
    data += ","
    data += str(len(equil_p))
    data += ","

for ms in xrange(0, len(equil_q)):
    data += str(equil_q[ms][0][0])
data += ","
data += str(len(equil_q))
print(str(r1) + "," + str(r2) + "," + str(K1) + "," + str(A1) + "," + str(lam1) + "," + data);
print("PT−EB set is = ")
print("p's: " + str(equil_p) + " q's: " + str(equil_q))
print("Run−time in secs: 
" + str(time.time() − start_time))
f.close()
import simplex
from copy import deepcopy

eps = 0.02
equils = []
bigN = 100
m_set1 = set([])
m_set2 = set([])
m1_sorted = []
m2_sorted = []
m_dict1 = {}
m_dict2 = {}
num_strat1 = 2
num_strat2 = 2
lam1 = 1.0
lam2 = 1.0
K1 = -55.0
K2 = -1010101.0
A1 = 2.0
A2 = 2.0
r1 = 1.0
r2 = 0.5
old = sys.stdout

# takes real b, nat e as inputs, computes unique real root
def r1_rt(b, e):
    e = int(e)
if b < 0.0:
    return float(mpmath.root(b, e, e/2))
else:
    return float(mpmath.root(b, e))

# fills m_set1, m_set2 with monetary payoff possibilities, m_dict1, m_dict2 w/ (s1,s2) -> m mappings

def fill_payoffs():
    for s1 in xrange(1, num_strat1+1):
        for s2 in xrange(1, num_strat2+1):
            m_set1.update([M1(s1,s2)])
            m_set2.update([M2(s1,s2)])
            m_dict1[(s1,s2)] = M1(s1,s2)
            m_dict2[(s1,s2)] = M2(s1,s2)

def sort_payoffs(p_num):
    if p_num == 1:
        return sorted(m_set1)
    elif p_num == 2:
        return sorted(m_set2)

# Cum/Decum Prob Fxns of monetary payoffs for each player
# Take as inputs p, q lists and relevant monetary reward

def CDF1(p, q, ml):
    tot_prob = 0.0
    for i in xrange(0,len(p)):
        for j in xrange(0,len(q)):
def DDF1(p, q, m1):
    tot_prob = 0.0
    for i in xrange(0,len(p)):
        for j in xrange(0,len(q)):
            if M1(i+1,j+1) >= m1:
                tot_prob += p[i] * q[j]
    return tot_prob

def DDF2(p, q, m2):
    tot_prob = 0.0
    for i in xrange(0,len(p)):
        for j in xrange(0,len(q)):
            if M2(i+1,j+1) <= m2:
                tot_prob += p[i] * q[j]
    return tot_prob

if M1(i+1,j+1) <= m1:
    tot_prob += p[i] * q[j]
return tot_prob

def CDF2(p, q, m2):
    tot_prob = 0.0
    for i in xrange(0,len(p)):
        for j in xrange(0,len(q)):
            if M2(i+1,j+1) <= m2:
                tot_prob += p[i] * q[j]
    return tot_prob

def DDF2(p, q, m2):
    tot_prob = 0.0
    for i in xrange(0,len(p)):
        for j in xrange(0,len(q)):
            if M2(i+1,j+1) >= m2:
                tot_prob += p[i] * q[j]
    return tot_prob
# dec takes a real r, returns its decimal part
def dec(r):
    d_part = abs(r - int(r))
    return d_part

# w1/2 takes a probability (& real K1/K2, real A1/2), returns weighted probability
def w1(p):
    K_dwn = math.floor(K1)
    w_dwn = 1.0 - dec(K1)
    K_up = K_dwn + 1.0
    w_up = 1.0 - w_dwn
    w_tot = 0.0

    if K_dwn >= 0.0:
        A_fac = 1.0 / (1.0 + math.pow(A1 - 1.0, 2.0 * K_dwn + 1.0))
        p_fac = math.pow(A1 * p + 1.0 - A1, 2.0 * K_dwn + 1.0) + math.pow(A1 - 1.0, 2.0 * K_dwn + 1.0)
        w_tot += w_dwn * A_fac * p_fac
    else:
        A_fac = 1.0 / (1.0 + r1_rt(A1 - 1.0, 2.0 * abs(K_dwn) + 1.0))
        p_fac = r1_rt(A1 * p + 1.0 - A1, 2.0 * abs(K_dwn) + 1.0) + r1_rt(A1 - 1.0, 2.0 * abs(K_dwn) + 1.0)
w_tot += w_dwn * A_fac * p_fac

