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Series Solutions of Polarized Gowdy Universes

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Series Solutions of Polarized Gowdy Universes

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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By Doniray Brusaferro, M.S.

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

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Einstein's field equations are a system of ten partial differential equations. For a special class of spacetimes known as Gowdy spacetimes, the number of equations is reduced due to additional structure of two dimensional isometry groups with mutually orthogonal Killing vectors. In this thesis, we focus on a particular model of Gowdy spacetimes known as the polarized $\mathbb{T}^3$ model, and provide an explicit solution to Einstein's equations.

Chapter 1

Introduction

Einstein's field equations are a system of ten partial differential equations given by:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = \frac{8\pi G}{c^4} T_{\mu\nu},$$

where μ and ν range over the indicies $\{0, 1, 2, 3\}$. Such a system of partial differential equations can be inherently difficult to solve; however a particular class of spacetimes, known as Gowdy spacetimes, greatly reduce the number of equations in the system to a more manageable size by enforcing symmetry in the underlying structure. Gowdy spacetimes are broadly categorized into groups based on their topological structure, for example the torus $\mathbb{S}^1 \times \mathbb{S}^2$, the 3-torus $\mathbb{T}^3$, or the 3-sphere which is defined as the unit sphere in four dimensions given by:

$$\mathbb{S}^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \sum_{i=1}^{4} x_i^2 = 1\}$$

Endowing each small neighborhood of a topological structure with a map into Euclidean space that is homeomorphism and is smooth whenever it is defined over a region that transitions from one mapping to another is what allows us to perform calculus on the new underlying structure, called a manifold. We then generalize the concept of the inner product from Euclidean space to the fundamental forms on the tangent space of a manifold, which allows for the familiar geometric notions of lengths and angles between vectors to be well defined in this new structure.

In examining the strong cosmic censorship hypothesis, Hans Ringström derived a metric on the collection of $\mathbb{T}^3$ Gowdy spacetimes using Fuchsian methods. Such methods describe properties of solutions of certain differential equations without giving an explicit expression for the solution. Solving this system using the method of separation of variables gives an explicit solution that satisfies Einstein's field equations for the given metric.

The solution that arises requires the use of the Fourier Series:

$$g(t,\theta) = \sum_{n\in\mathbb{Z}} c_n(t)e^{in\theta} = \frac{a_0}{2} + \sum_{n\in\mathbb{N}} a_n(t)\cos(n\theta) + b_n(t)\sin(n\theta).$$

with the usual Fourier coefficients

$$c_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t,\theta)e^{-in\theta}d\theta,$$

$$a_n(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t,\theta)\cos(n\theta)d\theta, \quad \text{and } b_n(t) = \tfrac{1}{\pi}\int_{-\pi}^{\pi} g(t,\theta)\sin(n\theta)d\theta;$$

and also with the Bessel functions $J_\alpha(t)$ and $Y_\alpha(t)$, which are the solutions to the ordinary differential equation given by:

$$t^2\frac{d^2}{dt^2}f(t) + t\frac{d}{dt}f(t) + (t^2-\alpha^2)f(t) = 0$$

where $\alpha \in \mathbb{C}$ is called the order of the Bessel function. The use of these functions also adds to the simplicity of the calculations, as derivatives of the Bessel functions with respect to t are linear combinations of Bessel functions of order $\alpha+1$, and derivative of the complex Fourier series with respect to θ is another complex Fourier series. More explicitly,

$$\frac{d}{dt}J_0(t) = -J_1(t), \quad \tfrac{d}{dt}J_1(t) = \tfrac{1}{2}(J_0(t) - J_2(t)), \quad \text{and } \frac{d}{d\theta}e^{in\theta} = ine^{in\theta}.$$

In theoretical physics, T-duality describes the relation between two physical theories consisting of distinct spacetime geometries. In a manner similar to duality from functional analysis, T-duality applies the concept of duality to physics, highlighting that two physical models can be related in a nontrivial manner. This subject plays a critical role in string theory, with the quintessential example being the attempt at describing how the relationship between a string in one dimension traveling on a circle of radius R relates to another one dimensional string traveling on a circle with radius $\tilde{R} = \frac{1}{R}$.

This thesis will begin witha review of the definitions necessary from Differential Geometry. We introduce the notion of a chart, which allows a neighborhood of a point on a manifold to be mapped into Euclidean space with a mapping of coordinates, and define a manifold in terms of a topological set and its charts that satisfy a collection of properties. We then move on to a brief description of the class of Gowdy spacetimes, before focusing specifically on the polarized $\mathbb{T}^3$ subclass. We then come up with an explicit solution for the metric in $\mathbb{T}^3$ Gowdy universes using Fourier analysis and the method of separation of variables. Finally, we will conclude by using the Büscher rules to calculate the dual metric, and the resulting Ricci tensor components.

Chapter 2

Einstein Manifolds

To begin, it is necessary to have some basic knowledge in the study of Differential Geometry. Much of the following material has been discussed in do Carmo's text [2], as wells as Gowdy's course in Gravitation [5].

Definition 1. *Let* $(\mathcal{M}, \tau_{\mathcal{M}})$ *be a second countable Hausdorff topological space. An* ***atlas*** *is a collection of maps* $\mathcal{A} = \{\{\varphi_i\}_{i\in\mathbb{I}} \mid \varphi_i : U_i \to \mathbb{R}^n\}$ *for some index set* $\mathbb{I}$ *such that:*

- $\forall p \in \mathcal{M}, \exists U_i \in \tau_{\mathcal{M}}$ *with* $p \in U_i$, *and* $\mathcal{M} = \bigcup_{i\in\mathbb{I}} U_i$.
- $\varphi_i : U_i \to \mathbb{R}^n$ *is a homeomorphism onto its image* $\varphi_i(U_i)$.
- $\forall i, j \in \mathbb{I}$, $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$ *is a diffeomorphism where defined.*

Each mapping $\varphi_i : U_i \to \mathbb{R}^n$ represents local coordinates in Euclidean space. Such a mapping is called a **coordinate chart.** An atlas, combined with a topological space, forms a structure on the set that is smooth, differentiable, and of dimension n. This structure allows for the familiar notions of calculus to apply.

Definition 2. *A topological space* $\mathcal{M}$ *together with an atlas forms a smooth* ***manifold****.*

As an example, consider the unit circle, $\mathbb{S}^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$. One open set of the circle can be defined as $U_1 = \mathbb{S}^1 \setminus \{(1,0)\}$, and the map $\varphi_1(x_1, x_2) = \frac{x_2}{x_1 - 1}$ with

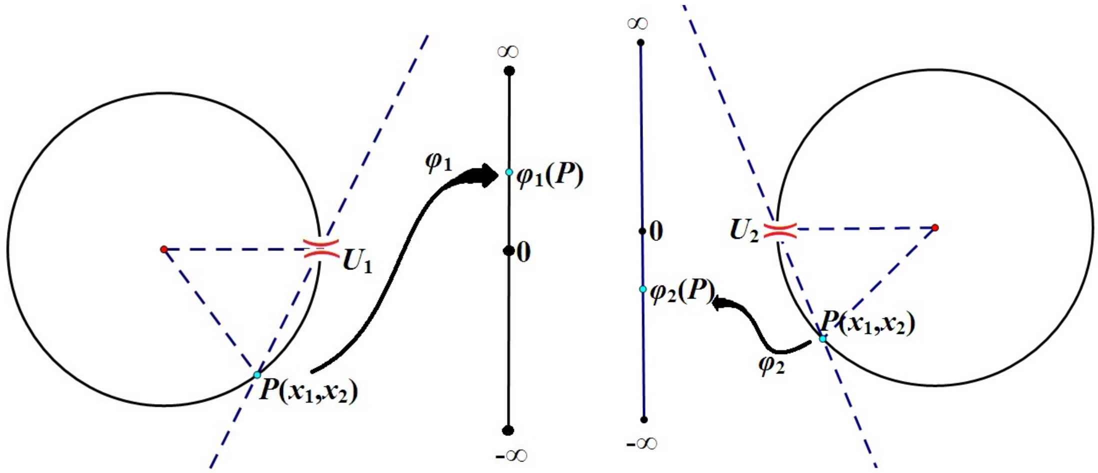

Figure 2.1: Mapping open subsets of $\mathbb{S}^1$ onto open subsets of $\mathbb{R}$