if K_up >= 0.0:
    A_fac = 1.0 / (1.0 + math.pow(A1 - 1.0, 2.0 * K_up + 1.0))
    p_fac = math.pow(A1 * p + 1.0 - A1, 2.0 * K_up + 1.0) + math.pow(A1 - 1.0, 2.0 * K_up + 1.0)
    w_tot += w_up * A_fac * p_fac
else:
    A_fac = 1.0 / (1.0 + rl rt(A1 - 1.0, 2.0 * abs(K_up) + 1.0))
    p_fac = rl rt(A1 * p + 1.0 - A1, 2.0 * abs(K_up) + 1.0) + rl rt(A1 - 1.0, 2.0 * abs(K_up) + 1.0)
    w_tot += w_up * A_fac * p_fac

return w_tot

def w2(q):
    K_dwn = math.floor(K2)
    w_dwn = 1.0 - dec(K2)
    K_up = K_dwn + 1.0
    w_up = 1.0 - w_dwn
    w_tot = 0.0

    if K_dwn >= 0.0:
        A_fac = 1.0 / (1.0 + math.pow(A2 - 1.0, 2.0 *
K_dwn + 1.0 )
q_fac = math.pow(A2 * q + 1.0 - A2, 2.0 * K_dwn + 1.0) + math.pow(A2 - 1.0, 2.0 * K_dwn + 1.0)
w_tot += w_dwn * A_fac * q_fac

else:
    A_fac = 1.0 / (1.0 + rl_rt(A2 - 1.0, 2.0 * abs(K_dwn) + 1.0))
    q_fac = rl_rt(A2 * q + 1.0 - A2, 2.0 * abs(K_dwn) + 1.0) + rl_rt(A2 - 1.0, 2.0 * abs(K_dwn) + 1.0)
w_tot += w_dwn * A_fac * q_fac

if K_up >= 0.0:
    A_fac = 1.0 / (1.0 + math.pow(A2 - 1.0, 2.0 * K_up + 1.0))
    q_fac = math.pow(A2 * q + 1.0 - A2, 2.0 * K_up + 1.0) + math.pow(A2 - 1.0, 2.0 * K_up + 1.0)
w_tot += w_up * A_fac * q_fac

else:
    A_fac = 1.0 / (1.0 + rl_rt(A2 - 1.0, 1.0/( 2.0 * abs(K_up) + 1.0 ) ) )
    q_fac = rl_rt(A2 * q + 1.0 - A2, 2.0 * abs(K_up) + 1.0) + rl_rt(A2 - 1.0, 2.0 * abs(K_up) + 1.0)
w_tot += w_up * A_fac * q_fac

return w_tot
# M1/2 takes two ints, returns corresponding monetary payoff;
these are given by the game's structure
# currently set up for Matching Pennies

def M1(s1, s2):
    if s1 == 1 and s2 == 1:
        return 9.0
    elif s1 == 1 and s2 == 2:
        return 0.0
    elif s1 == 2 and s2 == 1:
        return 0.0
    elif s1 == 2 and s2 == 2:
        return 1.0

def M2(s1, s2):
    if s1 == 1 and s2 == 1:
        return 0.0
    elif s1 == 1 and s2 == 2:
        return 1.0
    elif s1 == 2 and s2 == 1:
        return 1.0
    elif s1 == 2 and s2 == 2:
        return 0.0

# lilV1/2 takes ml/2 & ref point, returns corresponding (deterministic) PT value

def lilV1(ml, rl):

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if ml >= r1:
    return ml - r1
else:
    return lam1 * (ml - r1)

def lilV2(m2, r2):
    if m2 >= r2:
        return m2 - r2
    else:
        return lam2 * (m2 - r2)

# bigV1/2 takes two probability lists & ref point, returns corresponding (stochastic) PT value
def bigV1(p, q, r1):
    total_val = 0.0
    for i in xrange(0, len(m1_sorted)):
        if i > 0 and i < len(m1_sorted) - 1:
            if m1_sorted[i] >= r1:
                total_val += (wl/DDF1(p, q,
                                  m1_sorted[i])) -
                (wl/CDF1(p, q,
                          m1_sorted[i + 1]) ) *
                lilV1(m1_sorted[i], r1)
            else:
                total_val += (wl/CDF1(p, q,
                                        m1_sorted[i])) -
                (wl/CDF1(p, q,
                          m1_sorted[i - 1]) ) *
                lilV1(m1_sorted[i], r1)
elif i == 0:
    if ml_sorted[i] >= r1:
        total_val += (wl(DDFl(p,q,
            ml_sorted[i])) - wl(DDFl(p,q,
            ml_sorted[i+1])) ) * lilV1(
            ml_sorted[i], r1)
    else:
        total_val += (wl(CDF1(p,q,
            ml_sorted[i])) - 0.0 ) * lilV1
            (ml_sorted[i], r1)
elif i == len(ml_sorted) - 1:
    if ml_sorted[i] >= r1:
        total_val += (wl(DDFl(p,q,
            ml_sorted[i])) - 0.0 ) * lilV1
            (ml_sorted[i], r1)
    else:
        total_val += (wl(CDF1(p,q,
            ml_sorted[i])) - wl(CDF1(p,q,
            ml_sorted[i-1])) ) * lilV1(
            ml_sorted[i], r1)