$\varphi_1(x_1, x_2) \in (-\infty, \infty)$. To complete the circle, a second open set can be constructed by $U_2 = \mathbb{S}^1 \setminus \{(-1, 0)\}$, and accompanying map $\varphi_2(x_1, x_2) = \frac{x_2}{x_1+1}$, with $\varphi_2(x_1, x_2) \in (-\infty, \infty)$. Thus the manifold of $\mathbb{S}^1$ can be given by the atlas $\mathcal{A} = \{(U_1, \varphi_1), (U_2, \varphi_2)\}$ as defined above.

Proposition 1. *Let $(\mathcal{M}, \{\varphi_i\}_{i\in\mathbb{I}})$ be an m-dimensional manifold, and let $(\mathcal{N}, \{\psi_i\}_{i\in\mathbb{I}})$ be and n-dimensional manifold. Then, $(\mathcal{M} \times \mathcal{N}, \{\varphi_i \times \psi_j\}_{i,j\in\mathbb{I}})$ defines a manifold.*

Proof. Let $p \in \mathcal{M}$ and $q \in \mathcal{N}$ be points in their manifolds. Since $\mathcal{M}$ is a manifold, there is an open set U_i in the topology of $\mathcal{M}$ that contains the point p, such that $\varphi_i : U_i \to \mathbb{R}^m$ is a homeomorphism for some $i \in \mathbb{I}$. Similarly, since $\mathcal{N}$ is a manifold, then there exists some open set V_j in the topology of $\mathcal{N}$ that contains q, such that $\psi_j : V_j \to \mathbb{R}^n$ is also a homeomorphism for some $j \in \mathbb{I}$.

Now, define $(\varphi_i \times \psi_j)(x, y) : U_i \times V_j \to \mathbb{R}^{m+n}$ by

$$\big(\varphi_i \times \psi_j\big)(x, y) := \big(\varphi_i(x), \psi_j(y)\big);$$

Since φ_i and ψ_j are both homeomorphisms, then both are continuous and their inverses exist and are also continuous. Thus, since each of the components in $\varphi_i \times \psi_j$ is continuous, then $\varphi_i \times \psi_j$ is also continuous. Also, because the inverses of φ_i and ψ_j exist and are continuous, this gives that $(\varphi_i \times \psi_j)^{-1}(x, y) := (\varphi_i^{-1}(x), \psi_j^{-1}(y))$ exists and is also continuous. Finally, note that $\varphi_i \times \psi_j$ is also smooth since each of its components are smooth. So $\varphi_i \times \psi_j$ forms an atlas on $\mathcal{M} \times \mathcal{N}$, thus $\mathcal{M} \times \mathcal{N}$ is a manifold. $\square$

Another structure of interest is the 3-torus, $\mathbb{T}^3$, which is constructed by taking the Cartesian product of 3 unit circles. That is, $\mathbb{T}^3 \cong \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. The shape of $\mathbb{T}^3$ is similar to that of a donut, with the inside hollowed out and where a cross section of the outside portion forms another torus.

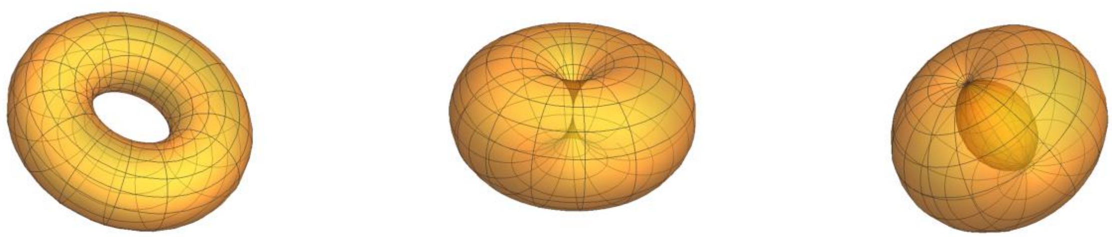

Figure 2.2: Ring Torus *(Left)*, Horn Torus *(Center)*, and Spindle Torus *(Right)*

Definition 3. *Let $\mathcal{M}$ be a manifold of dimension n, and let $\{f_i\}, \{g^i\} \in \mathbb{R}^{\mathbb{N}}$ be sequences of functions. Then, the notation where the index i appears in both a lower position and upper position is defined by:*

$$f_i g^i := \sum_{i=1}^{n} f_i g^i = f_1 g^1 + f_2 g^2 + \ldots + f_n g^n$$

is known as ***Einstein's summation convention****.*

Definition 4. *Let $\mathcal{M}$ be a manifold. Then,* ***the tangent space of the manifold at p*** *(denoted $T_p\mathcal{M}$) is the set of vectors tangent to a differential parameterized curveγ at a point in the manifold,*

p. *That is,*

$$T_p\mathcal{M} = \{\gamma'(t_0) \mid \gamma(t_0) = p \textit{ for some } t_0 \in \mathbb{R} \textit{ and some parametrized curve } \gamma\}$$

The definition of the tangent space at p forms a plane that is tangent to the manifold at the point p. This definition does not depend on the parameterization of γ since it was chosen to be arbitrary beyond that it passes through p; it is simply the collection of vectors that passes through a point on a manifold.

Definition 5. *Let $\mathcal{M}$ be a manifold of dimension n. A **metric** is a function $g_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ with the following properties:*

- $g_p(aX_p + bY_p, Z_p) = ag_p(X_p, Z_p) + bg_p(Y_p, Z_p)$, $\forall p \in \mathcal{M}$ *and* $\forall X_p, Y_p, Z_p \in T_p\mathcal{M}$ ***(bilinear)***
- $g_p(X_p, Y_p) = g_p(Y_p, X_p)$, $\forall p \in \mathcal{M}$ *and* $\forall X_p, Y_p \in T_p\mathcal{M}$ ***(symmetric)***
- $\forall X_p \in T_p\mathcal{M} \setminus \{0\}$, $\exists Y_p \in T_p\mathcal{M} \setminus \{0\}$ *such that* $g_p(X_p, Y_p) \neq 0$ $\forall p \in \mathcal{M}$. ***(nondegenerate)***
- $\forall U \in \tau_{\mathcal{M}}$, *and* $\forall X, Y$ *vector fields on* U, $g(X, Y)(p) = g_p(X_p, Y_p)$ *is smooth.*

The metric of a manifold generalizes the properties of the dot product to differential geometry. It should then be of no surprise that the dot product of any two vectors in the tangent space of a manifold is itself a metric. In a manner similar to how the dot product functions in vector spaces, we can use the metric to define lengths of tangent vectors, and the angles between any two. However, the analogy lacks one additional property that the definition of a metric tensor does not carry. First, to ensure that we arrive to a consistent definition of the metric tensor, we impose the requirement that g_p be a smooth function as we vary through points $p \in M$. We denote the metric tensor g as the collection of all g_p such that g_p is smooth for all points $p \in M$.

Definition 6. *Let M be a manifold and let g_p be a metric on M. If $g_p(X_p, X_p) > 0$ for all nonzero $X_p \in T_pM$ and $\forall p \in \mathcal{M}$, then g_p is said to be **positive definite**.*

By equipping our metric tensor with the property of positive definiteness, we complete the generalization of the dot product to differential geometry. This allows one to solve problems of differential geometry using the familiar methods from Euclidean geometry, and also gives rise to a special class of metric tensors.

Definition 7. *A metric g_p that is positive definite for all $p \in M$ is said to be a* ***Riemannian metric****. A smooth manifold paired with a Riemannian metric is called a* ***Riemannian manifold****.*

When a smooth manifold is equipped with a smooth positive definite metric, it allows for the familiar notions of geometry and calculus to carry forward into a branch known as Riemannian geometry. Einstein would later use language of Riemannian geometry to formulate the general theory of relativity.

Definition 8. *The* ***signature*** *of a metric, g, is the list of positive and negative eigenvalues of the metric tensor.*

A metric's signature can be written as a list of the signs of its eigenvalues. In the example of a 4 dimensional Riemannian manifold, since the metric is positive definite, its signature is written as $(+++ +)$. However, in modeling spacetime, Riemannian manifolds are generalized on by relinquishing the requirement to be positive definite. As a result, the time coordinate takes on a different sign than the spatial coordinates, and leads to a special subclass of pseudo-Riemannian manifolds.

Definition 9. *A metric with signature $(-+++)$ or $(+---)$ is called a* ***Lorentzian metric****. A smooth manifold paired with a Lorentzian metric is called a* ***Lorentzian manifold****.*

When a coordinate frame is applied to the manifold, then the metric takes the form of a symmetric matrix (g_{ij}). Due to the combined properties of the metric, the matrix (g_{ij}) must be nonsingular. As a result, the matrix for the metric can be inverted.

Definition 10. *Let $\mathcal{M}$ be a manifold, let $X = \{X_1, X_2, ..., X_n\}$ be a local basis of vector fields of a sufficiently small neighborhood of TM, and let (g_{ij}) be a metric tensor on M. The* ***inverse metric*** *(g^{ij}) is an n by n matrix such that $(g_{ij})(g^{jk}) = \delta_i^k$.*

With the above definitions, we are able to freely associate the components of a vector field to the components of its covector field in a process from tensor calculus known as "lowering the index." Similarly, we can associate the components of a covector field to the components of its vector field in a process know as "raising the index." The above process can be formalized by the following:

Remark 1. *Let* $\{v^1, v^2, ..., v^n\}$ *be the components of a vector field, and let* $\{a_1, a_2, ..., a_n\}$ *be the components of its covector field. Then* $g_{ij}v^j = a_i$*, and* $g^{ij}a_j = v^i$.

Given a metric g and two vectors $X, Y \in TM$, we ask that $g(X, Y)$ remain invariant as those vectors are parallel transported along a curve. With this condition, an $n \times n \times n$ array is formed to describe how the metric behaves as it transitions between tangent bundles. This will intrinsically lead to an extension of the directional derivative to vector fields.

Definition 11. *The* ***covariant derivative*** ∇ *of a generalized field* w *in the direction of a tangent vector field* v *is a function that maps* v *and* w *to a tangent vector (denoted* $\nabla_v w$*), with the following properties:*

- $\nabla_{au+bv}w = a\nabla_u w + b\nabla_v w$, $\forall a, b \in \mathbb{R}$ ***(locally linear)***
- $\nabla_v(w + z) = \nabla_v w + \nabla_v z$, *for any generalized field* z. ***(additive)***
- $\nabla_v(wz) = (\nabla_v w)\, z + w\, (\nabla_v z)$ ***(Leibniz's rule)***

When given a set of basis vectors for the generalized field w, the covariant derivative can be expressed in the terms of the basis vectors. The coefficients of these expanded derivatives form an $n \times n \times n$ array that represents the connection of the metric across tangent bundles.

Definition 12. *The* ***Christoffel symbol of the second kind***, Γ^i_{jk}, *is the collection of coefficients such that* $\nabla_{e_j} e_k = \Gamma^i_{jk} e_i$.

Using only the metric and its inverse, the Christoffel symbols can be calculated by:

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{jk} \right)$$

The covariant derivative extends the concept of the derivative to tensor and form fields. A noteworthy corollary to our definition of Christoffel symbols of the second kind is that it is symmetric in its lower indexes.

Corollary 1. *Let $\mathcal{M}$ be a manifold, and let g be a metric. Then, $\Gamma^i_{jk} = \Gamma^i_{kj}$.*

A proof of this follows simply by exchanging j and k in the calculations of the Christoffel symbols of the second kind using the metric. The first two terms are simply a reordering, while the third remains the same due to the symmetry of the metric. Specific details of this proof are left to the reader. The symmetry of the lower indexes reduces the number of total calculations needed for the Christoffel symbols of the second kind.

Definition 13. *Let $\mathcal{M}$ be a smooth manifold, let X, Y be smooth vector fields, and let $f \in \mathcal{C}^\infty(M)$ be a smooth function. The **commutator** of the vector fields X and Y is the vector field $[X, Y]$ such that:*

$$[\nabla_X, \nabla_Y]f = X(Y(f)) - Y(X(f))$$

For a vector field Z, the commutator of the covariant derivative acting on Z is $[\nabla_X, \nabla_Y]Z$, but this is not locally linear in any of the vector fields. However, the mapping $\mathcal{R}$ that takes a triple of vector fields (X, Y, Z) defined by:

$$\mathcal{R}(X, Y)Z := [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$

is locally linear, and is therefore a tensor field. The above mapping is called the *Riemannian curvature tensor*. Using the metric, Christoffel symbols, and Riemannian curvature tensor, the difference in the geometry of a Riemannian manifold from the usual Euclidean space of dimension n can be measured.

Definition 14. *The **Ricci curvature tensor** R_{ij} can be written expressly in terms of Christoffel symbols of the second kind by*

$$R_{ij} = \partial_k \Gamma^k_{ji} - \partial_j \Gamma^k_{ki} + \Gamma^k_{kl}\Gamma^l_{ji} - \Gamma^k_{jl}\Gamma^l_{ki}$$

An alternative definition for the Ricci tensor is given by the trace of the Riemannian curvature tensor. The Ricci curvature tensor is essential to the calculation of Einstein's field equations. Before delving in, additional terms are needed.

Definition 15. *The **Ricci scalar** is a real valued function R written in terms of the Ricci tensor and is given by*

$$R = g^{ij} R_{ij}$$

The Ricci scalar can be written explicitly in terms of Christoffel symbols of the second kind and the inverse metric g^{ij}, by substituting the explicit equation for the Ricci tensor. Thus we can write R as:

$$R = g^{ij}\left(\partial_k \Gamma^k_{ji} - \partial_j \Gamma^k_{ki} + \Gamma^k_{kl}\Gamma^l_{ji} - \Gamma^k_{jl}\Gamma^l_{ki}\right)$$

The final definition before getting to Einstein's field equations introduces a tensor that combines the Ricci tensor with the Ricci scalar.