return total_val

def bigV2(p,q,r2):
    total_val = 0.0
    for i in xrange(0,len(m2_sorted)):
        if i > 0 and i < len(m2_sorted) - 1:
            if m2_sorted[i] >= r2:
total_val += (w2(DDF2(p, q, m2_sorted[i])) - w2(DDF2(p, q, m2_sorted[i+1]))) * li1V2(m2_sorted[i], r2)

else:

total_val += (w2(CDF2(p, q, m2_sorted[i])) - w2(CDF2(p, q, m2_sorted[i-1]))) * li1V2(m2_sorted[i], r2)

elif i == 0:

if m2_sorted[i] >= r2:

total_val += (w2(DDF2(p, q, m2_sorted[i])) - w2(DDF2(p, q, m2_sorted[i+1]))) * li1V2(m2_sorted[i], r2)

else:

total_val += (w2(CDF2(p, q, m2_sorted[i])) - 0.0) * li1V2(m2_sorted[i], r2)

elif i == len(m2_sorted) - 1:

if m2_sorted[i] >= r2:

total_val += (w2(DDF2(p, q, m2_sorted[i])) - 0.0) * li1V2(m2_sorted[i], r2)

else:

total_val += (w2(CDF2(p, q, m2_sorted[i])) - w1(CDF2(p, q, m2_sorted[i]))) - w1(CDF2(p, q, m2_sorted[i]))
m2_sorted[i - 1]) * l1V1(m2_sorted[i], r2)

return total_val

# given q, r1, finds approx argmax of V1 over p
# currently set up for P1 having 2 strats
def argmax_V1(q, r1):
    argmax_set = set([])
    max_val = float("-inf")
    p = [1.0, 0.0]
    for i in xrange(0, bigN + 1):
        p[0] = float(i) / float(bigN)
        p[1] = 1.0 - p[0]
        if bigV1(p, q, r1) >= max_val:
            max_val = bigV1(p, q, r1)
    for i in xrange(0, bigN + 1):
        p[0] = float(i) / float(bigN)
        p[1] = 1.0 - p[0]
        if abs(bigV1(p, q, r1) - max_val) <= eps:
            argmax_set.add(tuple(p))

    return argmax_set

# given p, r2, maximizes V2 over q
# currently set up for P2 having 2 strats
def argmax_V2(p, r2):
    argmax_set = set([])
    max_val = float("-inf")
q = [1.0, 0.0]
for i in xrange(0, bigN+1):
    q[0] = float(i)/float(bigN)
    q[1] = 1.0 - q[0]
    if bigV2(p, q, r2) >= max_val:
        max_val = bigV2(p, q, r2)
for i in xrange(0, bigN+1):
    q[0] = float(i)/float(bigN)
    q[1] = 1.0 - q[0]
    if abs(bigV2(p, q, r2) - max_val) <= eps:
        argmax_set.add(tuple(q))
return argmax_set

# given a vector v and a finite set of vectors vecs, determines
# if v is in the convex hull of vecs
def co(v, vecs):
    vecs = list(vecs)

    t = simplex.Tableau([-1.0]*len(vecs))
    t.add_constraint([1.0]*len(vecs), 1.0)
    for i in xrange(0, len(vecs[0])):
        row_pos = []
        row_neg = []
        for j in xrange(0, len(vecs)):
            row_pos.append(vecs[j][i])
            row_neg.append(-1.0 * vecs[j][i])
        t.add_constraint(row_pos, v[i])
t.add_constraint(row_neg, -v[i])

t.solve()

wts = []
for i in xrange(0, len(vecs)):
    zeros = 0
    ones = 0
    val = 0.0
    for j in xrange(0, 2*len(vecs[0])+1):
        if t.rows[j][i+1] == 0.0:
            zeros += 1
        elif t.rows[j][i+1] == 1.0:
            ones += 1
        val = t.rows[j][len(t.rows[0])-1]
    if ones == 1 and zeros == 2*len(vecs[0]):
        wts.append(val)
    else:
        wts.append(0.0)