Definition 16. *The **Einstein tensor** G_{ij} is a second rank tensor defined by*

$$G_{ij} = R_{ij} - \frac{1}{2} R\left(g_{ij}\right)$$

The Einstein tensor allows us to condense Einstein's field equations into a compact formula.

$$G_{ij} + \Lambda g_{ij} = \frac{8\pi G}{c^4} T_{ij} \tag{2.1}$$

where G is Newton's gravitational constant, c is the speed of light, Λ is the cosmological constant, and T_{ij} is the stress-energy tensor. The terms on the left hand side of Einstein's field equations represents the curvature of spacetime, while the right hand side represents the matter and energy of spacetime. Originally, Einstein did not have the term with the cosmological constant in his derived field equations. Thus, to arrive at the original field equations of general relativity, we set $\Lambda = 0$ in (2.1).

In the context of a vacuum, which is just a region of empty space, there is consequentially no matter or energy within the region to affect the stress-energy tensor. Thus the vacuum field equations are those of (2.1) with T_{ij}, and subsequently the entire right hand side, equal to 0. In [6, Equation (2), page 12], Ringström develops a working definition for the metric in $\mathbb{T}^3$-Gowdy derived from the metric given by its coordinate system. The generalized metric is given by:

$$g = t^{-1/2}e^{\lambda/2}(-dt^2 + d\theta^2) + t[e^P d\sigma^2 + 2e^P Q d\sigma d\delta + (e^P Q^2 + e^{-P})d\delta^2].$$

In polarized $\mathbb{T}^3$ Gowdy spacetimes, additional conditions force that Q in the equation above vanishes, reducing the metric in $\mathbb{T}^3$ Gowdy universes to:

$$g = t^{-1/2}e^{\lambda/2}(-dt^2 + d\theta^2) + t[e^P d\sigma^2 + e^{-P} d\delta^2].$$

where P and λ are functions of t and θ.

Chapter 3

Gowdy Spacetimes

Originally introduced by Gowdy in [4], Gowdy spacetimes are a class of symmetric, four dimensional Einstein manifolds with two-dimensional Abelian isometry groups that admits no torsion (that is, all twist constraints vanish). Additionally, these manifolds have an associated topology, which further divides these models into subclasses. Specifically, these topological models give rise to coordinate systems, which are used further to derive the metric on each associated topology. These restrict the cases of Gowdy universes to the following topological models:

- $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^2$
- $\mathbb{R} \times \mathbb{S}^3$
- $\mathbb{R} \times \mathbb{T}^3$

We focus on the subclass of polarized Gowdy spacetimes with underlying topology $\mathbb{R} \times \mathbb{T}^3$, and begin by defining the coordinates as $x_0 = t, x_1 = \theta, x_2 = \sigma,$ and $x_3 = \delta$.

The metric for polarized $\mathbb{T}^3 \times \mathbb{R}$ is given by Ringström to be:

$$\begin{aligned} g &= t^{-1/2}e^{\lambda/2}(-dt^2 + d\theta^2) + t(e^P d\sigma^2 + e^{-P} d\delta^2) \\ &= \begin{pmatrix} -\sqrt{\frac{e^\lambda}{t}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{e^\lambda}{t}} & 0 & 0 \\ 0 & 0 & te^P & 0 \\ 0 & 0 & 0 & te^{-P} \end{pmatrix} \end{aligned}$$

where both P and λ are functions of t and θ.[6] That is, $P := P(t, \theta)$ and $\lambda := \lambda(t, \theta)$. The signature of this metric is given by $(- + ++)$, and thanks to the diagonal form of the metric, its inverse can be easily calculated to be:

$$\begin{aligned} g^{-1} &= t^{1/2}e^{-\lambda/2}(-dt^2 + d\theta^2) + t^{-1}(e^{-P} d\sigma^2 + e^P d\delta^2) \\ &= \begin{pmatrix} -\sqrt{\frac{t}{e^\lambda}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{t}{e^\lambda}} & 0 & 0 \\ 0 & 0 & \frac{e^{-P}}{t} & 0 \\ 0 & 0 & 0 & \frac{e^P}{t} \end{pmatrix} \end{aligned}$$

That the inverse is also a diagonal matrix lends to the ease in calculating the Christoffel symbols. Begin by noting that $\Gamma^i_{kl} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right)$ is zero if $i \neq m$. Thus, the above simplifies to $\Gamma^i_{kl} = \frac{1}{2}g^{ii}\left(\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i}\right)$. Because of the diagonal form metric, the first partial derivative term will be zero when $i \neq k$, the second term will be zero when $i \neq l$, and the third partial derivative term will be zero whenever $k \neq l$. Due to Corollary 1, we will omit the case of $i \neq l$, since this is equivalent to $i \neq k$.

Now, if $i = k$, then the Christoffel symbols take the form $\Gamma^i_{il} = \frac{1}{2}g^{ii}\left(\frac{\partial g_{ii}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^i} - \frac{\partial g_{il}}{\partial x^i}\right)$. So, if $i \neq l$, then the last two partial derivatives will be zero. Thus, we are left with $\Gamma^i_{il} = \frac{1}{2}g^{ii}\left(\frac{\partial g_{ii}}{\partial x^l}\right)$. Here, if l is either 2 or 3, then we are taking derivatives of a function with respect to variables not present in the function; hence these would be zero.

Observe that if $i = l$, then $\Gamma^i_{ii} = \frac{1}{2}g^{ii}\left(\frac{\partial g_{ii}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^i}\right) = \frac{1}{2}g^{ii}\left(\frac{\partial g_{ii}}{\partial x^i}\right)$. If i is either 2 or 3, then we get zero for the same reason as above. In the case where $k = l$, we have $\Gamma^i_{kk} = \frac{1}{2}g^{ii}\left(\frac{\partial g_{ik}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^k} - \frac{\partial g_{kk}}{\partial x^i}\right)$. If $i = k$, then we get the case from above, so consider when $i \neq k$. Then, the first two partial derivatives will be zero, leaving $\Gamma^i_{kk} = -\frac{1}{2}g^{ii}\left(\frac{\partial g_{kk}}{\partial x^i}\right)$. Note here that if i is 2 or 3, then the result will again be 0.

Thus, the only Christoffel symbols that must be calculated have been reduced to: Γ^0_{00}, Γ^0_{01}, Γ^0_{11}, Γ^0_{22}, Γ^0_{33}, Γ^1_{00}, Γ^1_{10}, Γ^1_{11}, Γ^1_{22}, Γ^1_{33}, Γ^2_{20}, Γ^2_{21}, Γ^3_{30}, and Γ^3_{31}. The calculations for these coefficients has also been greatly simplified. They are as follows:

$\Gamma^0_{00} = \frac{1}{2}g^{00}(\frac{\partial g_{00}}{\partial x^0}) = \frac{1}{2}(-\frac{t^{1/2}}{e^{\lambda/2}})\frac{\partial}{\partial t}(-\frac{e^{\lambda/2}}{t^{1/2}}) = \frac{1}{4}(\lambda_t - \frac{1}{t})$;

$\Gamma^0_{01} = \frac{1}{2}g^{00}(\frac{\partial g_{00}}{\partial x^1}) = \frac{1}{2}(-\frac{t^{1/2}}{e^{\lambda/2}})\frac{\partial}{\partial \theta}(-\frac{e^{\lambda/2}}{t^{1/2}}) = \frac{1}{4}\lambda_\theta$;

$\Gamma^0_{11} = -\frac{1}{2}g^{00}(\frac{\partial g_{11}}{\partial x^0}) = \frac{1}{2}(-\frac{t^{1/2}}{e^{\lambda/2}})\frac{\partial}{\partial t}(\frac{e^{\lambda/2}}{t^{1/2}}) = \frac{1}{4}(\lambda_t - \frac{1}{t})$;

$\Gamma^0_{22} = -\frac{1}{2}g^{00}(\frac{\partial g_{22}}{\partial x^0}) = \frac{1}{2}(-\frac{t^{1/2}}{e^{\lambda/2}})\frac{\partial}{\partial t}(te^P) = \frac{1}{2}t^{3/2}e^{P-\frac{\lambda}{2}}(P_t + \frac{1}{t})$;

$\Gamma^0_{33} = -\frac{1}{2}g^{00}(\frac{\partial g_{33}}{\partial x^0}) = \frac{1}{2}(-\frac{t^{1/2}}{e^{\lambda/2}})\frac{\partial}{\partial t}(te^{-P}) = -\frac{1}{2}t^{3/2}e^{-P-\frac{\lambda}{2}}(P_t - \frac{1}{t})$;

$\Gamma^1_{00} = -\frac{1}{2}g^{11}(\frac{\partial g_{00}}{\partial x^1}) = -\frac{1}{2}(\frac{t^{1/2}}{e^{\lambda/2}})\frac{\partial}{\partial \theta}(-\frac{e^{\lambda/2}}{t^{1/2}}) = \frac{1}{4}\lambda_\theta$;

$\Gamma^1_{10} = \frac{1}{2}g^{11}(\frac{\partial g_{11}}{\partial x^0}) = \frac{1}{2}(\frac{t^{1/2}}{e^{\lambda/2}})\frac{\partial}{\partial t}(\frac{e^{\lambda/2}}{t^{1/2}}) = \frac{1}{4}(\lambda_t - \frac{1}{t})$;

$\Gamma^1_{11} = \frac{1}{2}g^{11}(\frac{\partial g_{11}}{\partial x^1}) = \frac{1}{2}(\frac{t^{1/2}}{e^{\lambda/2}})\frac{\partial}{\partial \theta}(\frac{e^{\lambda/2}}{t^{1/2}}) = \frac{1}{4}\lambda_\theta$;

$\Gamma^1_{22} = -\frac{1}{2}g^{11}(\frac{\partial g_{22}}{\partial x^1}) = -\frac{1}{2}(\frac{t^{1/2}}{e^{\lambda/2}})\frac{\partial}{\partial \theta}(te^P) = -\frac{1}{2}t^{3/2}e^{P-\frac{\lambda}{2}}P_\theta$;

$\Gamma^1_{33} = -\frac{1}{2}g^{11}(\frac{\partial g_{33}}{\partial x^1}) = -\frac{1}{2}(\frac{t^{1/2}}{e^{\lambda/2}})\frac{\partial}{\partial \theta}(te^{-P}) = \frac{1}{2}t^{3/2}e^{-P-\frac{\lambda}{2}}P_\theta$;

$\Gamma^2_{20} = \frac{1}{2}g^{22}(\frac{\partial g_{22}}{\partial x^0}) = \frac{1}{2}(\frac{e^{-P}}{t})\frac{\partial}{\partial t}(te^P) = \frac{1}{2}(P_t + \frac{1}{t})$;

$\Gamma^2_{21} = \frac{1}{2}g^{22}(\frac{\partial g_{22}}{\partial x^1}) = \frac{1}{2}(\frac{e^{-P}}{t})\frac{\partial}{\partial \theta}(te^P) = \frac{1}{2}P_\theta$;

$\Gamma^3_{30} = \frac{1}{2}g^{33}(\frac{\partial g_{33}}{\partial x^0}) = -\frac{1}{2}(\frac{e^P}{t})\frac{\partial}{\partial t}(te^{-P}) = -\frac{1}{2}(P_t - \frac{1}{t})$;

$\Gamma^3_{31} = \frac{1}{2}g^{33}(\frac{\partial g_{33}}{\partial x^1}) = -\frac{1}{2}(\frac{e^P}{t})\frac{\partial}{\partial \theta}(te^{-P}) = -\frac{1}{2}P_\theta$.

Moving to calculating the Ricci curvature tensor, the formula is expanded over both k and l, and the nonzero components of the Ricci tensor evaluate to the following:

$R_{00} = \frac{1}{4}(\lambda_{\theta\theta} + \frac{1}{t}\lambda_t - \lambda_{tt} - 2P_t^2)$;

$R_{10} = \frac{1}{4t}(\lambda_\theta - 2tP_tP_\theta)$;

$R_{11} = -\frac{1}{4}(\lambda_{\theta\theta} - \frac{1}{t}\lambda_t - \lambda_{tt} + 2P_\theta^2)$;

$R_{22} = \frac{1}{2}t^{3/2}e^{P-\frac{\lambda}{2}}(P_{tt} + \frac{1}{t}P_t - P_{\theta\theta});$

$R_{33} = \frac{1}{2}t^{3/2}e^{-P-\frac{\lambda}{2}}(P_{tt} + \frac{1}{t}P_t - P_{\theta\theta}).$

For Einstein's equations to be satisfied, the above expressions for the components of the Ricci tensor must all vanish. Clearly, if $P_{tt} + \frac{1}{t}P_t - P_{\theta\theta} = 0$, then components R_{22} and R_{33} will vanish. Additionally, if $\lambda_\theta = 2tP_tP_\theta$, then components R_{10} and R_{01} vanish as well. What is not readily apparent is how to get R_{00} and R_{11} to vanish. Note that if R_{00} and R_{11} are summed, this gives: $R_{00} + R_{11} = \frac{1}{2t}\big(\lambda_t - t(P_t^2 + P_\theta^2)\big)$. Thus, to ensure these final two components vanish, we must have $\lambda_t = t(P_t^2 + P_\theta^2)$.

Chapter 4

Series Solutions to Ringström's Equations

As seen from the calculations of the Ricci tensor, and subsequently from Ringström's publication [6], the metric for $\mathbb{T}^3$ Gowdy universes depend on the functions $P(t, \theta)$ and $\lambda(t, \theta)$ to satisfy a system of partial differential equations. In particular, P and λ must satisfy:

$$\begin{aligned} \lambda_t &= t(P_t^2 + P_\theta^2) && (4.1) \\ \lambda_\theta &= 2tP_tP_\theta && (4.2) \\ P_{tt} + \frac{1}{t}P_t - P_{\theta\theta} &= 0 && (4.3) \end{aligned}$$

As the derivatives of λ are dependent on the derivatives of P, we will first solve this system for $P(t, \theta)$ using the method of separation of variables. A similar approach was taken by Banerjee and Date [1], however an explicit solution for λ is only implied. While separation of variables is known to work for a linear equation, like in the example of (4.3), it is not all obvious that it should work for the whole system, which is non-linear. In fact, this method does not work for the non-polarized case. Suppose we can write the function P as the product of two single-variable functions; that is $P(t,\theta) = T(t)\Theta(\theta)$, so that $P_t = T'(t)\Theta(\theta)$, $P_{tt} = T''(t)\Theta(\theta)$, and $P_{\theta\theta} = T(t)\theta''(\theta)$. These are then plugged into equation (4.3) to give:

$$T''(t)\Theta(\theta) + \frac{1}{t}T'(t)\Theta(\theta) - T(t)\Theta''(\theta) = 0.$$

which can then be rewritten as:

$$T''(t)\Theta(\theta) + \frac{1}{t}T'(t)\Theta(\theta) = T(t)\Theta''(\theta).$$

Next, divide both side of this equation by $P(t,\theta) = T(t)\Theta(\theta)$, which results in:

$$\frac{T''(t)}{T(t)} + \frac{1}{t}\frac{T'(t)}{T(t)} = \frac{\Theta''(\theta)}{\Theta(\theta)}.$$