for i in xrange(0, len(v)):
    co_sum = 0.0
    for j in xrange(0, len(vecs)):
        co_sum += vecs[j][i] * wts[j]
    if abs(co_sum - v[i]) > eps:
        return False

return True
# finds sample points in simplices, tests them; needs to be
# written differently for each M x N game; M/N change numbers of loops
# currently set up for 2 x 2 games

def find_equil():
    p = [1.0, 0.0]
    q = [1.0, 0.0]
    for i in xrange(0, bigN+1):
        p[0] = float(i) / float(bigN)
        p[1] = 1.0 - p[0]
        for j in xrange(0, bigN+1):
            sys.stdout = old
            print("Checking... (" + str(i) + "," + str(j) + ")")
            q[0] = float(j) / float(bigN)
            q[1] = 1.0 - q[0]
            if co(p, argmax_V1(q, r1)) and co(q, argmax_V2(p, r2)):
                equils.append(deepcopy([p, q]))

def set_ref(r, p):
    ref_pt = 0.0
    if r == 0:
        # set r to min of payoffs for player p
        if p == 1:
ref_pt = min(ml_sorted)

elif p == 2:
    ref_pt = min(m2_sorted)

elif r == 1:  # set r to mean of payoffs for player p
    if p == 1:
        for s1 in xrange(1, num_strat1+1):
            for s2 in xrange(1, num_strat2+1):
                ref_pt += M1(s1, s2)

        ref_pt = ref_pt / float(num_strat1 * num_strat2)

    elif p == 2:
        for s1 in xrange(1, num_strat1+1):
            for s2 in xrange(1, num_strat2+1):
                ref_pt += M2(s1, s2)

        ref_pt = ref_pt / float(num_strat1 * num_strat2)

elif r == 2:  # set r to max of payoffs for player p
    if p == 1:
        ref_pt = max(ml_sorted)

    elif p == 2:
        ref_pt = max(m2_sorted)

return ref_pt

fill_payoffs()

ml_sorted = sort_payoffs(1)
m2_sorted = sort_payoffs(2)

r1 = 2.5
r2 = 0.5
A1 = 2
A2 = 2
K1 = 1
K2 = 1
lam1 = 1
lam2 = 2

start_time = time.time()
for e in xrange(3, 7, 3):
    eps = float(e)/100.0
    for r in xrange(1, 2):
        r1 = set_ref(r, 1);
        r2 = set_ref(r, 2);
        for lam in xrange(1, 2):
            lam1 = float(lam);
            lam2 = float(lam);
            for k in xrange(-1, 2, 2):
                K1 = float(k);
                K2 = float(k);
                for A in xrange(2, 4):
                    A1 = float(A);
                    A2 = float(A);
                    f = file('SeltenChmura_PT
                             -EB_Game3.txt', 'a')
sys.stdout = f
equils = []
print("Selten & Chmura
Game 3")
print("r1 = " + str(r1) +
  " r2 = " + str(r2))
print("K1 = " + str(K1) +
  " K2 = " + str(K2))
print("A1 = " + str(A1) +
  " A2 = " + str(A2))
print("lam1 = " + str(lam1) + " lam2 = " +
  str(lam2))
print("bigN = " + str(
  bigN) + " eps = " +
  str(eps))
find_equil()
f.close()

Python2.7 Simplex Algorithm:

from __future__ import division
from numpy import *

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class Tableau:

def __init__(self, obj):
    self.obj = [1] + obj
    self.rows = []
    self.cons = []

def add_constraint(self, expression, value):
    self.rows.append([0] + expression)
    self.cons.append(value)

def _pivot_column(self):
    low = 0
    idx = 0
    for i in range(1, len(self.obj) - 1):
        if self.obj[i] < low:
            low = self.obj[i]
            idx = i
    if idx == 0: return -1
    return idx

def _pivot_row(self, col):
    rhs = [self.rows[i][col] for i in range(len(self.rows))]
    lhs = [self.rows[i][-1] for i in range(len(self.rows))]
    ratio = []
    for i in range(len(rhs)):
if \( \text{lhs}[i] \leq 0.0001 \):
    ratio.append(99999999 * abs(max(rhs)))
    continue
ratio.append(rhs[i]/lhs[i])
return argmin(ratio)

def display(self):
    print('
', matrix([self.obj] + self.rows))

def _pivot(self, row, col):
    e = self.rows[row][col]
    self.rows[row] /= e
    for r in range(len(self.rows)):
        if r == row: continue
    self.obj = self.obj - self.obj[col]*self.rows[row]

def _check(self):
    if min(self.obj[1:-1]) >= 0: return 1
    return 0

def solve(self):
    # build full tableau
    for i in range(len(self.rows)):
        self.obj += [0]
ident = [0 for r in range(len(self.rows))]
ident[i] = 1
self.rows[i] += ident + [self.cons[i]]
self.rows[i] = array(self.rows[i], dtype=float)
self.obj = array(self.obj + [0], dtype=float)