For this statement to be equivalent regardless of any t or θ chosen, both sides of this equality must be equal to some arbitrary constant, c. That is:

$$\frac{T''(t)}{T(t)} + \frac{1}{t}\frac{T'(t)}{T(t)} = \frac{\Theta''(\theta)}{\Theta(\theta)} = -c^2;$$

which are just two second order homogeneous ordinary differential equations. Equating both to zero yields:

$$\Theta''(\theta) + c^2\Theta(\theta) = 0 \quad \text{and} \quad T''(t) + \frac{1}{t}T'(t) + c^2T(t) = 0;$$

The solution for $\Theta(\theta)$ is given by the complex exponential function. Thus,

$$\Theta(\theta) = Ae^{ci\theta};$$

The solution to $T(t)$ is given by the Bessel function of order 0. That is, $T(t) = BJ_0(ct) +$

$CY_0(ct)$. However, notice the function $Y_0(nt) \to -\infty$ as $t \to 0$. This function is necessary to properly represent a singularity in this universe; however, we shall exclude such unbounded solutions for the sake of simplicity, so that the function for $T(t)$ reduces to simply:

$$T(t) = BJ_0(ct)$$

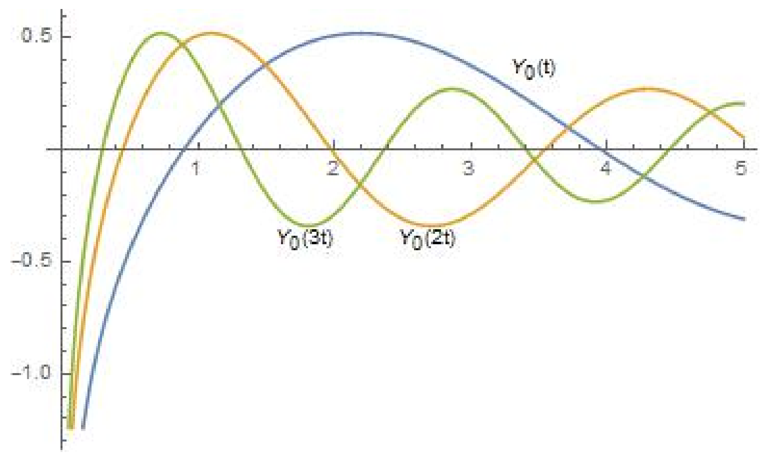

Figure 4.1: $Y_0(nt)$ for $n = 1, 2, 3$

We are then free to take linear combinations of the product of the two functions. Thus, a solution for $P(t, \theta)$ is given by:

$$P(t, \theta) = T(t)\Theta(\theta) = \sum_{n=-\infty}^{\infty} (A_n J_0(nt)) \left(B_n e^{in\theta}\right)$$

$$= \sum_{n=-\infty}^{\infty} D_n J_0(nt) e^{in\theta}$$

Note that derivatives of $P(t, \theta)$ with respect to t require derivatives of the Bessel function. Namely, $\frac{d}{dt} J_0(nt) = -nJ_1(nt)$, and $\frac{d}{dt} J_1(nt) = \frac{1}{2} n(J_0(nt) - J_2(nt))$. Due to the linearity of the derivative, we can exchange the summation with the derivative.

So, $\frac{d}{dt}P(t,\theta) = \frac{d}{dt}\left[\sum_{n=-\infty}^{\infty} D_n J_0(nt)e^{in\theta}\right] = \sum_{n=-\infty}^{\infty} D_n e^{in\theta}\frac{d}{dt}\left[J_0(nt)\right] =$

$$= \sum_{n=-\infty}^{\infty} D_n e^{in\theta}(-nJ_1(nt)) = -\sum_{n=-\infty}^{\infty} nD_n J_1(nt)e^{in\theta};$$

$\frac{d^2}{dt^2}P(t,\theta) = \frac{d^2}{dt^2}\left[\sum_{n=-\infty}^{\infty} D_n J_0(nt)e^{in\theta}\right] = \sum_{n=-\infty}^{\infty} D_n e^{in\theta}\frac{d^2}{dt^2}\left[J_0(nt)\right] =$

$$= \sum_{n=-\infty}^{\infty} D_n e^{in\theta}\left[-\frac{1}{2}n^2(J_0(nt) - J_2(nt))\right] = -\sum_{n=-\infty}^{\infty} \frac{1}{2}n^2 D_n\left[J_0(nt) - J_2(nt)\right]e^{in\theta};$$

and $\frac{d^2}{d\theta^2}P(t,\theta) = \frac{d^2}{d\theta^2}\left[\sum_{n=-\infty}^{\infty} D_n J_0(nt)e^{in\theta}\right] = \sum_{n=-\infty}^{\infty} D_n J_0(nt)\frac{d^2}{d\theta^2}\left[e^{in\theta}\right] =$

$$= \sum_{n=-\infty}^{\infty} D_n J_0(nt)\left[-n^2 e^{in\theta}\right] = -\sum_{n=-\infty}^{\infty} n^2 D_n J_0(nt)e^{in\theta}.$$

Placing the results into Ringström's equation yields the following:

$$P_{tt} + \frac{1}{t}P_t - P_{\theta\theta} =$$

$$= -\sum_{n=-\infty}^{\infty} \tfrac{1}{2}n^2 D_n\left[J_0(nt) - J_2(nt)\right]e^{in\theta} - \tfrac{1}{t}\sum_{n=-\infty}^{\infty} nD_n J_1(nt)e^{in\theta} + \sum_{n=-\infty}^{\infty} n^2 D_n J_0(nt)e^{in\theta}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2}D_n\left[n^2 J_0(nt) - \frac{2}{t}nJ_1(nt) + n^2 J_2(nt)\right]e^{in\theta} = \sum_{n=-\infty}^{\infty} \frac{1}{2}D_n\left[0\right]e^{in\theta} = 0.$$

Thus, $P(t,\theta)$ satisfies Ringström's equations. Using this constructed function, we can derive an explicit equation for $\lambda(t,\theta)$ as well. Starting with equation 4.2, we have:

$$\lambda_\theta = 2tP_tP_\theta = 2t\left(-\sum_{n=-\infty}^{\infty} nD_n J_1(nt)e^{in\theta}\right)\left(\sum_{m=-\infty}^{\infty} imD_m J_0(mt)e^{im\theta}\right) =$$

$$= -2t\sum_{m,n=-\infty}^{\infty} imnD_m D_n J_1(nt)J_0(mt)e^{i(m+n)\theta}.$$

This can be integrated with respect to θ as follows:

$$\int \lambda_\theta d\theta = \int \Big(-2t \sum_{m,n=-\infty}^{\infty} imnD_m D_n J_1(nt) J_0(mt) e^{i(m+n)\theta} \Big) d\theta =$$

$$= -2t \sum_{m,n=-\infty}^{\infty} imnD_m D_n J_1(nt) J_0(mt) \Big[\int e^{i(m+n)\theta} d\theta \Big]$$

$$= -2t \Bigg(\sum_{m \neq -n} imnD_m D_n J_1(nt) J_0(mt) \Big[-\frac{i}{m+n} e^{i(m+n)\theta} \Big] +$$

$$+ \sum_{m=-n} -in^2 D_{-n} D_n J_1(nt) J_0(nt)\theta \Bigg) + F(t);$$

So,

$$\lambda(t,\theta) = \sum_{m \neq -n} -\frac{2mn}{m+n} D_m D_n \big[tJ_1(nt) J_0(mt) \big] e^{i(m+n)\theta} + \sum_{m=-n} 2in^2 D_{-n} D_n \big[tJ_1(nt) J_0(nt) \big] \theta$$

$$+F(t);$$

where $F(t)$ is the function of t picked up through integrating λ_θ. Notice here that the second term vanishes. More specifically, note that this is being summed over all $n \in \mathbb{Z}$, so when $n = 0$ then the entire term is zero as well, thus this gives:

$$\sum_{n \in \mathbb{Z}} 2in^2 D_{-n} D_n J_1(nt) J_0(nt)\theta =$$

$$= \sum_{n>0} 2in^2 D_{-n} D_n J_1(nt) J_0(nt)\theta + \sum_{n<0} 2in^2 D_{-n} D_n J_1(nt) J_0(nt)\theta =$$

$$= \sum_{n>0} 2in^2 D_{-n} D_n J_1(nt) J_0(nt)\theta + \sum_{n>0} 2i(-n)^2 D_{-n} D_n J_1(-nt) J_0(-nt)\theta =$$

$$= \sum_{n>0} 2in^2 D_{-n} D_n J_1(nt) J_0(nt)\theta - \sum_{n>0} 2in^2 D_{-n} D_n J_1(nt) J_0(nt)\theta = 0;$$

as J_0 is an even function, and J_1 is an odd function. Now, taking the derivative of the function we found for λ with respect to t yields:

$$\frac{d}{dt}\lambda(t,\theta) = \frac{d}{dt}\left[\sum_{m\neq -n} -\frac{2mn}{m+n}D_m D_n\left[tJ_1(nt)J_0(mt)\right]e^{i(m+n)\theta} + F(t)\right] =$$

$$= \sum_{m\neq -n} -\frac{2mn}{m+n}D_m D_n e^{i(m+n)\theta}\frac{d}{dt}\left[tJ_1(nt)J_0(mt)\right] + \frac{d}{dt}\left[F(t)\right] =$$

$$= \sum_{m\neq -n} -\frac{2mn}{m+n}D_m D_n e^{i(m+n)\theta}\left[t\big(nJ_0(mt)J_0(nt) - mJ_1(mt)J_1(nt)\big)\right] + F'(t),$$

So, for the constructed function of λ, its derivative with respect to t is:

$$\lambda_t = \sum_{m\neq -n} -\frac{2mn}{m+n}D_m D_n e^{i(m+n)\theta}\left[t\big(nJ_0(mt)J_0(nt) - mJ_1(mt)J_1(nt)\big)\right] + F'(t).$$

Equation (4.1) gave an explicit solution for this partial derivative of λ. Specifically:

$$\lambda_t = t(P_t^2 + P_\theta^2) = t\left((-\sum_{n=-\infty}^{\infty} nD_n J_1(nt)e^{in\theta})^2 + (\sum_{n=-\infty}^{\infty} inD_m J_0(nt)e^{in\theta})^2\right) =$$

$$= t\left(\sum_{m,n=-\infty}^{\infty} mnD_m D_n J_1(nt)J_1(mt)e^{i(m+n)\theta} - \sum_{m,n=-\infty}^{\infty} mnD_m D_n J_0(nt)J_0(mt)e^{i(m+n)\theta}\right),$$

So that

$$\lambda_t = -t\left(\sum_{m,n=-\infty}^{\infty} mnD_m D_n\big(J_0(nt)J_0(mt) - J_1(nt)J_1(mt)\big)e^{i(m+n)\theta}\right);$$

Re-indexing so that this expression looks similar to the one derived gives:

$$\lambda_t = -t\left(\sum_{m\neq -n} mnD_m D_n\big(J_0(nt)J_0(mt) - J_1(nt)J_1(mt)\big)e^{i(m+n)\theta} + \right.$$

$$\left. + \sum_{m=-n} -n^2 D_{-n}D_n\big(J_0(nt)^2 + J_1(nt)^2\big)\right).$$

Comparing this expression with the one derived from λ_θ gives:

$$\sum_{m\neq -n} -mnD_mD_n\big(tJ_0(nt)J_0(mt) - tJ_1(nt)J_1(mt)\big)e^{i(m+n)\theta} +$$

$$+\sum_{m=-n} n^2D_{-n}D_n\big(tJ_0(nt)^2 + tJ_1(nt)^2\big) =$$

$$= \sum_{m\neq -n} -\frac{2mn}{m+n}D_mD_ne^{i(m+n)\theta}\Big[t\big(nJ_0(mt)J_0(nt) - mJ_1(mt)J_1(nt)\big)\Big] + F'(t);$$

Now, observe that

$$\sum_{m\neq -n} -\frac{2mn}{m+n}D_mD_ne^{i(m+n)\theta}\Big[t\big(nJ_0(mt)J_0(nt) - mJ_1(mt)J_1(nt)\big)\Big] =$$

$$= \sum_{m\neq -n} -mntD_mD_ne^{i(m+n)\theta}\left[\frac{2}{m+n}\big(nJ_0(mt)J_0(nt) - mJ_1(mt)J_1(nt)\big)\right] =$$

$$= \sum_{m\neq -n} -mntD_mD_n\big(J_0(nt)J_0(mt) - J_1(nt)J_1(mt)\big)e^{i(m+n)\theta};$$

if and only if

$$\frac{2}{m+n}\big(nJ_0(mt)J_0(nt) - mJ_1(mt)J_1(nt)\big) = J_0(nt)J_0(mt) - J_1(nt)J_1(mt).$$

Observe that through reindexing, we have that

$$\sum_{m\neq -n} \frac{2}{m+n}(nJ_0(mt)J_0(nt) - mJ_1(mt)J_1(nt)) =$$

$$= \sum_{m\neq -n} \frac{2}{m+n}\left(\frac{1}{2}\right)\big(nJ_0(mt)J_0(nt) - mJ_1(mt)J_1(nt) + mJ_0(nt)J_0(mt) - nJ_1(nt)J_1(nt)\big) =$$

$$= \sum_{m\neq -n} \frac{2}{m+n}\left(\frac{1}{2}\right)\big((m+n)(J_0(mt)J_0(nt) - J_1(mt)J_1(nt)\big) =$$

$$\sum_{m\neq -n}\big(J_0(mt)J_0(nt)-J_1(mt)J_1(nt)\big);$$

which leaves that $F'(t)=\sum_{m=-n} n^2D_{-n}D_n\big(tJ_0(nt)^2+tJ_1(nt)^2\big)$.
Finally, take the integral of $F'(t)$ with respect to t to get a function for $F(t)$. It is important to note $\int tJ_0(nt)^2dt=\frac{1}{2}t^2\big(J_0(nt)^2+J_1(nt)^2\big)$ and $\int tJ_1(nt)^2dt=\frac{1}{2}t^2\big(J_1(mt)^2-J_0(nt)J_2(nt)\big)$, so

$$\int F'(t)dt=\int\Big[\sum_{m=-n} n^2D_{-n}D_n\big(tJ_0(nt)^2+tJ_1(nt)^2\big)\Big]dt=$$

$$=\sum_{m=-n} n^2D_{-n}D_n\int\big[tJ_0(nt)^2+tJ_1(nt)^2\big]dt=$$

$$=\sum_{m=-n} n^2D_{-n}D_n\Big[\int tJ_0(nt)^2dt+\int tJ_1(nt)^2dt\Big]=$$

$$=\frac{1}{2}t^2\sum_{m=-n} n^2D_{-n}D_n\big[J_0(nt)^2+2J_1(nt)^2-J_0(nt)J_2(nt)\big]+E,$$

which gives λ the form:

$$\lambda(t,\theta)=\sum_{m\neq -n}-\frac{2mn}{m+n}D_mD_n\big[tJ_1(nt)J_0(mt)\big]e^{i(m+n)\theta}+$$

$$+\frac{1}{2}t^2\sum_{m=-n} n^2D_{-n}D_n\big[J_0(nt)^2+2J_1(nt)^2-J_0(nt)J_2(nt)\big]+E;$$