# solve
#self.display()
while not self._check():
    c = self._pivot_column()
    r = self._pivot_row(c)
    self._pivot(r, c)
    #print '\npivot column: %s\npivot row: %s'%(c+1,r+2)
    #self.display()

# if __name__ == '__main__':

    ""
    max z = 2x + 3y + 2z
    st
      2x + y + z <= 4
      x + 2y + z <= 7
      z <= 5
      x,y,z >= 0
    ""

# t = Tableau([-1,-1,-1])
#t.add_constraint([1, 1, 1], 1)
#t.add_constraint([-1, -1, -1], -2)
#t.add_constraint([0, 0, 0], 0.5)
#t.add_constraint([1, 1, 0], 0.25)
#t.add_constraint([0, 0, 1], 0.25)
#t.add_constraint([0, 0, 0], -0.5)
#t.add_constraint([-1, -1, 0], -0.25)
#t.add_constraint([0, 0, -1], -0.25)
#t.solve()

Diagnostics Charts for Regression of Data against NE and Best-Fitting eCVE:

Data partial correlation with Best-Fit eCVE after Controlling for NE and Points 1,19,24,31:
Partial correlation of 0.4361496 with $p = 7.270236e-05$.

Data partial correlation with NE after Controlling for Best-Fit eCVE and Points 1,19,24,31:
Partial correlation of 0.1937954 with $p = 0.1058918$.

Diagnostics Charts for Regression of Data against NE and Best-Fitting ePT-EB:
Data partial correlation with Best-Fit ePT-EB after Controlling for NE and Points 3,24,31,33: Partial correlation of 0.4555381 with $p = 2.80708e-05$.

Data partial correlation with NE after Controlling for Best-Fit ePT-EB and Points 3,24,31,33: Partial correlation of 0.2399349 with $p = 0.04306831$. 

Figure C.2:
Philip (Daniel) Leclerc is a US citizen, and was born June 17, 1986, in the sunny US state of California. A year hence, he and his family relocated to the east coast, living variously in New Jersey, Pennsylvania, and Richmond, Virginia. He received his B. A. from Christopher Newport University (CNU) in 2009, graduating Summa Cum Laude with a double major in Mathematical Economics and Psychology, and was also graduated from the CNU Honors Program. In his freshman year, he received CNU’s prestigious Klich Award, a $2,000 award given to the student with the highest GPA in his class; in his sophomore and junior years, he received $1,000 Outstanding Student awards from the CNU Honors program, one of five students in each year to receive these awards. In his fourth year at CNU, Leclerc served as a Rotary Ambassadorial Scholar to the east African nation of Tanzania, where he studied Kiswahili, mathematics, culture and history at the University of Dar es Salaam. Leclerc has volunteered avidly throughout his studies; he co-founded a service and political awareness student organization, Citizens of the World, in his sophomore year of undergraduate studies, and through this organization speakers about the conflict in Darfur, Sudan, as well as volunteered and coordinated other volunteering efforts with the local community of displaced emigrants from Darfur. During his graduate studies, Leclerc has served as a member of the service fraternity, Alpha Phi Omega, which requires at least 30 hours of community service per semester; he has been especially active in working with a local dog shelter, B.A.R.K. In addition to his academic and volunteer work, Leclerc has been active in a number of honors’ societies and extracurricular organization, including: The Captain’s Log, CNU’s student newspaper; and Phi Alpha Delta, the international pre-law fraternity.

Leclerc has taught a single, summer section of Calculus 1, differential calculus, in 2012. By all accessible accounts, both student and teacher enjoyed the experience, and found it deeply rewarding.
Leclerc has previously published (with co-author and former advisor Dr. Laura Mc Lay) a manuscript, “Modeling equity for allocating public resources” in the collection of works, *Community-Based Operations Research* (2014). Currently, he (and co-author Dr. Jason Merrick) have a version of chapter 2 of the present work under review with *Management Science*; chapters 3-4 are also expected to be submitted, though their destinations are not yet known. A final manuscript, “Application of the Matching Law to a Management Problem,” has been submitted for review to the *European Journal of Operations Research.*