Thus the functions which satisfy the given polarized $\mathbb{T}^3$ metric have the form:

$$P(t,\theta)=\sum_{n\in\mathbb{Z}}D_nJ_0(nt)e^{in\theta} \tag{4.4}$$

$$\begin{aligned}\lambda(t,\theta) &= -2t\sum_{\substack{m,n\in\mathbb{Z}\\ m\neq -n}}\frac{mn}{m+n}D_mD_n\big[J_1(nt)J_0(mt)\big]e^{i(m+n)\theta}+\\ &+\frac{1}{2}t^2\sum_{n\in\mathbb{Z}}n^2D_{-n}D_n\big[J_0(nt)^2+2J_1(nt)^2-J_0(nt)J_2(nt)\big]+E \end{aligned}\tag{4.5}$$

where E is the constant term picked up through integration.

Chapter 5

T-Duality

In this final section, we examine the effect of the metric in $\mathbb{T}^3$ Gowdy universes under its dual. The end goal is to determine whether the metric in the dual system will also satisfy Einstein's field equations in a manner similar to the metric given by Ringström. This chapter begins the analysis by computing several of the key components using the constructed dual metric. Using the Büscher rules, we can construct a dual metric from the original. We begin by clearly defining the Büscher rules.

Definition 17. *Let M be a manifold, let s be a coordinate on a circle T and let $\tilde{s}$ be the dual coordinate on the dual circle $\widetilde{T}$. Consider a generalized Riemannian metric (g, b) on $M \times T$ of the form*

$$\begin{aligned} g &= g_0(ds)^2 + g_1 \odot ds + g_2 \\ b &= b_1 \wedge ds + b_2 \end{aligned}$$

where $\odot$ is the symmetric tensor product, (g_2, b_2) is a Riemannian metric on M, g_1, b_1 are 1-forms on M and g_0 is a non-vanishing function on M. The T-dual of (g, b) is the generalized

Riemannian metric $(\tilde{g}, \tilde{b})$ *on* $M \times \widetilde{T}$ *defined by*

$$\begin{aligned}
\tilde{g} &= \frac{(d\tilde{s})^2}{g_0} - \frac{b_1}{g_0} \odot d\tilde{s} + g_2 + \frac{(b_1)^2 - (g_1)^2}{g_0} \\
\tilde{b} &= b_2 + \frac{g_1}{g_0} \wedge (b_1 - d\tilde{s})
\end{aligned}$$

Applying these rules to the original metric changes the third coordinate, and yields the following dual metric:

$$\tilde{g} = t^{-1/2} e^{\lambda/2}(-dt^2 + d\theta^2) + (\frac{e^{-P}}{t}) d\sigma^2 + t e^{-P} d\delta^2$$

$$= \begin{pmatrix} -\sqrt{\frac{e^{\lambda}}{t}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{e^{\lambda}}{t}} & 0 & 0 \\ 0 & 0 & \frac{e^{-P}}{t} & 0 \\ 0 & 0 & 0 & te^{-P} \end{pmatrix}$$

Thanks to the diagonalized matrix, the inverse is again simple to evaluate.

$$\tilde{g}^{-1} = t^{1/2} e^{-\lambda/2}(-dt^2 + d\theta^2) + t e^{P} d\sigma^2 + \frac{e^{P}}{t} d\delta^2$$

$$= \begin{pmatrix} -\sqrt{\frac{t}{e^{\lambda}}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{t}{e^{\lambda}}} & 0 & 0 \\ 0 & 0 & te^{P} & 0 \\ 0 & 0 & 0 & \frac{e^{P}}{t} \end{pmatrix}$$

So, the calculations for the Christoffel symbols of he second kind follows the same procedure as the original metric. Thus, the dual Christoffel symbols are given by:

$\tilde{\Gamma}^0_{00} = \frac{1}{4}(\lambda_t - \frac{1}{t})$;

$\tilde{\Gamma}^0_{10} = \frac{1}{4}\lambda_\theta$;

$\tilde{\Gamma}^0_{11} = \frac{1}{4}(\lambda_t - \frac{1}{t})$;

$\tilde{\Gamma}^0_{22} = -\frac{1}{2}t^{3/2}e^{(-P-\frac{\lambda}{2})}(P_t - \frac{1}{t});$

$\tilde{\Gamma}^0_{33} = -\frac{1}{2}t^{-1/2}e^{(-P-\frac{\lambda}{2})};$

$\tilde{\Gamma}^1_{00} = \frac{1}{4}\lambda_\theta;$

$\tilde{\Gamma}^1_{10} = \frac{1}{4}(\lambda_t - \frac{1}{t});$

$\tilde{\Gamma}^1_{11} = \frac{1}{4}\lambda_\theta;$

$\tilde{\Gamma}^1_{22} = \frac{1}{2}t^{-1/2}e^{(-P-\frac{\lambda}{2})}P_\theta;$

$\tilde{\Gamma}^1_{33} = \frac{1}{2}t^{3/2}e^{(-P-\frac{\lambda}{2})}P_\theta;$

$\tilde{\Gamma}^2_{20} = -\frac{1}{2}(P_t + \frac{1}{t});$

$\tilde{\Gamma}^2_{21} = -\frac{1}{2}P_\theta;$

$\tilde{\Gamma}^3_{30} = -\frac{1}{2}(P_t - \frac{1}{t});$

$\tilde{\Gamma}^3_{31} = -\frac{1}{2}P_\theta.$

The components of the dual Ricci tensor are given by:

$\tilde{R}_{00} = \frac{1}{4}\big(- P_t\lambda_t - P_\theta\lambda_\theta - 2P_t^2 + \frac{1}{t}P_t + 4P_{tt} + \lambda_{\theta\theta} - \lambda_{tt} - \frac{3}{t^2}\big);$

$\tilde{R}_{10} = -\frac{1}{4}\big(P_t\lambda_\theta + 2P_tP_\theta + \lambda_tP_\theta - \frac{1}{t}P_\theta - 4tP_{t\theta}\big);$

$\tilde{R}_{11} = \frac{1}{4}(-P_\theta\lambda_\theta - P_t\lambda_t - 2P_\theta^2 + 4P_{\theta\theta} + \frac{1}{t}P_t - \lambda_{\theta\theta} + \lambda_{tt} + \frac{1}{t^2});$

$\tilde{R}_{22} = \frac{1}{2}t^{-1/2}e^{-P-\frac{\lambda}{2}}(P_{\theta\theta} + \frac{1}{t}P_t - P_{tt} + P_t^2 - P_\theta^2 + \frac{1}{t^2});$

$\tilde{R}_{33} = -\frac{1}{2}t^{3/2}e^{-P-\frac{\lambda}{2}}(P_{tt} + \frac{1}{t}P_t - P_{\theta\theta} + P_\theta^2 - P_t^2 + \frac{1}{t^3}).$

What remains for future research is to determine if these components of the dual Ricci tensor can be further simplified by a factor of proportionality that will allow the dual metric to satisfy Einstein's vacuum equations with the same system of partial differential equations provided by Ringström. Such a result would confirm the equivalence of these physical systems, resulting in an alternate metric for Gowdy $\mathbb{T}^3$ universes.

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Vita

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